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Preface

Functional Analysis is a core branch of mathematical analysis which has wide application in various branches of mathematics such as differential equations, integral equations, approximation theory, classical theory of analytic functions etc. This subject deals with the study of vector spaces equipped with a distance function norm and hence the study endowed with topological structure.

In this course we study the theory of Banch spaces, functional spaces, Hilbert spaces, theory of operators, spectral theory etc. Content of this book is developed by taking in to account an actual classroom teaching. The material is self-explanatory and it is written keeping in mind the requirement of distance mode students. Thus the detailed explanations of theory provided with number of supporting examples. This self-instructional material is written according to the syllabus of Distance Education, Shivaji University, Kolhapur.

This book is divided into seven units. In Unit 1 normed spaces and Banach spaces are introduced. Unit 2 deals with bounded linear transformations and the well known theorems viz. open mapping theorem, Closed graph theorem and uniform boundedness principle. In unit 3, we study bounded linear functional, conjugate spaces, Hahn-Banach theorem and its consequences. Unit 4 is devoted to study second conjugate space, natural imbedding, equivalent norms and finite dimensional spaces. In unit 5, Inner product spaces are introduced. Properties of inner product spaces along with certain examples are discussed at the beginning. Hilbert spaces, orthogonal complements, orthonormal sets and Gram Schmidt orthogonalization procedure is discussed along with some examples. Conjugate spaces and Riesz representation theorem is discussed at end. Unit 6 deals with bounded operators on Hilbert spaces. Adjoint, self adjoint operator and their properties are discussed in detail. Normal, unitary operators and their properties are discussed along with certain examples. Projection and representation of spaces as sum of projection is proved at the end. In unit 5 is devoted to finite dimensional spectral theory.

This self instructional material is developed by taking into account the quarries of students in classroom. We feel that this book will find useful for the students to learn and understand the basic concepts in Functional Analysis.

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Functional Analysis

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M. Sc. (Mathematics)

Functional Analysis

Contents

Each Unit begins with the section Objectives -

Objectives are directive and indicative of :

1. What has been presented in the Unit and
2. What is expected from you
3. What you are expected to know pertaining to the specific Unit once you have completed working on the Unit.

The self check exercises with possible answers will help you to understand the Unit in the right perspective. Go through the possible answers only after you write your answers. These exercises are not to be submitted to us for evaluation. They have been provided to you as Study Tools to help keep you in the right track as you study the Unit.

UNIT - I

NORMED SPACES AND BANACH SPACES

In this unit we deal with the normed linear spaces, example and non-example of Banach spaces, nomed quotient space is defined and proved it is complete.

1.1 LINEAR SPACES

1.1.1 Definition

A linear space (or vector space) over the field $\mathbb{K}$ is a nonempty set L together with two algebraic operations.

$+: L\times L\to L$, called vector addition,

$\bullet : \mathbb{K}\times L\to L$, called scalar multiplication

satisfying the following conditions.

1) $(L, +)$ is an abelian group.

2) For all $x, y\in L$ and all $\alpha, \beta\in\mathbb{K}$, we have,

(a) $\alpha\cdot(x+y)=\alpha\cdot x+\alpha\cdot y$

(b) $(\alpha+\beta)\cdot x=\alpha\cdot x+\beta\cdot y$

(c) $(\alpha\beta)\cdot x=\alpha\cdot(\beta x)$

(d) $1\cdot x=x$, 1 is unity element of $\mathbb{K}$

1.1.2 Remark :

(i) Vector addition is mapping $(x, y)\to x+y$ which associate each pair of elements $x, y\in L$ to an element $x+y$ in L, called sum of x and y.

(ii) Scalar multiplication is a mapping $(\alpha, \mathrm{x})\to\alpha x$ which associate each element $\alpha\in\mathbb{K}$ and each $x\in L$ to an element αx in L.

(iii) The elements of a linear space are called vectors and the elements of the field $\mathbb{K}$ are called scalars.

(iv) If L is linear space over field $\mathbb{K}$ then L is called a linear space if $\mathbb{K} = \mathbb{R}$ (the field of real numbers), and a complex linear space if $\mathbb{K} = \mathbb{C}$ (the field of complex numbers).

