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# Complex Analysis

**(Mathematics)**

For

**M. Sc.-I**

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## Centre for Distance Education
## Shivaji University, Kolhapur

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**Complex Analysis**

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# Preface

The Shivaji University, Kolhapur has established the Distance Education Centre for external students from the year 2007-08, with the goal that, those students who are not able to complete their studies regularly, due to unavoidable circumstances, they must be involved in the main stream by appearing externally. The centre is trying hard to provide notes to those aspirants by entrusting the task to experts in the subjects to prepare the Self Instructional Material (SIM). Today we are extremely happy to present a book on Complex Analysis for M. Sc. Mathematics students as SIM prepared by us. The SIM is prepared strictly according to syllabus and we hope that the exposition of the material in the book will meet the needs of all students.

This book introduces the students the most interesting and beautiful analysis viz. Complex Analysis. As a matter of fact Complex Analysis is a hard analysis, but it is truly a beautiful Analysis. The first topic is an introduction to Complex analysis. The second unit deals with Mobius transformations. The third unit introduces the reader to the notion of complex integration. Fundamental theorem of algebra and maximum modulus theorem are the results covered in the unit four. Unit five and six cover concept of winding number, Cauchy's integral theorem, Open mapping theorem and Goursat theorem. Laurent series development, Residue theorem with its application to evaluation of Real integrals, Rouche's theorem and Maximum Modulus theorem are the results contained in last two units.

We owe a deep sense of gratitude to the Ag. Vice-Chancellor Dr. A. A. Dange who has given impetus to go ahead with ambitious projects like the present one. Dr. S. R. Chaudhari and Dr. U. H. Naik have to be profusely thanked for the ovation for they have poured to prepare the SIM on Complex Analysis (M.Sc. Mathematics).

We also thank Professor M. S. Chaudhary, Head of the Department of Mathematics, Shivaji University, Kolhapur, Director of Distance Education Mode Dr. Mrs. Cima Yeole and Deputy Director, Shri. Sanjay Ratnaparakhi for their help and keen interest in completion of the SIM. Thanks are also due to Mr. Girish Shelke who had taken pains in typing the manuscript and Mr. Sachin Kadam for providing printing copy of the manuscript neatly and correctly.

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**M. Sc. (Mathematics)**

# Complex Analysis

## Contents

UNIT - I

# COMPLEX NUMBERS

**Introduction**

We know that in the real number system $\mathbb{R}$ , the equation $x^2 + a = 0$ has no solution. This leads to introduction of complex number system in which equations of the form $x^2 + a = 0$ , where $a > 0$, have solutions. This chapter introduces complex numbers, their representation and basic properties.

**Definition 1** The complex numbers can be defined as pair of real numbers $\mathbb{C} = \{(x, y) : x, y \in \mathbb{R}\}$. Equipped with addition $(x, y) + (a, b) = (x + a, y + b)$ and multiplication being defined as $(x, y)(a, b) = (xa - yb, xb + ya)$.

One reason to believe that the definitions of these binary operations are "good" is that $\mathbb{C}$ is an extension of $\mathbb{R}$ , in the sense that the complex numbers of the form $(x, 0)$ behave just like real numbers; that is, $(x, 0) + (a, 0) = (x + a, 0)$ and $(x, 0)(a, 0) = (xa, 0)$. So we can think of the real numbers being embedded in $\mathbb{C}$ as those complex numbers whose second coordinate is zero. The following basic results states the algebraic structure that we established with our definitions. Its proof is straightforward but nevertheless a good exercise.

1. **Commutative law for addition :** $z_1 + z_2 = z_2 + z_1$.
2. **Associative law for addition :** $z_1 + (z_2 + z_3) = (z_1 + z_2) + z_3$.
3. **Additive identity :** There is a complex number $z'$ such that $z + z_0 = z$ for all complex number $z$. The number $z_0$ is an ordered pair $(0, 0)$.
4. **Additive inverse :** For any complex number $z$ there is a complex number $-z$ such that $z + (-z) = (0, 0)$. The number $-z$ is $(-x, -y)$.
5. **Commutative law for multiplication :** $z_1 z_2 = z_2 z_1$.
6. **Associative law for multiplication :** $z_1 (z_2 z_3) = (z_1 z_2) z_3$.

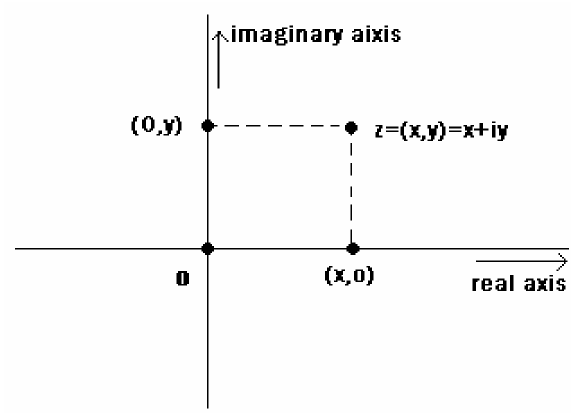
7. **Multiplicative identity :** There is a complex number $z'$ such that $zz' = z$ for all complex number $z$. The number $z'$ is an ordered pair $(1,0)$.
8. **Multiplicative inverse :** For any non-zero complex number $z$ there is a complex number $z^{-1}$ such that $zz^{-1} = (1,0)$. The number $z^{-1}$ is $\left(\frac{x}{x^2+y^2}, \frac{-y}{x^2+y^2}\right)$.
9. **The distributive law :** $z_1(z_2+z_3) = z_1z_2 + z_1z_3$.

If we write $x$ for the complex number $(x,0)$. This mapping $x \to (x,0)$ defines a field isomorphism of $\mathbb{R}$ into $\mathbb{C}$ so we may consider $\mathbb{R}$ as a subset of $\mathbb{C}$.

If we put $i = (0,1)$, then $z = (x,y) = (x,0) + (0,y) = (x,0) + (0,1)(y,0) = x + iy$.

Let $z = x + iy$, $x, y \in \mathbb{R}$, then $x$ and $y$ are called the real and imaginary parts of $z$ and denote this by $x = \operatorname{Re} z$, $y = \operatorname{Im} z$. If $x = 0$, the complex number $z$ is called purely imaginary and if $y = 0$, then $z$ is real. Note that zero is the only number which is at once real and purely imaginary. Two complex numbers are equal iff they have the same real part and the same imaginary part.

**Complex Plane or Argand plane :** The number $z = (x,y) = x + iy$ can be identified with the unique point $(x,y)$ in the plane $\mathbb{R}^2$. The plane $\mathbb{R}^2$ representing the complex numbers is called the complex plane. The x-axis is also called the real axis and the y-axis is called the imaginary axis.




**Definition 2** Let $z = x + iy$, $x, y \in \mathbb{R}$ then the complex number $x - iy$ is called the conjugate of $z$ and is denoted by $\bar{z}$.

Following are the basic properties of conjugates.

1. $\operatorname{Re} z = \frac{z+\bar{z}}{2}$ and $\operatorname{Im} z = \frac{z-\bar{z}}{2i}$.

2. $z$ is real iff $z = \bar{z}$.

3. $\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2$

4. $\overline{z_1 z_2} = \bar{z}_1 \, \bar{z}_2$.

5. $\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\bar{z}_1}{\bar{z}_2}$ if $z_2 \neq 0$.

6. $\bar{\bar{z}} = z$.

**Definition 3** Let $z = x + iy$, $x, y \in \mathbb{R}$ then modulus or absolute value of $z$ is a non-negative real number denoted by $|z|$ and is given by $|z| = \left(x^2 + y^2\right)^{\frac{1}{2}}$. The number $|z|$ is the distance between the origin and the point $(x, y)$.

Following are the basic properties of Modulus.

1. $|z|^2 = z\bar{z}$

2. $|z_1 z_2| = |z_1||z_2|$

3. $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$ if $z_2 \neq 0$.

4. $\overline{z_1 z_2} = \bar{z}_1 \, \bar{z}_2$.

5. $|\bar{z}| = |z|$.

6. $|x| = |\operatorname{Re}(z)| \leq |z|$ and $|y| = |\operatorname{Im}(z)| \leq |z|$.

7. $|z_1 + z_2| \leq |z_1| + |z_2|$.

8. $|z_1 - z_2| \geq |z_1| - |z_2|$.

9. Let $z_1 = x_1 + iy_1, z_2 = x_2 + iy_2$ then

$|z_1 - z_2| = |(x_1 - x_2) + i(y_1 - y_2)| = \left[(x_1 - x_2)^2 + (y_1 - y_2)^2\right]^{\frac{1}{2}}$ which is the distance between the points $(x_1, y_1), (x_2, y_2)$. Hence distance between the points $z_1$ and $z_2$ is given by $|\, z_1 - z_2 \,|$.

**Polar representation of complex numbers**

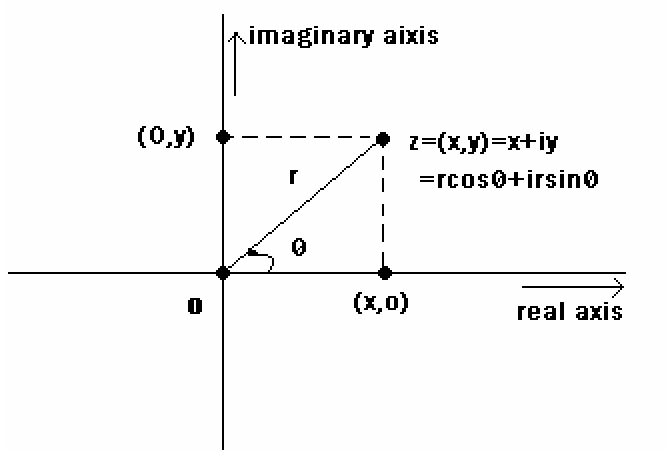
Consider the point $z = x + iy$ in the complex plane  . This point has polar co-ordinates $(r, \theta)$ where $x = r\cos\theta$ and $y = r\sin\theta$. Thus $z = x + iy = r(\cos\theta + i\sin\theta)$.

Clearly $r = |z| = \left(x^2 + y^2\right)^{\frac{1}{2}}$ which is **magnitude** of the complex number and $\theta$( undefined if $z = 0$ ) is the angle between the positive real axis and the line segment from $0$ to $z$ and is called the **argument** of $z$, denoted by $\theta = \arg z$.

We note that the value of argument of $z$ is not unique. If $\theta = \arg z$, then $\theta + 2\pi n$, where $n$ is an integer is also $\arg z$. The value of $\arg z$ that lies in the range $-\pi < \theta \le \pi$ is called the **principal value** of $\arg z$.

If $z_1, z_2$ are any two non-zero complex numbers then

1. $\arg z_1 = -\arg \overline{z_1}$
2. $\arg z_1 \, z_2 = \arg z_1 + \arg z_2$.
3. $\arg\left[\dfrac{z_1}{z_2}\right] = \arg z_1 - \arg z_2$.




We shall simply state

**De Moivre's Theorem :** For any real number n**,** $\cos n\theta + i\sin n\theta$ is one of the values of $(\cos\theta + i\sin\theta)^n$.

**$n^{th}$ Roots of Complex Numbers.**

Let $z = r(\cos\theta + i\sin\theta)$ be a non-zero complex number, then $w = \rho(\cos\varphi + i\sin\varphi)$ is $n^{th}$ root of $z$ if $w^n = z$, where n is a positive integer.

Therefore, $\rho^n(\cos\varphi + i\sin\varphi)^n = r(\cos\theta + i\sin\theta)$

$$\rho^n(\cos n\varphi + i\sin n\varphi) = r(\cos\theta + i\sin\theta)$$

$$\rho^n = r \text{ and } n\varphi = \theta + 2k\pi \text{, where } k \text{ is an integer.}$$

Thus $\rho = r^{\frac{1}{n}}$ and $\varphi = \dfrac{\theta + 2k\pi}{n}$, where $k$ is an integer.

However, only the values of $k = 0,1,2,...,(n-1)$ will give distinct values of $w$. Hence $z$ has n distinct $n^{th}$ roots and they are given by

$$w = r^{\frac{1}{n}}\left[\cos\left(\frac{\theta + 2k\pi}{n}\right) + i\sin\left(\frac{\theta + 2k\pi}{n}\right)\right] \text{ where } k = 0,1,2,...,(n-1).$$

**Some Topological aspects**

Note that $\mathbb{C}$ is a metric space with respect to usual metric $d(z,\zeta) = |z-\zeta|$.

By an open disc , we mean the set $\{z : |z-a| < \in\}$ and is denoted by $B(a;\in)$. And by closed disc, we mean, $\{z : |z-a| \le \in\}$ and is denoted by $\overline{B}(a;\in)$. Further an annulus is defined as the set $\{z : r < |z-a| < R\}$ and is denoted by $ann(a;r,R)$. The punctured disk of radius $\in$ centered at $a$ is defined by, $B(a;\in) - \{0\} = \{z : 0 < |z-a| \le \in\}$.

**Definition** 4 A subset $G \subseteq \mathbb{C}$ is open if, for each $z \in S$, there is an $\varepsilon > 0$ such that $B(z;\varepsilon) \subseteq G$. The point $z_0$ is said to be an **interior point** of the set $S \subseteq \mathbb{C}$ if there exists an $\varepsilon > 0$ such that $B(z;\varepsilon) \subseteq S$. Further, interior of $S$, written $\operatorname{int} S$, is the set $\overline{S} = \bigcap\{G : G \textit{ is open and } G \subseteq S\}$. The closure of $S \subseteq \mathbb{C}$ , denoted by $\overline{S}$, is the set $\overline{S} = \bigcap\{F : F \textit{ is closed and } F \supseteq S\}$. The boundary of $S$, denoted by $\partial S$ and defined by $\partial S = \overline{S} \cap \overline{(X-S)}$. Further, a subset $S$ is dense if $\overline{S} = \mathbb{C}$ .

**Definition 5** A metric space $(X,d)$ is *connected* if the only subsets of $X$ which are both open and closed are $X$ and the empty set. Further, a subset $S \subseteq X$ is connected if the metric space $(S,d)$ is connected.

**Definition 6** If $G$ is an open set in $\mathbb{C}$ and $f: G \to \mathbb{C}$, then $f$ is differentiable at a point $a$ in $G$ if $\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$ exists. It is denoted by $f'(a)$ and called derivative of $f$ at $a$.

**Definition 7** If $f$ is differentiable on $G$, then we define $f': G \to \mathbb{C}$. If $f'$ is continuous then we say that $f$ is continuously differentiable.

**Definition 8** A differentiable function such that each successive derivative is again differentiable is called infinitely differentiable.

**Definition 9** A function $f: G \to \mathbb{C}$ is analytic if $f$ is continuously differentiable on $G$.

**Power series**

**Definition 10** A series of the form $\sum_{n=0}^{\infty} a_n (z-a)^n$ where $a_1, a_2, ..., a_n, ...$ are constants, is called power series about $a$.

**Ex.** The geometrical series $\sum_{n=0}^{\infty} z^n$ is power series about $0$ and for $|z|<1$, $\sum_{n=0}^{\infty} z^n = \frac{1}{1-z}$.

**Theorem 11** For given power series $\sum_{n=0}^{\infty} a_n (z-a)^n$ define a number $0 \le R \le \infty$ by $\frac{1}{R} = \lim\sup |a_n|^{\frac{1}{n}}$, then

a) If $|z-a| < R$, the series converges absolutely.

b) If $|z-a| > R$, the term of series become unbounded and so series diverges.

c) If $0 < r < R$, then the series converges uniformly on $\{z : |z-a| \le r\}$.

Moreover the number $R$ is the only number having properties (a) and (b).

**Proof.** We may suppose that $a=0$.

a) If $|z|<R$, then there is an $r$ such that $|z|<r<R$, $\left(\frac{1}{r}>\frac{1}{R}\right)$. Thus by definition of limit sup, there is an integer $N$ such that

$$|a_n|^{1/n}<\frac{1}{r} \text{ for all } n \geq N.$$

$$|a_n|<\frac{1}{r^n} \text{ for all } n \geq N.$$

$$|a_n z^n|<\left(\frac{|z|}{r}\right)^n \text{ for all } n \geq N.$$

Thus the series $\sum_{n=0}^{\infty} a_n z^n$ is dominated by the series $\sum\left(\frac{|z|}{r}\right)^n$. Since the geometric series $\sum_{n=0}^{\infty}\left(\frac{|z|}{r}\right)^n$ converges for $|z|<r$, the series $\sum_{n=0}^{\infty} a_n z^n$ converges absolutely for each $|z|<R$.

b) Suppose $|z|>R$, and choose $r$ such that $|z|>r>R$, $\left(\frac{1}{r}<\frac{1}{R}\right)$. Thus by definition of limit sup, there are infinitely many integers $n$ such that $|a_n|^{1/n}>\frac{1}{r}$. It follows that $|a_n z^n|>\left(\frac{|z|}{r}\right)^n$ for all $n \geq N$ and , since $\left(\frac{|z|}{r}\right)>1$ these terms becomes unbounded and so the series diverges.

c) If $0<r<R$, choose $\rho$ such that $r<\rho<R$. As in (a) we have $|a_n|<\frac{1}{\rho^n}$ for all $n \geq N$. Thus if $|z|\leq r$, $|a_n z^n|\leq\left(\frac{|z|}{\rho}\right)^n \leq\left(\frac{r}{\rho}\right)^n$ and $\left(\frac{r}{\rho}\right)<1$. Hence, by Weierstrass M-test, the power series $\sum_{n=0}^{\infty} a_n z^n$ converges uniformly on $|z|\leq r$.

**Definition 12** The circle $|z-a|=R$ which includes in its interior $|z-a|<R$ , in which the power series $\sum_{n=0}^{\infty} a_n(z-a)^n$ converges, is called circle of convergence . Radius $R$ of this circle is called radius of convergence of power series and in view of above result it is given by $\frac{1}{R}=\limsup |a_n|^{\frac{1}{n}}$ .

**Theorem 13** If $\sum_{n=0}^{\infty} a_n(z-a)^n$ is given power series with radius of convergence $R$ , then

$R=\lim\left|\frac{a_n}{a_{n+1}}\right|$ if this limit exists.

**Proof.** We assume that $a=0$ and let $\alpha=\lim\left|\frac{a_n}{a_{n+1}}\right|$. Suppose that $|z|<r<\alpha$ and find an integer $N$ such that $r<\left|\frac{a_n}{a_{n+1}}\right|$ for all $n \geq N$. Let $B=|a_n|r^N$ then

$|a_{N+1}|r^{N+1}=|a_{N+1}|rr^N<|a_N|r^N=B$,

$|a_{N+2}|r^{N+2}=|a_{N+2}|rr^{N+1}<|a_{N+1}|r^{N+1}<B$.

Continuing this way we get, $|a_n r^n|\leq B$ for all $n \geq N$.

Then $|a_n z^n|=|a_n r^n|\frac{|z|^n}{r^n}\leq B\frac{|z|^n}{r^n}$ for all $n \geq N$.

Since $|z|<r$ we get that $\sum_{n=0}^{\infty}|a_n z^n|$ is dominated by convergent series and hence converges.

Since $r<\alpha$ was arbitrary this gives that $\alpha\leq R$.

Now if $|z|>r>\alpha$ then $|a_n|<r|a_{n+1}|$ for all $n \geq N$. As above we get $|a_n r^n|\geq B=|a_N r^N|$ for all $n \geq N$. This gives that $|a_n z^n|\geq B\frac{|z|^n}{|r|^n}\to\infty$ as $n\to\infty$. Hence $\sum_{n=0}^{\infty} a_n z^n$ diverges and so $R\leq\alpha$. Thus $R=\alpha$.

**Example 14** Find radius of convergence for the series

a) $\sum_{n=0}^{\infty}\frac{z^n}{n^n}$ b) $\sum_{n=0}^{\infty}\frac{z^n}{n!}$ c) $\sum_{n=0}^{\infty}\frac{(-1)^n}{n}(z-2i)^n$

**Solution. a)** Here $\sum_{n=0}^{\infty}\frac{z^n}{n^n}$ and $a_n=\frac{1}{n^n}$, $a=0$ .

Therefore $\frac{1}{R}=\lim\sup|a_n|^{\frac{1}{n}}=\lim\sup\left|\frac{1}{n^n}\right|^{\frac{1}{n}}=\lim\sup\frac{1}{n}=0$

Thus radius of convergence for the series $\sum_{n=0}^{\infty}\frac{z^n}{n^n}$ is $R=\infty$.

That is the series converges in whole complex plane.

**b)** Here $\sum_{n=0}^{\infty}\frac{z^n}{n!}$ and $a_n=\frac{1}{n!}$, $a=0$ .

Therefore $\frac{1}{R}=\lim\left|\frac{a_{n+1}}{a_n}\right|=\lim\left|\frac{n!}{(n+1)!}\right|=\lim\left|\frac{1}{(n+1)}\right|=0$

Thus radius of convergence for the series $\sum_{n=0}^{\infty}\frac{z^n}{n!}$ is $R=\infty$.

**c)** Here $\sum_{n=0}^{\infty}\frac{(-1)^n}{n}(z-2i)^n$ and $a_n=\frac{(-1)^n}{n}$, $a=2i$ .

Therefore $\frac{1}{R}=\lim\left|\frac{a_{n+1}}{a_n}\right|=\lim\left|\frac{n}{(n+1)}\right|=1$

Thus radius of convergence for the series $\sum_{n=0}^{\infty}\frac{(-1)^n}{n}(z-2i)^n$ is $R=1$ and circle of convergence is $|z-2i|=1$.

**Theorem 15** Let $f(z)=\sum_{n=0}^{\infty} a_n (z-a)^n$ have radius of convergence $R>0$. Then

a) For each $K \geq 1$ the series $\sum_{n=k}^{\infty} n(n-1)...(n-k+1)a_n (z-a)^{n-k}$ …(1) has radius of convergence $R$.

b) The function $f$ is infinitely differentiable on $B(a,R)$ and furthermore, $f^{(k)}(z)$ is given by the series (1) for all $K \geq 1$ and $|z-a|<R$.

c) For $n \geq 0$, $a_n = \frac{1}{n!} f^{(n)}(a)$.

**Proof.** With no loss of generality assume that $a=0$.

Therefore $f(z)=\sum_{n=0}^{\infty} a_n z^n$ …(2)

**a)** We first prove the result for $K=1$. That is the power series $\sum_{n=0}^{\infty} a_n z^n$ and $\sum_{n=1}^{\infty} na_n z^{n-1}$ have same radius of convergence.

Let $g(z)=\sum_{n=1}^{\infty} na_n z^{n-1}$ have radius of convergence $R'$ where $\frac{1}{R'} = \lim\sup |na_n|^{\frac{1}{n-1}}$

Since $R$ is radius of convergence of $f(z)=\sum_{n=0}^{\infty} a_n z^n$, $\frac{1}{R} = \lim\sup |a_n|^{\frac{1}{n}}$.

Now we have to show that $R=R'$.

$$\log\left(\lim n^{\frac{1}{n-1}}\right) = \lim\left(\log n^{\frac{1}{n-1}}\right) = \lim \frac{\log n}{n-1} = \lim \frac{1/n}{1} = 0$$

Therefore $\lim n^{\frac{1}{n-1}} = e^0 = 1$.

Thus $\lim\sup |na_n|^{\frac{1}{n-1}} = \left(\lim n^{\frac{1}{n-1}}\right)\left(\lim\sup |a_n|^{\frac{1}{n-1}}\right)$

$$= 1.\left(\lim\sup |a_n|^{\frac{1}{n-1}}\right) = \lim\sup |a_n|^{\frac{1}{n-1}}$$

Therefore the series $\sum_{n=1}^{\infty} na_n z^{n-1}$ and $\sum_{n=1}^{\infty} a_n z^{n-1}$ have same radius of convergence.

If $|z|<R'$, we write $\sum_{n=0}^{\infty} |a_n z^n| = |a_0| + |z| \sum_{n=1}^{\infty} |a_n z^{n-1}| < \infty$.

That is if $|z| < R'$, $\sum_{n=0}^{\infty} |a_n z^n|$ is convergent. Hence $R \geq R'$.

Also if $|z| < R$, we write $\sum_{n=1}^{\infty} |a_n z^{n-1}| = \frac{1}{|z|} \sum_{n=0}^{\infty} |a_n z^n| - \frac{|a_0|}{|z|} \leq \frac{1}{|z|} \sum_{n=0}^{\infty} |a_n z^n| + \frac{|a_0|}{|z|} < \infty$ for $z \neq 0$.

That is if $|z| < R$, $\sum_{n=1}^{\infty} a_n z^{n-1}$ is convergent. Hence $R \leq R'$.

Thus $R = R'$.

Thus $\sum_{n=0}^{\infty} a_n z^n$ and $\sum_{n=1}^{\infty} n a_n z^{n-1}$ have same radius of convergence.

Similarly $\sum_{n=1}^{\infty} n a_n z^{n-1}$ and $\sum_{n=2}^{\infty} n(n-1) a_n z^{n-2}$ have same radius of convergence $R$.

Therefore by method of induction for any $K \geq 1$ the series $\sum_{n=k}^{\infty} n(n-1)...(n-k+1) a_n (z-a)^{n-k}$ has radius of convergence $R$.

**b)** For $|z| < R$, let $S_n(z) = \sum_{k=0}^{n} a_k z^k$ and $R_n(z) = \sum_{k=n+1}^{\infty} a_k z^k$ so that $f(z) = S_n(z) + R_n(z)$.

Now fix a point $w \in B(0;R)$, then there is $0 < r < R$ such that $|w| < r < R$.

Let $\delta > 0$ be such that $\overline{B}(w;\delta) \subseteq B(0;r)$. Let $z \in B(w;\delta)$.

Consider

$$\frac{f(z)-f(w)}{z-w} - g(w) = \frac{S_n(z)-S_n(w)}{z-w} - S_n{}'(w) + S_n{}'(w) - g(w) + \frac{R_n(z)-R_n(w)}{z-w} \qquad \ldots(3)$$

And

$$\left| \frac{R_n(z)-R_n(w)}{z-w} \right| = \left| \frac{\sum_{k=n+1}^{\infty} a_k \left(z^k - w^k\right)}{z-w} \right| \leq \sum_{k=n+1}^{\infty} |a_k| \left| \frac{z^k - w^k}{z-w} \right|$$

$$= \sum_{k=n+1}^{\infty} |a_k| \left| z^{k-1} + z^{k-2} w + \ldots + z\, w^{k-2} + w^{k-1} \right|$$

$$\leq \sum_{k=n+1}^{\infty} |a_k| \left\{ |z|^{k-1} + |z|^{k-2} |w| + \ldots + |z||w|^{k-2} + |w|^{k-1} \right\}$$

$$\leq \sum_{k=n+1}^{\infty} |a_k| \left\{ r^{k-1} + r^{k-2} r + \ldots + r\, r^{k-2} + r^{k-1} \right\}$$

$$= \sum_{k=n+1}^{\infty} |a_k|\, k\, r^{k-1}$$

Since $r < R$, $\sum_{k=n+1}^{\infty} |a_k|\, k\, r^{k-1}$ converges.

Therefore for any $\in > 0$, there is an integer $N_1 > 0$ such that $\sum_{k=n+1}^{\infty} |a_k|\, k\, r^{k-1} < \in/3$ whenever $n \geq N_1$.

Thus for $n \geq N_1$

$$\left| \frac{R_n(z) - R_n(w)}{z - w} \right| < \in/3 \qquad \ldots(4)$$

Since $\lim_{n\to\infty} S_n'(w) = g(w)$, there is an integer $N_2 > 0$ such that $|S_n'(w) - g(w)| < \in/3$ $\ldots(5)$

whenever $n \geq N_2$.

Let $n = max\{N_1, N_2\}$.

Since $\lim_{z\to w} \frac{S_n(z) - S_n(w)}{z - w} = S_n'(w)$, for given $\in > 0$, we choose $\delta > 0$ such that

$$\left| \frac{S_n(z) - S_n(w)}{z - w} - S_n'(w) \right| < \in/3 \qquad \ldots(6)$$

whenever $0 < |z - w| < \delta$.

Thus for given $\in > 0$, there is $\delta > 0$ such that

$$\left| \frac{f(z) - f(w)}{z - w} - g(w) \right| < \in/3 + \in/3 + \in/3 = \in$$

whenever $0 < |z - w| < \delta$.

Hence $f$ is differentiable and $f'(w) = g(w)$ for all $w \in B(0; R)$.

That is $f'(z)=\sum_{n=1}^{\infty} na_n z^{n-1}$.

Similarly $f''(z)=\sum_{n=2}^{\infty} n(n-1)a_n z^{n-2}$

$$f'''(z)=\sum_{n=3}^{\infty} n(n-1)(n-2)a_n z^{n-3}$$

and so on $f^{(k)}(z)=\sum_{n=k}^{\infty} n(n-1)(n-1)...(n-k+1)a_n z^{n-k}$.

**c)** From part (b) we have

$f(0)=a_0, f''(0)=1.a_1,\ f'''(0)=1.2.a_2,\ldots,\ f^{(k)}(0)=1.2.3....k.a_2$.

Thus $a_k=\frac{1}{k!}f^{(k)}(0)$

Hence for $f(z)=\sum_{n=0}^{\infty} a_n(z-a)^n$, $\qquad a_n=\frac{1}{n!}f^{(n)}(a)$.

**Corollary 16** If the series $\sum_{n=0}^{\infty} a_n(z-a)^n$ has radius of convergence $R>0$ then, $f(z)=\sum_{n=0}^{\infty} a_n(z-a)^n$ is analytic in $B(a,R)$.

**Proof.** By above theorem if $\sum_{n=0}^{\infty} a_n(z-a)^n$ has radius of convergence $R>0$, then $f(z)=\sum_{n=0}^{\infty} a_n(z-a)^n$ is infinitely differentiable in $B(a,R)$.

Therefore $f', f''$ exists in $B(a,R)$ implies that $f'$ is continuous in $B(a,R)$.

Thus $f$ is continuously differentiable. Hence $f$ is analytic in $B(a,R)$.

**Result 17** A domain $G$ is connected iff its open as well as closed subset is either empty or $G$.

**Theorem 18** If $G$ is open and connected and $f : G \to \mathbb{C}$ is differentiable with $f'(z) = 0$ for all $z$ in $G$, then $f$ is constant.

**Proof.** Fix $z_0$ in $G$ and let $w_0 = f(z_0)$. Let $A = \{z \in G : f(z) = w_0\}$. Clearly $A \neq \emptyset$.

If $A$ is open as well as closed, then by connectedness of $G$, $A = G$. ( i.e. $f$ is constant )

First we prove that $A$ - is open.

Now for $a \in A$. Let $\in > 0$ be such that $B(a;\in) \subset G$.

If $z \in B(a;\in)$ we define $g : [0,1] \to G$ by $g(t) = f[\, t\,z + (1-t)\,a \,]$, $0 \le t \le 1$.

Then $\dfrac{g(t)-g(s)}{t-s} = \dfrac{f[\, t\,z+(1-t)\,a \,] - f[\, s\,z+(1-s)\,a \,]}{t-s}$

$$= \frac{f[\, t\,z+(1-t)\,a \,] - f[\, s\,z+(1-s)\,a \,]}{[t\,z+(1-t)\,a] - [s\,z+(1-s)\,a\,]} \cdot \frac{[t\,z+(1-t)\,a] - [s\,z+(1-s)\,a\,]}{t-s}$$

$$= \frac{f[\, t\,z+(1-t)\,a \,] - f[\, s\,z+(1-s)\,a \,]}{[t\,z+(1-t)\,a] - [s\,z+(1-s)\,a\,]} .(z-a)$$

$$\lim_{t\to s} \frac{g(t)-g(s)}{t-s} = \lim_{t\to s} \frac{f[\, t\,z+(1-t)\,a \,] - f[\, s\,z+(1-s)\,a \,]}{[t\,z+(1-t)\,a] - [s\,z+(1-s)\,a\,]} .(z-a)$$

$$g'(s) = f'[\, s\,z+(1-s)\,a \,].(z-a) = 0.(z-a) = 0$$

Therefore $g'(s) = 0$ for $0 \le s \le 1$ implies that $g$ is constant.

Hence $g(1) = g(0)$ implies that $f(z) = f(a) = w_0$. Therefore $z \in A$.

Thus if $z \in B(a;\in)$ then $z \in A$ that is $B(a;\in) \subset A$. Thus $A$ is open.

We now prove that $A$ - is closed.

Let $z$ be limit point of $A$ , then there is a sequence $\{z_n\}$ in $A$ such that $\lim\limits_{n\to\infty} z_n = z$.

Since $f$ is continuous , $f(z) = f\left(\lim\limits_{n\to\infty} z_n\right) = \lim\limits_{n\to\infty} f(z_n) = \lim\limits_{n\to\infty} w_0 = w_0$. Hence $z \in A$.

Thus $A$ contains all its limit points hence $A$ is closed.

# EXERCISES

1) Find the radius of convergence of the followings

a) $\sum \frac{1}{n^n} z^n$      b) $\sum \frac{1}{n!} z^n$      c) $\sum (3+4i)^n z^n$

d) $\sum \frac{1}{1+in^2} z^n$      e) $\sum_{n=0}^{\infty} a^n z^n \quad a \in \square$      f) $\sum_{n=0}^{\infty} k^n z^n \quad k \in \square$ .

UNIT - II

# MÖBIUS TRANSFORMATIONS

In this unit we study Möbius transformations and their properties. We begin with bilinear transformation.

**Definition 19** A mapping of the form $S(z) = \dfrac{az+b}{cz+d}$ is called bilinear or linear fractional transformation where $a, b, c, d \in \mathbb{C}$ .

