SOME FIXED POINT THEOREMS FOR WEAKLY COMPATIBLE MAPPINGS SATISFYING AN IMPLICIT RELATION.

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Abstract

The purpose of this paper is to establish some new common fixed point theorems for weakly compatible mappings. In this paper we prove some common fixed point theorems for six mappings under the condition of weakly compatible mappings satisfying an implicit relation. We point out that for the existence of the fixed point, continuity of the function is not required.

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1. Introduction

Jungck [1] introduced generalized commuting mappings called compatible mappings, which are more general than the concept of weakly commuting mappings [11]. Many authors have proved common fixed point theorems for compatible mappings for this we refer to [1]-[3], [4], [6], [13], [5], [14]. On other hand Wong et al. [18] proved some fixed point theorems on expansion mappings which correspond to some contractive mappings. Rhoades [10] generalized the above results for pairs of mappings. Some theorems on unique fixed point for expansion mappings were proved by Popa [7]. Popa [8] further extended results [7], [10] for compatible mappings.

Definition 1.1: Let S and T be two self-mappings of a metric space (X,d). Sessa [11] defines S and T to be weakly commuting if

$$d(STx, TSx) \leq d(Tx, Sx) \quad \text{for all } x \text{ in } X.$$

Jungck [1] defines S and T to be compatible if

$$\lim_{n \to \infty} d(STx_n, TSx_n) = 0$$

whenever, $\{x_n\}$ is a sequence in X such that

$$\lim_{n \to \infty} Sx_n = \lim_{n \to \infty} Tx_n = x \quad \text{for some } x \text{ in } X.$$

Clearly, commuting maps are weakly commuting and weakly commuting mappings are compatible but implications are not reversible.

Definition 1.2:

Two mappings S and T are said to be weakly compatible if they commute at their coincidence point. In 1999, Popa [9] proved some fixed point theorems for compatible mappings satisfying an implicit relation. Sharma and Deshpande [15] proved some common fixed point theorems for compatible mappings in Banach spaces, satisfying an implicit relation. Sharma and Rahurikar [17] improved result of Sharma and Choubey [16] and proved the theorem for five mappings under the condition of compatible mappings. In this paper, we prove common fixed point theorems for weakly compatible mappings in Banach spaces, satisfying an implicit relation. We extend the results of Sharma and Rahurikar [17].

2. Implicit relations:

Let ϕ be the set of all real continuous functions $\phi\ (t_1, \ldots, t_6)$**:** $R_+^6 \to R$ satisfying the following conditions:

ϕ_1: ϕ is non-increasing in variable t_6.

ϕ_2: there exists $h \in (0, 1)$ such that for every $u, v \geq 0$ with

(ϕ_a)**:** $\phi\ (u, v, v, u, (1/2)(u + v), 0) \leq 0$

Or

(ϕ_b): $\phi\ (u, v, u, v, (1/2)(u + v), u + v) \leq 0$

We have $u \leq hv$

ϕ_3: $\phi\ (u, u, 0, 0, 0, u) > 0$ for all $u > 0$

Example 2.1: $\phi\ (t_1, \ldots, t6) = t_1 - k \max \left\{ t_2, \frac{t_3 + t_4}{2}, t_5, \frac{t_6}{2} \right\}$, where $k \in (0, 1)$.

ϕ_1: obviously

ϕ_2: Let be $u > 0$, $\phi\ (u, v, v, u, (1/2)(u + v), 0) = u - k \max \{v, (1/2)(u + v), (1/2)(u + v), 0\}$.

If $u \geq v$ then $u \leq ku < u$, a contradiction. Thus $u < v$ and $u \leq kv = hv$, where $h = k \in (0, 1)$. Similarly,

if $u > 0$ then $\phi(u, v, u, v, (1/2)(u + v), u + v) \leq 0$ imply $u \leq hv$. If $u = 0$ then $u \leq hv$.