1.1.3 Theorem : Let L be a linear space over field $\mathbb{K}$. Then :

(a) $0x = 0$, $\forall x \in L$

where 0 in left side of equation is scalar zero and 0 in right side is zero vector.

(b) $\alpha 0 = 0$, $\forall \alpha \in \mathbb{K}$

where 0 in both side is zero vector.

(c) $(-1)x = -x$, $\forall x \in L$

(d) $\alpha x = 0 \Rightarrow \alpha = 0$ (scalar zero) or $x = 0$ (vector zero)

1.1.4 Definition : A nonempty subset M of a linear space L over the field $\mathbb{K}$ is said to be a linear subspace (or simply a subspace) if the following condition is satisfied :

$$\alpha x + \beta y \in M, \ \forall x, y \in M \text{ and } \forall \alpha, \beta \in \mathbb{K}.$$

Note : In what follows, the remaining related concepts of linear spaces we recall whenever it is needed.

1.2 NORMED LINEAR SPACES

1.2.1 Definition : Let $\mathbb{K} = \mathbb{R}$ (or $\mathbb{C}$), and X be a linear space over the field $\mathbb{K}$.

A function $\|.\| : X \to \mathbb{R}$ is said to be norm on X if for all $x, y \in X$ and all $\alpha \in \mathbb{K}$, we have,

(i) $\|x\| \geq 0$

(ii) $\|x\| = 0$ iff $x = 0$

(iii) $\|x + y\| \leq \|x\| + \|y\|$ (Triangle Inequality)

(iv) $\|\alpha x\| \leq |\alpha| \|x\|$ (Homogeneity of norm)

1.2.2 Definition : A linear space X over the field $\mathbb{K} = \mathbb{R}$ (or $\mathbb{C}$) with a norm $\|.\|$ defined on it is called a normed linear space over $\mathbb{K}$ (or simply a normed space).

We denote the normed linear space by pair $(X, \|\cdot\|)$ or simply by X. The normed linear space X is called real normed linear space if $\mathbb{K} = \mathbb{R}$, and complex normed linear space if $\mathbb{K} = \mathbb{C}$.

1.2.3 Remark : The real number $\|x\|$, $(x \in X)$ is called the norm of vector x.

The element of field $\mathbb{K} = \mathbb{R}$ (or $\mathbb{C}$) will be called Scalars.

1.2.4 Example : The linear space $\mathbb{R}$ over the field $\mathbb{R}$ is normed linear space with the norm defined by $\|x\| = |x|$, $x \in \mathbb{R}$.

1.2.5 Example : The linear space $\mathbb{C}$ over the field $\mathbb{R}$ (or $\mathbb{C}$) is normed linear space with the norm defined by $\|z\| = |z| = \sqrt{x^2 + y^2}$, $z = x + iy \in \mathbb{C}$.

We will see more examples of normed linear spaces in the topic Banach Spaces.

Note : Let X and Y be two linear space over the field $\mathbb{K} = \mathbb{R}$ (or $\mathbb{C}$).

Then the cartesian product $X \times Y$ is again a linear space over $\mathbb{K}$ under the algebraic operations given by,

$$(x, y) + (u, v) = (x + u, y + v)$$

and $$\alpha(x, y) = (\alpha x, \alpha y)$$

where (x, y), $(u, v) \in X \times Y$ and $\alpha \in \mathbb{K}$.

1.2.6 Problem : Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ are normed space. Prove that

$$\|(x, y)\| = \max\{\|x\|_X, \|y\|_Y\}, \ (x, y) \in X \times Y \qquad \text{..... (1)}$$

defines a norm on linear space $X \times Y$.

Proof : Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ are normed linear space over the same system of scalars.

Let any (x, y), $(u, v) \in X \times Y$ and α be any scalar.