**Definition 20** A bilinear transformation $S(z) = \dfrac{az+b}{cz+d}$ with $ad - bc \neq 0$ is called Möbius map or Möbius transformation.

**Remarks 21** 1) Möbius transformation is one-one and onto.

2) If $S(z) = \dfrac{az+b}{cz+d}$, then $S^{-1}(w) = \dfrac{-dw+b}{cw-a}$.

3) If $S$ and $T$ are Möbius transformations then $S \circ T$ is also Möbius transformation.

4) $S(z) = z + a$ ( Translation )

$S(z) = az$ ( Dilation/Magnification )

$S(z) = e^{i\theta} z$ ( Rotation )

$S(z) = \dfrac{1}{z}$ ( Inversion ).

**Theorem 22** If $S$ is a Möbius transformation then $S$ is composition of translation , dilation and inversion.

**Proof.** Let $S(z)=\dfrac{az+b}{cz+d}$ with $ad-bc\neq 0$ be Möbius transformation.

Case 1. When $c=0$ then $S(z)=\left(\dfrac{a}{d}\right)z+\left(\dfrac{b}{d}\right)$

Let $S_1(z)=\left(\dfrac{a}{d}\right)z,\quad S_2(z)=z+\left(\dfrac{b}{d}\right)$

Then $S_2\circ S_1(z)=S_2\left(S_1\left(z\right)\right)=S_2\left(\left(\dfrac{a}{d}\right)z\right)=\left(\dfrac{a}{d}\right)z+\left(\dfrac{b}{d}\right)=S(z)$

Thus $S=S_2\circ S_1$.

Case 2. When $c\neq 0$

Let $S_1(z)=z+\dfrac{d}{c}$, $S_2(z)=\dfrac{1}{z}$, $S_3(z)=\dfrac{bc-ad}{c^2}z$, $S_4(z)=z+\dfrac{a}{c}$.

Then $S_4\circ S_3\circ S_2\circ S_1(z)=S_4\circ S_3\circ S_2\left(S_1(z)\right)$

$$=S_4\circ S_3\circ S_2\left(z+\frac{d}{c}\right)$$

$$=S_4\circ S_3\left(S_2\left(z+\frac{d}{c}\right)\right)$$

$$=S_4\circ S_3\left(\frac{1}{z+\frac{d}{c}}\right)$$

$$=S_4\left(S_3\left(\frac{1}{z+\frac{d}{c}}\right)\right)$$

$$=S_4\left(\frac{bc-ad}{c^2}\left(\frac{1}{z+\frac{d}{c}}\right)\right)$$

$$=\left(\frac{bc-ad}{c(cz+d)}\right)+\frac{a}{c}$$

$$= \frac{az+b}{cz+d} = S(z).$$

Thus $S = S_4 \circ S_3 \circ S_2 \circ S_1$.

**Theorem 23** Every Möbius transformation can have at most two fixed points.

**Proof.** Let $S(z) = \frac{az+b}{cz+d}$ with $ad - bc \neq 0$ be Möbius transformation.

Let $z$ be fixed point of $S(z)$ then $S(z) = z$

$$\frac{az+b}{cz+d} = z$$

$$cz^2 + (d-a)z - b = 0$$

which is quadratic in $z$. Hence it can have at most two roots. Therefore every Möbius transformation can have at most two fixed points otherwise $S(z) = z$ for all $z$ (Identity map ).

**Theorem 24** Möbius map is uniquely determined by its action on any three distinct points in $\mathbb{C}_\infty$.

**Proof.** Let $z_1, z_2, z_3$ be three distinct points in $\mathbb{C}_\infty$. Let $S$ and $T$ be Möbius map such that $S(z_1) = w_1, S(z_2) = w_2, S(z_3) = w_3$ and $T(z_1) = w_1, T(z_2) = w_2, T(z_3) = w_3$.

Then

$$T^{-1} \circ S(z_1) = T^{-1}(S(z_1)) = T^{-1}(w_1) = z_1$$

$$T^{-1} \circ S(z_2) = T^{-1}(S(z_2)) = T^{-1}(w_2) = z_2$$

$$T^{-1} \circ S(z_3) = T^{-1}(S(z_3)) = T^{-1}(w_3) = z_3$$

Thus $T^{-1} \circ S$ is a Möbius map having three fixed points.

Hence $T^{-1} \circ S = I$ ( Identity map). Thus $T = S$.

**Definition 25** For $z \in \mathbb{C}_\infty$ the map denoted by $(z, z_2, z_3, z_4)$ where $z_2, z_3, z_4 \in \mathbb{C}_\infty$ is called cross ratio if it maps $z_2, z_3, z_4$ respectively to $1, 0, \infty$.

More precisely the map $z \mapsto (z, z_2, z_3, z_4)$ is a Möbius map that maps $z_2, z_3, z_4$ respectively to $1, 0, \infty$ and is given by

$$S(z) = (z, z_2, z_3, z_4) = \left( \frac{z - z_3}{z - z_4} \right) \Big/ \left( \frac{z_2 - z_3}{z_2 - z_4} \right).$$

**Remarks 26 1)** $(z_2, z_2, z_3, z_4) = 1, (z_3, z_2, z_3, z_4) = 0, (z_4, z_2, z_3, z_4) = \infty$.

**2)** $(z, 1, 0, \infty) = z$.

**3)** Let $M$ be any Möbius map such that $M(w_2) = 1, M(w_3) = 0, M(w_4) = \infty$ then $M(z) = (z, w_2, w_3, w_4)$.

**Theorem 27** If $z_2, z_3, z_4$ are distinct points and $T$ is any Möbius map then $(z_1, z_2, z_3, z_4) = (Tz_1, Tz_2, Tz_3, Tz_4)$ for any point $z_1$. ( Cross-ratio is invariant under any Möbius map )

**Proof.** Let $S(z) = (z, z_2, z_3, z_4)$ and $T$ is any Möbius map. Let $M = S \circ T^{-1}$, then $M \circ T = S$.

Now, $1 = S(z_2) = M \circ T(z_2) = M(T(z_2))$

$0 = S(z_3) = M \circ T(z_3) = M(T(z_3))$

$\infty = S(z_4) = M \circ T(z_4) = M(T(z_4))$

Thus $M(z) = (z, Tz_2, Tz_3, Tz_4)$ then $S \circ T^{-1}(z) = (z, Tz_2, Tz_3, Tz_4)$.

Let $T^{-1}(z) = z_1$ then $z = T(z_1)$. Therefore $S(z_1) = (Tz_1, Tz_2, Tz_3, Tz_4)$.

Thus $(z_1, z_2, z_3, z_4) = (Tz_1, Tz_2, Tz_3, Tz_4)$.

**Theorem 28** If $z_2, z_3, z_4$ are distinct points in $\mathbb{C}_\infty$ and $w_2, w_3, w_4$ are also distinct points of $\mathbb{C}_\infty$, then is one and only one Möbius map $S$ such that $S(z_2) = w_2, S(z_3) = w_3, S(z_4) = w_4$.

**Proof.** Let $T(z) = (z, z_2, z_3, z_4)$ and $M(z) = (z, w_2, w_3, w_4)$. Let $S = M^{-1} \circ T$ then

$$S(z_2) = M^{-1} \circ T(z_2) = M^{-1}(T(z_2)) = M^{-1}(1) = w_2$$

$$S(z_3) = M^{-1} \circ T(z_3) = M^{-1}(T(z_3)) = M^{-1}(0) = w_3$$

$$S(z_4) = M^{-1} \circ T(z_4) = M^{-1}(T(z_4)) = M^{-1}(\infty) = w_4$$

Thus we have a Möbius map $S$ such that $S(z_2)=w_2, S(z_3)=w_3, S(z_4)=w_4$.

**Uniqueness:**

Let $R$ be another Möbius map such that $R(z_2)=w_2, R(z_3)=w_3, R(z_4)=w_4$.

Then $R^{-1}\circ S(z_2)=R^{-1}\left(S(z_2)\right)=R^{-1}\left(w_2\right)=z_2$

$R^{-1}\circ S(z_3)=R^{-1}\left(S(z_3)\right)=R^{-1}\left(w_3\right)=z_3$

$R^{-1}\circ S(z_4)=R^{-1}\left(S(z_4)\right)=R^{-1}\left(w_4\right)=z_4$

Thus $R^{-1}\circ S$ has three fixed points $z_2,z_3,z_4$ implies that $R^{-1}\circ S=I$.

Therefore, $R=S$.

**Example 29** Evaluate following cross ratios a) $\left(7+i,1,0,\infty\right)$ b) $\left(2,1-i,1,1+i\right)$

**Sol.** We have, $S(z)=\left(z,z_2,z_3,z_4\right)=\left(\dfrac{z-z_3}{z-z_4}\right)\Big/\left(\dfrac{z_2-z_3}{z_2-z_4}\right)$

a) $\left(z,z_2,z_3,\infty\right)=\left(\dfrac{z-z_3}{z_2-z_3}\right)$

Therefore, $\left(7+i,1,0,\infty\right)=\left(\dfrac{7+i-0}{1-0}\right)=7+i$.

b) $\left(2,1-i,1,1+i\right)=\left(\dfrac{2-1}{2-(1+i)}\right)\Big/\left(\dfrac{(1-i)-1}{(1-i)-(1+i)}\right)=\left(\dfrac{1}{1-i}\right)\Big/\left(\dfrac{-i}{-2i}\right)=\dfrac{2}{1-i}=1+i$.

**Example 30** Find Möbius map which maps the points $z_2=2,z_3=i,z_4=-2$ onto $w_2=1,w_3=i,w_4=-1$ respectively.

**Solution** Let $S$ be the map that takes $z_i\mapsto w_i$ $\left(i=2,3,4\right)$. Since cross ratio is invariant under any Möbius map, $\left(z,z_2,z_3,z_4\right)=\left(Sz,Sz_2,Sz_3,Sz_4\right)$.

Therefore, $\left(z,z_2,z_3,z_4\right)=\left(w,w_2,w_3,w_4\right)$, where $S\left(z\right)=w$.

Thus $\left(z,2,i,-2\right)=\left(w,1,i,-1\right)$

$$\left(\frac{z-i}{z+2}\right)\Big/\left(\frac{2-i}{2+2}\right)=\left(\frac{w-i}{w+1}\right)\Big/\left(\frac{1-i}{1+1}\right)$$

$$\frac{4(z-i)}{(z+2)(2-i)}=\frac{2(w-i)}{(w+1)(1-i)}$$

$$\frac{2(z-i)}{2z-iz+4-2i}=\frac{(w-i)}{w-iw+1-i}$$

$$2(z-i)(w-iw+1-i)=(w-i)(2z-iz+4-2i)$$

$$w\big(2(z-i)-2(z-i)i\big)+2(z-i)(1-i)=w(2z-iz+4-2i)-i(2z-iz+4-2i)$$

$$w=\frac{-2(z-i)(1-i)-i(2z-iz+4-2i)}{2(z-i)-2(z-i)i-(2z-iz+4-2i)}=\frac{\big(-2(1-i)-i(2-i)\big)z+\big(2i(1-i)-i(4-2i)\big)}{\big(2-2i-(2-i)\big)z+\big(-2i+2i^2-(4-2i)\big)}$$

Therefore, $w=\dfrac{(-3)z+(-2i)}{(-i)z+(-6)}=\dfrac{3z+2i}{iz+6}$.

**Theorem 31** Let $z_1,z_2,z_3,z_4$ be four distinct points in $\mathbb{C}_\infty$, then $(z_1,z_2,z_3,z_4)$ is real number iff all four points lie on the circle.

**Proof.** Let $S(z)=(z,z_2,z_3,z_4)$ then $S$ is a Möbius map from $\mathbb{C}_\infty$ to $\mathbb{C}_\infty$. To prove this theorem we have to prove that $\{\, w\in\mathbb{C}_\infty : S(w)=\text{real}\,\}$ is a circle.

Suppose $S(w)=\text{real}$, then $S(w)=\overline{S(w)}$.

Let $S(w)=\dfrac{aw+b}{cw+d}$ with $ad-bc\neq 0$.

Thus, $\dfrac{aw+b}{cw+d}=\dfrac{\overline{a}\overline{w}+\overline{b}}{\overline{c}\overline{w}+\overline{d}}$

Therefore, $(a\overline{c}-\overline{a}c)|w|^2+(a\overline{d}-\overline{b}c)w+(b\overline{c}-\overline{a}d)\overline{w}+(b\overline{d}-\overline{b}d)=0$ …(1)

**Case 1**. When $a\overline{c}$ is real.

Therefore, $a\overline{c}=\overline{a}c$, then from (1) we have,

$(a\overline{d}-\overline{b}c)w+(b\overline{c}-\overline{a}d)\overline{w}+(b\overline{d}-\overline{b}d)=0$ …(2)

Let $\alpha=2(a\overline{d}-\overline{b}c),\ \beta=i(b\overline{d}-\overline{b}d)$ then (2) becomes,

$\frac{\alpha}{2}w+\frac{\overline{\alpha}}{2}\overline{w}+\frac{\beta}{i}=0$

$i\left(\alpha w+\overline{\alpha w}\right)+2\beta=0$

$i.2i.\operatorname{Im}\left(\alpha w\right)+2\beta=0$

$\operatorname{Im}\left(\alpha w\right)-\beta=0$ …(3)

Let $\alpha=p+iq, \quad w=x+iy$ then $\alpha w=px-qy+i(qx+py)$.

Therefore, $\operatorname{Im}\left(\alpha w\right)-\beta=(qx+py)-\beta=0$. Thus (3) represents a line $y=\left(\frac{-q}{p}\right)x+\beta$.

That is, $w$ lies on the line determined by (3) for fixed $\alpha$ and $\beta$. We know that straight line may be regarded as circle with infinite radius. Therefore, $w$ lies on the circle.

**Case 2**. When $a\overline{c}$ is not real.

Therefore, $a\overline{c}\neq\overline{a}c$, then from (1) we have,

$$|w|^2+\frac{(a\overline{d}-\overline{b}c)}{(a\overline{c}-\overline{a}c)}w+\frac{(b\overline{c}-\overline{a}d)}{(a\overline{c}-\overline{a}c)}\overline{w}+\frac{(b\overline{d}-\overline{b}d)}{(a\overline{c}-\overline{a}c)}=0$$

Let $\overline{\gamma}=\left(\frac{a\overline{d}-\overline{b}c}{a\overline{c}-\overline{a}c}\right), \delta=-\left(\frac{b\overline{d}-\overline{b}d}{a\overline{c}-\overline{a}c}\right)$.

Therefore, $|w|^2+\overline{\gamma}w+\gamma\overline{w}-\delta=0$

$$w\overline{w}+\overline{\gamma}w+\gamma\overline{w}+\gamma\overline{\gamma}=\delta+\gamma\overline{\gamma}$$

$$\left(w+\gamma\right)\left(\overline{w}+\overline{\gamma}\right)=\delta+\gamma\overline{\gamma}$$

$$|w+\gamma|^2=\delta+\gamma\overline{\gamma}$$

$$|w+\gamma|^2=\delta+\gamma\overline{\gamma}$$

Therefore, $|w+\gamma|=\lambda$ …(4)

where $\lambda=\left(\delta+\gamma\overline{\gamma}\right)^{1/2}=\left|\frac{ad-bc}{a\overline{c}-\overline{a}c}\right|>0$

Since $\gamma$ and $\lambda$ are independent of $w$, (4) represents a circle on which $w$ lies.

**Theorem 32** A Möbius transformation takes circles onto circles.

**Proof.** Let $S$ be a Möbius transformation. Let $\Gamma$ be a circle in $\mathbb{C}_\infty$ and $z_2, z_3, z_4$ are distinct points on $\Gamma$ such that $S(z_2) = w_2, S(z_3) = w_3, S(z_4) = w_4$. Then $w_2, w_3, w_4$ determine a circle $\Gamma'$.

We claim that $S(\Gamma) = \Gamma'$:

Since cross ratio is invariant under any Möbius transformation, for any $z$ in $\mathbb{C}_\infty$,

$$\begin{aligned}(z, z_2, z_3, z_4) &= (Sz, Sz_2, Sz_3, Sz_4)\\ &= (Sz, w_2, w_3, w_4)\end{aligned}$$

Now $z \in \Gamma \Leftrightarrow (z, z_2, z_3, z_4)$ is real.

$\Leftrightarrow (Sz, w_2, w_3, w_4)$ is real.

$\Leftrightarrow S(z) \in \Gamma'$

Thus $S(\Gamma) = \Gamma'$.

**Theorem 33** For any given circles $\Gamma$ and $\Gamma'$ in $\mathbb{C}_\infty$ there is a Möbius transformation $T$ such that $T(\Gamma) = \Gamma'$. Furthermore we can specify that $T$ takes any three points on $\Gamma$ onto any three points of $\Gamma'$. If we do specify $T(z_j) = w_j$ for $j = 2, 3, 4$ (distinct $z_j$ in $\Gamma$) then $T$ is unique.

**Proof.** Let $z_2, z_3, z_4$ be distinct points on $\Gamma$ and $w_2, w_3, w_4$ be points on $\Gamma'$. Let $S(z) = (z, z_2, z_3, z_4)$ and $M(z) = (z, w_2, w_3, w_4)$.

Let $T = M^{-1} \circ S$, then $T(z_2) = M^{-1} \circ S(z_2) = M^{-1}(S(z_2)) = M^{-1}(1) = w_2$

$$T(z_3) = M^{-1} \circ S(z_3) = M^{-1}(S(z_3)) = M^{-1}(0) = w_3$$

$$T(z_4) = M^{-1} \circ S(z_4) = M^{-1}(S(z_4)) = M^{-1}(\infty) = w_4$$

Thus $T$ is a Möbius transformation that takes $\Gamma$ onto $\Gamma'$.

Obviously Möbius transformation is unique.

## EXERCISES

**1)** Find fixed points of dilation, translation and inversion on $C_\infty$.

**2)** If $Tz = \dfrac{\alpha z + \beta}{\gamma z + \delta}$ and $Sz = \dfrac{az + b}{cz + d}$. Prove that T = S iff $\alpha = a\lambda,\ \beta = b\lambda\ \ \gamma = c\lambda,\ \delta = d\lambda$, for some complex number $\lambda$.

❍ ❍ ❍

**UNIT - III**

# COMPLEX INTEGRATION

In this section we shall study complex integrations of complex functions and established fundamental theorem of calculus for line integral. We show that an analytic function has a power series expansion as a Taylor's theorem. Form then we established Cauchy's estimate to prove Cauchy's theorem. We begin with elementary definitions

**Definition 1** A path in region $G \subset \mathbb{C}$ is a continuous function $\gamma : [a,b] \to G$ for some interval $[a,b]$ in $\mathbb{R}$.

If $\gamma'(t)$ exists for each $t$ in $[a,b]$ and $\gamma' : [a,b] \to \mathbb{C}$ is continuous then $\gamma$ is called **smooth path**. Also $\gamma$ is called **piecewise smooth** if there is partition of $[a,b]$, $a = t_0 < t_1 < ... < t_n = b$, such that $\gamma$ is smooth on each subinterval $[t_{j-1}, t_j],\ 1 \le j \le n$.

**Definition 2** Let $\gamma : [a,b] \to G$ be a path then trace of $\gamma$ is $\{\ \gamma(t) : t \in [a,b]\ \}$ and it is denoted by $\{\gamma\}$.

i.e. $\{\gamma\} = \{\ \gamma(t) : t \in [a,b]\ \}$. Note that trace of a path is always compact.

**Definition 3** A function $\gamma : [a,b] \to \mathbb{C}$, for $[a,b] \subset \mathbb{R}$, is of bounded variation if there is a constant $M > 0$ such that for any partition $P = \{a = t_0 < t_1 < ... < t_m = b\}$ of $[a,b]$

$$\upsilon(\gamma; P) = \sum_{k=1}^{m} |\gamma(t_k) - \gamma(t_{k-1})| \le M .$$

The **total variation** of $\gamma$, denoted by $V(\gamma)$ is defined as $V(\gamma) = \sup\{\upsilon(\gamma; P) : P \text{ a partition of [a,b] }\}$.

**Definition 4** A path $\gamma : [a,b] \to \mathbb{C}$ is rectifiable if $\gamma$ is a function of bounded variation.

**Theorem 5** Let $\gamma:[a,b]\to\mathbb{C}$ be of bounded variation. Then:

a) If $P$ and $Q$ are partitions of $[a,b]$ and $P\subset Q$ then $\upsilon(\gamma;P)\le\upsilon(\gamma;Q)$.

b) If $\sigma:[a,b]\to\mathbb{C}$ is also of bounded variation and $\alpha,\beta\in\mathbb{C}$ then $\alpha\gamma+\beta\sigma$ is of bounded variation and $V(\alpha\gamma+\beta\sigma)\le|\alpha|V(\gamma)+|\beta|V(\sigma)$.

**Theorem 6** If $\gamma:[a,b]\to\mathbb{C}$ is piecewise smooth then $\gamma$ is of bounded variation and $V(\gamma)=\int_a^b|\gamma'(t)|\,dt$.

**Proof.** Firstly we assume that $\gamma$ is smooth so that $\gamma'$ is continuous. Let $P=\{a=t_0<t_1<...<t_m=b\}$ then

$$\upsilon(\gamma;P)=\sum_{k=1}^{m}|\gamma(t_k)-\gamma(t_{k-1})|=\sum_{k=1}^{m}\left|\int_{t_{k-1}}^{t_k}\gamma'(t)dt\right|$$

$$\le\sum_{k=1}^{m}\int_{t_{k-1}}^{t_k}|\gamma'(t)|\,dt$$

$$=\int_a^b|\gamma'(t)|\,dt.$$

Hence, $V(\gamma)\le\int_a^b|\gamma'(t)|\,dt$, so that $\gamma$ is of bounded variation.

Since $\gamma'$ is continuous it is uniformly continuous. Thus for given $\in>0$ we can choose $\delta_1>0$ such that $|\gamma'(s)-\gamma'(t)|<\in$ whenever $|s-t|<\delta_1$. Also we choose $\delta_2>0$ such that if $P=\{a=t_0<t_1<...<t_m=b\}$ and $\|P\|=\max\{(t_k-t_{k-1}):1\le k\le m\}<\delta_2$ then

$$\left|\int_a^b|\gamma'(t)|\,dt-\sum_{k=1}^{m}\int_{t_{k-1}}^{t_k}|\gamma'(\tau_k)|(t_k-t_{k-1})\right|<\in \text{ where } \tau_k \text{ is any point in } [t_{k-1},t_k].$$

$$\int_a^b|\gamma'(t)|\,dt\le\ \in+\sum_{k=1}^{m}\int_{t_{k-1}}^{t_k}|\gamma'(\tau_k)|(t_k-t_{k-1})$$

$$=\in+\sum_{k=1}^{m}\left|\int_{t_{k-1}}^{t_k}\gamma'(\tau_k)dt\right|$$

$$\leq \in + \sum_{k=1}^{m}\left|\int_{t_{k-1}}^{t_k}\left[\gamma'(\tau_k)-\gamma'(t)\right]dt\right| + \sum_{k=1}^{m}\left|\int_{t_{k-1}}^{t_k}\gamma'(t)dt\right|$$

If $\|P\| < \delta = \min\left(\delta_{1,}\delta_2\right)$ then $|\gamma'(\tau_k)-\gamma'(t)| < \in$ for $t$ in $[t_{k-1}, t_k]$ and

$$\int_a^b |\gamma'(t)|\,dt \leq \in + \in\left(b-a\right) + \sum_{k=1}^{m}|\gamma(t_k)-\gamma(t_{k-1})|$$

$$= \in\left[1+\left(b-a\right)\right] + \upsilon\left(\gamma;P\right)$$

$$\leq \in\left[1+\left(b-a\right)\right] + V\left(\gamma\right).$$

Letting $\in \to 0+$ we get $\int_a^b |\gamma'(t)|\,dt \leq V\left(\gamma\right)$.

Thus $V\left(\gamma\right) = \int_a^b |\gamma'(t)|\,dt$.

**Theorem 7** Let $f$ and $g$ be continuous functions on $[a,b]$ and let $\gamma$ and $\sigma$ be functions of bounded variation on $[a,b]$. Then for any scalar $\alpha$ and $\beta$:

a) $\int_a^b \left(\alpha f + \beta g\right) d\gamma = \alpha\int_a^b fd\gamma + \beta\int_a^b gd\gamma$

b) $\int_a^b fd\left(\alpha\gamma + \beta\sigma\right) = \alpha\int_a^b fd\gamma + \beta\int_a^b gd\sigma$.

**Theorem 8** If $\gamma$ is piecewise smooth and $f:[a,b]\to\mathbb{C}$ is continuous then $\int_a^b f\ d\gamma = \int_a^b f(t)\ \gamma'(t)dt$.

**Proof.** To prove this theorem we consider real and imaginary parts of $\gamma$, we reduce the proof to the case where $\gamma\left([a,b]\right)\subset\mathbb{R}$. For any $\in > 0$ choose $\delta > 0$ such that if $P = \left\{a = t_0 < t_1 < ... < t_n = b\right\}$ has $\|P\| < \delta$ then

$$\left|\int_a^b f\ d\gamma - \sum_{k=1}^{n}\int_{t_{k-1}}^{t_k} f(\tau_k)\left[\gamma\left(t_k\right)-\gamma\left(t_{k-1}\right)\right]\right| < \frac{\in}{2} \text{ and}$$

$$\left|\int_a^b f(t)\gamma'(t)dt-\sum_{k=1}^{n}\int_{t_{k-1}}^{t_k} f(\tau_k)\gamma'(\tau_k)\left(t_k-t_{k-1}\right)\right|<\frac{\in}{2} \text{ for any choice of } \tau_k \text{ in } [t_{k-1},t_k]$$

Now by Mean Value Theorem

$$\sum_{k=1}^{n} f(\tau_k)\left[\gamma\left(t_k\right)-\gamma\left(t_{k-1}\right)\right]=\sum_{k=1}^{n} f(\tau_k)\gamma'(\tau_k)\left(t_k-t_{k-1}\right) \text{ for some } \tau_k \text{ in } [t_{k-1},t_k].$$

Combining this with above two inequalities we get

$\left|\int_a^b f\,d\gamma-\int_a^b f(t)\,\gamma'(t)dt\right|<\in$. Since $\in>0$ was arbitrary we have, $\int_a^b f\,d\gamma=\int_a^b f(t)\,\gamma'(t)dt$.

**Definition 9** If $\gamma:[a,b]\to\mathbb{C}$ is a rectifiable path and $f$ is a function defined and continuous on the trace of $\gamma$ then (line) **integral of $f$ along** $\gamma$ is $\int_a^b f\left(\gamma(t)\right)d\gamma\left(t\right)$. This line integral is also denoted by $\int_\gamma f=\int_\gamma f(z)dz$.

**Definition 10** Let $\gamma:[a,b]\to\mathbb{C}$ and $\sigma:[c,d]\to\mathbb{C}$ be rectifiable paths. The path $\sigma$ is *equivalent* to $\gamma$ if there is a continuous function $\varphi:[c,d]\to[a,b]$, which is strictly increasing, and with $\varphi(c)=a, \varphi(d)=b$, such that $\sigma=\gamma\circ\varphi$.

The idea is to recognize all the paths having same trace as identical. The above definition brings about a partition of the class of all paths. Thus we are prompted to define.

A **curve** is an equivalence class of paths. The trace of a curve is the trace of any one of its members. A curve is smooth (piecewise smooth) if and only if one of its representative is smooth (piecewise smooth). A curve C is called **simple** if it does not cross over itself. That is $\gamma\left(t_1\right)\neq\gamma\left(t_2\right)$ whenever $t_1\neq t_2$. A curve C is called a **simple closed curve** if i) $\gamma(a)=\gamma(b)$ ii) $\gamma\left(t_1\right)\neq\gamma\left(t_2\right)$ whenever $t_1\neq t_2$, except when $t_1=a$ and $t_2=b$. It is also called **Jordan curve**.

**Theorem 11** Let $\gamma$ be a rectifiable curve and suppose that $f$ is a function continuous on $\{\gamma\}$. Then:

a) $\int\limits_{\gamma} f = -\int\limits_{-\gamma} f$;

b) $\left|\int\limits_{\gamma} f\right| \le \int\limits_{\gamma} |f||dz| \le V(\gamma)\sup\left[|f(z)| : z \in \{\gamma\}\right]$;

c) If $c \in \mathbb{C}$ then $\int\limits_{\gamma} f(z)dz = \int\limits_{\gamma+c} f(z-c)dz$.

We shall conclude this chapter with

**Theorem 12** Let $G$ be an open set in $\mathbb{C}$ and let $\gamma$ be a rectifiable path in $G$ with initial and end points $\alpha$ and $\beta$ respectively. If $f : G \to \mathbb{C}$ is a continuous function with primitive $F : G \to \mathbb{C}$, then $\int\limits_{\gamma} f = F(\beta) - F(\alpha)$.

We now prove Leibnitz theorem:

**Theorem 13** Let $\varphi : [a,b] \times [c,d] \to \mathbb{C}$ be a continuous function. Define $g : [c,d] \to \mathbb{C}$ by

$$g(t) = \int_a^b \varphi(s,t)\, ds \ , \ \forall\, t \in [c,d]$$

Then,

i) g is continuous function and

ii) If $\dfrac{\partial \varphi}{\partial t}$ exists and continuous, then g is continuously differentiable.

Moreover, $g'(t) = \int_a^b \dfrac{\partial}{\partial t}\varphi(s,t)\, ds$.

**Proof :** i) Let $t_0 \in [c,d]$ and $\in > 0$.

Since $\varphi$ is continuous on $[a,b]\times[c,d]$, we have $\varphi$ is uniformly continuous on $[a,b]\times[c,d]$.

Therefore, there is $\delta > 0$ such that for each $s \in [a,b]$, we have,

$\left| \varphi(s,t_0+h)-\varphi(s,t_0) \right| < \dfrac{\in}{b-a}$, whenever $\left| t_0+h-t_0 \right| < \delta$

Now, for $\left| h \right| < \delta$ we have

$$\left| g(t_0+h)-g(t_0) \right| = \left| \int_a^b \varphi(s,t_0+h)\,ds - \int_a^b \varphi(s,t_0)\,ds \right| = \left| \int_a^b [\ \varphi(s,t_0+h)-\varphi(s,t_0)\ ]\,ds \right|$$

$$\leq \int_a^b \left| \varphi(s,t_0+h)-\varphi(s,t_0) \right| ds \leq \int_a^b \frac{\in}{b-a}\,ds = \frac{\in}{b-a}\,(b-a)$$

$= \in$

Thus, $\left| g(t_0+h)-g(t_0) \right| \leq \in$, whenever $\left| h \right| < \delta$

i.e. $g$ is continuous at $t_0 \in [c,d]$

Since $t_0$ is arbitrary element of $[c,d]$, we have $g$ is continuous on $[c,d]$.

ii) Suppose that $\dfrac{\partial \varphi}{\partial t}$ exists and continuous

Let $t_0 \in [c,d]$ and $\in > 0$

Denote, $\varphi_2(s,t) = \dfrac{\partial}{\partial t}\varphi(s,t)$

Again since $\varphi_2(s,t)$ is uniformly continuous on $[a,b]\times[c,d]$, $\exists\ \delta > 0$ such that for each $s \in [a,b]$, we have

$\left| \varphi_2(s,t)-\varphi_2(s,t_0) \right| < \in$, whenever $\left| t-t_0 \right| < \delta$

Thus, for $\left| t-t_0 \right| < \delta$ and $s \in [a,b]$, we have $\left| \displaystyle\int_{t_0}^{t} [\ \varphi_2(s,\tau)-\varphi_2(s,t_0)\ ]\,d\tau \right| < \left| \displaystyle\int_{t_0}^{t} \in d\tau \right|$

i.e $\left| \displaystyle\int_{t_0}^{t} [\ \varphi_2(s,\tau)-\varphi_2(s,t_0)\ ]\,d\tau \right| < \in \left| t-t_0 \right|$, whenever $\left| t-t_0 \right| < \delta$ and $s \in [a,b]$

… (1)

Let $\Phi(t) = \varphi(s,t) - t\,\varphi_2(s,t_0)$, for some fixed $s \in [a,b]$. Then $\Phi'(t) = \varphi_2(s,t)-\varphi_2(s,t_0)$

i.e. $\Phi(t)$ is primitive of $\varphi_2(s,t)-\varphi_2(s,t_0)$

Then by Fundamental Theorem of Calculus for Line Integrals, we have

$$\int_{t_0}^{t}[\ \varphi_2(s,\tau)-\varphi_2(s,t_0)\ ]\,d\tau \ = \ \Phi(t)-\Phi(t_0)$$

$$=\left[\varphi(s,t)-\varphi(s,t_0)\right]-\left(t-t_0\right)\varphi_2(s,t_0)$$

Inequality (1) becomes,

$$\left|\ \left[\varphi(s,t)-\varphi(s,t_0)\right]-\left(t-t_0\right)\varphi_2(s,t_0)\ \right| \ < \in \left|\ t-t_0\ \right| \qquad \ldots(2)$$

Now,

$$\left|\ \frac{g(t)-g(t_0)}{t-t_0}-\int_a^b \varphi_2(s,t_0)\,ds\ \right| \ = \ \left|\frac{1}{t-t_0}\left[\ \int_a^b \varphi(s,t)\,ds-\int_a^b \varphi(s,t_0)\,ds\ \right]-\int_a^b \varphi_2(s,t_0)\,ds\right|$$

$$=\left|\ \int_a^b\left[\frac{\varphi(s,t)-\varphi(s,t_0)-\left(t-t_0\right)\varphi_2(s,t_0)}{t-t_0}\right]ds\right|$$

$$\leq \ \int_a^b\left|\frac{\varphi(s,t)-\varphi(s,t_0)-\left(t-t_0\right)\varphi_2(s,t_0)}{t-t_0}\right|ds$$

$$< \ \in \int_a^b \ ds \ = \ \in\left(b-a\right) \quad \text{(By equation (2))}$$

Thus, for each $\in>0$, $\exists\,\delta>0$ such that

$$\left|\ \frac{g(t)-g(t_0)}{t-t_0}-\int_a^b \varphi_2(s,t_0)\,ds\ \right| \ < \ \in\left(b-a\right), \text{ whenever } 0<\ \left|\ t-t_0\ \right|<\delta$$

i.e. $g$ is differentiable at $t_0$ and hence on [c, d].