ϕ_3: $\phi(u, u, 0, 0, 0, u) = u(1 - k) > 0$. For all $u > 0$.

Example 2.2: $\phi(t_1,\ldots, t_6) = t^2_1 - k \max\{t^2_2, t_3 t_5, (1/2) t_4 t_6\}$, where $k \in (0,1)$.

ϕ_1: obviously.

ϕ_2: Let be $u > 0$, $\phi(u, v, v, u, (1/2)(u + v), 0) = u^2 - k \max\{v^2, (1/2) v (u + v), 0\} \leq 0$. If $u \geq v$ then $u \leq \sqrt{k}\, u < u$, a contradiction. Thus $u < v$ and $u \leq \sqrt{k}\, v = hv$, where $h = \sqrt{k} \in (0,1)$. Similarly,

if $u > 0$ then $\phi(u, v, u, v, (1/2)(u + v), u + v) \leq 0$. Implies $u \leq hv$. If $u = 0$ then $u \leq hv$

ϕ_3: $\phi(u, u, 0, 0, u) = u^2 (1- k) > 0$, for all $u > 0$.

Example 2.3: $\phi(t_{1,\ldots,} t_6) = t^2_1 - a\, t^2_2 - t_3 \left(\dfrac{t_3 + t_4}{2} - t_5\right) - bt_6 (t_2 - t_4)$, where $a, b \in (0, 1/2)$.

ϕ_1: obviously

ϕ_2: Let be $u > 0$, $\phi(u, v, v, u, (1/2)(u + v), 0) = u^2 - av^2 \leq 0$, which implies $u \leq \sqrt{a}\, v = hv$, where $h = \sqrt{a} \in (0, 1)$. Similarly,

if $u > 0$ then $\phi(u, v, u, v, (1/2)(u + v), u + v) \leq 0$. Which implies $u \leq hv$. If $u = 0$ then $u \leq hv$.

ϕ_3: $\phi(u, u, 0, 0, 0, u) = u^2 [1- (a + b)] > 0$, for all $u > 0$.

Sharma and Choubey [16] proved the following results for Banach spaces.

Theorem A:

Let $(X, \|\cdot\|)$ be a Banach space and $A, B, S, T: X \to X$ be four mappings satisfying the conditions.

(i) $\phi(\|Ax - By\|, \|Sx - Ty\|, \|Sx - Ax\|, \|Ty - By\|, (1/2)(\|Sx - Ax\| + \|Ty - By\|), \|Ty - Ax\|) \leq 0$. For all x,y in X, where $\phi \in \Phi$,

(ii) $A(X) \subset T(X)$ and $B(X) \subset S(X)$

(iii) $\{A, S\}$ and $\{B, T\}$ are compatible pairs.

Then A, B, S and T have a unique common fixed point.

Theorem B: Let S, T and $\{A_i\}_{i \in N}$ be mappings from a Branch space (X, ‖ . ‖) into itself such that

(i) $A_2(X) \subset S(X)$ and $A_1(X) \subset T(X)$

(ii) the pairs $\{A_1, S\}$ and $\{A_2, T\}$ are compatible.

(iii) the inequality

$\phi(\| A_i x - A_{i+1} y \|, \| Sx - Ty \|, \| Sx - A_i x \|, \| Ty - A_{i+1} y \|, (1/2) (\| Sx - A_i x \| + \| Ty - A_{i+1} y \|), \| Ay - A_i x \|) \leq 0$

holds for each $x,y \in X$, for all $i \in N$ and $\phi \in \Phi$.

Then S, T and $\{A_i\}_{i \in N}$ have a unique common fixed point.

Sharma and Rahurikar [17] proved the following results for Branch spaces.

Theorem C: Let (X, ‖ . ‖) be Banach spaces and A, B, S, T, P: $X \to X$ be five mappings satisfying the following conditions:

(i) $\phi(\| Px - Py \|, \| STx - ABy \|, \| STx - Px \|, \| ABy - Py\|, ½(\| STx - Px \| + \| ABy - Py\|), \| ABy - Px \|) \leq 0$. For all x,y in X, where $\phi \in \Phi$,

(ii) $P(X) \subset AB(X)$, $P(X) \subset ST(X)$,

(iii) {P, ST} and {P, AB} are compatible pairs.