(i) Since $\|x\|_X \geq 0$ and $\|y\|_Y \geq 0$

$\|(x, y)\| = \max\{\|x\|_X, \|y\|_Y\} \geq 0$

(ii) Let $(x, y) = (0, 0)$ (zero vector in $X \times Y$)

Then $\|(x, y)\| = \|(0, 0)\| = \max\{\|0\|_X, \|0\|_Y\} = 0$

Conversely, let $\|(x, y)\| = 0$. Then,

$\max\{\|x\|_X, \|y\|_Y\} = 0$

$\Rightarrow \|x\|_X = 0$, $\|y\|_Y = 0$

$\Rightarrow x = y = 0$

$\Rightarrow (x, y) = (0, 0)$, zero vector in $X \times Y$.

(ii) $\|\alpha(x, y)\| = \|(\alpha x, \alpha y)\| = \max\{\|\alpha x\|_X, \|\alpha y\|_Y\}$

$= \max\{|\alpha|\|x\|_X, |\alpha|\|y\|_Y\}$

$= |\alpha| \max\{\|x\|_X, \|y\|_Y\}$

$= |\alpha|\|(x, y)\|$

(iii) $\|x + u\|_X \leq \|x\|_X + \|u\|_X$

$\leq \max\{\|x\|_X, \|y\|_Y\} + \max\{\|u\|_X, \|v\|_Y\}$

$= \|(x, y)\| + \|(u, v)\|$

i.e. $\|x + u\|_X \leq \|(x, y)\| + \|(u, v)\|$

on the same line,

$$\|y+v\|_Y \le \|(x,y)\| + \|(u,v)\|$$

Therefore,

$$\max\left\{\|x+u\|_x, \|y+v\|_y\right\} \le \|(x,y)\| + \|(u,v)\|$$

$$\Rightarrow \|(x+u, y+v)\| \le \|(x,y)\| + \|(u,v)\|$$

$$\Rightarrow \|(x,y)+(u,v)\| \le \|(x,y)\| + \|(u,v)\|$$

From (i) - (iii), $X \times Y$ is normed space with the norm defined by (1)

Exercise : Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be two normed spaces. Prove that $\|(x,y)\| = \|x\|_X + \|y\|_Y$, $(x,y) \in X \times Y$ defines norm on linear space $X \times Y$.

1.2.7 Theorem : Let $(X, \|\cdot\|)$ be a normed linear space. Define $d : X \times X \to \mathbb{R}$ by

$$d(x,y) = \|x-y\|,\ x, y \in X.$$

Then d is metric on X.

Proof : Let $(X, \|\cdot\|)$ be a normed linear space. Let any $x, y, z \in X$. Then,

(a) $\|x-y\| \ge 0 \Rightarrow d(x,y) \ge 0$

(b) $d(x,y) = \|x-y\| = \|(-1)(y-x)\|$

$$= |(-1)|\|y-x\| = \|y-x\| = d(y,x)$$

(c) $d(x,y) = \|x-y\|$

$$= \|(x-z)+(z-y)\|$$

$$\le \|x-z\| + \|z-y\|$$

$$= d(x,z) + d(z,y)$$

Therefore, d is metric on X. Hence (X, d) is metric space.

1.2.8 Remark : Let $(X, \|\cdot\|)$ be a normed linear space. A metric d on X given by,

$$d(x, y) = \|x - y\|, \ x, y \in X$$

is called the metric induced by the norm.

With this metric, a normed linear space become a metric space and hence a topological space.

1.2.9 Theorem : A metric d induced by a norm $\|.\|$ on a normed linear space X satisfies

(a) $d(x+z, y+z) = d(\mathrm{x}, y)$

(b) $d(\alpha x, \alpha y) = |\alpha| d(\mathrm{x}, y)$

for all $x, y, z \in X$ and every scalar α.

Proof : Let $(X, \|\cdot\|)$ be a normed linear space over the field $\mathbb{K} = \mathbb{R}$ (or $\mathbb{C}$).

Let any $x, y, z \in X$ and $\alpha \in \mathbb{K}$ and d is metric induced by the norm $\|.\|$.