Next, $g'(t_0)=\int_a^b \varphi_2(s,t_0)\,ds=\int_a^b \frac{\partial}{\partial t}\varphi(s,t_0)ds$

As $\varphi_2(s,t)=\dfrac{\partial\varphi}{\partial t}$ is continuous, we have $g'$ is continuous

Hence, $g$ is continuously differentiable.

**Example 14** Prove that $\displaystyle\int_0^{2\pi} \frac{e^{is}}{e^{is}-z}\,ds = 2\pi$, when $|\,z\,|<1$

**Solution:** Define $g:[0,1]\to\mathbb{C}$ by $\displaystyle g(t)=\int_0^{2\pi}\frac{e^{is}}{e^{is}-tz}\,ds$, $\forall\ t\in[0,1]$

By Leibnitz rule,

$$g'(t_0)=\int_0^{2\pi}\frac{\partial}{\partial t}\left(\frac{e^{is}}{e^{is}-tz}\right)ds \;=\int_0^{2\pi}\frac{z\,e^{is}}{\left(e^{is}-tz\right)^2}\,ds$$

Let $\displaystyle \Phi(s)=\frac{z\,i}{e^{is}-tz}$ . Then $\displaystyle \Phi'(s)=\frac{z\,e^{is}}{\left(e^{is}-tz\right)^2}$

Thus, $g'(t)=\Phi(2\pi)-\Phi(0)$ (By fundamental theorem of calculus for line integrals)

$$=\frac{z\,i}{e^{2\pi i}-tz}-\frac{z\,i}{e^{0i}-tz}$$

$$=0$$

i.e. $g'(t)=0,\quad \forall\ t\in[0,1]$

Therefore, $g$ is constant function

In particular, we have $g(1)=g(0)$

i.e. $\displaystyle\int_0^{2\pi}\frac{e^{is}}{e^{is}-z}\,ds=\int_0^{2\pi}ds$

i.e. $\displaystyle\int_0^{2\pi}\frac{e^{is}}{e^{is}-z}\,ds=2\pi$ .

**Exercise 15** Show that $\displaystyle\int_{|z|=1}\frac{dz}{z-a}=2\pi i,$ when $|a|<1$.

(Hint: put $z = e^{it}$ and use above example) .

The following theorem is known as Cauchy Integral Formula.

**Theorem 16** Let $f:G\to\mathbb{C}$ be analytic and suppose $\overline{B}(a,r)\subset G(r>0)$. If $\gamma(t)=a+re^{it}$, $0\le t\le 2\pi$, then $f(z)=\dfrac{1}{2\pi i}\displaystyle\int_\gamma \frac{f(w)}{w-z}dw$, for $|z-a|<r$.

**Proof.** Without loss of generality assume $a=0$ and $r=1$

That is we may assume that $\overline{B}(0,1)\subset G$

Let $z\in \overline{B}(0,1)$ ; we have to show that

$$f(z)=\frac{1}{2\pi i}\int_\gamma \frac{f(w)}{w-z}dw$$

$$=\frac{1}{2\pi i}\int_0^{2\pi}\frac{f(e^{is})e^{is}i}{e^{is}-z}ds$$

i.e. $2\pi\ f(z)=\displaystyle\int_0^{2\pi}\frac{f(e^{is})e^{is}}{e^{is}-z}ds$

i.e. $\displaystyle\int_0^{2\pi}\left[\frac{f(e^{is})e^{is}}{e^{is}-z}-f(z)\right]ds=0$

Let $\varphi(s,t)=\dfrac{f(z+t(e^{is}-z))e^{is}}{e^{is}-z}-f(z)$ where $0\le s\le 2\pi$ and $0\le t\le 1$

Since $|z+t(e^{is}-z)| = |z(1-t)+te^{is}| \le (1-t)|z|+t|e^{is}| < 1$, $\varphi$ is well defined and continuously differentiable

Now define $g:[0,1]\to\mathbb{C}$ by

$$g(t)=\int_0^{2\pi}\varphi(s,t)\,ds$$

Then by Leibnitz rule $g$ ha s continuous derivative

$$g'(t)=\int_0^{2\pi}\frac{\partial}{\partial t}\varphi(s,t)\,ds$$

$$= \int_0^{2\pi} \frac{\partial}{\partial t}\left( \frac{f(z+t(e^{is}-z))e^{is}}{e^{is}-z} - f(z) \right) ds$$

$$= \int_0^{2\pi} f'(z+t(e^{is}-z))e^{is} ds$$

Let $\Phi(s) = -i\, t^{-1}\ f(z+t(e^{is}-z))$ then $\Phi'(s) = f'(z+t(e^{is}-z))e^{is}$ for $0 < t \le 1$

$g'(t) = \Phi(2\pi) - \Phi(0)$ (By fundamental theorem of calculus for line integrals)

$$= -i\, t^{-1}\ f(z+t(e^{2\pi i}-z)) + i\, t^{-1}\ f(z+t(e^{0i}-z))$$

$$= 0$$

$g'(t) = 0$ for $0 < t \le 1$

Since $g$ is continuous on $[0,1]$ we have $g'(0) = 0$

$g'(t) = 0$ for $0 \le t \le 1$

$g$ is constant on $[0,1]$

$g(1) = g(0)$

$$\int_0^{2\pi} \left[ \frac{f(z+e^{is}-z)e^{is}}{e^{is}-z} - f(z) \right] ds = \int_0^{2\pi} \left[ \frac{f(z)e^{is}}{e^{is}-z} - f(z) \right] ds$$

$$= f(z)\left[ \int_0^{2\pi} \frac{e^{is}}{e^{is}-z} ds - \int_0^{2\pi} ds \right]$$

$$= f(z)\left[2\pi - 2\pi\right] = 0 .$$

**Lemma 17** Let $\gamma$ be a rectifiable curve in $\mathbb{C}$ and suppose that $F_n$ and $F$ are continuous functions on $\{\gamma\}$ . If $F = u - \lim F_n$ on $\{\gamma\}$ then $\int_\gamma F = \lim \int_\gamma F_n$ .

**Proof.** Since $F = u - \lim F_n$ , for given $\in > 0$ there is an integer $n_0 > 0$ such that

$\left| F_n(w) - F(w) \right| < \in / V(\gamma)$ for all $w \in \{\gamma\}$ and $n \ge n_0$ .

Therefore $\left| \int_\gamma F - \int_\gamma F_n \right| = \left| \int_\gamma (F-F)_n \right|$

$$\leq \int_\gamma |F(w) - F_n(w)||dw|$$

$$\leq \in$$

when $n \geq n_0$.

The following theorem gives the Taylor's series expansion of an analytic function:

**Theorem 18** Let $f$ be analytic in $B(a,R)$. Then $f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n$ for $|z-a| < R$ where $a_n = \frac{1}{n!} f^{(n)}(a)$ and this series has radius of convergence $\geq R$.

**Proof.** Since $f$ is analytic in $B(a,R)$, then there is $0 < r < R$ such that $\overline{B}(a,r) \subset B(a,R)$. Let $\gamma(t) = a + re^{it}$, $0 \leq t \leq 2\pi$ then by Cauchy integral formula , we have

$$f(z) = \frac{1}{2\pi i} \int_\gamma \frac{f(w)}{w-z} dw \quad \text{for } |z-a| < r. \qquad \ldots(1)$$

Now

$$\frac{1}{w-z} = \frac{1}{(w-a)-(z-a)} = \frac{1}{w-a} . \frac{1}{1-\left[\frac{z-a}{w-a}\right]}$$

$$= \frac{1}{w-a}\left(1-\left[\frac{z-a}{w-a}\right]\right)^{-1}$$

$$= \frac{1}{w-a}\left(1-\left[\frac{z-a}{w-a}\right]\right)^{-1}.$$

$$\frac{1}{w-z} = = \frac{1}{(w-a)} \sum_{n=0}^{\infty} \left(\frac{z-a}{w-a}\right)^n. \qquad \ldots(2)$$

Since $\{\gamma\}$ is compact and $f$ is continuous on $\{\gamma\}$, $f$ bounded on $\{\gamma\}$

Let $M = \sup\{\ |f(w)| \ : w \in \{\gamma\}\ \}$

Then

$$\left|\frac{f(w)(z-a)^n}{(w-a)^{n+1}}\right| = \frac{|f(w)||(z-a)^n|}{|(w-a)^{n+1}|} \le \frac{M}{|(w-a)|}\frac{|(z-a)^n|}{|(w-a)^n|}$$

$$\le \frac{M}{r}\left(\frac{|(z-a)|}{r}\right)^n$$

Let $M_n = \frac{M}{r}\left(\frac{|(z-a)|}{r}\right)^n$ , then $\sum_{n=0}^{\infty} M_n < \infty$ as $\frac{|z-a|}{r} < 1$

By Weierstrass M- test the series $\sum_{n=0}^{\infty}\frac{f(w)(z-a)^n}{(w-a)^{n+1}}$ converges uniformly to $\frac{f(w)}{w-z}$ for $w \in \{\gamma\}$

In view of (1), (2) and Lemma

$$f(z) = \frac{1}{2\pi i}\int_{\gamma}\frac{f(w)}{w-z}dw = \frac{1}{2\pi i}\int_{\gamma}\sum_{n=0}^{\infty}\frac{f(w)(z-a)^n}{(w-a)^{n+1}}dw.$$

$$= \sum_{n=0}^{\infty}\frac{1}{2\pi i}\int_{\gamma}\frac{f(w)(z-a)^n}{(w-a)^{n+1}}dw$$

$$= \sum_{n=0}^{\infty}\left(\frac{1}{2\pi i}\int_{\gamma}\frac{f(w)}{(w-a)^{n+1}}dw\right)(z-a)^n$$

$$= \sum_{n=0}^{\infty} a_n (z-a)^n \quad \text{where} \quad a_n = \frac{1}{2\pi i}\int_{\gamma}\frac{f(w)}{(w-a)^{n+1}}dw,\ |z-a| < r$$

Thus $f$ has power series expansion in $B(a, R)$

$a_n = \frac{1}{n!}f^{(n)}(a)$, so that value of $a_n$ is independent of $\gamma$, hence independent of $r$

Moreover , as $0 < r < R$ is arbitrary , we have

$$f(z)=\sum_{n=0}^{\infty} a_n (z-a)^n \text{ for } |z-a|<R$$

clearly radius of convergence $\geq R$ .

**Corollary 19** If $f: G \to \mathbb{C}$ is analytic and $a \in G$ then $f(z)=\sum_{n=0}^{\infty} a_n (z-a)^n$ for $|z-a|<R$ where $R = d(a, \partial G)$.

**Proof.** Since $R = d(a,\partial G) = inf\{d(a,\partial G) : z \in \partial G\} = inf\{|z-a| : z \in \partial G\}$, $B(a,R) \subset G$

Since $f$ is analytic in $G$ , $f$ is analytic on $B(a,R)$

Hence by Taylor's theorem,

$$f(z)=\sum_{n=0}^{\infty} a_n (z-a)^n \quad \text{for} \quad |z-a|<R$$

**Corollary 20** If $f: G \to \mathbb{C}$ is analytic then $f$ is infinitely differentiable.

**Proof.** Suppose $f: G \to \mathbb{C}$ is analytic, then for any $a \in G$, $f$ has Taylor's series expansion about $a$

$$f(z)=\sum_{n=0}^{\infty} a_n (z-a)^n \quad \text{for} \quad |z-a|<R \ , \ R = d(a,\partial G)$$

Then by theorem, $f$ is infinitely differentiable.

**Corollary 21** If $f: G \to \mathbb{C}$ is analytic and $\overline{B}(a,r) \subset G$ then $f^{(n)}(a) = \frac{n!}{2\pi i}\int_{\gamma} \frac{f(w)}{(w-a)^{n+1}} dw$

where $\gamma(t) = a + re^{it}$ , $0 \leq t \leq 2\pi$

**Proof.** Suppose $f: G \to \mathbb{C}$ is analytic and $\overline{B}(a,r) \subset G$

Let $\gamma(t) = a + re^{it}$ , $0 \leq t \leq 2\pi$ .

Then by Taylor's theorem,

$$f(z)=\sum_{n=0}^{\infty} a_n (z-a)^n \quad \text{where } a_n = \frac{1}{2\pi i}\int_{\gamma} \frac{f(w)}{(w-a)^{n+1}} dw$$

We also have,

$$a_n = \frac{1}{n!} f^{(n)}(a)$$

$$f^{(n)}(a) = \frac{n!}{2\pi i} \int_\gamma \frac{f(w)}{(w-a)^{n+1}} dw .$$

**Example 22** Evaluate the integral $\int_\gamma \frac{e^{iz}}{z^2} dz$ where $\gamma(t) = re^{it}$, $0 \le t \le 2\pi$ .

**Solution:** Let $f(z) = e^{iz}$ , then $f$ is analytic function

We have

$$f^{(1)}(0) = \frac{1!}{2\pi i} \int_\gamma \frac{f(z)}{(z-0)^{1+1}} dz$$

$$ie^{i0} = \frac{1!}{2\pi i} \int_\gamma \frac{e^{iz}}{z^2} dz$$

$$\int_\gamma \frac{e^{iz}}{z^2} dz = -2\pi .$$

**Example 28** Evaluate the integral $\int_\gamma \frac{\sin z}{z^3} dz$ where $\gamma(t) = re^{it}$, $0 \le t \le 2\pi$ .

**Solution:** Let $f(z) = \sin z$ , then $f$ is analytic function.

We have

$$f^{(2)}(0) = \frac{2!}{2\pi i} \int_\gamma \frac{f(z)}{(z-0)^{2+1}} dz$$

$$-\sin(0) = \frac{1}{\pi i} \int_\gamma \frac{\sin z}{z^3} dz$$

Therefore $\int_\gamma \frac{\sin z}{z^3} dz = 0$ .

**Example 23** Evaluate the integral $\int_{\gamma} \frac{1}{z-a} dz$ where $\gamma(t) = a + re^{it}$, $0 \le t \le 2\pi$.

**Solution:** Let $f(z) = 1$, then $f$ is analytic function.

We have

$$f^{(0)}(a) = \frac{0!}{2\pi i} \int_{\gamma} \frac{f(z)}{(z-a)^{0+1}} dz$$

$$1 = \frac{1}{2\pi i} \int_{\gamma} \frac{1}{z-a} dz$$

Therefore, $\int_{\gamma} \frac{1}{z-a} dz = 2\pi i$.

**Example 24** Evaluate the integral $\int_{\gamma} \frac{e^z + \sin z}{z} dz$ where $\gamma(t) = re^{it}$, $0 \le t \le 2\pi$.

**Solution:** Let $f(z) = e^z + \sin z$, then $f$ is analytic function.

We have

$$f^{(0)}(0) = \frac{1!}{2\pi i} \int_{\gamma} \frac{f(z)}{(z-0)^{0+1}} dz$$

$$e^0 + \sin 0 = \frac{0!}{2\pi i} \int_{\gamma} \frac{e^z + \sin z}{z} dz$$

Therefore $\int_{\gamma} \frac{e^z + \sin z}{z} dz = 2\pi i$.

The following theorem is known as Cauchy's Estimate

**Theorem 25** Let $f$ be analytic in $B(a,R)$ and suppose that $|f(z)| \le M$ for all $f$ in $B(a,R)$.

Then $\left|f^{(n)}(a)\right| \le \frac{n!M}{R^n}$

**Proof.** Since $f$ is analytic in $B(a,R)$, then there is $0 < r < R$ such that $\overline{B}(a,r) \subset B(a,R)$.

Then by corollary,

$$f^{(n)}(a) = \frac{n!}{2\pi i} \int_{\gamma} \frac{f(w)}{(w-a)^{n+1}} dw \quad \text{where } \gamma(t) = a + re^{it},\ 0 \le t \le 2\pi .$$

Therefore $\left|f^{(n)}(a)\right| = \left|\frac{n!}{2\pi i}\int_\gamma \frac{f(w)}{(w-a)^{n+1}}dw\right|$

$$\le \frac{n!}{2\pi}\int_\gamma \frac{|f(w)|}{\left|(w-a)^{n+1}\right|}|dw|$$

$$\le \frac{n!}{2\pi}\frac{M}{r^{n+1}}\int_\gamma |dw|$$

$$\le \frac{n!}{2\pi}\frac{M}{r^{n+1}}2\pi r.$$

$\left|f^{(n)}(a)\right| \le \frac{n!M}{r^n}$.

Since $0 < r < R$ is orbitrary , letting $r \to R^-$ we get

$\left|f^{(n)}(a)\right| \le \frac{n!M}{R^n}$.

Cauchy's estimate leads us to Cauchy's Theorem:

**Theorem 26** Let $f$ be analytic in the disk $B(a,R)$ and suppose that $\gamma$ is a closed rectifiable curve in $B(a,R)$. Then $\int_\gamma f = 0$.

**Proof.** Since $f$ is analytic in $B(a,R)$, it has power series expansion

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n \qquad \text{for } |z-a| < R. \qquad \ldots(1)$$

Let

$$F(z) = \sum_{n=0}^{\infty}\left(\frac{a_n}{n+1}\right)(z-a)^{n+1} = (z-a)\sum_{n=0}^{\infty}\left(\frac{a_n}{n+1}\right)(z-a)^n \qquad \ldots(2)$$

Since $\lim\left(n+1\right)^{1/n} = 1$

$$\lim Sup\left|\frac{a_n}{n+1}\right|^{1/n} = \lim Sup|a_n|^{1/n}\lim\left(\frac{1}{n+1}\right)^{1/n} = \lim Sup|a_n|^{1/n}.$$

Thus series (1) and (2) have same radius of convergence.

Therefore, $F$ is defined and analytic in $B(a,R)$.

Hence

$$F'(z)=\sum_{n=0}^{\infty}\left(\frac{a_n}{n+1}\right)(n+1)(z-a)^n=\sum_{n=0}^{\infty}a_n(z-a)^n=f(z).$$

i.e. $F$ is primitive of $f$.

If $\gamma:[a,b]\to\mathbb{C}$ then

$\int_\gamma f=\int_\gamma F'=F(\gamma(b))-F(\gamma(a))=0$, Since $\gamma$ is a closed curve $\gamma(a)=\gamma(b)$.

# EXERCISES

1) **1)** Evaluate the integral $\int_\gamma \frac{1}{z}dz$, where $\gamma(t)=e^{int}$, $0\le t\le 2\pi$, n is some positive integer.
2) **2)** Evaluate the integral $\int_\gamma z^n dz$,, where $\gamma$ is a closed polygonal curve [1-i, 1+i, -1+i, -1-i, 1-i].
3) **3)** Let $\gamma$, $\delta$ be polygons [1, 1+i, i] and [1, i] respectively. Evaluate the integral $\int|z^2|dz$ over $\gamma$ *as well as* $\delta$.
4) **4)** **Evaluate** $\int_\gamma \frac{e^{iz}}{z^2}dz$, where $\gamma(t)=e^{it}$, $0\le t\le 2\pi$.
5) **5)** **Evaluate** $\int_\gamma \frac{\sin(z)}{z^3}dz$, where $\gamma(t)=e^{it}$, $0\le t\le 2\pi$.
6) **6)** Let G be connected set and $f:G\to\mathbb{C}$ be analytic function. If f(z) is real for all z in G, the prove that f is constant function.
7) **7)** Prove above exercise for a) f(z) is imaginary number for all z and b) f(z) with constant modulus.

UNIT - IV

# FUNDAMENTAL THEOREM OF ALGEBRA AND MAXIMUM MODULUS THEOREM

In this unit we prove Liouville's theorem use it to prove fundamental theorem of algebra. We also prove maximum modulus theorem.

**Definition 1** An entire (integral) function is a function which is defined and analytic in the whole complex plane $\mathbb{C}$ .

**Note 1.** $f(z) = e^z, \sin z, \cos z$ are entire functions.

**2.** All polynomials are entire functions.

**Theorem 2** If $f$ is entire function then $f$ has power series expansion $f(z) = \sum_{n=0}^{\infty} a_n z^n$ with infinite radius of convergence.

**Proof.** For any $R > 0$ , $B(a,R) \subset \mathbb{C}$ .Then $f$ is analytic in $B(a,R)$.

By Taylor's theorem,

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \quad , \text{ for } |z| < R.$$

Since $R > 0$ is arbitrary, radius of convergence is infinite.

Following theorem is known as Liouville's theorem

**Theorem 3** If $f$ is bounded and entire function then $f$ is constant.

**Proof.** Since $f$ is bounded and entire function, $|f(z)| \le M \quad \forall\, z \in \mathbb{C}$ and for $a \in \mathbb{C}$ , $R > 0$ , $f$ is analytic in $B(a,R)$.

By Cauchy's Estimate, we have $\left|f^{(n)}(a)\right| \le \dfrac{n!M}{R^n}$

In particular for $n=1$ we have $\left|f^{'}(a)\right| \leq \frac{n!M}{R}$

Since $R$ is arbitrary , as $R \to \infty$, we get $\left|f^{'}(a)\right| \leq 0$ .

Therefore, $f^{'}(a)=0$ for any $a \in \mathbb{C}$ .

Thus $f$ is constant.

Thus we can prove the Fundamental theorem of Algebra:

**Theorem 3** If $p(z)$ is a non constant polynomial then there is a complex number $a$ with $p(a)=0$.

**Proof.** Let $p(z)$ is a non constant polynomial and that $p(z) \neq 0$ for any $z \in \mathbb{C}$ .

Let $f(z)=\frac{1}{p(z)}$ then $f$ is entire function . ( $p(z)$ is entire and $p(z) \neq 0$ )

Since $p(z)$ is non constant polynomial, assume that $p(z)=z^n+a_1z^{n-1}+a_2z^{n-2}+\ldots++a_n$

$\lim\limits_{z\to\infty} p(z)=\lim\limits_{z\to\infty} z^n\left(1+a_1z^{-1}+a_2z^{-2}+\ldots+a_nz^{-n}\right)=\infty$

$\lim\limits_{z\to\infty} f(z)=\lim\limits_{z\to\infty}\frac{1}{p(z)}=0$

Therefore, for $\in=1$ there is $R>0$ such that $\left|f(z)-0\right|<1$, whenever $|z|>R$.

That is $\left|f(z)\right|<1$, whenever $|z|>R$.

Since $f$ is continuous on closed bounded disk $\overline{B}(0,R) \subset \mathbb{C}$ , $f$ is bounded on $\overline{B}(0,R)$.

Therefore, there is $M>0$ such that $\left|f(z)\right| \leq M$ for $z \in \overline{B}(0,R)$

That is $\left|f(z)\right| \leq M$ , whenever $|z| \leq R$.

Thus $\left|f(z)\right| \leq \max\left\{1,M\right\}$, for all $z \in \mathbb{C}$ .

This $f$ is bounded entire function.

Hence by Liouville's theorem $f$ is constant and consequently $p$ is constant.

Which contradicts our assumption.

Hence the theorem.

**Definition 4** Let $f : G \to \mathbb{C}$ be analytic and $a \in G$ satisfies $f(a) = 0$ then $a$ is zero of $f$ of multiplicity $m \geq 1$ if there is an analytic function $g : G \to \mathbb{C}$ such that $f(z) = (z-a)^m g(z)$, where $g(a) \neq 0$.

**Corollary 5** If $p(z)$ is a polynomial and $a_1, a_2, ..., a_m$ are its zeros with $a_j$ having multiplicity $k_j$ then $p(z) = c(z-a_1)^{k_1} ...(z-a_m)^{k_m}$ for some constant $c$ and $k_1 + k_2 + ... + k_m$ is the degree of $p$.

**Proof.** Since $a_1, a_2, ..., a_m$ are zeros of $p(z)$ having multiplicities $k_1, k_2, ..., k_m$ respectively, there exists a polynomial $g(z)$ such that $p(z) = (z-a_1)^{k_1} ...(z-a_m)^{k_m} g(z)$, where $g(a_j) \neq 0, \quad (1 \leq j \leq m)$.

Therefore by fundamental theorem of algebra, $g(z)$ is forced to be constant.

Let $g(z) = c$ for some $c \in \mathbb{C}$

$p(z) = c\,(z-a_1)^{k_1} ...(z-a_m)^{k_m}$ .

Obviously, degree of $p(z)$ is $k_1 + k_2 + ... + k_m$.

**Theorem 6** Let $G$ be a connected open set and let $f : G \to \mathbb{C}$ be analytic function. Then the following are equivalent statements:

(a) $f \equiv 0$;

(b) $\{ z \in G : f(z) = 0 \}$ has limit point in $G$;

(c) there is a point $a$ in $G$ such that $f^n(a) = 0$ for each $n \geq 0$.

**Proof.**

$(a) \Rightarrow (b)$

suppose $f \equiv 0$, then $\{ z \in G : f(z) = 0 \} = G$, which is open.

Hence, every point of $G$ is a limit point of $G$.

Thus $\{ z \in G : f(z) = 0 \}$ has limit point in $G$.

$(b) \Rightarrow (c)$

Suppose that, $A = \{ z \in G : f(z) = 0 \}$ has limit point $a$ in $G$, then there is a sequence $\{z_n\}$ of points in $A$ such that $a = \lim z_n$.

Since $f$ is continuous, $f(a) = \lim f(z_n) = 0$ $\quad ( z_n \in A \ \therefore \ f(z_n) = 0 \ )$

Now suppose that, there is an integer $n \geq 1$ such that $f(a) = f'(a) = ... = f^{(n-1)}(a) = 0$ and $f^{(n)}(a) \neq 0$.

Since $f$ is analytic in $G$, there is $R > 0$ such that $f$ is analytic in $B(a;R) \subset G$.

By Taylor's theorem

$$f(z) = \sum_{k=0}^{\infty} a_k (z-a)^k \quad \text{for } |z-a| < R \quad \text{where} \quad a_k = \frac{1}{k!} f^{(k)}(a).$$

Since $f(a) = f'(a) = ... = f^{(n-1)}(a) = 0$ and $f^{(n)}(a) \neq 0$ we have

$$f(z) = \sum_{k=n}^{\infty} a_k (z-a)^k = (z-a)^n \sum_{k=n}^{\infty} a_k (z-a)^{k-n}$$

Let $g(z) = \sum_{k=n}^{\infty} a_k (z-a)^{k-n}$, then $g$ is analytic in $B(a;R)$ and

$f(z) = (z-a)^n g(z)$ and $g(a) = a_n \neq 0$.

Since $g$ is continuous in $B(a;R)$, there is $R > r > 0$ such that $g(z) \neq 0$ in $B(a;r)$.

As $a$ is limit point of $A$, $B(a;r) \cap A - \{a\} \neq \phi$.

Let $b \in B(a;r) \cap A - \{a\}$, then $b \in B(a;r)$ and $b \in A - \{a\} \subset A$.

Therefore, $b \in B(a;r) \Rightarrow \quad g(b) \neq 0$ and $b \in A - \{a\} \subset A \Rightarrow \quad f(b) = 0 \Rightarrow g(b) = 0$

which gives the contradiction to our assumption.

Hence no such integer $n \geq 1$ can be found.

Thus $f^n(a) = 0$ for each $n \geq 0$.

$(b) \Rightarrow (c)$

Suppose there is $a$ in $G$ such that $f^n(a) = 0$ for each $n \geq 0$.

Let $H = \{ z \in G : f^{(n)}(z) = 0 \}$ then $H \neq \phi$.

Now we claim that $H$ is both open and closed :

Let $a \in H$, then there is $R > 0$ such that $B(a;R) \subset G$ $R > 0$. Then $f$ is analytic in $B(a;R) \subset G$.

By Taylor's theorem

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n \qquad \text{for } |z-a| < R \qquad \text{where} \quad a_n = \frac{1}{n!} f^{(n)}(a).$$

Since $f^n(a) = 0$ for each $n \geq 0$, each $a_n = 0$.

$f(z) = 0$ in $B(a;R)$ i.e. $f \equiv 0$ in $B(a;R)$.

$f^n(z) = 0$ in $B(a;R)$

$B(a;R) \subseteq H$

Thus for $a \in H$, then there is $R > 0$ such that $B(a;R) \subseteq H$.

Hence $H$ is open.

Now let $z$ be limit point of $H$, then there is a sequence $\{z_m\}$ in $H$ such that $z = \lim z_m$

Since $f^{(n)}$ is continuous , $f^{(n)}(z) = \lim f^{(n)}(z_n) = 0$

Therefore $f^{(n)}(z) = 0$ implies $z \in H$.

Thus $\overline{H} \subseteq H$.

Hence $H$ is closed.

Thus $H$ is open as well as closed subset of connected set $G$.

Hence by property of connectedness $H = G$.

Therefore, $f^{(n)}(z) = 0 \quad \forall\, z \in G, n \geq 0$

That is $f(z) = 0 \quad \forall\, z \in G$

Hence $f \equiv 0$ on $G$.

**Corollary 7** If $f$ and $g$ are analytic on a region $G$, then $f \equiv g$ iff $\{\, z \in G : f(z) = g(z) \,\}$ has a limit point in $G$.

**Proof.** Let $h(z) = f(z) - g(z) \quad \forall\, z \in G$, which is analytic in $G$.

Then $h \equiv 0$ on $G \Leftrightarrow \{ z \in G : h(z) = 0 \}$ has a limit point in $G$.

i.e. $f(z) - g(z) = 0$ on $G \Leftrightarrow \{ z \in G : f(z) - g(z) = 0 \}$ has a limit point in $G$.

i.e. $f(z) = g(z)$ on $G \Leftrightarrow \{ z \in G : f(z) = g(z) \}$ has a limit point in $G$.

i.e. $f \equiv g$ on $G \Leftrightarrow \{ z \in G : f(z) = g(z) \}$ has a limit point in $G$.

**Corollary 8** If $f$ is analytic on an open connected set $G$ and $f$ is not identically zero then for each $a$ in $G$ with $f(a) = 0$ there is an integer $n \geq 1$ and an analytic function $g : G \to \mathbb{C}$ such that $g(a) \neq 0$ and $f(z) = (z-a)^n g(z)$ for all $z$ in $G$. That is each zero of $f$ has finite multiplicity.

**Proof.** Let $f$ be analytic on an open connected set $G$. Since $f \neq 0$ and $f(a) = 0$ for some $a$ in $G$, there is positive integer $n \geq 1$ such that $f(a) = f'(a) = \ldots = f^{(n-1)}(a) = 0$ and $f^{(n)}(a) \neq 0$.

Now we define $g : G \to \mathbb{C}$, by

$$g(z) = \frac{f(z)}{(z-a)^n} \qquad \text{for} \qquad z \neq a$$

$$= \frac{f^{(n)}(a)}{n!} \qquad \text{for} \qquad z = a \qquad \ldots(1)$$

Therefore $g$ is analytic on $G - \{a\}$.

Now to show that $g$ is analytic on $G$ it need only to show $g$ is analytic in a neighborhood of $a$.

Since $f$ is analytic in $G$, there is $R > 0$ such that $f$ is analytic in $B(a;R) \subset G$.

By Taylor's theorem

$$f(z) = \sum_{k=0}^{\infty} a_k (z-a)^k \qquad \text{for } |z-a| < R \qquad \text{where} \quad a_k = \frac{1}{k!} f^{(k)}(a).$$

Since $f(a) = f'(a) = \ldots = f^{(n-1)}(a) = 0$ and $f^{(n)}(a) \neq 0$ we have

$$f(z) = \sum_{k=n}^{\infty} a_k (z-a)^k = (z-a)^n \sum_{k=n}^{\infty} a_k (z-a)^{k-n}$$

Let $h(z)=\sum_{k=n}^{\infty} a_k (z-a)^{k-n}$, then $h$ is analytic in $B(a;R)$ and

$f(z)=(z-a)^n h(z)$ and $h(a)=a_n=\dfrac{f^{(n)}(a)}{n!}$.

Thus from (1) $g \equiv h$ in $B(a;R)$

Therefore $g$ is analytic in $B(a;R)$.

Hence $g$ is analytic in $G$ with $f(z)=(z-a)^n g(z)$ and $g(a)=h(a)=a_n \neq 0$.

**Corollary 9** If $f:G\to\mathbb{C}$ is analytic and not constant, $a\in G$ and $f(a)=0$ then there is an $R>0$ such that $B(a;R)\subset G$ and $f(z)\neq 0$ for $0<|z-a|<R$. That is zeros of are isolated.

**Proof.** Let $f:G\to\mathbb{C}$ be non-constant analytic function with $f(a)=0$ for some $a\in G$. Then by corollary there is an analytic function $g:G\to\mathbb{C}$ and an integer $n\geq 1$ such that $f(z)=(z-a)^n g(z)$ and $g(a)\neq 0$.

Since $g$ is analytic, g is continuous on $G$.