(iv) AB = BA, ST = TS, PT = TP, PB = BP.

Then, A, B, S, T and P have a unique common fixed point.

3. **Main results**

Theorem 3.1: Let (X, ‖ . ‖) be Banach space and A, B, S, T, P, Q: $X \to X$ be six mappings satisfying the following conditions:

(3.1) $\phi(\| Px - Qy \|, \| STx - ABy \|, \| STx - Px \|, \|ABy-Qy \|, 1/2(\| STx - Px \| + \|ABy-Qy\|), \| ABy - Px \|) \leq 0$. For all x,y in X, where $\phi \in \Phi$.

(3.2) $P(X) \subset AB(X)$, $Q(X) \subset ST(X)$,

(3.3) {Q, AB} and {P, ST} are weakly compatible.

(3.4) AB = BA, QB = BQ, PT = TP and ST = TS,

Then A, B, S, T, P and Q have a unique common fixed point.

Proof : By (3.2), since $P(X) \subset AB(X)$, for an arbitrary point $x_0 \in X$ there exists a point $x_1 \in X$ such that $Px_0 = ABx_1$. Since $Q(X) \subset ST(X)$, for this point $x_1 \in X$, we can choose a point

$\in X$ such that $Qx_1 = STx_2$ and so on. Inductively we can define a sequence $\{y_n\}$ in X such that

$$y_{2n} = Px_{2n} = ABx_{2n+1}$$

$$y_{2n+1} = Qx_{2n+1} = STx_{2n+2}, \text{ for every } n = 0,1,2,\ldots$$

By (3.1) we have

$\phi (\| Px_{2n} - Qx_{2n+1} \|, \| STx_{2n} - ABx_{2n+1}\|, \| STx_{2n} - Px_{2n} \|, \| ABx_{2n+1} - Qx_{2n+1}\|, (1/2) (\| STx_{2n} - Px_{2n} \| + \| ABx_{2n+1} - Qx_{2n+1} \|), \| ABx_{2n+1} - Px_{2n} \|) \leq 0$

$\phi (\| y_{2n} - y_{2n+1}\|, \| y_{2n-1} - y_{2n} \|, \| y_{2n-1} - y_{2n} \|, \| y_{2n} - y_{2n+1} \|, ½(\| y_{2n-1} - y_{2n} \| + \| y_{2n}- y_{2n+1} \|) , \| y_{2n} - y_{2n} \|) \leq 0$

$\phi(\| y_{2n} - y_{2n+1}\|, \| y_{2n-1} - y_{2n} \|, \| y_{2n-1} - y_{2n} \|, \| y_{2n} - y_{2n+1}\|, ½(\| y_{2n-1} - y_{2n} \| + \| y_{2n} - y_{2n+1}\|), 0) \leq 0$

By (ϕ_a) we have $\| y_{2n} - y_{2n+1} \| \leq h \| y_{2n-1} - y_{2n} \|$.Similarly by ϕ_1, we have

$$\| y_{2n-1} - y_{2n} \| \leq h \| y_{2n-2} - y_{2n-1} \|$$

and so

$$\| y_{2n}- y_{2n+1} \| \leq h^{2n} \| y_0 - y_1\| \qquad \text{for } n = 0, 1, 2\ldots$$

By routine calculations it follow that $\{y_n\}$ is a Cauchy sequence in X and hence it converges to a point z in X.

Consequently, sub sequences $\{Px_{2n}\}$, $\{Qx_{2n+1}\}$, $\{ABx_{2n+1}\}$ and $\{STx_{2n+1}\}$ of $\{y_n\}$ also converges to the point z. Since $Q(X) \subset ST(X)$, there exists a point $u \in X$ such that $STu = z$. Then using (3.1), we write.