Therefore, there is $R>0$ such that $g(z)\neq 0$ in $B(a;R)\subset G$ i.e for $|z-a|<R$

Hence $f(z)=(z-a)^n g(z)\neq 0$ for $0<|z-a|<R$.

We now prove Maximum Modulus Theorem:

**Theorem 10** Let $G$ is a region and $f:G\to\mathbb{C}$ is an analytic function such that there is a point $a$ in $G$ with $|f(a)|\geq|f(z)|$ for all $z$ in $G$, then $f$ is constant.

**Proof.** Since $f:G\to\mathbb{C}$ is analytic function, there is $r>0$ such that $\overline{B}(a;r)\subset G$.

Then by Cauchy Integral formula

$$f(a)=\frac{1}{2\pi i}\int_{\gamma}\frac{f(w)}{w-a}dw \quad \text{for } |z-a|<r \text{ and } \gamma(t)=a+re^{it},\ 0\leq t\leq 2\pi .$$

$$=\frac{1}{2\pi i}\int_0^{2\pi}\frac{f(a+re^{it})}{a+re^{it}-a}rie^{it}dt$$

$$=\frac{1}{2\pi}\int_0^{2\pi} f(a+re^{it})\,dt$$

$$|f(a)| \leq \frac{1}{2\pi}\int_0^{2\pi} \left|f(a+re^{it})\right| dt$$

$$\leq \frac{1}{2\pi}\int_0^{2\pi} |f(a)|\, dt \qquad \text{as } |f(a)| \geq |f(z)| \text{ for all } z \text{ in } G.$$

$$= |f(a)|$$

Therefore

$$|f(a)| \leq \frac{1}{2\pi}\int_0^{2\pi} \left|f(a+re^{it})\right| dt \leq |f(a)|$$

$$|f(a)| = \frac{1}{2\pi}\int_0^{2\pi} \left|f(a+re^{it})\right| dt$$

$$\frac{1}{2\pi}\int_0^{2\pi} \left[|f(a)| - \left|f(a+re^{it})\right|\right] dt = 0 \qquad \ldots(1)$$

Since $|f(a)| - \left|f(a+re^{it})\right| \geq 0$ for all $t$

Therefore from (1) we have $|f(a)| = \left|f(a+re^{it})\right|$ for all $t$.

Let $f(a) = \alpha$, then $\left|f(a+re^{it})\right| = |\alpha|$ for all $t$.

Since $r > 0$ is arbitrary, we have $|f(z)| = |\alpha|$ for all $z$ in $B(a;r)$.

That is $f$ maps whole disk $B(a;r) \subset G$ into the circle $|z| = |\alpha|$ where $f(a) = \alpha$.

Therefore $f$ has constant modulus on $B(a;r)$ and hence $f$ is constant on $B(a;r)$.

Let $f(z) = c$ on $B(a;r) \subset G$ then $\{\ z \in G : f(z) = c\ \} \supseteq B(a;r)$ has a limit point in $G$.

Thus $f(z) = c$ on $G$, i.e. $f$ is constant on $G$.

**Theorem 11** If $\gamma : [0,1] \to \mathbb{C}$ is closed rectifiable curve $a \notin \{\gamma\}$, then

$$\frac{1}{2\pi i}\int_\gamma \frac{1}{z-a}\, dz \quad \text{is an integer.}$$

**Proof.** Define $g : [0,1] \to \mathbb{C}$ by

$$g(t) = \int_0^t \frac{\gamma'(s)}{\gamma(s)-a}\, ds$$

where $\gamma$ is closed rectifiable curve so that $g$ is well defined .

Therefore $g(0)=0$ and $g(1)=\int_0^1 \frac{\gamma'(s)}{\gamma(s)-a}ds=\int_\gamma \frac{1}{z-a}dz$ .

Also $g'(t)=\frac{\gamma'(t)}{\gamma(t)-a}$ for $0\le t\le 1$ .

Then $$\frac{d}{dt}\left[e^{-g(t)}(\gamma(t)-a)\right]=e^{-g(t)}\gamma'(t)-e^{-g(t)}g'(t)\,(\gamma(t)-a)$$

$$=e^{-g(t)}\left[\gamma'(t)-\frac{\gamma'(t)}{\gamma(t)-a}(\gamma(t)-a)\right]$$

$$=0$$

Therefore $e^{-g(t)}(\gamma(t)-a)$ is constant function.

Hence $$e^{-g(0)}(\gamma(0)-a)=e^{-g(1)}(\gamma(1)-a)$$

$$e^0=e^{g(1)} \qquad (\ \gamma(0)=\gamma(1)\ )$$

$$e^{2\pi ik}=e^{g(1)} \qquad (\ k \text{ is integer })$$

Therefore $g(1)=2\pi ik$

Therefore, $2\pi ik=\int_\gamma \frac{1}{z-a}dz$

Hence $\frac{1}{2\pi i}\int_\gamma \frac{1}{z-a}dz=k$ .

**Definition 12** If $\gamma$ is a closed rectifiable curve in $\mathbb{C}$ then for $a\notin\{\gamma\}$ . Then the integer

$\frac{1}{2\pi i}\int_\gamma \frac{1}{z-a}dz$ is called the index of $\gamma$ with respect to the point $a$ .

**Definition 13** A subset D of a metric space X is called component of X , if it is maximal connected subset of X.

**Theorem 14** Let $\gamma$ be a closed rectifiable curve in $\mathbb{C}$. Then

a) $n(\gamma;a)$ is constant for $a$ belonging to a component of $G=\mathbb{C}-\{\gamma\}$.

b) $n(\gamma;a)=0$ for $a$ belonging to the unbounded component of $G$.

**Proof.** Define $f:G\to\mathbb{C}$ by

$$f(a)=\frac{1}{2\pi i}\int_{\gamma}\frac{1}{z-a}dz=n(\gamma,a) \qquad \forall \text{ in } G.$$

Claim: $f$ is continuous.

Let $a\in G$ and $r=d\left(a,\{\gamma\}\right)>0$.

For any $\in>0$ we choose $\delta>0$ such that $|b-a|<\delta<r/2$.

Therefore

$$|f(b)-f(a)|=\left|\frac{1}{2\pi i}\int_{\gamma}\left[\frac{1}{z-b}-\frac{1}{z-a}\right]dz\right|$$

$$\leq\frac{1}{2\pi}\int_{\gamma}\left|\frac{b-a}{(z-b)(z-a)}\right|dz$$

$$\leq\frac{|b-a|}{2\pi}\int_{\gamma}\frac{|dz|}{|(z-b)||(z-a)|}$$

Now $|z-a|\geq r>r/2$ and $|z-b|=|(z-a)-(b-a)|\geq|z-a|-|b-a|>r-r/2=r/2$.

Thus

$$|f(b)-f(a)|<\frac{\delta}{2\pi}\int_{\gamma}\frac{2}{r}.\frac{2}{r}|dz|=\frac{2\delta}{\pi r^2}V(\gamma).$$

Therefore, for any $\in>0$ there is $0<\delta<\min\left\{r/2,\frac{\in\pi r^2}{2V(\gamma)}\right\}$, such that $|f(b)-f(a)|<\in$,

whenever $|b-a|<\delta$.

Thus $f$ is continuous .

**a)** Let $D$ be component of $G$ then $D$ is open and connected .

Since $f$ is continuous, $f(D)$ is connected.

Since $f$ is integer valued and subset of set of all integers which are connected are precisely singleton sets .

Therefore $f(D)=\{k\}$ for some integer $k$.

That is $f(a)=k \quad \forall$ in $D$.

Hence $n(\gamma;a)$ is constant for $a$ belonging to $D$.

**b)** Let $U$ be unbounded component of $G=\mathbb{C}-\{\gamma\}$, then there is $R>0$ such that $\{\ z\in G:|z|>R\ \}\subseteq U$.

For $\in>0$ choose $a$ such that $|a|>R$ and $|z-a|>\dfrac{V(\gamma)}{2\pi\in}$ for all $z$ on $\{\gamma\}$ then

$$|n(\gamma;a)|=\left|\frac{1}{2\pi i}\int_{\gamma}\frac{1}{z-a}dz\right|$$

$$\leq\frac{1}{2\pi}\int_{\gamma}\frac{1}{|z-a|}|dz|\leq\frac{1}{2\pi}\frac{2\pi\in}{V(\gamma)}\int_{\gamma}|dz|=\in$$

Therefore $|n(\gamma;a)|<\in$

Since $n(\gamma;a)$ is an integer $|n(\gamma;a)|<\in\Rightarrow n(\gamma;a)=0$ for some $a$ in $U$.

Since $f(a)=n(\gamma;a)$ is constant on $U$, we must have $n(\gamma;a)=0$ for all $a$ in $U$.

## EXERCISES

**1)** Let $f : C \to C$ be entire function. Suppose for some $R > 0, |z| > R$ implies $|f(z)| \leq M|z|^n$, for some constant M. Then prove that f is a polynomial of degree at most n.

**2)** Let $f : G \to C$ be analytic function defined on a region G with $|f(a)| \leq f(z)|$, for all z in G. Show that either $f \equiv 0$ or f is constant function.

**3)** Let f and g be nalytic functions defined on the region G. If f.g = 0 on G, prove that either $f \equiv 0$ or $g \equiv 0$.

**4)** Show by an example that $n(\gamma; a) = k$ for a closed rectifiable curve $\gamma$ in C, where $a \notin \{\gamma\}$.

UNIT - V

# WINDING NUMBERS AND CAUCHYS INTEGRALTHEOREM

In the last unit we prove that $\frac{1}{2\pi i}\int_{\gamma}\frac{1}{z-a}dz$ is an integer. We shall denote this integer by $n(\gamma;a)$ and called it is a winding number or Index of a closed curve $\gamma$ around $a$. In this unit we discuss Cauchy's integral formulae.

**Lemma 1** Let $\gamma$ be rectifiable curve and suppose $\phi$ is a function defined and continuous on $\{\gamma\}$. For each $m \geq 1$ let $F_m(z)=\int_{\gamma}\frac{\phi(w)}{(w-z)^m}dw$ for $z \notin \{\gamma\}$. Then each $F_m$ is analytic on $\mathbb{C}-\{\gamma\}$ and $F_m^{'}(z)=mF_{m+1}(z)$.

**Proof.** Let $F_m(z)=\int_{\gamma}\frac{\phi(w)}{(w-z)^m}dw$ for $z \notin \{\gamma\}$ where $\phi$ is continuous function and $\gamma$ is rectifiable curve .

First we claim that each $F_m$ is continuous:

Let $a \in G = \mathbb{C}-\{\gamma\}$ and $r = d\left(a,\{\gamma\}\right) > 0$.

For any $\in > 0$ we choose $\delta > 0$ such that $|z-a| < \delta < r/2$.

Therefore

$$|F_m(z)-F_m(a)| = \left|\int_{\gamma}\left[\frac{\phi(w)}{(w-z)^m}-\frac{\phi(w)}{(w-a)^m}\right]dw\right|$$

$$\leq \int_{\gamma}|\phi(w)|\left|\frac{1}{(w-z)^m}-\frac{1}{(w-a)^m}\right||dw| \qquad \ldots(1)$$

Since $\phi$ is continuous function on compact set $\{\gamma\}$, we have $M = \sup\{|\phi(w)| : w \in \{\gamma\}\}$.

And

$$\frac{1}{(w-z)^m}-\frac{1}{(w-a)^m}=\left[\frac{1}{(w-z)}-\frac{1}{(w-a)}\right]\sum_{k=0}^{m-1}\frac{1}{(w-z)^{m-k-1}(w-a)^k}$$

$$=(z-a)\sum_{k=0}^{m-1}\frac{1}{(w-z)^{m-k}(w-a)^{k+1}}$$

Also $|w-a|\geq r> r/2$ and $|w-z|=|(w-a)-(z-a)|\geq|w-a|-|z-a|>r-r/2=r/2$.

Therefore $\left|\frac{1}{(w-z)^m}-\frac{1}{(w-a)^m}\right|=\left|(z-a)\sum_{k=0}^{m-1}\frac{1}{(w-z)^{m-k}(w-a)^{k+1}}\right|$

$$\leq|(z-a)|\sum_{k=0}^{m-1}\frac{1}{|(w-z)|^{m-k}|(w-a)|^{k+1}}$$

$$<\delta\sum_{k=0}^{m-1}\frac{1}{\left(r/2\right)^{m-k}\left(r/2\right)^{k+1}}=\delta\left(2/r\right)^{m+1}m$$

Thus (1) gives

$$|F_m(z)-F_m(a)|<\int_\gamma M\delta\left(2/r\right)^{m+1}m|dw|=mM\delta\left(2/r\right)^{m+1}V(\gamma).$$

Therefore, for any $\in>0$ there is $0<\delta<\min\left\{r/2,\frac{\in\left(r/2\right)^{m+1}}{mM\,V(\gamma)}\right\}$, such that $|F_m(z)-F_m(a)|<\in$, whenever $|z-a|<\delta$.

Thus $F_m$ is continuous on $G=\mathbb{C}-\{\gamma\}$ for any $m\geq 1$.

Now to show $F_m^{'}(z)=mF_{m+1}(z)$ :

Consider

$$F_m(z)-F_m(a)=\int_\gamma\left[\frac{\phi(w)}{(w-z)^m}-\frac{\phi(w)}{(w-a)^m}\right]dw$$

$$= (z-a)\int_{\gamma}\sum_{k=0}^{m+1}\frac{\phi(w)}{(w-z)^{m-k}(w-a)^{k+1}}dw$$

Thus

$$\frac{F_m(z)-F_m(a)}{(z-a)} = \sum_{k=0}^{m+1}\int_{\gamma}\frac{\left[\phi(w)\big/(w-a)^{k+1}\right]}{(w-z)^{m-k}}dw.$$

Since $a \notin \{\gamma\}$ and $\phi(w)\big/(w-a)^{k+1}$ continuous on $\{\gamma\}$ for each $k$, each integral $\int_{\gamma}\frac{\left[\phi(w)\big/(w-a)^{k+1}\right]}{(w-z)^{m-k}}dw$ is continuous .

Hence letting $z \to a$ we get

$$\lim_{z\to a}\frac{F_m(z)-F_m(a)}{(z-a)} = \lim_{z\to a}\sum_{k=0}^{m-1}\int_{\gamma}\frac{\left[\phi(w)\big/(w-a)^{k+1}\right]}{(w-z)^{m-k}}dw$$

$$= \sum_{k=0}^{m-1}\lim_{z\to a}\int_{\gamma}\frac{\left[\phi(w)\big/(w-a)^{k+1}\right]}{(w-z)^{m-k}}dw = \sum_{k=0}^{m-1}\int_{\gamma}\frac{\left[\phi(w)\big/(w-a)^{k+1}\right]}{(w-a)^{m-k}}dw$$

$$\lim_{z\to a}\frac{F_m(z)-F_m(a)}{(z-a)} = \sum_{k=0}^{m-1}\int_{\gamma}\frac{\phi(w)}{(w-a)^{m+1}}dw = m\int_{\gamma}\frac{\phi(w)}{(w-a)^{m+1}}dw$$

Therefore,

$F_m^{'}(a) = mF_{m+1}(a)$ for all $a \in \mathbb{C} - \{\gamma\}$.

Thus, $F_m$ is differentiable for any $m \geq 1$.

Since $F_{m+1}$ is continuous, $F'_m$ is continuous .

Therefore $F_m$ is continuously differentiable.

Hence $F_m$ is analytic on $\mathbb{C} - \{\gamma\}$ .

We now prove Cauchy' s Integral formulae:

**Theorem 2 ( First Version)** Let $G$ be an open subset of the plane and $f: G \to \mathbb{C}$ be an analytic function. If $\gamma$ is a closed rectifiable curve in $G$ such that $n(\gamma; w) = 0$ for all $w$ in $\mathbb{C} - G$, then for $a$ in $G - \{\gamma\}$

$$n(\gamma; a)\, f(a) = \frac{1}{2\pi i} \int_{\gamma} \frac{f(z)}{z-a} dz$$

**Proof.** Define $\phi: G \times G \to \mathbb{C}$ by

$$\phi(z,w) = \frac{f(w)-f(z)}{w-z} \quad \text{if } z \neq w$$

$$= f'(z) \qquad \text{if } z = w.$$

Then $\phi$ is continuous .

Let $H = \{w \in \mathbb{C} : n(\gamma; w) = 0\}$. Since $n(\gamma; w)$ is a continuous integer valued function , $H$ is open. Moreover $\mathbb{C} - G \subseteq H$. Thus $\mathbb{C} = G \cup H$.

Now, define $g: \mathbb{C} \to \mathbb{C}$ by

$$g(z) = \int_{\gamma} \phi(z,w)\, dw \quad \text{if } z \in G$$

$$= \int_{\gamma} \frac{f(w)}{w-z} dw \quad \text{if } z \in H.$$

If $z \in G \cap H$ then

$$\int_{\gamma} \phi(z,w)\, dw = \int_{\gamma} \frac{f(w)-f(z)}{w-z} dw$$

$$= \int_{\gamma} \frac{f(w)}{w-z} dw - f(z) \int_{\gamma} \frac{1}{w-z} dw$$

$$= \int_{\gamma} \frac{f(w)}{w-z} dw - f(z)\, n(\gamma; z)\, 2\pi i$$

$$= \int_{\gamma} \frac{f(w)}{w-z} dw - f(z).\, 0\, .2\pi i$$

$$= \int_{\gamma} \frac{f(w)}{w-z} dw.$$

Therefore $g$ is well defined function.

Thus by lemma $g$ is analytic on $\mathbb{C}$, hence $g$ is entire function.

Since $H$ contains neighborhood of infinity, we have $\lim\limits_{z\to\infty}\dfrac{1}{w-z}=0$ uniformly for $w\in\{\gamma\}$.

Since $\{\gamma\}$ is compact, $f$ is bounded on $\{\gamma\}$.

Hence there is $M>0$ such that $|f(w)|\le M$ for all $w\in\{\gamma\}$.

Therefore, $\left|\int\limits_{\gamma}\dfrac{f(w)}{w-z}\,dw\right|\le\int\limits_{\gamma}\dfrac{|f(w)|}{|w-z|}|dw|$

$$\le M\int\limits_{\gamma}\frac{1}{|w-z|}|dw|$$

Hence $\lim\limits_{z\to\infty}g(z)=\lim\limits_{z\to\infty}\int\limits_{\gamma}\dfrac{f(w)}{w-z}\,dw=0$

Therefore, there is $R>0$ such that $|g(z)|\le 1$ for $|z|>R$.

Since, $g$ continuous on compact set $\overline{B}(0,R)$, $g$ is bounded on $\overline{B}(0,R)$.

Thus $g$ is a bounded entire function. Hence by Liouville's Theorem $g$ is constant.

Since $\lim\limits_{z\to\infty}g(z)=0$ we must have $g\equiv 0$.

Thus $\int\limits_{\gamma}\dfrac{f(w)-f(a)}{w-a}\,dw=0$ for all $a$ in $G-\{\gamma\}$.

$$\int\limits_{\gamma}\frac{f(w)}{w-a}\,dw=f(a)\int\limits_{\gamma}\frac{1}{w-a}\,dw \quad \text{for all } a \text{ in } G-\{\gamma\}.$$

$$\int\limits_{\gamma}\frac{f(w)}{w-a}\,dw=f(a).2\pi i.n(\gamma;a) \quad \text{for all } a \text{ in } G-\{\gamma\}.$$

Thus

$$n(\gamma;a)\,f(a)=\frac{1}{2\pi i}\int\limits_{\gamma}\frac{f(z)}{z-a}dz \quad \text{for all } a \text{ in } G-\{\gamma\}.$$

**Theorem 3 (Second Version)** Let $G$ be an open subset of the plane and $f: G \to \mathbb{C}$ be an analytic function. If $\gamma_1, \gamma_2, \ldots, \gamma_m$ are closed rectifiable curves in $G$ such that $n(\gamma_1; w) + n(\gamma_2; w) + \ldots + n(\gamma_m; w) = 0$ for all $w$ in $\mathbb{C} - G$, then for $a$ in $G - \{\gamma\}$

$$f(a)\sum_{k=1}^{m} n(\gamma_k; a) = \sum_{k=1}^{m} \frac{1}{2\pi i} \int_{\gamma_k} \frac{f(z)}{z-a} dz$$

**Proof.** Define $\phi: G \times G \to \mathbb{C}$ by

$$\phi(z, w) = \frac{f(w) - f(z)}{w - z} \quad \text{if } z \neq w$$

$$= f'(z) \qquad \text{if } z = w.$$

Then $\phi$ is continuous .

Let $H = \{w \in \mathbb{C} : n(\gamma_1; w) + n(\gamma_2; w) + \ldots + n(\gamma_m; w) = 0\}$. Since $n(\gamma; w)$ is a continuous integer valued function , $H$ is open. Moreover $\mathbb{C} - G \subseteq H$ . Thus $\mathbb{C} = G \cup H$ .

Now, define $g: \mathbb{C} \to \mathbb{C}$ by

$$g(z) = \sum_{k=1}^{m} \int_{\gamma_k} \phi(z, w)\, dw \quad \text{if } z \in G$$

$$= \sum_{k=1}^{m} \int_{\gamma_k} \frac{f(w)}{w - z} dw \quad \text{if } z \in H .$$

If $z \in G \cap H$ then

$$\sum_{k=1}^{m} \int_{\gamma_k} \phi(z, w)\, dw = \sum_{k=1}^{m} \int_{\gamma_k} \frac{f(w) - f(z)}{w - z} dw$$

$$= \sum_{k=1}^{m} \left[ \int_{\gamma_k} \frac{f(w)}{w - z} dw - f(z) \int_{\gamma_k} \frac{1}{w - z} dw \right]$$

$$= \sum_{k=1}^{m} \left[ \int_{\gamma_k} \frac{f(w)}{w - z} dw - f(z)\, n(\gamma_k; z)\, 2\pi i \right]$$

$$= \sum_{k=1}^{m} \int_{\gamma_k} \frac{f(w)}{w-z}\, dw - f(z) 2\pi i \sum_{k=1}^{m} n(\gamma_k; z) \;= \sum_{k=1}^{m} \int_{\gamma_k} \frac{f(w)}{w-z}\, dw - f(z) 2\pi i .0$$

$$= \sum_{k=1}^{m} \int_{\gamma_k} \frac{f(w)}{w-z}\, dw .$$

Therefore $g$ is well defined function.

Thus by lemma $g$ is analytic on $\mathbb{C}$, hence $g$ is entire function.

Since $H$ contains neighborhood of infinity, we have $\lim\limits_{z\to\infty} \dfrac{1}{w-z} = 0$ uniformly for $w \in \{\gamma_k\}$.

Since each $\{\gamma_k\}$ is compact , $f$ is bounded on $\{\gamma_k\}$.

Hence $\lim\limits_{z\to\infty} g(z) = 0$

Therefore, there is $R > 0$ such that $|g(z)| \le 1$ for $|z| > R$.

Since $g$ continuous on compact set $\overline{B}(0,R)$, $g$ is bounded on $\overline{B}(0,R)$.

Thus $g$ is a bounded entire function. Hence by Liouville's Theorem $g$ is constant.

Since $\lim\limits_{z\to\infty} g(z) = 0$ we must have $g \equiv 0$.

Thus $\displaystyle\sum_{k=1}^{m} \int_{\gamma_k} \frac{f(w)-f(a)}{w-a}\, dw = 0$ for all $a$ in $G - \overset{m}{\underset{k=1}{\cup}} \{\gamma_k\}$.

$$\sum_{k=1}^{m} \int_{\gamma_k} \frac{f(w)}{w-a}\, dw = f(a) \sum_{k=1}^{m} \int_{\gamma_k} \frac{1}{w-a}\, dw \quad \text{for all } a \text{ in } G - \overset{m}{\underset{k=1}{\cup}} \{\gamma_k\}.$$

$$\sum_{k=1}^{m} \int_{\gamma_m} \frac{f(w)}{w-a}\, dw = f(a).2\pi i \sum_{k=1}^{m} n(\gamma_k; a) \quad \text{for all } a \text{ in } G - \overset{m}{\underset{k=1}{\cup}} \{\gamma_k\}.$$

Thus

$$f(a) \sum_{k=1}^{m} n(\gamma_k; a) \;= \sum_{k=1}^{m} \frac{1}{2\pi i} \int_{\gamma_k} \frac{f(z)}{z-a}\, dz \quad \text{for all } a \text{ in } G - \overset{m}{\underset{k=1}{\cup}} \{\gamma_k\}.$$

**Theorem 4** Let $G$ be an open subset of the plane and $f : G \to \mathbb{C}$ be an analytic function. If $\gamma_1, \gamma_2, ..., \gamma_m$ are closed rectifiable curves in $G$ such that $n(\gamma_1; w) + n(\gamma_2; w) + ... + n(\gamma_m; w) = 0$ for all $w$ in $\mathbb{C} - G$, then

$$\sum_{k=1}^{m} \int_{\gamma_k} f = 0$$

**Proof.** Let $a \in \mathbb{C} - \bigcup_{k=1}^{m} \{\gamma_k\}$ and $F(z) = (z-a) f(z)$.

Then $F$ is analytic in $G$.

Hence by Cauchy's integral theorem,

$$F(a) \sum_{k=1}^{m} n(\gamma_k; a) = \sum_{k=1}^{m} \frac{1}{2\pi i} \int_{\gamma_k} \frac{F(z)}{z-a} dz$$

$$2\pi i.0. \sum_{k=1}^{m} n(\gamma_k; a) = \sum_{k=1}^{m} \int_{\gamma_k} \frac{(z-a) f(z)}{z-a} dz$$

$$\sum_{k=1}^{m} \int_{\gamma_k} f(z) dz = 0$$

Thus $\sum_{k=1}^{m} \int_{\gamma_k} f = 0$ .

**Theorem 5** Let $G$ be an open subset of the plane and $f : G \to \mathbb{C}$ be an analytic function. If $\gamma_1, \gamma_2, ..., \gamma_m$ are closed rectifiable curves in $G$ such that $n(\gamma_1; w) + n(\gamma_2; w) + ... + n(\gamma_m; w) = 0$ for all $w$ in $\mathbb{C} - G$, then for $a$ in $G - \{\gamma\}$

$$f^{(n)}(a) \sum_{k=1}^{m} n(\gamma_k; a) = n! \sum_{k=1}^{m} \frac{1}{2\pi i} \int_{\gamma_k} \frac{f(z)}{z-a} dz$$

**Proof.** By Cauchy Integral formula we have

$$f(a) \sum_{k=1}^{m} n(\gamma_k; a) = \sum_{k=1}^{m} \frac{1}{2\pi i} \int_{\gamma_k} \frac{f(z)}{z-a} dz \quad \text{for all } a \text{ in } G - \bigcup_{k=1}^{m} \{\gamma_k\} .$$

Hence

$$f^{(n)}(a)\sum_{k=1}^{m} n(\gamma_k;a) \;=\sum_{k=1}^{m}\frac{d^n}{da^n}\left[\frac{1}{2\pi i}\int_{\gamma_k}\frac{f(z)}{z-a}dz\right]$$

$$=\sum_{k=1}^{m}\frac{1}{2\pi i}\int_{\gamma_k}\frac{\partial^n}{\partial a^n}\left(\frac{f(z)}{z-a}\right)dz$$

$$=\sum_{k=1}^{m}\frac{n!}{2\pi i}\int_{\gamma_k}\frac{f(z)}{\left(z-a\right)^{n+1}}dz$$

Therefore,

$$f(a)\sum_{k=1}^{m} n(\gamma_k;a) \;=\sum_{k=1}^{m}\frac{1}{2\pi i}\int_{\gamma_k}\frac{f(z)}{z-a}dz\,.$$

**Corollary 6** Let $G$ be an open subset of the plane and $f:G\to\mathbb{C}$ be an analytic function. If $\gamma$ is a closed rectifiable curve in $G$ such that $n(\gamma;w)=0$ for all $w$ in $\mathbb{C}-G$, then for $a$ in $G-\{\gamma\}$

$$f^{(n)}(a)\, n(\gamma;a)=\frac{n!}{2\pi i}\int_{\gamma}\frac{f(z)}{\left(z-a\right)^{n+1}}dz\,.$$

**Definition 7** A closed polygonal path having three sides is called triangular path.

**Theorem 8 Morera's Theorem** Let $G$ be a region and let $f:G\to\mathbb{C}$ be a continuous function such that $\int_T f=0$ for every triangular path $T$ in $G$; then $f$ is analytic in $G$.

**Proof.** To prove that $f$ is analytic in $G$ we have to prove that $f$ is analytic on each open disk contained in $G$. Hence without loss of generality we may assume that $G=B(a;R)$.

Now define $F:G\to\mathbb{C}$ by

$$F(z)=\int_{[a.z]} f(w)dw \qquad \text{where } [a,z] \text{ is the line segment joining } a \text{ to } z\,.$$

Fix $z_0\in B(a;R)$, then for any $z$ in $G$, $T=[a,z,z_0,a]$ be a triangular path in $G$.

Therefore by hypothesis

$$\int_T f = 0 \Rightarrow \int_{[a,z]} f + \int_{[z,z_0]} f + \int_{[z_0,a]} f = 0$$

$$\Rightarrow \int_{[a,z]} f = \int_{[z_0,z]} f + \int_{[a,z_0]} f$$

$$\Rightarrow F(z) = \int_{[z_0,z]} f(w)dw + F(z_0)$$

Thus

$$\frac{F(z)-F(z_0)}{z-z_0} - f(z_0) = \frac{1}{z-z_0}\int_{[z_0,z]} f(w)dw - f(z_0)$$

$$= \frac{1}{z-z_0}\int_{[z_0,z]} \left[f(w)-f(z_0)\right]dw$$

Since $f$ is continuous in $G$, for any $\in > 0$ there is $\delta > 0$, such that

$|f(w)-f(z_0)| < \in$ whenever $|w-z_0| < \delta$.

Therefore

$$\left|\frac{F(z)-F(z_0)}{z-z_0} - f(z_0)\right| \le \frac{1}{|z-z_0|}\int_{[z_0,z]} |f(w)-f(z_0)||dw| < \frac{\in}{|z-z_0|}\int_{[z_0,z]} |dw| = \in$$

Thus

$$\lim_{z\to z_0}\frac{F(z)-F(z_0)}{z-z_0} = f(z_0)$$

$$F'(z_0) = f(z_0)$$

Since $z_0$ is arbitrary , we have $F' \equiv f$ in $G$.

Since $f$ is continuous, $F'$ is continuous on $G$

Therefore $F$ is continuously differentiable, that is $F$ is analytic.

Hence $F' \equiv f$ is also analytic in $G$.

**Singularities**

**Definition 9** A function $f$ has an singularity at $z=a$ if $f$ is not analytic at $z=a$.

**Ex.** $\frac{1}{z}, \frac{\sin z}{z}, e^{\frac{1}{z}}$ has an singularity at $z=0$.

**Ex.** $\frac{1}{\cos\left(\frac{1}{z}\right)}$ has an singularity at the points $z=\frac{2}{(2n+1)\pi}$ $,n=0,\pm1,\pm2,...$

**Definition 10** A function $f$ has an isolated singularity at $z=a$ if there is $R>0$ such that $f$ is analytic in $B(a;R)-\{a\}$ ,otherwise $z=a$ is non-isolated singularity of $f$.

**Ex.** $\frac{1}{\sin\left(\frac{\pi}{z}\right)}$ , $z=\frac{1}{n}$ are isolated singularities and $z=0$ is non-isolated singularity.

**Ex.** $\frac{1}{(z-1)(z-2)}$ , $z=1,2$ are isolated singularities.

There are three kinds of isolated singularities

A) Removable singularity

B) Pole

C) Essential Singularity

**Definition 11** An isolated singularity at $z=a$ of a function $f$ is removable singularity if there is $R>0$ and an analytic function $g:B(a,R)\to\mathbb{C}$ such that $g(z)=f(z)$ in $0<|z-a|<R$.

**Ex.** $f(z)=\frac{\sin z}{z}$ has removable singularity at $z=0$.

**Ex.** $f(z)=\frac{z}{e^z-1}$ has removable singularity at $z=0$.

**Theorem 12** If $f$ has an isolated singularity at $z=a$, then the point $z=a$ is removable singularity iff $\lim_{z\to a}(z-a)f(z)=0$.

**Proof.** Suppose $z=a$ is removable singularity, then there is $R>0$ and an analytic function $g: B(a,R) \to \mathbb{C}$ such that $g(z)=f(z)$ in $0<|z-a|<R$.

Therefore, $\lim_{z\to a}(z-a)f(z)=\lim_{z\to a}(z-a)g(z)=0.g(a)=0$. ( since $g$ is continuous )

Conversely suppose that $\lim_{z\to a}(z-a)f(z)=0$. Since $f$ has an isolated singularity at $z=a$, there is $R>0$ such that $f$ is analytic in $B(a;R)-\{a\}$.

Define $h(z)=\begin{cases}(z-a)f(z) & \text{if } z\neq a\\ 0 & \text{if } z=a\end{cases}$

Clearly $h$ is analytic in $B(a;R)-\{a\}$ and continuous at $z=a$.

Now to prove $f$ has removable singularity at $z=a$ we have to prove that $h$ is analytic in $B(a;R)$.

Claim: $h$ is analytic in $B(a;R)$.

To prove this we use Moreras theorem. Let $T$ be the triangle in $B(a;R)$ and $\Delta$ denote inside of $T$ along with $T$.

Case 1: When $a\notin\Delta$.

Since $h$ is analytic in $B(a;R)-\{a\}$ and $T \sim 0 \Rightarrow T\approx 0$, by Cauchy theorem $\int_T h=0$.

Case 2: When $a$ is vertex of $T$.

Let $T=[a,b,c,a]$ be a triangle with $a$ as one of the vertex. For $x\in[a,b]$ and $y\in[a,c]$ let $P=[x,b,c,y,x]$, then $P \sim 0$ and by Cauchy theorem $\int_P h=0$.

Let $T_1=[a,x,y,a]$ then $\int_T h=\int_{T_1} h+\int_P h=\int_{T_1} h$.

Since $h$ is continuous and $h(a)=0$, for any $\in>0$ there is $\delta>0$ such that $|h(z)|<\frac{\in}{l(T_1)}$ for $|z-a|<\delta$.

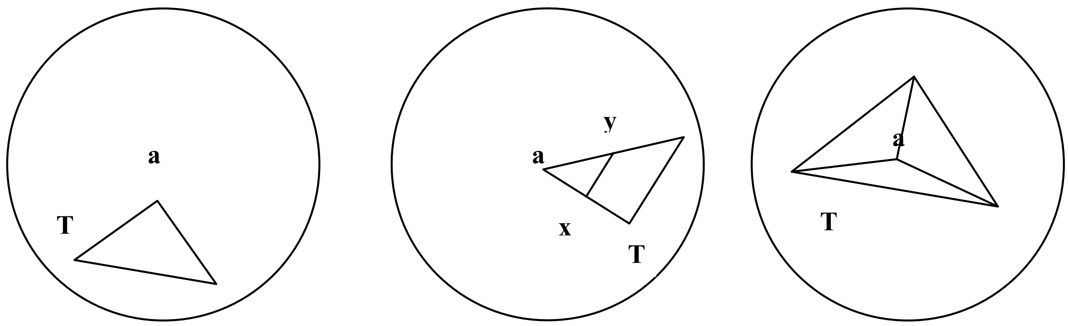
Now we choose $x,y$ such that $x,y\in B(a;\delta)$. Therefore $\left|\int_T h(z)dz\right|=\left|\int_{T_1} h(z)dz\right|\le\int_{T_1}|h(z)||dz|<\frac{\in}{l(T_1)}l(T_1)$.

Hence $\int_T h = 0$.

Case 3. When $a$ lies on or inside $T$

In this case we can construct triangle as shown with  belonging to vertex of each constructed triangle. Therefore using case 2 we must have $\int_T h = 0$.

Thus $\int_T h = 0$ for any triangular path $T$ in $B(a;R)$. Hence by Moreras theorem $h$ is analytic in $B(a;R)$.




**Case 1 : a lies outside T** **Case 2 : a is vertex of T** **Case 3 : a lies inside T**

**Corollary 13** An isolated singularity of a function $f$ at $z = a$ is removable singularity iff $f$ is bounded in the neighborhood of $z = a$.

**Proof.** Let $z = a$ is removable singularity of $f$ then there is $R > 0$ and an analytic function $g : B(a;R) \to \mathbb{C}$ such that $g(z) = f(z)$ in $0 < |z - a| < R$.

Since $g$ is continuous at $z = a$, $g(a)$ is finite. Hence $g$ is bounded in neighborhood of $z = a$. Therefore $f$ is bounded in the neighborhood of $z = a$.

Conversely, suppose $f$ is bounded in the neighborhood of $z = a$, then there is $M > 0$ such that $|f(z)| \le M$ in $0 < |z - a| < \delta$.

Therefore $|(z-a)f(z)| \le |z - a| M \to 0$ as $z \to a$.

Thus $\lim_{z \to a}(z-a)f(z) = 0$. Hence $f$ has removable singularity at $z = a$.

**Corollary 14** An isolated singularity of a function $f$ at $z=a$ is removable singularity iff $\lim_{z\to a} f(z)=c$ .

**Proof.** Let $z=a$ is removable singularity at $z=a$ then there is $R>0$ and an analytic function $g:B(a;R)\to\mathbb{C}$ such that $g(z)=f(z)$ in $0<|z-a|<R$.

Therefore $\lim_{z\to a} f(z)=\lim_{z\to a} g(z)=g(a)=c$ (say).

Conversely, suppose $\lim_{z\to a} f(z)=c$ then $\lim_{z\to a}(z-a)f(z)=\lim_{z\to a}(z-a)\lim_{z\to a} f(z)=0.c=0$

Thus $\lim_{z\to a}(z-a)f(z)=0$. Hence $f$ has removable singularity at $z=a$.

**Definition 15** An isolated singularity at $z=a$ of a function $f$ is pole if $\lim_{z\to a}|f(z)|=\infty$.

**Ex.** $f(z)=\dfrac{1}{z}$ has pole at $z=0$.

**Ex.** $f(z)=\dfrac{z^2}{(z-1)(z-i)^2}$ has pole at $z=1,i$.

**Theorem 16** If $G$ is a region with $a$ in $G$ and if $f$ is analytic on $G-\{a\}$ with pole at $z=a$, then there is a positive integer $m$ and an analytic function $g:G\to\mathbb{C}$ such that

$$f(z)=\frac{g(z)}{(z-a)^m}.$$

**Proof.** Suppose $z=a$ is pole of $f$ then $\lim_{z\to a}|f(z)|=\infty$. Therefore $\lim_{z\to a}\left|\dfrac{1}{f(z)}\right|=0$.

Then $\dfrac{1}{f(z)}$ has removable singularity at $z=a$. Then there is an analytic function $h:B(a;R)\to\mathbb{C}$ such that $h(z)=\dfrac{1}{f(z)}$ when $0<|z-a|<R$.

Now we define $h(z)=\begin{cases}\dfrac{1}{f(z)} & \text{if } z\neq a\\ 0 & \text{if } z=a\end{cases}$

Since $h(a)=0$, there is an analytic function $h_1$ such that $h(z)=(z-a)^m h_1(z)$ and $h_1(a)\neq 0$ for some integer $m\geq 1$.

For $z\neq a$, $\dfrac{1}{f(z)}=(z-a)^m h_1(z)$

Therefore $\dfrac{1}{h_1(z)}=(z-a)^m f(z)$

Hence $\lim\limits_{z\to a}(z-a)^m f(z)=\lim\limits_{z\to a}\dfrac{1}{h_1(z)}=\dfrac{1}{h_1(a)}<\infty$

Then $(z-a)^m f(z)$ has removable singularity at $z=a$.

By definition there is an analytic function $g:B(a;R)\to\mathbb{C}$ such that $g(z)=(z-a)^m f(z)$, $z\neq a$.

Hence $f(z)=\dfrac{g(z)}{(z-a)^m}$, $z\neq a$.

**Definition 17** If $f$ has pole at $z=a$ and $m$ is smallest positive integer such that $(z-a)^m f(z)$ has removable singularity at $z=a$, then $f$ has a pole of order $m$ at $z=a$.

**Ex.** $f(z)=\dfrac{e^z}{(1-\cos z)}$ has pole of order 2.

**Definition 18** An isolated singularity at $z=a$ of a function $f$ is essential singularity if it is neither pole nor removable singularity.

**Ex.** $f(z)=e^{\frac{1}{z}}$ has essential singularity at $z=0$.

**Ex.** $f(z)=\sin\left(\dfrac{1}{z}\right)$ has essential singularity at $z=0$.

**Theorem 19 Casorati-Weierstrass Theorem** If $f$ has an essential singularity at $z=a$, then for every $\delta>0$ $\overline{f\left(ann\left(a;0,\delta\right)\right)}=\mathbb{C}$, that is $f\left(ann\left(a;0,\delta\right)\right)$ is dense in $\mathbb{C}$.

**Proof.** Suppose $z=a$ is essential singularity of $f$, then we have to prove that for any $c\in\mathbb{C}$ and $\delta>0$, $c$ is the limit point of $f\left(ann\left(a;0,\delta\right)\right)$. In another words we have to prove that, for any given $\in,\delta>0$, there is $z\in ann\left(a;0,\delta\right)$ such that $\left|f(z)-c\right|<\in$.

On contrary suppose the theorem is false. Then there exists $\in>0$ such that for any $\delta>0$, $\left|f(z)-c\right|\geq\in$ , for all $z\in ann\left(a;0,\delta\right)$.

Hence $\lim\limits_{z\to a}\left|\dfrac{f(z)-c}{z-a}\right|=\infty$. Therefore $\dfrac{f(z)-c}{z-a}$ has a pole at $z=a$. Let $m$ be the order of pole, then $\left(z-a\right)^m\dfrac{\left(f(z)-c\right)}{\left(z-a\right)}$ has removable singularity at $z=a$.

Therefore $\lim\limits_{z\to a}\left(z-a\right)\left[\left(z-a\right)^m\dfrac{\left(f(z)-c\right)}{\left(z-a\right)}\right]=0$

That is $\lim\limits_{z\to a}\left(z-a\right)^m\left(f(z)-c\right)=0$

Therefore $\lim\limits_{z\to a}\left(z-a\right)^m f(z)=\lim\limits_{z\to a}\left(z-a\right)^m\left(f(z)-c+c\right)$

$$=\lim_{z\to a}\left(z-a\right)^m\left(f(z)-c\right)+\lim_{z\to a}\left(z-a\right)^m c$$

$$=0$$

Thus $f$ will have either removable singularity or a pole , which is a contradiction.

Hence the theorem.

# EXERCISES

**1)** Discuss the singularities of the following functions:

a) $f(z)=\dfrac{\sin(z)}{z}$ b) $f(z)=\dfrac{\cos(z)}{z}$ c) $f(z)=\dfrac{z^2+1}{z(z-1)}$ d) $f(z)=\dfrac{\log(z+1)}{z^2}$.

**2)** Discuss the singularities of the following function and classify them

a) $\dfrac{z+i}{z^2+1}$ b) $\dfrac{1}{(z^2+1)(z-6)}$ c) $ze^{\frac{1}{z^2}}$

**3)** Prove that an entire function has a pole at $\infty$ of order m if and only if it is a polynomial of degree m.

**4)** Prove that an entire function has removable singularity if and only if it is a constant function.

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## UNIT - VI

# OPEN MAPPING THEOREM AND GOURSAT THEOREM

In this unit we shall discuss that zeros of an analytic function which are used to evaluate some complex integrals. We prove open mapping theorem and prove that a differentiable complex valued function defined on an open set is analytic on the set.

**Definition 1** Let $\gamma_1:[0,1]\to\mathbb{C}$ , $\gamma_2:[0,1]\to\mathbb{C}$ be two closed rectifiable curves in a region $G$; then $\gamma_1$ is homotopic to $\gamma_2$ in $G$ , written as $\gamma_1 \sim \gamma_2$, if there is a continuous function $\Gamma:[0,1]\times[0,1]\to G$ such that

1) $\Gamma(s,0)=\gamma_1(s)$ and $\Gamma(s,1)=\gamma_2(s)$ $\quad (0\le s\le 1)$

2) $\Gamma(0,t)=\Gamma(1,t)$ $\quad (0\le t\le 1)$.

**Definition 2** A closed rectifiable curve $\gamma$ is homotopic to zero, if $\gamma$ is homotopic to a constant curve and is written as $\gamma \sim 0$.

**Definition 3** A closed rectifiable curve $\gamma$ is homologous to zero, if $n(\gamma,w)=0$ for $w\in\mathbb{C}-G$ and is written as $\gamma\approx 0$.

**Theorem 4** Let $G$ be a region and let $f:G\to\mathbb{C}$ be an analytic function on $G$ with zeros $a_1,...,a_m$ (repeated according to multiplicity). If $\gamma$ is a closed rectifiable curve in $G$ which does not pass through any point $a_k$ and if $\gamma\approx 0$ then

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}dz=\sum_{k=1}^{m} n(\gamma;a_k) \ .$$

**Proof.** Since $a_1,...,a_m$ are zeros of $f$, there exists an analytic function $g$ such that $f(z)=(z-a_1)...(z-a_m).g(z)$, where $g(a_k)\neq 0$ for $k=1,...,m$ and that, $g$ is non-vanishing on $G$.

Taking logarithmic differentiation on both sides we get

$$\frac{f'(z)}{f(z)}=\frac{1}{(z-a_1)}+...+\frac{1}{(z-a_m)}+\frac{g'(z)}{g(z)}$$

Therefore

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}dz=\sum_{k=1}^{m}\frac{1}{2\pi i}\int_\gamma \frac{1}{(z-a_k)}dz+\int_\gamma \frac{g'(z)}{g(z)}dz$$

Since $g$ is non-vanishing $\frac{g'(z)}{g(z)}$ is analytic in $G$. Hence $\int_\gamma \frac{g'(z)}{g(z)}dz=0$.

Thus

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}dz=\sum_{k=1}^{m}n(\gamma;a_k) \ .$$

**Corollary 5** Let $G$ be a region and let $f:G\to\mathbb{C}$ be an analytic function on $G$ such that $a_1,...,a_m$ satisfies $f(z)=\alpha$ (repeated according to multiplicity). If $\gamma$ is a closed rectifiable curve in $G$ which does not pass through any point $a_k$ and if $\gamma\approx 0$ then

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)-\alpha}dz=\sum_{k=1}^{m}n(\gamma;a_k) \ .$$

**Proof.** Let $F(z)=f(z)-\alpha$, which is analytic and $a_1,...,a_m$ are zeros of $F$.

Therefore by theorem

$$\frac{1}{2\pi i}\int_\gamma \frac{F'(z)}{F(z)}dz=\sum_{k=1}^{m}n(\gamma;a_k)$$

Thus $$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)-\alpha}dz=\sum_{k=1}^{m}n(\gamma;a_k) \ .$$

**Example 6** Evaluate $\int\limits_{|z|=2} \frac{2z+1}{z^2+z+1}dz$.

**Solution.** Let $f(z)=z^2+z+1$ , $f'(z)=2z+1$.

Here zeros of $f$ $w_1=\frac{-1+i\sqrt{3}}{2}$ and $w_2=\frac{-1-i\sqrt{3}}{2}$ are lies inside $|z|=2$

Then $n(\gamma;w_1)=1$ and $n(\gamma;w_2)=1$.

Therefore

$$\frac{1}{2\pi i}\int\limits_{|z|=2}\frac{f'(z)}{f(z)}dz=\sum_{k=1}^{2}n(\gamma;a_k)$$

$$\frac{1}{2\pi i}\int\limits_{|z|=2}\frac{2z+1}{z^2+z+1}dz=1+1$$

$$\int\limits_{|z|=2}\frac{2z+1}{z^2+z+1}dz=4\pi i.$$

**Theorem 7** Suppose $f$ is analytic $B(a,R)$ in and let $\alpha=f(a)$. If $f(z)-\alpha$ has a zero of order $m$ at $z=a$ then there is an $\in>0$ and $\delta>0$ such that for $|\zeta-a|<\delta$; the equation $f(z)=\zeta$ has exactly $m$ simple roots in $B(a,\in)$.

**Proof.** Let $F(z)=f(z)-\alpha$, then $f$ is analytic and $z=a$ is zero of order $m$ in $B(a,R)$. Since zeros of analytic functions are isolated, there is $0<\in<R/2$ such that $F(z)=f(z)-\alpha\neq 0$ in $0<|z-a|<2\in$ and $f'(z)\neq 0$ in $0<|z-a|<2\in$.

Let $\gamma(t)=a+\in e^{2\pi it}$, $(0\leq t\leq 2\pi)$.

Let $\sigma=fo\gamma$ then $\sigma$ is a closed rectifiable curve in $f\left(B(a,R)\right)$. Since $a\notin\{\gamma\}$, $\alpha=f(a)\notin\{\sigma\}$, then there is $\delta>0$ such that $B(\alpha,\delta)\cap\{\sigma\}=\phi$. Thus $B(\alpha,\delta)$ lies in the same component of $\mathbb{C}-\{\sigma\}$. Therefore for any $\zeta$ such that $0<|\alpha-\zeta|<\delta$, $n(\sigma;\alpha)=n(\sigma;\zeta)$.

Therefore $\frac{1}{2\pi i}\int_{\sigma}\frac{1}{w-\alpha}dw=\frac{1}{2\pi i}\int_{\sigma}\frac{1}{w-\zeta}dw$

$$\frac{1}{2\pi i}\int_{\gamma}\frac{f'(z)}{f(z)-\alpha}dz=\frac{1}{2\pi i}\int_{\gamma}\frac{f'(z)}{f(z)-\zeta}dz$$

Therefore $\sum_k n(\gamma;z_k(\alpha))=\sum_k n(\gamma;z_k(\zeta))$ where $z_k(\alpha)$ and $z_k(\zeta)$ are zeros of $f(z)-\alpha$ and $f(z)-\zeta$ respectively.

Since $z=a$ is zero of order $m$ of $f(z)-\alpha$, $z_k(\alpha)=a$ for each $k$, $\sum_k n(\gamma;z_k(\alpha))=m$.

Therefore $\sum_k n(\gamma;z_k(\zeta))=m$.

Since $n(\gamma;z_k(\zeta))$ must be either 0 or 1, there are exactly $m$ zeros of $f(z)=\zeta$ in $B(a,\in)$.

Since $f'(z)\neq 0$ in $0<|z-a|<\in$, all these zeros must be simple in $0<|z-a|<\in$.

Thus $f(z)=\zeta$ has exactly $m$ zeros in $B(a,\in)$ for any $\zeta\in B(\alpha;\delta)-\{\sigma\}$.

We now prove open mapping theorem:

**Theorem 8** Let $G$ be a region and suppose that $f$ is a non-constant analytic function on $G$. Then for any open set $U$ in $G$, $f(U)$ is open.

**Proof.** Let $\alpha=f(a)\in f(U)$ for some $a\in U$.

Since $U$ is open, there is $R>0$ such that $B(a,R)\subseteq U$ and $f(B(a,R))\subseteq f(U)$.

Since $f$ is analytic on $G$, $f$ is analytic on $B(a,R)$. Hence there exists $\in>0$ and $\delta>0$ such that $B(a,\in)\subseteq B(a,R)$ and that $B(\alpha,\delta)\subseteq f(B(a,\in))\subseteq f(B(a,R))\subseteq f(U)$.

Hence for any $\alpha\in f(U)$, there is $\delta>0$ such that $B(\alpha,\delta)\subseteq f(U)$.

Therefore $f(U)$ is open.

**Theorem 9 Goursat's Theorem** Let $G$ be an open set and $f:G\to\mathbb{C}$ be differentiable function, then $f$ is analytic.

**Proof.** Let $f: G \to \mathbb{C}$ be differentiable function, then $f$ is continuous function on $G$.

To prove that $f$ is analytic we shall use Morera's theorem , it is sufficient to prove that $\int_T f = 0$ for every triangular path $T$ in $G$.

Let $T = [a,b,c,a]$ be a triangular path in $G$. Let $\Delta$ denote the inside of $T$ along with its boundary.

Let $d =$ diameter $(\Delta)$ and $l =$ length $(T)$.

Let $T_1, T_2, T_3, T_4$ be four triangle formed by midpoints of sides of $T$.

Then we have

$$\int_T f = \int_{T_1} f + \int_{T_2} f + \int_{T_3} f + \int_{T_4} f \qquad \ldots(1)$$

Let $T^{(1)}$ denote a triangle amongst $T_1, T_2, T_3, T_4$ such that,

$$\left|\int_{T_j} f\right| \le \left|\int_{T^{(1)}} f\right| \qquad (j = 1,2,3,4)$$

Then $\left|\int_T f\right| \le 4\left|\int_{T^{(1)}} f\right| \qquad \ldots(2)$

Let $\Delta^{(1)}$ denote inside of $T^{(1)}$ along with its boundary.

Then diam $(\Delta^{(1)}) = \frac{1}{2}d$ and $l(T^{(1)}) = \frac{1}{2}l$.

Repeating above process we obtain sequence of triangular paths $T^{(1)}, T^{(2)}, \ldots, T^{(n)}$ such that

$$\left|\int_{T^{(n-1)}} f\right| \le 4\left|\int_{T^{(n)}} f\right| \qquad \ldots(3)$$

and closed sets $\Delta^{(1)}, \Delta^{(2)}, \ldots \Delta^{(n)}$ such that $\Delta^{(1)} \supseteq \Delta^{(2)} \supseteq \ldots$

diam $(\Delta^{(n)}) = \frac{1}{2^n}d$ and $l(T^{(n)}) = \frac{1}{2^n}l$.

Therefore

$$\left|\int_T f\right| \le 4^n \left|\int_{T^{(n)}} f\right|$$

Since $\Delta^{(1)} \supseteq \Delta^{(2)} \supseteq \ldots$ is a descending series of closed sets such that, diam $(\Delta^{(n)}) = \frac{1}{2^n} d \to 0$ as $n \to \infty$.

Therefore by Cantor's theorem, $\bigcap_n \Delta^{(n)} = \{z_0\}$ for some $z_0 \in G$ .

Since $f$ is differentiable in $G$ , $f$ is differentiable at $z_0$.

Hence , given $\in > 0$ there is $\delta > 0$ such that,

$$\left|\frac{f(z) - f(z_0)}{z - z_0} - f'(z_0)\right| < \in \text{ whenever } 0 < |z - z_0| < \delta$$

Equivalently, $\left|f(z) - f(z_0) - (z - z_0) f'(z_0)\right| < \in |z - z_0|$, $0 < |z - z_0| < \delta$ .

Since diam $(\Delta^{(n)}) = \frac{1}{2^n} d \to 0$ as $n \to \infty$, there is $n_0 > 0$ such that diam $(\Delta^{(n)}) < \delta$ .

Since $z_0 \in \Delta^{(n)}$ for each $n$, $z_0 \in \Delta^{(n_0)}$ also. Hence $\Delta^{(n_0)} \subseteq B(z_0;\delta)$.

Hence for $n \ge n_0$

$$\left|\int_{T^{(n)}} f\right| = \left|\int_{T^{(n)}} \left[f(z) - f(z_0) - (z - z_0) f'(z_0)\right] dz\right|$$

$$\le \int_{T^{(n)}} \left|f(z) - f(z_0) - (z - z_0) f'(z_0)\right| |dz|$$

$$\le \int_{T^{(n)}} \in |z - z_0| |dz|$$

$$\le \in \text{diam } (\Delta^{(n)})\, l(T^{(n)}) = \in \frac{dl}{4^n}$$

Thus $\left|\int_T f\right| \le 4^n \left|\int_{T^{(n)}} f\right| \le \in d\, l \qquad n \ge n_0$,

Therefore $\int_T f = 0$.

## EXERCISE

**1)** Let p(z) be a polynomial of degree n and let R > 0 be sufficiently large so that p never vanishes in $\{z : |z| \geq R\}$. If $\gamma(t) = \mathrm{Re}^{it}$, $0 \leq t \leq 2\pi$, show that $\int_{\gamma} \frac{p'(z)}{p(z)} dz = 2\pi i n$.

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UNIT - VII

# LAURENT SERIES DEVELOPMENT AND RESIDUE THEOREM

In this unit we shall discuss Laurent's series expansion of complex valued functions and prove residue theorem.

**Definition 1** A series of the form $\sum_{n=-\infty}^{\infty} a_n (z-a)^n$ is called double series about $z = a$.

**Definition 2** The double series $\sum_{n=-\infty}^{\infty} a_n (z-a)^n$ is said to be absolutely convergent, if both the series $\sum_{n=0}^{\infty} a_n (z-a)^n$ and $\sum_{n=-1}^{-\infty} a_n (z-a)^n$ are absolutely convergent.

**Definition 3** The double series $\sum_{n=-\infty}^{\infty} a_n (z-a)^n$ is said to be uniformly convergent, if both the series $\sum_{n=0}^{\infty} a_n (z-a)^n$ and $\sum_{n=-1}^{-\infty} a_n (z-a)^n$ are uniformly convergent.

**Theorem 4** Let $f$ be analytic in annulus $ann\left(a; R_1, R_2\right)$. Then

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z-a)^n \qquad \ldots(1)$$

where the convergence is absolute and uniform over $ann\left(a; r_1, r_2\right)$ if $R_1 < r_1 < r_2 < R_2$.

Also coefficients $a_n$ are given by the formula

$$a_n = \frac{1}{2\pi i} \int_{\gamma} \frac{f(z)}{(z-a)^{n+1}} dz \qquad \ldots(2)$$ where $\gamma$ is the circle $|z-a| = r$ for any $r$, $R_1 < r < R_2$. Moreover the series is unique.

**Proof.** We shall begin by showing that the integral in (2) is independent of $r$, so that for each integer $n$, $a_n$ is constant.

Let $R_1 < r_1 < r_2 < R_2$ and $\gamma_1 = a + r_1 e^{it}$, $\gamma_2 = a + r_2 e^{it}$ $(0 \le t \le 2\pi)$.

Let $\gamma = \gamma_2 + [z_2, z_1] - r_1 + [z_1, z_2]$ then $\gamma \approx 0$. Since $f$ be analytic in annulus $ann(a; R_1, R_2)$, by Cauchy's theorem we have $\int_\gamma \frac{f(w)}{(w-a)^{n+1}} dw = 0$.

Therefore $\int_{\gamma_2} \frac{f(w)}{(w-a)^{n+1}} dw + \int_{[z_2,z_1]} \frac{f(w)}{(w-a)^{n+1}} dw + \int_{-\gamma_1} \frac{f(w)}{(w-a)^{n+1}} dw + \int_{[z_1,z_2]} \frac{f(w)}{(w-a)^{n+1}} dw = 0$

Thus $\int_{\gamma_2} \frac{f(w)}{(w-a)^{n+1}} dw = \int_{\gamma_1} \frac{f(w)}{(w-a)^{n+1}} dw$.

Now let $z \in ann(a; r_1, r_2)$ then by Cauchy integral formula we have,

$$f(z) = \frac{1}{2\pi i} \int_{\gamma_2} \frac{f(w)}{(w-z)} dw - \frac{1}{2\pi i} \int_{\gamma 1} \frac{f(w)}{(w-z)} dw$$

We define $f_1(z) = \frac{-1}{2\pi i} \int_{\gamma_1} \frac{f(w)}{(w-z)} dw$ and $f_2(z) = \frac{1}{2\pi i} \int_{\gamma_2} \frac{f(w)}{(w-z)} dw$.

Therefore, $f(z) = f_1(z) + f_2(z)$. …(3)

Now

$$f_2(z) = \frac{1}{2\pi i} \int_{\gamma_2} \frac{f(w)}{(w-z)} dw = \frac{1}{2\pi i} \int_{\gamma_2} \frac{f(w)}{(w-a)-(z-a)} dw$$

$$= \frac{1}{2\pi i} \int_{\gamma_2} \frac{f(w)}{(w-a)\left[1 - \frac{(z-a)}{(w-a)}\right]} dw$$

$$= \frac{1}{2\pi i} \int_{\gamma_2} \frac{f(w)}{(w-a)} \sum_{n=0}^{\infty} \left( \frac{z-a}{w-a} \right)^n dw \qquad \left| \frac{z-a}{w-a} \right| < 1 \text{ on } \gamma_2$$

$$= \sum_{n=0}^{\infty} \left\{ \frac{1}{2\pi i} \int_{\gamma_2} \frac{f(w)}{(w-a)^{n+1}} dw \right\} (z-a)^n$$

$$= \sum_{n=0}^{\infty} a_n (z-a)^n \,. \qquad \ldots(4)$$

where $a_n = \dfrac{1}{2\pi i} \displaystyle\int_{\gamma_2} \frac{f(w)}{(w-z)^{n+1}} dw$.

Also

$$f_1(z) = \frac{-1}{2\pi i} \int_{\gamma_1} \frac{f(w)}{(w-z)} dw = \frac{1}{2\pi i} \int_{\gamma_1} \frac{f(w)}{(z-a)-(w-a)} dw$$

$$= \frac{1}{2\pi i} \int_{\gamma_1} \frac{f(w)}{(z-a)\left[1 - \dfrac{(w-a)}{(z-a)}\right]} dw$$

$$= \frac{1}{2\pi i} \int_{\gamma_1} \frac{f(w)}{(z-a)} \sum_{m=0}^{\infty} \left( \frac{w-a}{z-a} \right)^m dw \qquad \left| \frac{w-a}{z-a} \right| < 1 \text{ on } \gamma_1$$

$$= \sum_{m=0}^{\infty} \left\{ \frac{1}{2\pi i} \int_{\gamma_1} (w-a)^m f(w) dw \right\} (z-a)^{-m-1}$$

$$= \sum_{m=0}^{\infty} b_m (z-a)^{-m-1} \,.$$

where $b_m = \dfrac{1}{2\pi i} \displaystyle\int_{\gamma_1} (w-a)^m f(w) dw$.

$$= \frac{1}{2\pi i} \int_{\gamma_1} \frac{f(w)}{(w-a)^{-m}} dw$$

$$= \frac{1}{2\pi i} \int_{\gamma_1} \frac{f(w)}{(w-a)^{(-m-1)+1}} dw$$

$$= a_{-m-1}$$

Therefore $f_1(z) = \sum_{m=0}^{\infty} a_{-m-1} (z-a)^{-m-1}$

$$= \sum_{n=-1}^{-\infty} a_n (z-a)^n \qquad ...(5)$$

Therefore from (3), (4) and (5) we have

$f(z) = \sum_{n=-1}^{-\infty} a_n (z-a)^n + \sum_{n=0}^{\infty} a_n (z-a)^n = \sum_{n=-\infty}^{\infty} a_n (z-a)^n$ for $z \in ann(a; r_1, r_2)$.

**Uniqueness**:

Let $f(z) = \sum_{n=-\infty}^{\infty} c_n (z-a)^n$ be another Laurent series of $f$ where

$$a_n = \frac{1}{2\pi i} \int_{\gamma} \frac{f(z)}{(z-a)^{n+1}} dz$$

$$= \frac{1}{2\pi i} \int_{\gamma_1} \frac{\sum_{k=-\infty}^{\infty} c_k (z-a)^k}{(z-a)^{n+1}} dz$$

$$= \frac{1}{2\pi i} \sum_{k=-\infty}^{\infty} c_k \int_{\gamma_1} (z-a)^{k-n-1} dz \qquad \left[ \int_{|z-a|=r} (z-a)^n dz = \begin{cases} 2\pi i & \text{if } n = -1 \\ 0 & \text{if } n \neq -1 \end{cases} \right]$$

$$= c_n$$

Hence the uniqueness.

**Corollary 5** Let $z = a$ be an isolated singularity of a function $f$ and let $f(z) = \sum_{n=-\infty}^{\infty} a_n (z-a)^n$ be its Laurent series expansion in $ann(a; 0, R)$. Then :

a) $z=a$ is a removable singularity iff $a_n = 0$ for $n \le -1$.

b) $z=a$ is a pole of order $m$ iff $a_{-m} \ne 0$ and $a_n = 0$ for $n \le -(m+1)$.

c) $z=a$ is an essential singularity iff $a_n \ne 0$ for infinitely many negative integers.

**Proof. a)** Suppose $z=a$ is removable singularity of $f$ then there is $R>0$ and an analytic function $g: B(a;R) \to \mathbb{C}$ such that $g(z) = f(z)$ in $0<|z-a|<R$. Since $g$ is analytic in $B(a;R)$ by Taylor's theorem $g(z) = \sum_{n=0}^{\infty} b_n (z-a)^n$ for $|z-a|<R$.

Therefore $f(z) = g(z) = \sum_{n=0}^{\infty} b_n (z-a)^n$ for $0<|z-a|<R$.

This is Laurent series expansion of $f$ in $ann(a;0,R)$ then by uniqueness of Laurent series we must have $a_n = b_n$ for all $n$.

Therefore, $a_n = 0$ for $n \le -1$.

Conversely, suppose that $a_n = 0$ for $n \le -1$ then Laurent series expansion of $f$ is

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n = a_0 + a_1(z-a) + a_2(z-a)^2 + \ldots + .$$

Therefore, $\lim_{z\to a}(z-a)f(z) = \lim_{z\to a}(z-a)\left[\sum_{n=0}^{\infty} a_n (z-a)^n\right] = 0.a_0 = 0$.

Thus $z=a$ is removable singularity of $f$.

**b)** Suppose $z=a$ is a pole of order $m$ then $(z-a)^m f(z)$ has removable singularity at $z=a$.

$$\text{Therefore } (z-a)^m f(z) = (z-a)^m \sum_{n=0}^{\infty} a_n (z-a)^n$$

$$= \sum_{n=0}^{\infty} a_n (z-a)^{m+n}$$

From part (a) $a_n = 0$ for all $m+n \le -1$ .

That is $a_{-m} \ne 0$ and $a_n = 0$ for $n \le -(m+1)$.

Conversely, suppose $a_{-m} \ne 0$ and $a_n = 0$ for $n \le -(m+1)$ then Laurent series expansion of $f$ is $f(z) = \sum_{n=-m}^{\infty} a_n (z-a)^n = a_{-m}(z-a)^{-m} + \ldots + a_{-1}(z-a)^{-1} + a_0 + a_1(z-a)^1 + \ldots$.

Then $(z-a)^m f(z) = a_{-m} + \ldots + a_{-1}(z-a)^{m-1} + a_0(z-a)^m + a_1(z-a)^{m+1} + \ldots$.

Thus $(z-a)^m f(z)$ has Laurent series expansion which does not contains negative powers of $(z-a)$. Therefore from part (a) $(z-a)^m f(z)$ has removable singularity at $z=a$.