$\phi (\| Pu - Qx_{2n+1}\|, \| Stu - ABx_{2n+1} \|, \| Stu – Pu \|, \| ABx_{2n+1} - Qx_{2n+1}\|, ½(\| Stu – Pu \| + \| ABx_{2n+1} - Qx_{2n+1}\|) , \| ABx_{2n+1} – Pu \|) \leq 0.$

Taking the limit as $n \to \infty$, we have

$\phi (\| Pu – z \|, \| z – z \|, \| z – Pu \|, \| z – z \|, \| z + Pu \| + \| z – z \| ,1/2 (\| z + Pu \| + \| z – z \|), \| z - Pu \|) \leq 0$

$\phi (\| Pu – z \| , 0 , \| z – Pu \| , 0 ,(½) \| z - Pu \| , \| z - Pu \|) \leq 0$

which implies by (ϕ_b), that $Pu = z$. Therefore $Pu = STu = z$. Similarly since $P(X) \subset$

AB(X), there exists a point $v \in X$ such that $ABv = z$. Then again using (3.1), we have

$\phi(\|Px_{2n} - Qv\|, \|STx_{2n} – ABv\|, \|STx_{2n} - Px_{2n}\|, \|ABv- Qv\|, 1/2 (\|STx_{2n} - Px_{2n}\| + \|ABv - Qv\|) , \|ABv - Px_{2n}\|) \le 0$

Taking the limit as $n \to \infty$, we have

$\phi (\|z – Qv\|, \|z – z\|, \|z – z\|, \|z – Qv\|,1/2 (\|z – z\| + \|z – Qv\|) , \|z – z\|) \le 0$

$\phi (\|z – Qv\|, 0, 0, \|z – Qv\|, (1/2) \|z – Qv\|, 0) \le 0$

which implies by (ϕ_a)) that Qv = z .Therefore ABv = Qv = z.Since the pair{P,ST} is weak compatible. Therefore P and ST commute at their coincidence point i.e. P (STu) = (ST)Pu or Pz = STz.

Similarly Q (ABv) = (AB)Qv, or Pz = STz. Similarly Q (ABv) = (AB) Qv or Qz = ABz.

Now, we prove that Pz = z, by (3.1), we write.

$\phi(\|Pz - Qx_{2n+1}\|, \|STz - ABx_{2n+1}\|, \|STz - z\|, \|ABx_{2n+1} - Qx_{2n+1}\|, ½ (\|STz - z\| + \|ABx_{2n+1} - Qx_{2n+1}\|), \|ABx_{2n+1} - Pz\|) \le 0$

Taking the limit $n \to \infty$, we have

$\phi(\|Pz – z\|, \|Pz -z\|, \|Pz – Pz\|, \|z –z\|, 1/2 (\|Pz – Pz\| + \|z – z\|), \|z – Pz\|) \le 0.$

$\phi (\|Pz – z\|, \|Pz – z\|, 0, 0, 0, \|z – Pz\|) \le 0$

which is a contradiction to (ϕ_3), if $\|Pz – z\| \neq 0$. Thus Pz = z. Therefore z = Pz = STz.

Now, we show that Qz = z, by (3.1), we write,

$\phi (\|Px_{2n} – Qz\|, \|STx_{2n} – ABz\|, \|STx_{2n} - Px_{2n}\|, \|ABz – Qz\|, 1/2(\|STx_{2n} - Px_{2n}\| + \|ABz – Qz\|), \|ABz - Px_{2n}\|) \le 0$

Taking the limit $n \to \infty$, we have

$\phi (\|z – Qz\|, \|z – Qz\|, \|z-z\|, \|Qz – Qz\|, 1/2 (\|z –z\| + \|Qz –Qz\|), \|Qz –z\| \le 0$