**c)** (a) and (b) together implies (c).

**Example 6** Find Laurent series expansion of $f(z) = e^{\frac{1}{z}}$ at $z=0$.

**Solution:** we have $e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \ldots$

Therefore,

$$e^{\frac{1}{z}} = 1 + \frac{1}{z} + \frac{\left(\frac{1}{z}\right)^2}{2!} + \frac{\left(\frac{1}{z}\right)^3}{3!} + \ldots \text{ if } |z| > 0.$$

$$= 1 + \frac{1}{z} + \frac{1}{2!z^2} + \frac{1}{3!z^3} + \ldots .$$

$$e^{\frac{1}{z}} = \sum_{n=0}^{\infty} \frac{1}{n!\,z^n} \text{ for } ann(0;0,\infty).$$

**Ex ample 7** Find Laurent series expansion of $\frac{1}{z(z-1)(z-2)}$ in

a) $ann(0;0,1)$ b) $ann(0;2,\infty)$

**Solution:** Let $f(z) = \frac{1}{z(z-1)(z-2)}$.

a) Consider, $ann(0;0,1) = \{z : 0 < |z| < 1\}$

Here $|z| < 1$ and $\left|\frac{z}{2}\right| < \frac{1}{2} < 1$.

Therefore $f(z) = \frac{1}{z}\left[\frac{1}{z-2} - \frac{1}{z-1}\right]$

$$=\frac{1}{z}\left[\frac{1}{-2\left(1-z/2\right)}+\frac{1}{\left(1-z\right)}\right]$$

$$=\frac{1}{z}\left[\frac{1}{-2}\left(1-z/2\right)^{-1}+\left(1-z\right)^{-1}\right]$$

$$=\frac{1}{z}\left[\frac{1}{-2}\sum_{n=0}^{\infty}\left(z/2\right)^{n}+\sum_{n=0}^{\infty}\left(z\right)^{n}\right], \qquad |z|<1 \text{ and } \left|\frac{z}{2}\right|<\frac{1}{2}<1$$

$$=\sum_{n=0}^{\infty}\left[1-1/2^{n+1}\right]z^{n-1}.$$

b) $ann\left(0;2,\infty\right)=\left\{z:2<|z|<\infty\right\}$

Here $|z|>2$ implies that $\left|\frac{1}{z}\right|<\frac{1}{2}<1$ and $\left|\frac{2}{z}\right|<1$.

Therefore $f(z)=\frac{1}{z}\left[\frac{1}{z-2}-\frac{1}{z-1}\right]$

$$=\frac{1}{z}\left[\frac{1}{z\left(1-2/z\right)}-\frac{1}{z\left(1-1/z\right)}\right]$$

$$=\frac{1}{z^{2}}\left[\left(1-2/z\right)^{-1}-\left(1-1/z\right)^{-1}\right]$$

$$=\frac{1}{z^{2}}\left[\sum_{n=0}^{\infty}\left(2/z\right)^{n}-\sum_{n=0}^{\infty}\left(1/z\right)^{n}\right]$$

$$=\sum_{n=0}^{\infty}\left(2^{n}-1\right)\frac{1}{z^{n+2}}.$$

**Definition 8** Let $z=a$ be an isolated singularity of a function $f$. Then residue of $f(z)$ at $z=a$ is defined to be coefficient of $\dfrac{1}{(z-a)}$ ( that is $a_{-1}$) in Laurent series expansion of $f(z)$ about $z=a$ and is denoted by $\operatorname{Res}(f;a)$.

$$\operatorname{Res}(f;a)=a_{-1}=\frac{1}{2\pi i}\int_{\gamma} f(z)\,dz.$$

**Example 9** Let $f(z)=\dfrac{1-\cos z}{z^4}=\dfrac{1-\left(1-\dfrac{z^2}{2!}+\dfrac{z^4}{4!}-\dfrac{z^6}{6!}+...\right)}{z^4}=\dfrac{1}{2!z^2}-\dfrac{1}{4!}+\dfrac{z^2}{6!}-...$

Therefore ,

$$\operatorname{Res}(f;0)=a_{-1}=\text{coefficient of } \frac{1}{z}=0 .$$

We shall now prove residue theorem:

**Theorem 10** Let $f$ be analytic in a region $G$ except for isolated singularities $a_1,a_2,..,a_m$. If $\gamma$ is a close rectifiable curve in $G$ which does not pass through any of the points $a_k$ and if $\gamma\approx 0$ in $G$ then $\dfrac{1}{2\pi i}\displaystyle\int_{\gamma} f(z)\,dz=\sum_{k=1}^{m} n(\gamma;a_k).\operatorname{Res}(f;a_k)$

**Proof.** Let $m_k=n(\gamma;a_k)$, $(1\le k\le m)$. Let $r_1,r_2,..,r_m$ be positive numbers such that no two disks $\overline{B}(a_k;r_k)$ intersect, none of them intersects $\gamma$ and each disk is contained in $G$. Let $\gamma_k(t)=a_k+r_k e^{-2\pi i m_k t}\quad 0\le t\le 1$.

Then for each $k=1,2,...,m$,

$n(\gamma;a_k)+n(\gamma_k;a_k)=m_k-m_k=0$ and that $n(\gamma_j;a_k)=0$ for $k\neq j$.

Therefore

$$n(\gamma;a_k)+\sum_{j=1}^{m} n(\gamma_j;a_k)=0 \qquad k=1,2,...,m. \qquad \ldots(1)$$

Since $\gamma\approx 0$, $n(\gamma;w)=0,\quad w\notin G$ also $n(\gamma_k;w)=0,\quad w\notin G$.

Therefore for $w\notin G$

$$n(\gamma;w)+\sum_{j=1}^{m} n(\gamma_j;w)=0. \qquad \ldots(2)$$

Thus from (1) and (2) we have

$$n(\gamma; w) + \sum_{k=1}^{m} n(\gamma_k; w) = 0 \text{ for } w \notin G - \{a_1, a_2, ..., a_m\}.$$

Since $f$ be analytic in $G - \{a_1, a_2, ..., a_m\}$, by Cauchy's theorem we have

$$\int_{\gamma} f + \sum_{k-1}^{m} \int_{\gamma_k} f = 0.$$

Now consider $\int_{\gamma_k} f(z)dz = \int_{\gamma_k} \sum_{j=-\infty}^{\infty} b_j (z - a_k)^j dz$

where $f(z) = \sum_{j=-\infty}^{\infty} b_j (z - a_k)^j$ be Laurent series expansion about $z = a_k$

$$= \sum_{j=-\infty}^{\infty} b_j \int_{\gamma_k} (z - a_k)^j dz$$

$$= b_{-1} \int_{\gamma_k} (z - a_k)^{-1} dz$$

$$= \text{Res}(f; a_k)\, 2\pi i n(\gamma_k; a_k)$$

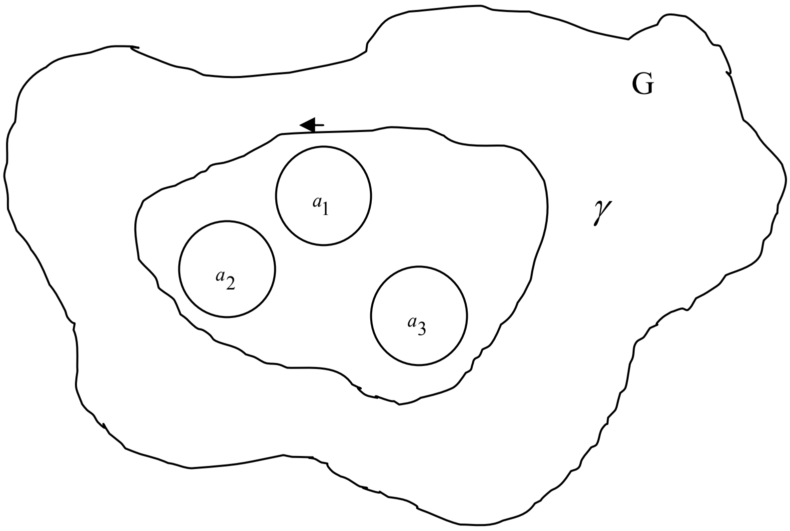
$$= -2\pi i n(\gamma; a_k).\text{Res}(f; a_k)$$

Thus

$$\int_{\gamma} f(z) + \sum_{k-1}^{m} -2\pi i \;\; n(\gamma; a_k).\text{Res}(f; a_k) = 0.$$

Hence,

$$\frac{1}{2\pi i} \int_{\gamma} f(z)\, dz = \sum_{k=1}^{m} n(\gamma; a_k).\text{Res}(f; a_k).$$




**Theorem 11** Suppose $f$ has a pole of order $m$ at $z=a$ and $g(z)=(z-a)^m f(z)$ then

$$\operatorname{Res}(f;a)=\frac{1}{(m-1)!}g^{(m-1)}(a)$$

**Proof.** Since $f$ has a pole of order $m$ at $z=a$, $g(z)=(z-a)^m f(z)$ has removable singularity at $z=a$. Then $g$ has Laurent series expansion of the form $g(z)=\sum_{n=0}^{\infty} b_n(z-a)^n$ where $b_n=\frac{1}{(n)!}g^{(n)}(a)$, in $0<|z-a|<R$ for some $R>0$.

Therefore, $f(z)=\frac{b_0}{(z-a)^m}+\frac{b_1}{(z-a)^{m-1}}+\ldots+\frac{b_{m-1}}{(z-a)}+\sum_{n=m}^{\infty} b_n(z-a)^{n-m}$.

Thus $\operatorname{Res}(f;a)=b_{m-1}=\frac{1}{(m-1)!}g^{(m-1)}(a)$.

**Corollary 12** If $f$ has a simple pole at $z=a$ then $\operatorname{Res}(f;a)=g(a)=\lim_{z\to a}(z-a)f(z)$.

**Proof.** We have $\operatorname{Res}(f;a)=\frac{1}{(1-1)!}g^{(1-1)}(a)$

$$=g(a)$$

$$=\lim_{z\to a} g(z)$$

$$=\lim_{z\to a}(z-a)f(z).$$

**Corollary 13** If $f$ has a simple pole at $z=a$ and $f(z)=\frac{h(z)}{k(z)}$ then $\operatorname{Res}(f;a)=\frac{h(a)}{k'(a)}$.

**Proof.** Since $f$ has a simple pole at $z=a$, $k$ has simple zero at $z=a$, $k(a)=0$.

So that, $k(z)=(z-a)g(z)$, where $g$ is analytic and non-vanishing at $z=a$. Moreover, $k'(a)=g(a)$ Thus

$$\operatorname{Res}(f;a)=\lim_{z\to a}(z-a)f(z)$$

$$=\lim_{z\to a}(z-a)\frac{h(z)}{k(z)}$$

$$=\lim_{z\to a}h(z)\lim_{z\to a}\frac{(z-a)}{(z-a)g(z)}$$

$$=h(a).\frac{1}{g(a)}=\frac{h(a)}{k'(a)}.$$

**Example 14** Calculate residue of $\frac{z^2}{(z-1)(z-2)^2}$.

**Sol.** Let $f(z)=\frac{z^2}{(z-1)(z-2)^2}$ then $f$ has simple pole at $z=1$ and pole of order 2 at $z=2$.

a) $\operatorname{Res}(f;1)=\lim_{z\to 1}(z-1)f(z)$

$$=\lim_{z\to 1}(z-1)\frac{z^2}{(z-1)(z-2)^2}$$

$$=1$$

Alternatively, $\operatorname{Res}(f;1)=\frac{h(1)}{k'(1)}$ where $h(z)=z^2$ , $k(z)=(z-1)(z-2)^2=z^3-5z^2+8z-4$

and $k'(z)=3z^2-10z+8$

Therefore $\operatorname{Res}(f;1)=\frac{h(1)}{k'(1)}=\frac{1^2}{3.1^2-10.1+8}=1$.

b) $z=2$ is pole of order 2. Let $g(z)=(z-2)^2 f(z)=\dfrac{z^2}{(z-1)}$ then

$$g'(z)=\frac{(z-1).2z-1.z^2}{(z-1)^2}=\frac{z^2-2.z}{(z-1)^2}$$

Therefore

$$\begin{aligned}\operatorname{Res}(f;2)&=\frac{1}{(2-1)!}g^{(2-1)}(2)\\ &=g'(2)\\ &=0.\end{aligned}$$

**Ex ample 15** Calculate residue of $\dfrac{z+1}{z^2-2z}$.

**Sol.** Let $f(z)=\dfrac{z+1}{z^2-2z}=\dfrac{h(z)}{k(z)}$ then $f$ has two simple polesnamely, at $z=0$ and $z=2$.

$\operatorname{Res}(f;0)=\dfrac{h(0)}{k'(0)}=\dfrac{0+1}{2.0-2}=-\dfrac{1}{2}$ and

$\operatorname{Res}(f;2)=\dfrac{h(2)}{k'(2)}=\dfrac{2+1}{2.2-2}=\dfrac{3}{2}$.

**Evaluation of real integrals using residue theorem:**

We shall now discuss the methods of evaluating real integrals using residue calculus.

**Type-I :** Integration of the type $\int_{-\infty}^{\infty} f(x)dx$ where $f(x)=\dfrac{h(x)}{g(x)}$ and $h(x)$, $g(x)$ are polynomials in $x$ and $\deg(g(x))-\deg(h(x))\geq 2$ then $\int_C f(z)dz\to 0$ as $R\to\infty$, where $C$ is the semicircle $|z|=R$ in the upper half of the plane.

**Example 16** Evaluate $\displaystyle\int_0^{\infty}\frac{1}{1+x^2}dx$.

**Solution.** Let $I=\displaystyle\int_0^{\infty}\frac{1}{1+x^2}dx=\frac{1}{2}\int_{-\infty}^{\infty}\frac{1}{1+x^2}dx$.

Let $f(z)=\dfrac{1}{1+z^2}$ and $\gamma=[-R,R]\cup C$ where $C$ is the semicircle $|z|=R$ lie in the upper half of the plane. Here we choose $R$ so that all pole in the upper half of the plane are in the interior of $\gamma$.

Here poles of $f(z)=\dfrac{1}{1+z^2}$ are $z=i,-i$ and $z=i$ lies in upper half of the plane .

Since $|z|=|i|=1$, the pole $z=i$ lies inside $\gamma$ if we choose $R>1$.

Therefore by residue theorem $\int\limits_{\gamma} f(z)\,dz=\int\limits_{\gamma}\dfrac{1}{1+z^2}\,dz=2\pi i.\operatorname{Res}\left(f;i\right)$ .

Since $z=i$ is a simple pole of $f(z)$

$$\operatorname{Res}\left(f;i\right)=\lim_{z\to i}(z-i)\frac{1}{1+z^2}=\frac{1}{2i}$$

Thus $\int\limits_{\gamma} f(z)\,dz=\int\limits_{\gamma}\dfrac{1}{1+z^2}\,dz=2\pi i.\dfrac{1}{2i}=\pi$ …(1)

Also $\int\limits_{\gamma} f(z)\,dz=\int\limits_{-R}^{R} f(x)dx+\int\limits_{C} f(z)dz$

$$\pi=\int_{-R}^{R}\frac{1}{1+x^2}dx+\int_{C}\frac{1}{1+z^2}dz \qquad \text{…(2)}$$

Consider,

$$\int_{C} f(z)dz=\int_{0}^{\pi}\frac{1}{1+R^2e^{2it}}R\,\mathrm{e}^{it}\,idt=i\int_{0}^{\pi}\frac{R\,\mathrm{e}^{it}}{1+R^2e^{2it}}dt\,.$$

Since,

$|1+z^2|\geq||1|-|z^2||=|1-R^2|=R^2-1$,

we have

$$\left|\int_{C} f(z)dz\right|\leq\int_{0}^{\pi}\frac{R}{R^2-1}dt\,.$$

Since $\frac{R}{R^2-1} \to 0$, as $R \to \infty$, we get $\lim_{R\to\infty} \int_C \frac{1}{1+z^2} dz = 0$

On taking limit as $R \to \infty$ in (2), we obtain

$$\pi = \int_{-\infty}^{\infty} \frac{1}{1+x^2} dx + 0.$$

Therefore, $\int_0^{\infty} \frac{1}{1+x^2} dx = \frac{1}{2} \int_{-\infty}^{\infty} \frac{1}{1+x^2} dx = \frac{\pi}{2}$.

**Example 17** Evaluate $\int_{-\infty}^{\infty} \frac{x^2}{1+x^4} dx$.

**Solution.** Let $I = \int_{-\infty}^{\infty} \frac{1}{1+x^2} dx$.

Let $f(z) = \frac{z^2}{1+z^4}$ and $\gamma = [-R, R] \cup C$ where $C$ is the semicircle $|z| = R$ lie in the upper half of the plane. Here we choose $R$ so that all pole lies in the upper half of the plane are in the interior of $\gamma$.

Here poles of $f(z) = \frac{z^2}{1+z^4}$ are $z = e^{\frac{\pi i}{4}}, e^{\frac{3\pi i}{4}}, e^{\frac{5\pi i}{4}}, e^{\frac{7\pi i}{4}}$ and $z = e^{\frac{\pi i}{4}}, e^{\frac{3\pi i}{4}}$ are lies in upper half of the plane and inside $\gamma$ if we choose $R > 1$.

Therefore by residue theorem $\int_\gamma f(z)\, dz = \int_\gamma \frac{z^2}{1+z^4} dz = 2\pi i. \left[ \operatorname{Res}\left( f; e^{\frac{\pi i}{4}} \right) + \operatorname{Res}\left( f; e^{\frac{3\pi i}{4}} \right) \right]$.

$$\operatorname{Res}\left( f; e^{\frac{\pi i}{4}} \right) = \frac{h\left( e^{\frac{\pi i}{4}} \right)}{g'\left( e^{\frac{\pi i}{4}} \right)} = \frac{\left( e^{\frac{\pi i}{4}} \right)^2}{4\left( e^{\frac{\pi i}{4}} \right)^3} = \frac{1}{4} e^{-\frac{\pi i}{4}} = \frac{1}{4}\left[ \cos\left( \frac{\pi i}{4} \right) - i \sin\left( \frac{\pi i}{4} \right) \right] = \frac{(1-i)}{4\sqrt{2}}$$

and

$$\operatorname{Res}\left( f; e^{\frac{3\pi i}{4}} \right) = \frac{h\left( e^{\frac{3\pi i}{4}} \right)}{g'\left( e^{\frac{3\pi i}{4}} \right)} = \frac{\left( e^{\frac{3\pi i}{4}} \right)^2}{4\left( e^{\frac{3\pi i}{4}} \right)^3} = \frac{1}{4} e^{-\frac{3\pi i}{4}} = \frac{1}{4}\left[ \cos\left( \frac{3\pi i}{4} \right) - i \sin\left( \frac{3\pi i}{4} \right) \right] = \frac{(-1-i)}{4\sqrt{2}}$$

Thus $\int\limits_{\gamma} f(z)\, dz = \int\limits_{\gamma} \frac{z^2}{1+z^4}\, dz = 2\pi i.\left[\frac{(1-i)}{4\sqrt{2}} + \frac{(-1-i)}{4\sqrt{2}}\right] = \frac{\pi}{\sqrt{2}}$ …(1)

Also $\int\limits_{\gamma} f(z)\, dz = \int\limits_{-R}^{R} f(x)dx + \int\limits_{C} f(z)dz$

$$\frac{\pi}{\sqrt{2}} = \int\limits_{-R}^{R} \frac{x^2}{1+x^4}\, dx + \int\limits_{C} \frac{z^2}{1+z^4}\, dz \qquad \text{…(2)}$$

Consider,

$$\int\limits_{C} f(z)dz = \int\limits_{0}^{\pi} \frac{R^2\, \mathrm{e}^{2it}}{1+R^4 e^{4it}}\, R\, \mathrm{e}^{it}\, idt = i\int\limits_{0}^{\pi} \frac{R^3\, \mathrm{e}^{3it}}{1+R^4 e^{4it}}\, dt\,.$$

Since,

$|1+z^4| \geq ||1|-|z^4|| = |1-R^4| = R^4-1$,

we have

$$\left|\int\limits_{C} f(z)dz\right| \leq \int\limits_{0}^{\pi} \frac{R^3}{R^4-1}\, dt\,.$$

Since $\frac{R^3}{R^4-1} \to 0$, as $R \to \infty$, we get $\lim\limits_{R\to\infty} \int\limits_{C} \frac{z^2}{1+z^4}\, dz = 0$

On taking limit as $R \to \infty$ in (2) we get

$$\frac{\pi}{\sqrt{2}} = \int\limits_{-\infty}^{\infty} \frac{x^2}{1+x^4}\, dx + 0\,.$$

Therefore, $\int\limits_{-\infty}^{\infty} \frac{x^2}{1+x^4}\, dx = \frac{\pi}{\sqrt{2}}$.

**Type-II :** Integration of the type $\int_0^{2\pi} f(\cos\theta,\sin\theta)d\theta$ where $f(\cos\theta,\sin\theta)$ is rational function of $\cos\theta$ and $\sin\theta$.

Here we substitute $z=e^{i\theta}$ $(0\le\theta\le 2\pi)$ that is $z=e^{i\theta}$ describes the unit circle $|z|=1$.

Also $\cos\theta=\dfrac{e^{i\theta}+e^{-i\theta}}{2}=\dfrac{z+z^{-1}}{2}=\dfrac{z^2+1}{2z}$ and $\sin\theta=\dfrac{e^{i\theta}-e^{-i\theta}}{2i}=\dfrac{z-z^{-1}}{2i}=\dfrac{z^2-1}{2iz}$.

Then $\int_0^{2\pi} f(\cos\theta,\sin\theta)d\theta=\int_\gamma f\left(\dfrac{z^2+1}{2z},\dfrac{z^2-1}{2iz}\right)\dfrac{dz}{iz}$ where $\gamma$ is positively oriented unit circle $|z|=1$ can be evaluated using residue theorem.

**Example 18** Evaluate $\int_0^{2\pi}\dfrac{1}{1+a\sin\theta}d\theta$ $(-1<a<1)$.

**Solution.** Let $I=\int_0^{2\pi}\dfrac{1}{1+a\sin\theta}d\theta$. …(1)

Put $z=e^{i\theta}$ $(0\le\theta\le 2\pi)$ then $dz=izd\theta$ and $\sin\theta=\dfrac{z^2-1}{2iz}$.

Therefore $\int_0^{2\pi}\dfrac{1}{1+a\sin\theta}d\theta=\int_\gamma\dfrac{1}{1+a\left(\dfrac{z^2-1}{2iz}\right)}\dfrac{dz}{iz}$ where $\gamma$ is unit circle $|z|=1$.

$$=\frac{2}{a}\int_\gamma\frac{1}{\left(z^2+\dfrac{2i}{a}z-1\right)}dz \qquad \text{…(2)}$$

Let $f(z)=\dfrac{1}{\left(z^2+\dfrac{2i}{a}z-1\right)}=\dfrac{1}{(z-z_1)(z-z_2)}$ where $z_1=\dfrac{-i+i\sqrt{1-a^2}}{a}$ and $z_2=\dfrac{-i-i\sqrt{1-a^2}}{a}$ are simple poles of $f(z)$.

Note that $|z_2|=\left|\dfrac{-i-i\sqrt{1-a^2}}{a}\right|=\dfrac{1+\sqrt{1-a^2}}{|a|}>1$ $(-1<a<1)$.

Since $|z_1 z_2| = 1$, $|z_1| < 1$.

Thus the pole $z_1$ lies inside $\gamma$ and $z_2$ lies outside.

Therefore $\operatorname{Res}(f; z_1) = \lim_{z \to z_1} (z - z_1) f(z) = \lim_{z \to z_1} (z - z_1) \frac{1}{(z - z_1)(z - z_2)} = \frac{1}{(z_1 - z_2)} = \frac{a}{2i\sqrt{1-a^2}}$

Then by residue theorem

$$\frac{1}{2\pi i} \int_\gamma \frac{1}{\left(z^2 + \frac{2i}{a} z - 1\right)} dz = \operatorname{Res}(f; z_1).n(\gamma; z_1) = \frac{a}{2i\sqrt{1-a^2}}$$

Therefore $\displaystyle\int_\gamma \frac{1}{\left(z^2 + \frac{2i}{a} z - 1\right)} dz = \frac{a\pi}{\sqrt{1-a^2}}$

Thus from (2) we have

$$\int_0^{2\pi} \frac{1}{1 + a\sin\theta} d\theta = \frac{2}{a} \int_\gamma \frac{1}{\left(z^2 + \frac{2i}{a} z - 1\right)} dz = \frac{2}{a} \frac{a\pi}{\sqrt{1-a^2}} = \frac{2\pi}{\sqrt{1-a^2}}.$$

**Example 19** Evaluate $\displaystyle\int_0^{\pi} \frac{1}{a + \cos\theta} d\theta \; (a > 1)$.

**Solution.** Let $\displaystyle I = \int_0^{\pi} \frac{1}{a + \cos\theta} d\theta = \frac{1}{2} \int_0^{2\pi} \frac{1}{a + \cos\theta} d\theta.$ …(1)

Put $z = e^{i\theta}$ $(0 \le \theta \le 2\pi)$ then $dz = iz d\theta$ and $\cos\theta = \frac{z^2 + 1}{2z}$.

Therefore $\displaystyle I = \frac{1}{2} \int_0^{2\pi} \frac{1}{a + \cos\theta} d\theta = \frac{1}{2} \int_\gamma \frac{1}{a + \left(\frac{z^2 + 1}{2z}\right)} \frac{dz}{iz}$ where $\gamma$ is unit circle $|z| = 1$.

$$= \frac{1}{i} \int_{\gamma} \frac{1}{\left(z^2 + 2az + 1\right)} dz \qquad \ldots(2)$$

Let $f(z) = \frac{1}{\left(z^2 + 2az + 1\right)} = \frac{1}{(z - z_1)(z - z_2)}$ where $z_1 = -a + \sqrt{a^2 - 1}$ and $z_2 = -a - \sqrt{a^2 - 1}$ are simple poles of $f(z)$.

Note that $|z_2| = \left|-a - \sqrt{a^2 - 1}\right| = a + \sqrt{a^2 - 1} > 1 \quad (\text{as } a > 1)$.

Since $|z_1 z_2| = 1$, $|z_1| < 1$.

Thus the pole $z_1$ lies inside $\gamma$ and $z_2$ lies outside.

Therefore $\operatorname{Res}\left(f; z_1\right) = \lim_{z \to z_1} (z - z_1) f(z) = \lim_{z \to z_1} (z - z_1) \frac{1}{(z - z_1)(z - z_2)} = \frac{1}{(z_1 - z_2)} = \frac{1}{2\sqrt{a^2 - 1}}$

Then by residue theorem

$$\frac{1}{2\pi i} \int_{\gamma} \frac{1}{\left(z^2 + 2az + 1\right)} dz = \operatorname{Res}\left(f; z_1\right).n\left(\gamma; z_1\right) = \frac{1}{2\sqrt{a^2 - 1}}$$

Therefore $\displaystyle\int_{\gamma} \frac{1}{\left(z^2 + 2az + 1\right)} dz = \frac{\pi i}{\sqrt{a^2 - 1}}$

Thus from (2) we have

$$\int_0^{\pi} \frac{1}{a + \cos\theta} d\theta == \frac{1}{i} \int_{\gamma} \frac{1}{\left(z^2 + 2az + 1\right)} dz = \frac{1}{i} \frac{\pi i}{\sqrt{a^2 - 1}} = \frac{\pi}{\sqrt{a^2 - 1}}.$$

## EXERCISES

1. Show that

   i) $\int\limits_{-\infty}^{\infty} \frac{dx}{1+x^4} = \frac{\pi}{\sqrt{2}}$, ii) $\int\limits_{0}^{\infty} \frac{xdx}{1+x^4} = \frac{\pi}{4}$,

   iii) $\int\limits_{0}^{\infty} \frac{dx}{1+x^2+x^4} = \frac{\pi}{2\sqrt{3}}$ iv) $\int\limits_{0}^{\infty} \frac{x^2dx}{1+x^6} = \frac{\pi}{6}$

2. Show that

   i) $\int\limits_{0}^{\pi/2} \frac{dx}{a+\sin^2 x} = \frac{\pi}{2a\sqrt{a+1}}$, ii) $\int\limits_{0}^{2\pi} \frac{\cos 3xdx}{5-4\cos x} = \frac{\pi}{12}$

   iii) $\int\limits_{-\pi}^{\pi} \frac{\cos xdx}{3+4\cos x} = -\frac{\pi}{3}$ iv) $\int\limits_{0}^{\pi} \frac{\cos 2xdx}{1-2a\cos x+a^2} = \frac{\pi}{12}$ $(a^2 < 1)$.

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UNIT - VIII

# ROUCHE'S THEOREM AND MAXIMUM MODULUS THEOREM

In this unit we shall prove Argument Principle, Rouche's theorem and Maximum modulus theorem. Rouche's theorem is found to be very useful in finding number of zeros inside a given closed curve.

**Definition(Meromorphic function)1.** A function $f$ that is analytic except for finite number of points is meromorphic function.

e.g. i) $\dfrac{1}{(z-2)(z+5)}$ ii) $\dfrac{1}{\exp(z)-1}$.

**Theorem(Argument Principle)2.** Let $f$ be meromorhic in G with pole $p_1, p_2, \ldots, p_m$ and zeros $z_1, z_2, \ldots, z_n$ counted according to the multiplicity. If $\gamma$ is a closed rectifiable value in G with $\gamma \cong 0$ and not passing through $p_1, p_2, \ldots, p_m$ and $z_1, z_2, \ldots, z_n$ then

$$\frac{1}{2\pi_i}\int_{\gamma}\frac{f'(z)}{f(z)}dz = \sum_{k=1}^{n} n(\gamma_j z_k) - \sum_{k=1}^{m}(\gamma_j p_k)$$

**Proof** : Since $p_1, p_2, \ldots, p_m$ are poles and $z_1, z_2, \ldots, z_n$ are zeros of $f$, there is an analytic function $g$ such that

$$f(z) = \left[\frac{\prod_{k=1}^{n}(z - z_k)}{\prod_{k=1}^{n}(z - p_k)}\right] \cdot g(z) \qquad (1)$$

where $g$ is non vanishing.

Logarithmic differention of $f$ gives us

$$\frac{f'(z)}{f(z)} = \sum_{k=1}^{n} \frac{1}{(z-z_k)} - \sum_{k=1}^{n} \frac{1}{(z-p_k)} + \frac{g'(z)}{g(z)} \tag{2}$$

Since g is non vanishing analytic function

$\int_\gamma \frac{g'}{g} = 0$ . Hence, (2) implies

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}dz = \sum_{k=1}^{n} \frac{1}{2\pi i}\int_\gamma \frac{dz}{(z-z_k)} - \sum_{k=1}^{n} \frac{1}{2\pi i}\int_\gamma \frac{dz}{(z-p_k)} + 0$$

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}dz = \sum_{k=1}^{n} n(\gamma; z_k) - \sum_{k=1}^{n} n(\gamma; p_k).$$

**Note 3** If $\gamma(t) = a + re^{it}$, then $\frac{1}{2\pi i}\int \frac{f'(z)}{f(z)}dz = Z_f - P_f$

where

$Z_f$ : Number of zeros of $f$ inside $B(a; r)$ counted according to multiplicity

and

$P_f$ : Number of poles of $f$ inside $B(a; r)$ counted according to multiplicity.

e. g. $\quad f(z) = \frac{(z-1)^3(z-4)^4}{(z+i)^2(z-i)(z+1-i)^3}, \qquad \gamma : |z| = 2$

Then $\quad \int_\gamma \frac{f'(z)}{f(z)}dz = 2\pi i[3-(2+1+3)] = -6\pi i.$

**Rouche's theorem** 4 Suppose $f$ and $g$ are meromorphic in a neighborhood of $\overline{B}(a;R)$ with no zeros or poles on the circle $\gamma = \{z / |z-a| = R\}$. If $Z_f$ ,$Z_g$ ($P_f$, $P_g$) are the numbers of zeros (poles) of $f$ & $g$ inside $\gamma$ counted according to their multiplicities and if $|f(z) + g(z)| < |f(z)| + |g(z)|$ on $\{\gamma\}$, then $Z_f - Z_g = P_f - P_g$ or $Z_f - P_f = Z_g - P_g$.

**Proof** : If $f/g = \lambda$ is real, then $| f(z) + g(z) | < | f(z)| + | g(z)|$ gives us

$$\left|\frac{f(z)}{g(z)}+1\right| < \left|\frac{f(z)}{g(z)}+1\right| \text{ i.e. } | \lambda + 1 | < | \lambda | + 1.$$

Further, if $f/g = \lambda \geq 0$ then (1) gives us, $\lambda + 1 < \lambda + 1$which is absurd.

Hence $f/g$ does not take any value in $[0, \infty)$. Therefore, $f/g$ has well defined logarithm log $(f/g)$ in $C - [0, \infty)$. Moreover, log $(f/g)$ is primitive of

$(f/g)' / (f/g)$. Hence $\int_\gamma \frac{f'(z)}{f(z)} dz = 0$ so that

$$\int_\gamma \left(\frac{gf' - g'f}{g^2}\right)\left(\frac{g}{f}\right) = 0 \quad \Rightarrow \quad \int_\gamma \left(\frac{f'}{f} - \frac{g'}{g}\right) = 0$$

$$\Rightarrow \quad \frac{1}{2\pi i}\int_\gamma \frac{f'}{f} \quad = \quad \frac{1}{2\pi i}\int_\gamma \frac{g'}{g}$$

$$\Rightarrow Z_f - P_f = Z_g - P_g.$$

**Corollary 5** If $f$ and $g$ are analytic in neighborhood of $\bar{B}(a; R)$ with no zeros on $\{\gamma\}$ where $\gamma : | z - a | = R$ and if $| f(z) + g(z) | < | f(z)|$, then $f$ and $g$ have same number of zeros in $B(a; R)$.

**Proof** : We have $| g(z)| < | f(z)|$ on $\{\gamma\} \Rightarrow | f(z) + g(z) | < | f(z)| + | f(z) + g(z) |$ on $\{\gamma\}$, hence f and f + g have same number of zeros in $B(a; r)$.

**Corollary** 6 If $f$ and $g$ are analytic in neighborhood of $\bar{B}(a; R)$ with no zeros on $\{\gamma\}$ where $\gamma : | z - a | = R$ and if $| g(z) | < | f(z)|$ on $\{\gamma\}$, then $f$ and $f+g$ have same number of zeros in $B(a; R)$.

**Proof:** We have | *g(z)* | < | *f(z)*| on {γ}. Thus we have $|f(z)+g(z)-f(z)| < |f(z)|+|f(z)+g(z)|$, hence, $f$ and $f+g$ have same number of zeros.

**Fundamental theorem of Algebra** 7 Every non constant poly. has a zero. OR A polynomial P($z$) = $z^n + a_1 z^{n-1} + \dots + a_n$. has precisely $n$ – zeros.

**Proof** : Consider P($z$) = $z^n + a_1 z^{n-1} + \dots + a_n$

$$\Rightarrow P(z) / z^n = 1 + a_1/z + \dots + a_n/z^n .$$

hence P($z$) / $z^n \to 1$ as $n \to \infty$. Clearly P($z$) / $z^n$ is analytic in ____ – {0}. Hence is analytic in the neighborhood of infinity. i.e. $\{z / |z| \geq R\}$.

Let γ : $\{z / |z| = R\}$, then $\left|\dfrac{P(z)}{z^n} - 1\right| < 1$ or | P($z$) – $z^n$ | < | $z^n$ | on {γ} for suitable choice of R.

Hence Rouche's theorem, P($z$) & $z^n$ have same number of zeros in $B(a; r)$. Since $z^n$ has n zeros (counted according to multiplicities) in $B(a; r)$. P($z$) also has precisely $n$- zeros in $B(a; r)$ for some R > 0.

**Example 8** Prove that there are 3 zeros of $z^3 - 6z + 8$ in an $B(0; 3)$.

**Proof** : Let $f_1(z) = z^3 - 6z + 8$, We shall prove that $f_1$ has 3 zeros in $B(0; 3)$and no zeros in $B(0; 1)$.

Let $f(z) = z^3$ and $g(z) = -6z + 8$

Now, on $|z| = 3$, $|f(z)| = |z|^3 = 27$, and $|g(z)| = |-6z + 8| \leq 6|z| + 8 = 26$.

Thus $|g(z)| \leq 26 < 27 = |f(z)|$ on $|z| = 3$.

Hence $f$ and $f + g$ have same nos. of zeros in $B(0; 3)$. Since $f(z) = z^3$ has 3 zeros in $B(0; 3)$. $f + g = z^3 - 6z + 8$ also has 3 zeros in $B(0; 3)$.

Now consider $| z | = 1$ .

Let $f(z) = z^3 + 8$ and $g(z) = - 6z$ .

Now $|f(z)| = | z^3 + 8 | \geq 8 - | z |^3 = 8 - 1 = 7.$

& $| g(z) | = | - 6z | = 6 | z| = 6.$

Thus $|f(z)| = 6 < 7 \leq |f(z)|$ on $| z | = 1$.

Hence $f$ and $f + g$ have same number of zeros in $B(0; 3)$.

Since $f(z) = z^3 + 8$ has 3 zeros on $| z | = 2$, $f$ has no zeros in $B(0; 1)$.

Hence $f(z)+ g(z) = z^3 - 6z + 8$ has no zeros in $B(0; 1)$.

Thus all the three zeros of $z^3 - 6z + 8$ lie in of ann(0; 1,3).

**Example 9** Let > 1, and show that the equation of $\lambda - z - e^{-z} = 0$ has exactly one solution in the half plane $\{ z / Re(z) > 0 \}$. Prove that this solution must be real.

**Solution** Let r > 1, let $f(z) = - \lambda + z$ and $g(z) = e^{-z}$. Then on [–iR, iR],

we have, $z = iy$ & $|f(z)| = | - \lambda + iy | = \sqrt{\lambda^2 + y^2} \geq \lambda > 1$ and $| g(z)| = e^{-z}$.

$$= | e^{-iy}| = 1.$$

Thus $| g(z)| < |f(z)|$ on [–iR, iR].

Now consider the semi circle $\{ z / | z | = R : Re(z) > 0\}$.

i.e., $z = Re^{it}$ , $- \pi/2 < t < \pi/2$.

Here $|f(z)| = | - \lambda + Re^{it} | \geq R - \lambda$, and $| g(z)| = | e^{-z} | | e^{-R \cos t} | \leq 1 \leq R - \lambda \leq |f(z)|$.

Thus $| g(z)| < |f(z)|$ on the semi circle & [–iR, iR].

Thus $f$ and $f + g$ have the same number of zeros, inside the circle

$\{ z / | z | = R ; Re(z) > 0\} \cup [-iR, iR]$.

Since $f(z) = - \lambda + z$ has precisely one zero so does $f(z) + g(z) = \lambda - z - e^{-z}$.

**Example 10** Consider $z^3 - 6z^2 + 3z + 1$,

Solution Let $f(z) = 6z^2 - 3z$ & $g(z) = z^3 + 1$

Let $\gamma : |z| = 3$ and $\gamma = \{ z / |z| = 3 \}$ .

Now $|f(z)| = |6z^2 - 3z| \geq 6|z|^2 - 3|z| = 6\times9 - 3\times3 = 45$

& $|g(z)| = |z^3 - 1| = |z|^3 + 1 = 28$

Thus $|g(z)| = |f(z)|$ for $z \in \{\gamma\}$

Hence $f$ and $f + g$ have the same no of zeros in $B(0; 3)$

Since $f(z) = 3z(2z - 1)$ both the zeros viz., 0 and ½ of $f$ lies inside $|z| = 3$

Hence $z^3 - 6z^2 + 3z + 1 = 0$ has precisely 2 zeros in $B(0 ; 3)$.

**Example 11** Prove that $e^{-iz} = 2 + z^2$ has only one root in the upper half plane.

**Solution** Let $f(z) = 2 + z^2$ & $g(z) = e^{-iz}$ ,

Consider $\gamma = [-R, R] \cup \{z / \text{Re}(z) > 0, |z| = R\}$

for $z \in [-R, R]$, $z = x$ real,

$$f(z) = f(x) = 2 + x^2 > 1,$$

$$= |e^{-ix}|$$

$$= |e^{-iz}|$$

$$= |g(z)|$$

$\therefore$ $g(z) < f(z)$ for $z \in [-R, R]$,

$\therefore$ let $z = Re^{-it}$ $0 \leq t \leq 2\pi$

Then $|f(z)| = |2z + z^2| \geq R^2 - 2$

By choosing $R > \sqrt{3}$ we have

$$|f(z)| \geq R^2 - 2 > 1 > e^{-R\sin t} = \left| e^{iR(\cos t + i\sin t)} \right|$$

$$= |e^{-iz}| \text{ on } \{z / \text{Re}(z) > 0, |z| = R\}$$

Thus $| g(z) | \leq | f( z) |$

Hence by Rouches theorem $f$ and $f + g$ have the same no of zeros in $\gamma$,

Since $f$ has only one zero viz. $\sqrt{2}i$ inside $\gamma$ and above real axis i.e. upper half plane $f + g$ has precisely one zero in the upper half plane.

**Maximum Modulus Theorems**

**Theorem 12** If $f$ is analytic in a region G and $a$ is a point in G with $| f(x) | \geq | g(x) | \ \forall \ z \in$ G, then $f$ must be a constant function.

**Proof** : Let $\alpha = f(a)$ & $\Omega = f(G)$. By hypothesis

$| f(a) | \geq | f(z) | \ \forall \ z \in$ G $\Rightarrow | \alpha | \geq | \xi | \ \forall \ z \in \Omega$. Hence $\Omega \cap \partial\Omega \neq \phi$, since $\alpha | \geq | \xi | \ \forall \ \xi \in \Omega \Rightarrow \alpha \in \Omega$. Therefore, $\Omega$ cannot be an open set. Since f is analytic and non constant $\Omega = f(G)$ is necessarily open by open mapping theorem. Hence $f$ must be a constant.

**Maximum Modulus Theorem 13** (2nd version) : Let G be a neighbourhood open set in G and suppose F is continuous function on $\overline{G}$ which is analytic in G. Then max

$\{| f(z) | : z \in \overline{G} \} = \{| f(z) | : z \in \partial G\}$.

Proof : Since G is bounded, $\overline{G}$ id bounded closed set. Hence, $\overline{G}$ is compact. Since f is continuous on compact set $\overline{G}$, it attains maximum at some $a \in \overline{G}$, i.e., there is $a \in \overline{G}$ such that $| f(z) | \leq | f(a) | \ \forall \ z \in \overline{G}$. If $f$ were non-constant and $a \in$ G, then we are lead to contradiction by 1st version of MMT, hence $a \in \overline{G}$ but $a \notin$ G i.e., $a \in \partial$G.

**Definition 14** Let $f$ be real valued function G $\subseteq$ **. Let $a \in \overline{G}$ or $a = \infty$ then the limit superior of f(z) as z $\to \infty$ is defined as

$$\limsup_{r \to a} f(z) = \lim_{r \to o+} \sup f(z) / z \in G \cap B(a;r).$$

Similarly, $\liminf_{r \to a} f(z) = \lim_{r \to o+} \inf f(z) / z \in G \cap B(a;r)$.

By $\partial_\infty$G, we mean extended boundary of G defined as $\partial_\infty G = \partial G \cup \{\infty\}$, if g is unbounded & $\partial_\infty G = \partial G$ if G is bounded.

**Maximum Modulus Theorem 15 (3rd version)** : Let G be region in C & f an analytic function on G. Suppose there is constant M s. t. $\limsup_{r \to a} |f(z)| \leq M$ $\forall$ $a \in$ G. Then $|f(z)| \leq$ M $\forall$ $z \in$ G.

**Proof**: Let $\delta > 0$ be arbitrary and $H = \{z \in G : |f(z)| > M + \delta\}$. Now, it is enough to prove that $H = \phi$. Since $f$ is analytic, its real and imaginary parts are continuous and hence $|f|$, is continuous. Therefore, $H$ is open . Now, given that $\limsup_{r \to a} |f(z)| \leq M \quad \forall a \in \partial_\infty G$. Hence, there is $r > 0$ such that $|f(z)| < M + \delta \quad \forall z \in G \cap B(a;r)$. Hence, $\overline{H} \cap \partial_\infty G = \phi$. In other words, $\overline{H} \subseteq G$, regardless of whether $G$ is bounded or unbounded. Hence, by 2nd version of MMT, there is $z \in \partial H$ , such that $|f(z)| = M + \delta$, which is absurd, hence, $H = \phi$ or $f$ is constant. Now if $f$ is constant, $H = \phi$ is hypothesis. Thus in any case $H = \phi$.

**Schwarz' Lemma 16** Let $D = \{z : |z| < 1\}$ and suppose $f$ is analytic in $D$ with

(a) $|f(z)| \leq 1$ for $z \in D$

(b) $f(0) = 0$.

Then $|f'(0)| \leq 1, \quad |f(z)| \leq |z| \quad \forall z \in D$.

Moreover, if $|f'(0)| = 1$ or $|f(z)| = |z|$ for some $z \neq 0$. Then there is a constant $c, |c| = 1$ such that $f(w) = cw, \forall w \in D$.

**Proof.** Define $g : D \to \mathbb{C}$ by $g(z) = \frac{f(z)}{z}$ *if* $z \neq 0$ and $g(0) = f'(0)$. Note that $\lim_{z \to 0} g(z) = \lim_{z \to 0} \frac{f(z)}{z} = \lim_{z \to 0} \frac{f(z) - f(0)}{z - 0} = f'(0)$. Thus $g$ is continuous at $z = 0$ and consequently $g$ is analytic in $D$. Now by maximum modulus theorem, for any $r < 1$, there is a point $z_0$ such that $|z_0| = r < 1$ and $|g(z)| = \left|\frac{f(z)}{z}\right| \leq \left|\frac{f(z_0)}{z_0}\right| \leq \frac{1}{r}$. Letting $r \to 1^-$, we obtain $|g(z)| \leq 1 \quad \forall z \in D$. Hence, $\left|\frac{f(z)}{z}\right| \leq 1$ or $|f(z)| \leq |z|$ for all $z \neq 0$ and $|g(0)| = |f'(0)| \leq 1$. Since $f(0) = 0$, we have $|f(z)| \leq |z|$ for all $z \in D$ and $|f'(0)| \leq 1$. Let $|f'(0)| = 1$, then $|g(0)| = 1$, which implies that g attains maximum in the disc D. Therefore, g must be constant. Therefore, there is $c \in \mathbb{C}$ such that $g(z) = c$. That is, $\frac{f(z)}{z} = c$ or $f(z) = cz$, since $f(z) = cz$ holds trivially for $z = 0$, we have $f(z) = cz$ for all $z \in D$. Further, $|c| = |g(z)| = 1$. In other words $|c| = 1 \Rightarrow c = e^{i\theta}$ for some real $\theta$. Hence, $f(z) = e^{i\theta} z$ for all $z \in D$.

# EXERCISES

1. Find the number of zeros of the following polynomials

   i) $z^4 + 6z^2 + z + 2$ in the unit disk and in the $ann(0;1,3)$

   ii) $z^5 - 12z^2 + 14$ in $ann(0;1,5/2)$ and also in $\{z : |z| < 2\}$

2. Suppose $R > 0$, is sufficiently large, then if $p$ is a polynomial of degree n>0. Then $\int_{|z|=R} \frac{p'(z)}{p(z)} dz = 2i\pi n$.

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We, the authors, are not claiming any originality of the content of this book. The content is taken form the following references.

# REFERENCES


1) J. B. Conway Foundations of One Complex Variable, Springer Int. Student Edition 1973; Indian Reprint by Narosa Publishing House, New Delhi, 1997.
2) Ponnusamy. Foundations of Complex Analysis, Narosa Publishing House, New Delhi.
3) B. Chauhdary. The Elements of Complex Analysis, Newage International.
4) Shanti Narayan Theory of Functions of A Complex Variables, S. Chand and Company Ltd. New Delhi, 1986.
5) L. V. Ahlfors Complex Analysis, Mc Graw Hill Book Co. 1966.

## UNIT - IX

# SCHWARZ'S LEMMA AND ITS CONSEQUENCES

**Theorem 9.1 (Schwarz's Lemma)**

Let $D=\{z:|z|<1\}$ and suppose $f$ is analytic on D with

(a) $|f(z)|\leq 1$ for z in D

(b) $f(0)=0$

Then $|f'(0)|\leq 1$ and $|f(z)|\leq|z|$ for all z in the disk D. Moreover if $|f'(0)|=1$ or if $|f(z)|=|z|$ for some $z\neq 0$ then there is a constant c, $|c|=1$, such that $f(w)=cw$ for all w in D.

**Proof :** Let $D=\{z:|z|<1\}$ and $f:D\to\mathbb{C}$ is analytic with

(a) $|f(z)|\leq 1$ for $z\in D$

(b) $f(0)=0$

Define $g:D\to\mathbb{C}$ by

$$g(z)=\begin{cases}\dfrac{f(z)}{z} & \text{for } z\neq 0\\ f'(0) & \text{for } z=0\end{cases}$$

Then $g$ is analytic in D, and hence analytic on $B(0;r)=\{z:|z|r\}$ for every $0<r<1$.

Applying maximum modulus theorem to $g$ on $B(0, r)$, for all $z\in\overline{B}(0,r)$ we have

$$|g(z)|\leq\max_{|z|\leq r}|g(z)|=\max_{|z|=r}|g(z)|$$

$$=\max_{|z|=r}\left|\frac{f(z)}{z}\right|\leq\frac{1}{r},\qquad \text{by condition (a).}$$

Letting $r\to 1$ we have

$$|g(z)|\leq 1,\ \text{for all } z\in B(0,1)=D \qquad \text{.... (1)}$$

$$\Rightarrow\left|\frac{f(z)}{z}\right|\leq 1,\ \forall z\in D,\ z\neq 0$$

Since $f(0)=0$, we have

$$\left|f(z)\right| \le |z|, \ \forall z \in D$$

Further, from (1) we have

$$\left|f'(0)\right| = \left|g(0)\right| \le 1$$

Now, if $\left|f'(0)\right| = 1$ or $\left|f(z)\right| = \left|f(z)\right|$ for some $z \ne 0$ in D then

$$\left|g(0)\right| = \left|f'(0)\right| = 1$$

or $\quad \left|g(z)\right| = \dfrac{\left|f(z)\right|}{|z|} = 1$, for some $z \ne 0$ in D.

Thus by (1)

$$\left|g(w)\right| \le \left|g(0)\right|, \ \forall w \in D$$

or $\quad \left|g(w)\right| \le \left|g(z)\right|; \ \forall w \in D$ and some $z \ne 0$ in D.

Therefore by maximum modulus theorem $g$ must be constant function on D.

This implies $g(w) = c$, $\forall w \in D$ and some constant c.

$$\Rightarrow \frac{f(w)}{w} = c, \ \forall w \in D, \ w \ne 0.$$

$$\Rightarrow f(w) = cw, \ \forall w \in D, \ w \ne 0.$$

Since $0 = f(0) = c(0)$, we have

$$f(w) = cw, \ \forall w \in D$$

where $\quad |c| = \left|g(0)\right| = \left|f'(0)\right| = 1$

**Theorem 9.2 :**

Let $D = \{z : |z| < 1\} = B(0,1)$ be the unit disk, and $\partial D = \{z : |z| < 1\}$.

Fix $a \in \mathbb{C}$ such that $|a| < 1$. Define the Mobius transformation.

$$f_a |z| = \frac{z-a}{1-\overline{a}z}$$

Then :

(a) $f_a$ is a one-one map of D onto itself,

(b) $f_a$ is analytic in an open disk containing the closure of D,

(c) the inverse of $f_a$ is $f_{-a}$,

(d) $f_a$ maps $\partial D$ onto $\partial D$,

(e) $f_a(a) = 0$, $f_a(0) = -a$

(f) $f_a(a) = 0$, $f_a{}'(0) = 1 - |a|^2$ and $f_a^{'}(a) = \dfrac{1}{1-|a|^2}$.

**Proof :**

Let $D = \{z : |z| < 1\}$ and $\partial D = \{z : |z| = 1\}$.

Fix $a \in \mathbb{C}$ such that $|a| < 1$.

Consider the Mobius transformation

$$f_a(z) = \frac{z-a}{1-\overline{a}z}$$

(a) We know Mobius transformation is composition of translations, dilations and the inversion. Since each function involved in the composition of Mobius transformation is bijective on $\mathbb{C}_\infty$, it follows that $f_a$ is bijective on D.

(b) The function $f_a$ is well defined everywhere except at $z = \dfrac{1}{\overline{a}}$.

As $|a| < 1$, we have $\left|\dfrac{1}{\overline{a}}\right| = \left|\dfrac{1}{a}\right| > 1$.

This implies $f_a$ is well defined on an open disk at the origin containing not only D but $\partial D$ also. Thus $f_a$ is analytic in an open disk containing the closure of D, as $\overline{D} = D \cup \partial D$.

(c) If $|a| < 1$ then $|-a| < 1$. For any $z \in D$

$$f_{-a}(f_a(z)) = f_{-a}\left(\frac{z-a}{1-\overline{a}z}\right)$$

$$= \frac{\left(\dfrac{z-a}{1-\overline{a}z}\right) - (-a)}{1 - \overline{(-a)}\left(\dfrac{z-a}{1-\overline{a}z}\right)}$$

$$= \frac{\left(\frac{z-a}{1-\overline{a}z}\right)+a}{1+\overline{a}\left(\frac{z-a}{1-\overline{a}z}\right)}$$

$$= \frac{z-a+a-a\overline{a}z}{1-\overline{a}z+\overline{a}z-a\overline{a}}$$

$$= \frac{z-|a|^2 z}{1-|a|^2} = \frac{z\left(1-|a|^2\right)}{1-|a|^2}$$

$$= z$$

Similarly one can show that

$$f_a\left(f_{-a}(z)\right) = z$$

Thus

$$f_a\left(f_{-a}(z)\right) = z = f_{-a}\left(f_a(z)\right)$$

This proves the inverse of $f_a$ is $f_{-a}$. Further $f_a$ maps D onto itself in a one-one fashion.

(d) Let any $z \in \partial D$, then $|z| = 1$.

Let $z = e^{iq}$, for some real $q$.

Then

$$\left|f_a(z)\right| = \left|f_a\left(e^{iq}\right)\right|$$

$$= \left|\frac{e^{iq}-a}{1-\overline{a}e^{iq}}\right| = \frac{\left|e^{iq}-a\right|}{\left|e^{iq}\right|\left|e^{-iq}-\overline{a}\right|}$$

$$= \frac{\left|e^{iq}-a\right|}{\left|\overline{\left(e^{iq}-a\right)}\right|} = 1 \qquad [\because |w| = |\overline{w}| \text{ for any } w \in \mathbb{C}]$$

$$\Rightarrow f_a(z) \in D$$

This proves $f_a$ maps $\partial D$ on to $\partial D$.

(e) From the definition of $f_a$, it follows that $f_a(a) = 0$, $f_a(0) = -a$.

(f) Already we have proved that $f_a$ is analytic in an open disk containing the closure of D.

Thus,

$$f_a'(z) = \frac{d}{dz}\left(\frac{z-a}{1-\bar{a}z}\right)$$

$$= \frac{(1-\bar{a}z)(1)-(z-a)(-\bar{a})}{(1-\bar{a}z)^2}$$

$$= \frac{1-\bar{a}z+\bar{a}z-|a|^2}{(1-\bar{a}z)^2}$$

$$= \frac{1-|a|^2}{(1-\bar{a}z)^2}$$

In particular,

$$f_a'(0) = 1-|a|^2$$

$$f_a'(a) = \frac{1-|a|^2}{(1-\bar{a}a)^2} = \frac{1-|a|^2}{\left(1-|a|^2\right)^2} = \frac{1}{1-|a|^2}$$

The following theorem help to find an upper bound for $|f'(z)|$ of any analytic map $f : D \to D$, $D = \{z : |z| < 1\}$.

**Theorem 9.3 :** Let *f* is analytic on D with $|f(z)| \leq 1$ and let $f(a) = a$ for

$a \in D = \{z : |z| < 1\}$ then $|f'(a)| \leq \frac{1-|a|^2}{1-|a|^2}$

Furthermore if $|f'(a)| = \frac{1-|a|^2}{1-|a|^2}$ then there is a constanmt c with $|c| = 1$ and $f(z) = f_{-a}\left(cf_a(z)\right)$ for $z \in D$.

**Proof :** Let *f* is analytic on D with $|f(z)| \leq 1$.

Let $a \in D = \{z : |z| < 1\}$ and $f(a) = a$, so $|a| < 1$ unless *f* is constant.

Define $g = f_a \circ f \circ f_{-a}$. Then $g$ maps D into D and also satisfies

$$g(0) = (f_a \circ f \circ f_{-a})(0)$$

$$= f_a\left(f\left(f_{-a}(0)\right)\right)$$

$$= f_a\left(f(a)\right)$$

$$= f_a(a)$$

$$= 0$$

Thus by applying Schwarz's Lemma we obtain

$$|g'(0)| \leq 1$$

Now by applying chain rule

$$g'(0) = (f_a \circ f \circ f_{-a})(0)$$

$$= (f_a \circ f)'(f_{-a}(0)) f_{-a}'(0)$$

$$= (f_a \circ f)'(a)\left(1-|a|^2\right)$$

$$= f_a'(f(a)) f'(a)\left(1-|a|^2\right)$$

$$= f_a'(a) f'(a)\left(1-|a|^2\right)$$

$$= \frac{1}{1-|a|^2} f'(a)\left(1-|a|^2\right)$$

Therefore,

$$f'(a) = \frac{1-|a|^2}{1-|a|^2} g'(0)$$

Using the fact $|g'(0)| \leq 1$, we obtain

$$|f'(a)| \leq \frac{1-|a|^2}{1-|a|^2}, \; a \in D.$$

Further observe that, equality holds exactly when $|g'(0)| = 1$.

Thus applying Schwarz's Lemma to a function $g$, there is a constant c, $|c| = 1$, such that.

$$g(z) = cz \text{ for all } z \in D.$$

$$\Rightarrow (f_a \circ f \circ f_{-a})(z) = cz, \forall z \in D$$

$$\Rightarrow f_a \circ f \circ f_{-a} = cI \text{ where } I : D \to D,\ I(z) = z$$

$$\Rightarrow f_a \circ f = cI \circ f_a = cf_\infty$$

$$\Rightarrow f = f_{-a} \circ (cf_a)$$

$$\Rightarrow f(z) = f_{-a}(cf_a(z)) \text{ for } z \in D$$

**Theorem 9.4 :** Let $f : D \to D$, be a one-one analytic map of $D = \{z : |z| < 1\}$ onto itself and suppose $f(a) = 0$. Then there is a complex number c with $|c| = 1$ such that $f = cf_a$.

**Proof :** Let $f : D \to D$ be a bijective map, where $D = \{z : |z| < 1\}$.

Then there is an analytic function $g : D \to D$ such that

$$g(f(z)) = z \text{ for } \forall z \in D \qquad \text{.... (1)}$$

Let $f(a) = 0$. Then by Theorem 9.3

$$|f'(a)| \le \frac{1-|0|^2}{1-|a|^2} = \frac{1}{1-|a|^2} \qquad \text{..... (2)}$$

Further by (1), we have

$$g(f(a)) = a$$

$$\Rightarrow g(0) = a, \text{ as } f(a) = 0$$

Again applying the Theorem 9.3 to $g$ we obtain

$$|g'(0)| \le \frac{1-|a|^2}{1-|0|^2} = 1-|a|^2$$

Differentiating (1), we get

$$g'(f(z))\, f'(z) = 1$$

Therefore,

$$1 = g'(f(a))\, f'(a)$$

$$= g'(0)\, f'(a) \qquad [\because f(a) = 0]$$

$$\Rightarrow |f'(a)| = \left|\frac{1}{g'(0)}\right| \geq \frac{1}{1-|a|^2} \qquad \text{.... (3)}$$

By (2) and (3) we have

$$|f'(a)| = \frac{1}{1-|a|^2} = \frac{1-|0|^2}{1-|a|^2}$$

Therefore by Theorem 9.3, there is a complex number c with $|c| = 1$ such that

$$f(z) = f_{-0}(cf_a(z))$$

$$= f_0(cf_a(z))$$

$$= cf_a(z), \quad z \in D$$

$$\Rightarrow f = cf_a$$

**Problem 9.1 :** Let $f : D \to \mathbb{C}$ is analytic with Re $f(z) \geq 0$ for all z in $D = \{z : |z| < 1\}$, and $f(0) = 1$. Show that Re $f(z) > 0$.

and $\quad \frac{1-|z|}{1+|z|} \leq |f(z)| \leq \frac{1+|z|}{1-|z|}, \; z \in D$

**Solution :** Let $D = \{z : |z| < 1\}$, $f : D \to \mathbb{C}$ is analytic with Re $f(z) \geq 0$ for all $z \in D$, and $f(0) = 1$.

Define $f(z) = \frac{z-1}{z+1}$. Then $f$ maps $\{z : \text{Re}\, z > 0\}$ onto D, so the function $g = f \circ f$ maps D to itself and

$$g(0) = f(f(0)) = f(1) = 0$$

Applying Schwarz's Lemma to *g*, we obtain

$$|g(z)| \leq |z|, \; \forall z \in D$$

$$\Rightarrow |f(f(z))| \leq |z|, \; \forall z \in D$$

$$\Rightarrow \left|\frac{f(z)-1}{f(z)+1}\right| \leq |z|; \; \forall z \in D$$

Therefore

$$\frac{|f(z)|-1}{|f(z)|+1} \le \left|\frac{f(z)-1}{f(z)+1}\right| \le |z|, \ \forall z \in D$$

$$\Rightarrow |f(z)|-1 \le |z||f(z)|+|z|$$

$$\Rightarrow |f(z)|(1-|z|) \le 1+|z|$$

$$\Rightarrow |f(z)| \le \frac{1+|z|}{1-|z|}; \forall z \in D$$

On the same line

$$\Rightarrow \frac{1-|f(z)|}{1+|f(z)|} \le \left|\frac{f(z)-1}{f(z)+1}\right| \le |z|, \ \forall z \in D$$

implies $\frac{1-|z|}{1+|z|} \le |f(z)|, \ \forall z \in D$

Therefore

$$\frac{1-|z|}{1+|z|} \le |f(z)| \le \frac{1+|z|}{1-|z|}, \ \forall z \in D$$

**Problem 9.2 :** Suppose $|f(z)| \le 1$ for $|z| < 1$ and *f* is analytic. Prove that

$$|f(z)| \le \frac{|f(0)|+|z|}{1+|f(0)||z|} \text{ for } |z| < 1,$$

**Proof :** Let $|f(z)| \le 1$ for $|z| < 1$ and *f* is analytic on $D = \{z : |z| < 1\}$.

Let $f(0) = a$ and consider the function $g = f_a \circ f$.

Then $g(0) = f_a(f(0)) = f_a(a) = 0$

Thus by applying Schwarz's Lemma, we have

$$|g(z)| \le |z|, \ \forall z \in D \qquad \text{..... (1)}$$

$$\Rightarrow |f_a(f(z))| \le |z|; \ \forall z \in D$$

$$\Rightarrow \left|\frac{f(z)-a}{1-\bar{a}f(z)}\right| \le |z|; \ \forall z \in D$$

For any $z \in D$, $g(z) = \dfrac{f(z) - a}{1 - \bar{a}f(z)}$

$$\Rightarrow g(z) - \bar{a}g(z)\, f(z) = f(z) - a$$

$$\Rightarrow g(z) + a = f(z)\left(1 + \bar{a}g(z)\right)$$

$$\Rightarrow f(z) = \frac{a + g(z)}{1 + \bar{a}g(z)}$$

Thus $f(z)$ is obtained from $w = g(z)$ by a bilinear transformation, which maps circles onto circles and centre of a circle onto centre of its image. From (1) we see that for any $z \in D$, $g(z)$ is in the disk $B(0,|z|)$. This disk is mapped on a disk $D_1 \subseteq D$ with centre $f(a) = a$.

That is, $f(z) \in D_1$ and $|f(z)| \le |p - 0|$, where p is the point on the closure of $D_1$.

Then P is given by $P = \dfrac{a + |z|e^{it}}{1 + \bar{a}|z|e^{it}}$

Therefore, $$|f(z)| \le \left|\frac{a + |z|e^{it}}{1 + \bar{a}|z|e^{it}}\right|$$

$$\le \frac{|a| + |z|}{1 + |a||z|} \qquad [\because |\bar{a}| = |a|]$$

$$= \frac{|f(0)| + |z|}{1 + |f(0)||z|}, \text{ for } |z| < 1$$

## EXERCISE :

1. Suppose $|f(z)| \le 1$ for $|z| < 1$ and *f* is analytic on $D = \{z : |z| < 1\}$.

Prove that :

(a) $|f(z)| \le \dfrac{|f(0)| + |z|}{1 - |f(0)||z|}$, for $|z| < 1$.

(b) $\dfrac{1 - |f(0)||z|}{1 + |f(0)||z|} \le |f(z)|$, for $|z| < 1$.

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UNIT - X

# SPACES OF ANALYTIC FUNCTIONS AND THE RIEMANN MAPPING THEOREM

## The Space of Continuous Functions :

Let G is an open set in $\mathbb{C}$ and $(\Omega, d)$ is a complete metric space then $C(G,\Omega)$ denotes the set of all continuous function from G to $\Omega$.

The set $C(G,\Omega)$ is non-empty, as it always contains the constant functions.

**Theoerm 10.1 :** If G is an open set in $\mathbb{C}$ then there is a sequence $\{K_n\}$ of compact subsets of G such that $G = \bigcup_{n=1}^{\infty} K_n$.

Moreover, the sets can $K_n$ be chosen to satisfy the following conditions :

(a) $K_n \subseteq K_{n+1}$;

(b) $K \subset G$ and K compact implies $K \subset K_n$ for some n;

(c) Every component of $\mathbb{C}_\infty - K_n$ contains a component of $\mathbb{C}_\infty - G$.

**Definition 10.1 :**

Let $G = \bigcup_{n=1}^{\infty} K_n$, where each $K_n$ is compact and $K_n \subset \operatorname{int} K_{n+1}$.

Define, $s_{\mathrm{n}}(f,g) = \sup\{d(f(z), g(z)) : z \in K_n\}$

for all functions *f* and *g* in $C(G,\Omega)$.

Also define, $$s(f,g) = \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n \frac{s_{\mathrm{n}}(f,g)}{1+s_{\mathrm{n}}(f,g)} \mathbb{Z} \qquad \text{....... (1)}$$

As $\frac{t}{1+t} \le 1$ for all $t \ge 0$, the series in (1) is dominated by $\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n$ and hence it is convergent.

**Theorem 10.2 :** $(C(G,\Omega), s)$ is a complete metric space.

## Spaces of Analytic Functions :

Let G be an open subset of the complex plane. Let $H(G)$ is the collection of analytic functions on G. Then we consider $H(G)$ as a subset of $C(G,\mathbb{C})$, and the metric on $H(G)$ is the metric which it inherits as subset of $C(G,\mathbb{C})$.