$\phi (\|z – Qz\|, \|z – Qz\|, 0, 0, 0, \|Qz – z\|) \le 0$

which is a contradiction to (ϕ_3), if $\|Qz – z\| \neq z$. Thus Qz = z. Hence Pz = STz = Qz = ABz = z. If putting x = z and y = Bz in (3.1) we write

$\phi (\|Pz - Q(Bz)\|, \|STz - AB(Bz)\|, \|STz – Pz\|, \|AB(Bz) - Q(Bz)\|, 1/2(\|STz – Pz\| + \|AB(Bz) - Q(Bz)\|), \|AB(Bz) – Pz\|) \le 0$

$\phi(\|z – Bz\|, \|z – Bz\|, \|z – z\|, \|Bz – Bz\|, ½(\|z –z\| + \|Bz – Bz\|), \|Bz – z\|) \le 0$

$\phi(\|z – Bz\|, \|z – Bz\|, 0, 0, 0, \|Bz – z\|) \le 0$

which is a contradiction to (ϕ_3) if $\|Bz – z\|) \neq 0$. Thus Bz = z. Since ABz = z, therefore Az = z. By putting x = Tz and y = z in (3.1) we write

$\phi(\| P(Tz) - Qz \|, \| ST(Tz) - ABz\|, \| ST(Tz) - P(Tz) \|, \| ABz - Qz \|, ½(\| ST(Tz) - P(Tz) \| + \| ABz - Qz \|), \| ABz - P(Tz) \|) \leq 0$

Taking the limit $n \to \infty$, we have

$\phi(\| Tz - z \|, \| Tz - z \|, \| Tz - Tz \|, \| z - z \|, ½(\| Tz - Tz \|+ \| z - z \|), \| z - Tz \|) \leq 0$

$\phi(\| Tz - z \|, \| Tz - z \|, 0,0,0,\| z - Tz \|) \leq 0$

which is a contradiction to (ϕ_3) if $\| Tz - z \| \neq 0$. Thus $Tz = z$. Since $STz = z$ therefore $Sz = z$. Hence $Az = Bz = Sz = Tz = Pz = Qz = z$. Thus z is a common fixed point of A, B, S, T, P and Q.

For uniqueness, let w ($z \neq w$) be another common fixed point of A, B, S, T, P and Q. Then using (3.1), we write

$\phi(\| Pz - Qw \|, \| STz - ABw\|, \| STz - Pz \|, \| ABw - Qw\|, 1/2(\| STz - Pz \| + \| ABw - Qw\|), \| ABw - Pz \|) \leq 0,$

$\phi(\| z - w \|, \| z - w \|, \| z - z \|, \| w - w \|, ½(\| z - z \| + \| w - w \|), \| w - z \|) \leq 0$

$\phi(\| z - w \|, \| z - w \|, 0,0,0, \| w - z \|) \leq 0$

which is a contradiction to (ϕ_3) if $\| z - w\| \neq 0$. Thus $z = w$.

Theorem 3.2: Let $(X, \| . \|)$ be a Banach space and A, B, S, T, P, Q: $X \to X$ be six mappings satisfying the conditions (3.1),(3.2), (3.4) and the following

(3.5) the pair {P, AB} is weakly compatible.

(3.6) $\| x - STx \| \leq \| x - ABx\|$ for all $x \in X$

Then A, B, S, T, P and Q have a unique common fixed point.

Corollary 3.1: If we put P = Q in Theorem 3.1 our theorem reduces to the result of Sharma and Rahurikar [17]

If we put $P = Q = \{P_i\}_{i \in N}$, we have the followings :

Theorem 3.3: Let A, B, S, T and $\{P_i\}_{i \in N}$, be mappings from a Banach space $(X, \| . \|)$ into itself such that.

(3.7) the inequality

$\phi(\| P_i x - P_i y \|, \| STx - ABy\|, \| STx - P_i x\|, \| ABy - P_i y \|, (1/2)(\| STx - P_i x \| + \| ABy - P_i y \|, \| ABy - P_i x \|) \leq 0$

holds for all $x, y \in X$, for all $i \in N$, where $\phi \in \Phi$, (3.8) $P_i(X) \subset AB(X) \cap ST(X)$,

(3.9) the pairs $\{P_i, ST\}$ and $\{P_i, AB\}$ are weakly compatible.