**Theoerm 10.3 :** If $\{f_n\}$ is a sequence in H (G) and $f$ belongs to $C(G,\mathbb{C})$ such that $f_n \to f$ then $f$ is analytic and $f_n^{(k)} \to f^{(k)}$ for each integer $k \geq 1$.

**Proof :** Let $\{f_n\}$ is a sequence in $H(G)$ and $f \in C(G,\mathbb{C})$ such that $f_n \to f$ .

To prove *f* is analytic we use Morera's theorem. Let T be a triangle contained inside a disk $D \subset G$.

Since T is compact, $f_n \to f$ uniformly over T.

Therefore,

$$\lim_{n\to\infty} \int_T f_n = \int_T f \qquad \text{...... (1)}$$

As each $f_n$ is analytic on G and T is a closed rectifiable curve in a disk D, by Cauchy's theorem.

$$\int_T f_n = 0, \text{ for each n.}$$

Therefore from (1) we have,

$$\int_T f = 0$$

Thus *f* must be analytic in every disk $D \subset G$, by Morera's theorem, and hence *f* is analytic in G. Now we prove that $f_n^{(k)} \to f^{(k)}$ for each integer $k \geq 1$.

Let $D = \overline{B}(a;r) \subset G$. Choose R > r such that $\overline{B}(a;R) \subset G$.

Let $g$ is the circle $|z-a| = R$, then by Cauchy's integral formula, for each integer $k \geq 1$, and each n, we have

$$f_n^{(k)}(z) - f^{(k)}(z) = \frac{k!}{2p\,i}\int_r \frac{f_n(w) - f(w)}{(w-z)^{k+1}}\,dw, \quad z \in D \qquad \text{...... (2)}$$

Since $\{g\}$ is compact, uniformly on $\{g\}$.

Let $M_n = \sup\{|f_n(w) - f(w)| : w \in \{g\}\}$, then $M_n \to 0$, as $f_n \to f$.

Further for $z \in D$ and $w \in \{g\}$, we have

$$|w-z| = |(w-a)+(a-z)|$$

$$\geq ||w-a|-|z-a||$$

$$\geq |w-a|-|z-a|$$

$$\geq R-r$$

$$\Rightarrow \frac{1}{|w-z|} \leq \frac{1}{R-r} \qquad \text{.... (3)}$$

Thus from (2) and (3), for $z \in D$, we have

$$\left|f_n^{(k)}(z) - f^{(k)}(z)\right| \leq \frac{k!}{2p}\int_r \frac{|f_n(w)-f(w)|}{|w-z|^{K+1}}|dw|$$

$$\leq \frac{k!M_n}{2p\,(R-r)^{k+1}}\int_r |dw|$$

$$= \frac{k!M_n}{2p\,(R-r)^{k+1}} \cdot 2pR$$

$$= \frac{k!M_nR}{(R-r)^{k+1}}$$

Therefore for each $Z \in D$,

$$\lim_{n\to\infty}\left|f_n^{(k)}(z) - f^{(k)}(z)\right| = \frac{k!R}{(R-r)^{k+1}}\lim_{n\to\infty} M_n = 0$$

Hence $f_n^{(k)} \to f^{(k)}$ uniformly on $D = \overline{B}(a,r)$.

Let $k$ be an arbitrary compact subset of G. Let $0 < r < d(K, \partial G)$, then there are $a_1$, $a_2$, ...., $a_n$ in K such that

$$K \subseteq \bigcup_{j=1}^{n} B(a_j; r)$$

Since $f_n^{(k)} \to f^{(k)}$ uniformly on each $B(a_j; r)$, the convergence is uniform on K.

As K is an arbitrary compact subset of G, $f_n^{(k)} \to f^{(k)}$ uniformly on G.

**Corollary 10.1 :** $H(G)$ is a complete metric space.

**Proof :** As $H(G)$ is collection of analytic functions on G and $C(G,\mathbb{C})$ is the set of all continuous functions, $H(G)$ is subset of $C(G,\mathbb{C})$.

Thus $H(G)$ is a subspace of comlete metric space $C(G,\mathbb{C})$ with metric induced on $H(G)$ by metric on $C(G,\mathbb{C})$.

To prove $H(G)$ is complete metric space it suffices to show that $H(G)$ is closed in $C(G,\mathbb{C})$.

Let $\{f_n\}$ be any sequence in $H(G)$ converging to $f$, $f \in C(G,\mathbb{C})$.

By Theorem 10.3 $f : G \to \mathbb{C}$ is analytic on G, and hence $f \in H(G)$.

This proves $H(G)$ is closed in complete metric space $C(G,\mathbb{C})$, hence $H(G)$ is complete.

**Corollary 10.2 :**

If $f_n : G \to \mathbb{C}$ is analytic and $\sum_{n=1}^{\infty} f_n(z)$ converges uniformly on compact sets to $f(z)$ then

$$f^{(k)}(z) = \sum_{n=1}^{\infty} f_n^{(k)}(z)$$

**Proof :** Let $f_n : G \to \mathbb{C}$ is qualatic

Let $\sum_{n=1}^{\infty} f_n(z)$ converges uniformly on compact sets to $f(z)$.

Define $S_n(z) = \sum_{j=1}^{n} f_j(z)$, then by assumption $S_n \to f$ uniformly on compact sets.

Therefore, by Theorem 10.3

$S_n^{(k)} \to f^{(k)}$ uniformly on compact set.

$$\Rightarrow f^{(k)}(z) = \lim_{n\to\infty} S_n^{(k)}(z) = \lim_{n\to\infty} \sum_{j=1}^{n} f_j(z) = \sum_{j=1}^{\infty} f_j(z)$$

**Theoerm 10.4 (Hurwitz's Theorem) :**

Let G be a region and suppose the sequence $\{f_n\}$ in $H(G)$ converges to *f*. If $f \not\equiv 0$, $\overline{B}(a;R) \subset G$, and $f(z) \neq 0$ for $|z-a| = R$ then there is an integer N such that for $n \geq N$, $f$ and $f_n$ have the same number of zeros in $B(a;R)$.

**Proof :** Let G be a region and let the sequence $\{f_n\}$ in $H(G)$ converges to *f*.

Let $f \not\equiv 0$, $B(a;R) \subset G$ and $f(z) \neq 0$ for $|z-a| = R$.

Define $d = \inf\{|f(z)| : |z-a| = R\}$

As $f(z) \neq 0$ for $|z-a| > R$, we have $d > 0$.

Since $f_n \to f$ uniformly on $\{z : |z-a| = R\}$, corrosponding to $\frac{d}{2}$ there is an $N \in \mathbb{N}$ such that if $n \geq N$ and $|z-a| = R$ then

$$|f(z) - f_n(z)| < \frac{d}{2} < d \leq |f(z)| \leq |f(z)| + |f_n(z)|$$

Therefore, by Rouche's theorem *f* and $f_n$ have the same number of zero's in $B(a;R)$.

**Corollary 10.3 :**

If $\{f_n\} \subset H(G)$ converges to *f* in $H(G)$ and each $f_n$ never vanishes on G then either $f \equiv 0$ or *f* never vanishes.

**Proof :** Let $\{f_n\} \subseteq H(G)$, $f \in H(G)$ and $f_n \to f$.

Let each $f_n$ never vanishes on G..

If *f* is not identically zero, then $f(a) = 0$ for some $a \in G$. Since zeros of an analytic function are isolated, there is R > 0 such that $\overline{B}(a,R) \subseteq G$ such that $f \not\equiv 0$ on $\overline{B}(a,R)$.

Therefore by Hurwitz's Theorem there is an integer N such that for $n \geq N$, *f* and $f_n$ have the same number of zeros in $B(a;R)$.

This is contradiction to the assumption each $f_n$ never vanishes on G..

**Definition 10.3 (Normal Family) :** A set $\mathcal{F} \subset C(G,\mathbb{C})$ is normal if each sequence in $\mathcal{F}$ has a subsequence which converges to a function $f$ in $C(G,\mathbb{C})$.

**Definition 10.4 (Equicontinuous Family) :**

A set $\mathcal{F} \subseteq C(G, \mathbb{C})$ is equicontinuous at a point $z_0$ in G iff for every $e > 0$ there is a $d > 0$ such that

if $|z - z_0| < d$ then $d(f(z), f(z_0)) < e$, for all $f \in \mathcal{F}$.

We say that $\mathcal{F}$ is equicontinuous on $E \subset G$ if for every $e > 0$ there is a $d > 0$ such that

if $z, w \in E$ and $|z - w| < d$ then $d(f(z), f(w)) < e$, for all $f$ in $\mathcal{F}$.

**Remark 10.1 :**

1. If $\mathcal{F}$ consists of single function $f$ then the statement that $\mathcal{F}$ is equicontinuous at $z_0$ is only the statement that $f$ is continuous at $z_0$.
2. $\mathcal{F} = \{f\}$ is equicontinuous over E is equivalent to saying that $f$ is uniformly continuous on E.

**Definition 10.5 (Locally bounded family) :**

A family $\mathcal{F} \subset H(G)$ is locally bounded if for each $a \in G$, there are constants M and r > 0 such that for all $f \in \mathcal{F}$,

$$|f(z)| \le M, \text{ for } |z - a| < r.$$

Alternatively, $\mathcal{F}$ is locally bounded if there is an r > 0 such that,

$$\sup\{|f(z)| : |z - a| < r, f \in \mathcal{F}\} < \infty.$$

That is, $\mathcal{F}$ is locally bounded if about each point a in G there is a disk on which $\mathcal{F}$ is uniformly bounded.

We state few theorems without proof which are required to prove Montel's theorem and its consequences.

**Theorem 10.5 :**

A set $\mathcal{F} \subset C(G, \mathbb{C})$ is normal iff its closure is compact.

**Theorem 10.6 (Arzela-Ascoli Theorem) :**

As set $\mathcal{F} \subseteq C(G, \mathbb{C})$ is normal iff the following two conditions are satisfied.

(a) for each $z$ in G, $\{f(z) : f \in \mathcal{F}\}$ has compact closure in $\Omega$;

(b) $\mathcal{F}$ is equicontinuous at each point of G..

**Theorem 10.7 :** A set $\mathcal{F}$ in H (G) is locally bounded iff for each compact set $K \subset G$ there is a constant M such that $|f(Z)| \le M$ for all $f \in \mathcal{F}$ and $z$ in K.

**Theorem 10.8 (Montel's Theorem) :** A family $\mathcal{F}$ in H (G) is normal iff $\mathcal{F}$ is locally bounded.

**Proof :** Let $\mathcal{F} \subseteq H(G)$.

Let $\mathcal{F}$ is normal.

We have to prove that $\mathcal{F}$ is locally bounded. If possible $\mathcal{F}$ is not locally bounded, then there is a compact set $K \subset G$ such that

$$\sup\{|f(z)| : z \in K, f \in \mathcal{F}\} = \infty$$

This implies for each $n$, there is $f_n \in \mathcal{F}$ such that $|f_n(z)| \ge n$ for each $z \in K$. That is, there is a sequence $\{f_n\}$ in $\mathcal{F}$ such that

$$\sup\{|f_n(z)| : z \in K\} \ge n \qquad .... (1)$$

But as $\mathcal{F}$ is normal there is a function $f \in H(G)$ and a subsequence $\{f_{n_k}\}$ of $\{f_n\}$ such that $f_{n_k} \to f$ .

Since $K \subset G$ and $f \in H(G)$, $f$ is continuous on compact set K, and hence $f$ is bounded on K. Thus there is M > 0 such that

$$\sup\{|f(z)| : z \in K\} \le M \qquad .... (2)$$

From (1) and (2) we have

$$n_k \le \sup\{|f_{n_k}(z)| : z \in K\}$$

$$\le \sup\{|f_{n_k}(z) - f(z)| : z \in K\} + \sup\{|f(z)| : z \in K\}$$

$$\le \sup\{|f_{n_k}(z) - f(z)| : z \in K\} + M \qquad .... (3)$$

Since $f_{n_k} \to f$, we have

$$\lim_{k\to\infty} \sup\left\{\left|f_n(z) - f(z)\right| : z \in K\right\} = 0 \qquad \text{.... (4)}$$

Usince (4) we have

$\lim\limits_{k\to\infty} n_k \le M$ , a contradiction to the fact $\lim\limits_{k\to\infty} n_k = \infty$ .

Therefore $\mathcal{F}$ must be locally bounded.

Conversely, let $\mathcal{F}$ is locally bounded.

To prove $\mathcal{F}$ is normal we use Arzela-Ascoli theorem.

(a) As $\mathcal{F}$ is normal, for each $a \in G$ there are constants M and r > 0 such that for all $f \in \mathcal{F}$, $|f(z)| \le M$, for all $z \in B(a,r)$.

In particular, $|f(a)| \le M$, $\forall f \in \mathcal{F}$.

$$\Rightarrow A = \{f(a) : f \in \mathcal{F}\} \subseteq B(0;M)$$

$$\Rightarrow \overline{A} \subseteq \overline{B}(0;M)$$

Thus $\overline{A}$ is closed bounded subset of $\mathbb{C}$ and hence compact by Heine Borel theorem.

We have proved that for each $a \in G$, $\{f(a) : f \in \mathcal{F}\}$ has compact closure in $\mathbb{C}$.

(b) We now prove that $\mathcal{F}$ is equicontinuous at each point of G..

Fix any $a \in G$.

Let $e > 0$ be given.

Since $\mathcal{F}$ is locally bounded there are constants M and r > 0 such that $\overline{B}(a;r) \subset G$ and $|f(z)| \le M$ for all $z \in \overline{B}(a;r)$, and all $f \in \mathcal{F}$.

Let $|z-a| < \dfrac{r}{2}$ and $f \in \mathcal{F}$ then using Cauchy's formula with $g(t) = a + re^{it}$, $0 \le t \le 2p$, we have

$$f(a) - f(z) = \frac{1}{2p\,i}\int_g \frac{f(w)}{w-a}dw - \frac{1}{2p\,i}\int_g \frac{f(w)}{w-z}dw$$

$$= \frac{1}{2p\,i}\int_g \left[\frac{1}{w-a} - \frac{1}{w-z}\right] f(w)\,dw$$

$$= \frac{1}{2\pi i}\int_{r}\left[\frac{(w-z)-(w-a)}{(w-a)(w-z)}\right] f(w)\,dw$$

$$= \frac{1}{2\pi i}\int_{r}\frac{(a-z)}{(w-a)(w-z)} f(w)\,dw$$

Therefore,

$$\left|f(a)-f(z)\right| \le \frac{1}{2\pi}\int_{r}\frac{|a-z|}{|w-a||w-z|}\left|f(w)\right||dw|$$

$$\le \frac{1}{2\pi}\frac{M|a-z|}{\frac{r}{2}\cdot\frac{r}{2}}2\pi r$$

$$= \frac{4M}{r}|a-z|$$

Choose $\delta < \left\{\frac{r}{2}, \frac{r}{4M}\epsilon\right\}$, then for $|a-z| < \delta$ , we have $\left|f(a)-f(z)\right| < \epsilon$ for all $f \in \mathcal{F}$ .

Thus $\mathcal{F}$ is equicontinuous at each $a \in G$.

We have proved that $\mathcal{F}$ satisfies the conditions of Arzela-Ascoli theorem. Therefore $\mathcal{F}$ is normal.

**Corollary 10.4 :** A set $\mathcal{F} \subset H(G)$ is compact iff it is closed and locally bounded.

**Proof :** We know the theorems

1) A set $\mathcal{F} \subset C(G,\mathbb{C})$ is normal iff $\overline{\mathcal{F}}$ is compact.

2) $\mathcal{F} \subset H(G)$ is normal iff $\mathcal{F}$ is locally bounded.

Therefore $\mathcal{F}$ is locally bounded iff $\overline{\mathcal{F}}$ is compact.

This implies $\mathcal{F}$ closed and locally bounded iff $\mathcal{F} = \overline{\mathcal{F}}$ is compact.

## The Riemann Mapping Theorem

**Definition 10.6 :**

A region $G_1$ is conformally equivalent to the region $G_2$ if there is analytic function $f : G_1 \to \mathbb{C}$ such that $f$ is one-one and $f(G_1) = G_2$.

This is an equivalence relation.

**Definition 10.7 :**

An open set G is simply connected if G is connected and every closed rectifiable curve in G is homotopic to zero.

**Theorem 10.9 :** Let G be an open connected subset of $\mathbb{C}$. Then the following are equivalent.

(a) G is simply connected.

(b) $n(g;q) = 0$ for every closed rectifiable curve $g$ in G and every point a in $G - \mathbb{C}$.

(c) $\mathbb{C}_\infty - G$ is connected.

(d) For any $f \in H(G)$ such that $f(z) \neq 0$ for all $z$ in G , there is a function $g \in H(G)$ such that $f(z) = [g(z)]^2$.

**Theorem 10.10 (Open Mapping Theorem)**

Let G be a region and suppose that $f$ is a non-constant analytic function on G. Then for any open set U in G, $f(U)$ is open.

**Theorem 10.11 (Identity Theorem)**

Let G be a connected open set and let $f : G \to \mathbb{C}$ be an analytic function. Then the following are equivalent statements.

(a) $f \equiv 0$

(b) there is a point a in G such that $f^{(n)}(a) = 0$ for each $n \geq 0$.

(c) $\{z \in G : f(z) = 0\}$ has a limit point in G..

With this basics in our hand we prove the well known Riemann mapping theorem.

**Theorem 10.12 (Riemann Mapping Theorem)**

Let G be a simply connected region which is not the whole plane and let $a \in G$. Then there is a unique analytic function $f : G \to \mathbb{C}$ having the properties :

(a) $f(a) = 0$ and $f'(a) > 0$;

(b) $f$ is one-one.

(c) $f(G) = \{z : |z| < 1\}$.

**Proof :** Let G be a simply connected region which is not the whole plane and let $a \in G$.

Define,

$\mathcal{F} = \{ f \in H(G) : f$ is one-one, $f(a) = 0, f'(a) > 0$ and $f(G) \subset D\}$

where $D = \{z : |z| < 1\}$

We give the proof in the following steps.

**Step 1 :** $\mathcal{F}$ is nonempty**.**

**Step 2 :** $\overline{\mathcal{F}} = \mathcal{F} \cup \{0\}$, that is, if $f \in \overline{\mathcal{F}}$ then either $f \in \mathcal{F}$ or $f \equiv 0$ on G..

**Step 3 :** There exists $f \in \mathcal{F}$ such that $f(G) = D$.

**Setp 4 :** $f$ is unique satisfying the conditions in (a), (b) and (c).

Let us proceed towards the first step.

**Step 1 :** We prove $\mathcal{F}$ is non-empty. As $G \neq \mathbb{C}$, G is proper subset of $\mathbb{C}$. Thus there is $b \in \mathbb{C}$ such that $b \notin G$.

Since G is simply connected, $z - b \in H(G)$ and $z - b \neq 0$ for all $z$ in G, there is a function $g \in H(G)$ such that

$$[g(z)]^2 = z - b$$

If $z_1$ and $z_2$ be any point in G then

$$g(z_1) = \pm g(z_2) \Rightarrow [g(z_1)]^2 = [g(z_2)]^2$$

$$\Rightarrow z_1 - b = z_2 - b$$

$$\Rightarrow z_1 = z_2$$

In particular, g is one-one and

$$g(z_1) = -g(z_2) \Rightarrow z_1 = z_2, \ \forall z_1, z_2 \in G \qquad \text{.... (1)}$$

As $g$ is non constant analytic function on G, by open mapping theorem, $g(G)$ is open in $\mathbb{C}$. Further, $a \in G$ implies $g(a) \in g(G)$.

Therefore there exists $r > 0$ such that

$$B(g(a);r) \subset g(G) \qquad \text{..... (2)}$$

We claim that $|g(z) + g(a)| \geq r, \ \forall z \in G$, that is,

$$B(-g(a);r) \cap g(G) = f$$

If $B(-g(a);r) \cap g(G) \neq f$, there is $z \in G$ such that $g(z) \in B(-g(a);r)$, that is,

$$|g(z) + g(a)| < r$$

$$\Rightarrow |-g(z) - g(a)| < r$$

$$\Rightarrow -g(z) \in B(g(a);r)$$

$$\Rightarrow -g(z) \in g(G) \qquad [\because \text{ By (2)}]$$

Therefore, $\exists w \in G$ such that $-g(z) = g(w)$.

But (1) implies that $z = w$.

Thus $g(z) = g(w) = -g(z)$.

$$\Rightarrow 2g(z) = 0$$

$$\Rightarrow [g(z)]^2 = 0$$

$$\Rightarrow z - b = 0$$

$$\Rightarrow z = b$$

But $z = w$ gives that $b = w \in G$, a contradiction.

Thus we must have

$$B(-g(a);r) \cap g(G) = f$$

$$\Rightarrow |g(z) + g(a)| \geq r, \ \forall z \in G \qquad \text{.... (3)}$$

Inparticular for $z = a \in G$, we have

$$|g(a)+g(a)| \geq r \Rightarrow 2|g(a)| \geq r$$

$$\Rightarrow |g(a)| \geq \frac{r}{2} \qquad \text{.... (4)}$$

Define, $h: G \to \mathbb{C}$ by

$$h(z) = \frac{r}{4} \cdot \frac{|g'(a)|}{g'(a)} \cdot \frac{g(a)}{|g(a)|^2} \cdot \frac{g(z)-g(a)}{g(z)+g(a)}, \quad z \in G \qquad \text{..... (3)}$$

As $g \in H(G)$, we have $h \in H(G)$.

Further $h(a) = 0$.

Let $z_1, z_2 \in G$ and $h(z_1) = h(z_2)$

then $\dfrac{g(z_1)-g(a)}{g(z_1)+g(a)} = \dfrac{g(z_2)-g(a)}{g(z_2)+g(a)}$

$\Rightarrow g(z_1) = g(z_2)$

$\Rightarrow z_1 = z_2$, as *g* is one-one.

This proves *h* is one-one.

From (3), we have

$$h'|z| = \frac{r}{4} \cdot \frac{|g'(a)|}{g'(a)} \cdot \frac{g(a)}{|g(a)|^2} \cdot \frac{(g(z)+g(a))g'(z)-(g(z)-g(a))g'(z)}{[g(z)+g(a)]^2}$$

$$= \frac{r}{4} \cdot \frac{|g'(a)|}{g'(a)} \cdot \frac{g(a)}{|g(a)|^2} \cdot \frac{2g'(z)g(z)}{[g(z)+g(a)]^2}$$

Therefore,

$$h'(a) = \frac{r}{4} \cdot \frac{|g'(a)|}{g'(a)} \cdot \frac{g(a)}{|g(a)|^2} \cdot \frac{2g'(a)g(a)}{4[g(a)]^2}$$

$$= \frac{r}{8} \cdot \frac{|g'(a)|}{|g(a)|^2} > 0$$

Thus $h'(a) > 0$.

Now

$$|h(z)| = \frac{r}{4} \cdot \frac{|g'(a)|}{|g'(a)|} \cdot \frac{|g(a)|}{|g(a)|^2} \cdot \left|\frac{g(z)-g(a)}{g(z)+g(a)}\right|$$

$$= \frac{r}{4} \cdot \frac{1}{|g(a)|} \left|\frac{g(z)-g(a)}{g(z)+g(a)}\right|$$

$$= \frac{r}{4} \left|\frac{g(z)-g(a)}{g(a)[g(z)+g(a)]}\right|$$

$$= \frac{r}{4} \left|\frac{[g(z)+g(a)]-2g(a)}{g(a)[g(z)+g(a)]}\right|$$

$$= \frac{r}{4} \left|\frac{1}{g(a)} - \frac{2}{g(z)+g(a)}\right|$$

$$\leq \frac{r}{4} \left(\frac{1}{|g(a)|} + \frac{2}{|g(z)+g(a)|}\right)$$

$$\leq \frac{r}{4} \left(\frac{2}{r} + \frac{2}{r}\right) \qquad [\because \text{ By (3) and (4)}]$$

$$= 1$$

Thus $|h(z)| \leq 1, \ \forall z \in G$.

But $h$ being non consant analytic function G, it cannot attains its maximum on G.

Hence $|h(z)| < 1, \ \forall z \in G$.

$$\Rightarrow h(G) \subset D$$

We have proved that $h \in H(G)$, $h$ is one-one, $h(a) = 0$, $h'(a) > 0$ and $h(G) \subset D$.

Therefore $h \in \mathcal{F}$, and consequently, $\mathcal{F}$ is nonempty.

**Step 2 :** In this step we prove that $\overline{\mathcal{F}} = \mathcal{F} \cup \{0\}$.

Let any $f \in \mathcal{F}$.

Then there is a sequence $\{f_n\}$ in $\mathcal{F}$ such that $f_n \to f$ on G.

Therefore,

$$f(a)=\lim_{h\to\infty} f_n(a) \text{ and } f'(a)=\lim_{h\to\infty} f_n'(a)$$

But for each *n*, $f_n\in\mathcal{F}$ implies $f_n(a)=0$ and $f_n'(a)>0$.

Thus $f(a)=0$ and $f'(a)\geq 0$ .... (5)

We claim that $f'(a)\neq 0$.

Fix any $z_1\in G$ and let $z_2\in G$ such that $z_1\neq z_2$.

Then $\exists e>0$ such that $z_1\notin \overline{B}(z_2;e)=K$.

Let $f(z_1)=x$ and $f_n(z_1)=x_n$, for each *n*.

Since $f_n$ is one-one for each *n*, $f_n-x_n$ is never vanishing on K.

As $f_n\to f$, $f_n-x_n\to f-x$ on K.

As K is compact $f_n-x_n\to f-x$ uniformly on K, so Hurwitz's theorem gives that $f(z)-x$ never vanishes on K or $f(z)-x\equiv 0$.

If $f(z)-x\equiv 0$ on K then $f(z)=x$ on G, that is, *f* is the constant function $x$ throughout G..

Since $f(a)=0\Rightarrow x=0$, and hence we have $f\equiv 0$ on G.

On the other hand if $f(z)-x$ is never vanishing on K, then

$$z_1\neq z_2\Rightarrow f(z_1)-x\neq f(z_2)-x$$

$$\Rightarrow f(z_1)\neq f(z_2)$$

Thus *f* is one-one

But if *f* is one-one than *f'* can never vanish, so (5) implies $f'(a)>0$.

Next, for each *n*, $f_n\in\mathcal{F}$ implies $f_n(G)\subset D$, that is $|f_n(z)|<1$.

Therefore,

$$|f_n(z)|=\left|\lim_{n\to\infty} f_n(z)\right|=\lim_{n\to\infty}|f_n(z)|\leq 1$$

If $|f(z)|=1$ for some $z\in G$, maximum modulus theorem forces *f* to be constant.

Thus we must have

$$\left|f(z)\right| < 1,\ \forall z \in G$$

$$\Rightarrow f(G) \subset D$$

Therefore $f$ is analytic on G, $f(a) = 0$, $f'(a) > 0$, $f$ is one-one and $f(G) \subset D$.

This proves $f \in \mathcal{F}$.

We have proved that for any $f \in \overline{\mathcal{F}}$, either $f = 0$ or $f \in \mathcal{F}$.

Hence $\overline{\mathcal{F}} = \mathcal{F} \bigcup \{0\}$.

**Step 3 :** We prove that there exists $f \in \mathcal{F}$ such $f(G) = D$.

Consider the function $f : H(G) \to \mathbb{C}$, defined by

$$f(f) = f'(a)$$

Let $\{f_n\}$ be any sequence in $H(G)$ and $f \in H(G)$ such that $f_n \to f$ then $f$ is analytic and $f_n^{(k)} \to f^{(k)}$ for each integer $k \geq 1$.

Inparticular $f_n' \to f' \Rightarrow f_n'(a) \to f'(a)$.

$$\Rightarrow f(f_n) \to f(f)$$

This proves $f$ is continuous.

Further, $f(G) \subset D$, $\forall f \in \mathcal{F}$ implies that

$$\sup\left\{\left|f(z)\right| : z \in D, \forall f \in \mathcal{F}\right\} \leq 1$$

Hence, $\mathcal{F}$ is locally bounded an consequently, by Montel's theorem $\overline{\mathcal{F}}$ is compact.

As $f$ is continuous on compact set $\overline{\mathcal{F}}$, there is $f \in \overline{\mathcal{F}}$ such that

$$\left|f(f)\right| = \max\left\{\left|f(g)\right| : g \in \overline{\mathcal{F}}\right\}$$

$$\Rightarrow \left|f'(a)\right| = \max\left\{\left|g'(a)\right| : g \in \overline{\mathcal{F}}\right\}$$

$$\Rightarrow f'(a) = \max\left\{g'(a) : g \in \overline{\mathcal{F}}\right\}$$

$$\Rightarrow f'(a) \geq g'(a),\ \forall g \in \overline{\mathcal{F}} \qquad \text{.... (6)}$$

As $\mathcal{F} \neq f$ and $\overline{\mathcal{F}} = \mathcal{F} \bigcup \{0\}$ implies $f \in \mathcal{F}$.

For this *f* we prove that $f(G)=D$.

If $f(G)\neq D$, then there is $w\in D$ such that $w\notin f(G)$.

Thus $w\neq f(z)$, $\forall z\in G$.

This implies $\dfrac{f(z)-w}{1-\overline{w}f(z)}\in H(G)$ and $\dfrac{f(z)-w}{1-\overline{w}f(z)}\neq 0$ for $z\in G$.

Therefore, there is an analytic function $h:G\to\mathbb{C}$ such that

$$[h(z)]^2=\frac{f(z)-w}{1-\overline{w}f(z)} \qquad \text{..... (7)}$$

Since the Mobius transformation $Tx=\dfrac{x-w}{1-\overline{w}x}$ maps D on D, we have $h(G)\subset D$.

Define $g:G\to\mathbb{C}$ by

$$g(z)=\frac{|h'(a)|}{h'(a)}\cdot\frac{h(z)-h(a)}{1-\overline{h(a)}h(z)}$$

then *g* is analytic and $g(G)\subset D$.

Further $g(a)=0$, *g* is one-one and

$$g'(z)=\frac{|h'(a)|}{h'(a)}\cdot\frac{\left(1-\overline{h(a)}h(z)\right)h'(z)-\left(h(z)-h(a)\right)\overline{h(a)}h'(z)}{\left[1-\overline{h(a)}h(z)\right]^2}$$

$$=\frac{|h'(a)|}{h'(a)}\cdot\frac{h'(z)\left(1-\overline{h(a)}h(z)-h(z)\overline{h(a)}+h(a)\overline{h(a)}\right)}{\left[1-\overline{h(a)}h(z)\right]^2}$$

Therefore,

$$g'(a)=\frac{|h'(a)|}{h'(a)}\cdot\frac{h'(a)\left[1-|h(a)|^2\right]}{\left[1-|h(a)|^2\right]^2}$$

$$\frac{|h'(a)|}{1-|h(a)|^2} \qquad \text{.... (8)}$$

But,

$$|h(a)|^2 = \left|\frac{f(a)-w}{1-\overline{w}f(a)}\right| = |-w| = |w| \qquad \text{.... (9)}$$

Differentiating (7) we obtain

$$2h(z)h'(z) = \frac{[1-\overline{w}f(z)](f'(z)) - [f(z)-w](-\overline{w}f'(z))}{[1-\overline{w}f(z)]^2}$$

$$= \frac{f'(z)\left[1-\overline{w}f(z)+\overline{w}f(z)-|w|^2\right]}{[1-\overline{w}f(z)]^2}$$

$$\Rightarrow 2h(a)\, h'(a) = \frac{f'(a)\left[1-|w|^2\right]}{[1-\overline{w}f(z)]^2} = f'(a)\left[1-|w|^2\right], \text{ as } f(a)=0$$

$$\Rightarrow h'(a) = \frac{f'(a)\left(1-|w|^2\right)}{2h(a)} \qquad \text{.... (10)}$$

Using (9), (10) in (8) we obtain,

$$g'(a) = \frac{f'(a)\left(1-|w|^2\right)}{2\sqrt{|w|}} \cdot \frac{1}{(1-|w|)}$$

$$= \frac{f'(a)}{2\sqrt{|w|}} \cdot \frac{(1-|w|)(1+|w|)}{1-|w|}$$

$$g'(a) = f'(a)\left(\frac{1+|w|}{2\sqrt{|w|}}\right) > f'(a) > 0$$

Thus $g'(a) > 0$ and $g'(a) > f'(a)$.

This gives that $g \in \mathcal{F}$ and contradicts to the choice of *f*, given in (6).

Hence, we must have $f(G) = D$.

**Step 4 :** We prove that *f* is unique satisfying conditions (a), (b) and (c).

Suppose there is analytic function $g: G \to \mathbb{C}$ satisfying the conditions (a), (b) and (c).

Then $fog^{-1}: D \to D$ is analytic, one-one and onto.

Also $\left(fog^{-1}\right)(0)=f\left(g^{-1}(0)\right)=f(a)=0$.

Hence, by Schwartz's lemma, there is a constant c with $|c|=1$ and $fog^{-1}(z)=cz$ for all $z\in D$.

As $g(G)=D$, $z\in G\Rightarrow g(z)\in D$.

Thus $\left(fog^{-1}\right)(g(z))=cg(z),\ \forall z\in G$

$$\Rightarrow f\left(g^{-1}(g(z))\right)=cg(z),\forall z\in G$$

$$\Rightarrow f(z)=cg(z),\ \forall z\in G \qquad \text{..... (11)}$$

$\Rightarrow\ 0<f'(a)=cg'(a)$.

But $g'(a)>0$ implies $c > 0$.

Thus $1=|c|=c$.

Therefore from (11) we have,

$$f(z)=g(z),\ \forall z\in G.$$

Hence $f=g$ on G. This proves the uniqueness.

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