(3.10) $AB = BA$, $ST = TS$, $P_iT = TP_i$, $P_iB = BP_i$.

Then A, B, S, T and $\{P_i\}_{i\in N}$ have a unique common fixed point.

If we put P = Q = B = T = 1 in Theorem 3.1 and Theorem 3.3, we have the following

Corollary 3.2: Let $(X, \|\ .\ \|)$ be a Banach space and A, S, P: $X \to X$ be three mapping satisfying the conditions:

(3.12) $\phi(\| Px - Py \|, \| Sx - Ay\|, \| Sx - Px\|, \|Ay - Py \|, (1/2)(\| Sx - Px \| + \| Ay - Py\|), \| Ay - Px \|) \leq 0$

for all x, y in X,

where $\phi \in \Phi$,

(3.13) $P(X) \subset A(X) \cap S(X)$,

(3.14) the pairs $\{P, S\}$ and $\{P, A\}$ are weakly compatible

Then A, S, and P have a unique common fixed point.

Corollary 3.3: Let $(X, \|\ .\ \|)$ be a Banach space and A, S, P: $X\to X$ be three mappings satisfying the conditions (3.12), (3.13) and

(3.15) $\| x - Sx \| \leq \| x - Ax \|$, for all x in X,

(3.16) the pair $\{P, A\}$ is weak compatible.

Then A, S, and P have a unique common fixed point.

Corollary 3.4: Let A, S, and $\{P_i\}_{i\in N}$ be mappings from a Banach space $(X, \|\ .\ \|)$ into itself such that

(3.17) the inequality

$\phi(\| P_ix - P_iy \|, \| Sx - Ay \|, \| Sx - P_i x \|, \| Ay - P_iy \|, (1/2)(\| Sx - P_ix \| + \| Ay - P_iy \|, \|Ay - P_ix \| \leq 0$

holds for all $x, y \in X$ and for all $i\in N$,

where $\phi \in \Phi$,

(3.18) $P_i(X) \subset A(X) \cap S(X)$,

(3.19) the pairs $\{P_i, S\}$ and $\{P_i,T\}$ are weak compatible.

Then A, S and $\{P_i\}_{i\in N}$ have a unique common fixed point.

Corollary 3.5: Let A, S and $\{P_i\}_{i\in N}$, be mappings from a Banach space (X, $\|\cdot\|$) into itself satisfying the conditions (3.15), (3.17), (3.18) and (3.19) the pair {Pi, A} is weak compatible. Then A, S and $\{P_i\}_{i\in N}$ have a unique common fixed point If we put A = B = S = T = 1 in Theorem 3.1 and Theorem 3.3 we have

Corollary 3.6: Let (X, $\|\cdot\|$) be a Banach space and P: X $\to$ X and Q: X $\to$ X be mapping satisfying the following condition:

(3.20) $\phi(\| Px - Qy \|, \| x- y\|, \| x - Px \|,\| y - Qy \|, (1/2)(\| x - Px \| + \| y - Qy \|), \| y - Px \|) \leq 0$

for all x,y$\in$ X where $\phi \in \Phi$,

Then P and Q have a unique common fixed point.

Corollary 3.7: Let $\{P_i\}_{i\in N}$, be mappings from a Banach space (X, $\|\cdot\|$) into itself such that

(3.21) the inequality

$\phi(\| P_ix - P_iy \|,\| x -y \|, \| x - P_i x \|, \|y - P_iy \|, (1/2)(\| x - P_ix \| + \| y - P_iy \|,\| y-P_ix \| \leq 0$

holds for all x, y $\in$ X and for all i$\in$N, where $\phi \in \Phi$.

Then $\{P_i\}_{i\in N}$ has a unique common fixed point.

Theorem 3.1 and Examples 2.1 to 2. 3 imply the following:

Corollary 3.8: Let (X, $\|\cdot\|$) be a Banach space and A, B, S, T, P and Q: X $\to$ X be six mappings satisfying the conditions (3.2) - (3.4) and the following:

(3.22) $\| Px - Qy \| \leq k \max \{\| STx - ABy \|, (1/2) (\| STx - Px \| + \| ABy -Qy \|), (1/2) \| ABy - Px \|\}$

for all x,y in X, where k $\in$ (0,1)

or

(3.23) $\| Px - Qy \|^2 \leq k \max \{ \| STx - ABy \|^2, (1/2)\| STx - Px \| (\| STx - Px \| + \| ABy - Qy \|), (1/2) \| ABy - Qy \|. \| ABy - Px \|\}$

for all x, y in X where k $\in$ (0, 1)

or

(3.24) $\| Px - Qy \|^2 \leq a\| STx -ABy \|^2 - b\| ABy - Px \| (\| STx - ABy \|-\| ABy - Qy\|$ for all x, y in X, where a, b $\in$ (0, 1/2)

Then A, B, S, T, P and Q have a unique common fixed point.

Example 3.1: Let the set X = [0, 1] with the ‖ . ‖ defined by d (x, y) = ‖ x – y ‖, for all x, y $\in$ X.

Clearly (X, ‖ . ‖) is a complete Banach Space. Let A, B, S, T, P and Q be defined as

Ax = x, Bx = x/2, Sx = x/5, Tx = x/3, Px = x/6 and Qx = 0, for all x $\in$ X.

Then P(X) = [0, 1/6] $\subset$ [0, 1/2] = AB(X) and Q(X) $\subset$ {0} $\subset$ [0, 1/15] = ST(X).

We see that the condition (3.2) of Theorem 3.1 is satisfied. Clearly (3.3) and (3.4) is also satisfied.

Example 3.2: Consider X = R with the usual norm. Define A, B, S, T and Q by

Px = 0, Qx = 0, Bx = x; for all x $\in$ R

$$Ax = \begin{cases} 0 & if -\infty < x \le 0 \\ 1 & if \ \ 0 < x < \infty \end{cases}$$

$$Tx = \begin{cases} x & if -\infty < x < 1 \\ 1 & if \ \ 1 \le x < \infty \end{cases}$$

$$Sx = \begin{cases} 0 & if \quad -\infty < x \le 0 \\ -1 & if \quad 0 < x < 1 \\ 1 & if \quad 1 \le x < \infty \end{cases}$$

and so

$$ABx = \begin{cases} 0 & if -\infty < x \le 0 \\ 1 & if \ \ 0 < x < \infty \end{cases}$$

and

$$STx = \begin{cases} 0 & if \quad -\infty < x \le 0 \\ -1 & if \quad 0 < x < 1 \\ 1 & if \quad 1 \le x < \infty \end{cases}$$

Then A, B, S, T, P and Q satisfy conditions (3.2) and (3.4) of the Theorem 3.1. Let us consider a decreasing sequence $\{x_n\}$ such that

$$\lim_{n \to \infty} x_n = -3$$

Then

$$\lim_{n \to \infty} Px_n = \lim_{n \to \infty} ABx_n = 0$$

and

$$\lim_{n \to \infty} \left\| PABx_n - ABPx_n \right\| = 0$$

Thus the pair {P, AB} is compatible. Similarly the pair {P, ST} is also compatible. We see that condition (3.1) is satisfied if ϕ is similar to that of Example 2.1 or Example 2.2 or Example 2.3. Here we take k $\in(0, 1)$ in Example 2.1 and Example 2.2 and a = 0.4, b = 0.3, in Example 2.3. Thus A, B, S, T, P and Q satisfy all the conditions of Theorem 3.1 and Corollary 3.8 and have a unique common fixed point x=0.

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