Then we have :

(a) $$d(x+z, y+z) = \|(x+z) - (y+z)\|$$
$$= \|x - y\|$$
$$= d(x, y)$$

(b) $$d(\alpha x, \alpha y) = \|\alpha x - \alpha y\|$$
$$= \|\alpha(x - y)\| = |\alpha| \|x - y\|$$
$$= |\alpha| d(x, y)$$

1.2.10 Remark : Every norm on a normed linear space induces a metric but every metric on a linear space cannot be obtained from a norm.

The above theorem gives the conditions under which a metric on a linear space can be obtained by a norm on it.

1.2.11 Example : Let X be a space of all (bounded or unbounded) sequences of complex numbers and define

$$d(x,y)=\sum_{n=1}^{\infty}\frac{1}{2^n}\frac{|x_n-y_n|}{1+|x_n-y_n|}$$

where $x=\{x_n\}_{n=1}^{\infty}$ and $y=\{y_n\}_{n=1}^{\infty}$ belongs to X.

Then (X, d) is metric space.

If d is metric obtained from some norm then it must satisfy

$$d(x+z,y+z)=d(x,y) \text{ and } d(\alpha x,\alpha y)=|\alpha|d(x,y)$$

for any $x,y,z\in X$ and $\alpha\in\mathbb{K}$.

However, we see that

$$d(\alpha x,\alpha y)=\sum_{n=1}^{\infty}\frac{1}{2^n}\frac{|\alpha x_n-\alpha y_n|}{1+|\alpha x_n-\alpha y_n|}$$

$$\neq|\alpha|\sum_{n=1}^{\infty}\frac{1}{2^n}\frac{|x_n-y_n|}{1+|x_n-y_n|}$$

$$=|\alpha|d(x,y)$$

i.e. $d(\alpha x,\alpha y)\neq|\alpha|d(x,y)$

Thus metric d cannot be obtained from any norm.

1.2.12 Example :

Let d be the discrete metric on set of real numbers $\mathbb{R}$. Then,

$$d(x,y)=\begin{cases}1 & \text{if } x\neq y\\ 0 & \text{if } x=y\end{cases}$$

Then for $x,y\in\mathbb{R}$ with $x\neq y$ we have

$$d(5x,5y)=1, \text{ as } 5x\neq 5y$$

But, $5d(x,y)=5(1)=5$

Therefore, $d(5x,5y) \neq 5d(x,y)$

Thus the discrete metric on $\mathbb{R}$ cannot be obtained from any norm.

Exercise :

1) Let $(X,\|\cdot\|)$ be a normed space.

Define $\rho : X \times X \to \mathbb{R}$ by

$$\rho(x,y) = \min\{1, \|x-y\|\}, \; x, y \in X$$

Prove that there is a no norm on X which generates metric ρ on X.

2) Let d be a metric induced by a norm $\|.\|$ on a linear space $X \neq \{0\}$.

Define $$\rho(x,y) = \begin{cases} 0 & \text{if } x = y \\ 1 + d(x,y) & \text{if } x \neq y \end{cases}$$

Prove that ρ cannot be obtained from a norm on X.

1.3 PROPERTIES OF NORM

As every normed linear space is metric space with induced metric, the concept of open sets, closed sets, convergence of sequences and related concepts of metric spaces naturally enter into normed linear spaces.

1.3.1 Definitions : Let $(X,\|\cdot\|)$ be a normed linear space.

1) Let any $x_0 \in X$ and $r > 0$. Then the set $S_r(x_0) = \{x \in X : \|x - x_0\| < r\}$ is called open sphere with centre x_0 and radius r, and the set $S_r[x_0] = \{x \in X : \|x - x_0\| \leq r\}$ is called closed sphere with centre x_0 and radius r.

2) A sequence $\{x_n\}$ in X is said to be convergent to $x \in X$ if for given $\varepsilon > 0$, $\exists\, n_0 \in \mathbb{N}$ such that

$$\|x_n - x\| < \varepsilon, \; \forall n \geq n_0$$

We write, $\lim\limits_{n\to\infty} \|x_n - x\| = 0$ or $\lim\limits_{n\to\infty} x_n = x$.

3) A sequence $\{x_n\}$ in X is said to be Cauchy sequence if for given $\varepsilon > 0$, $\exists\, n_0 \in \mathbb{N}$ such that $\|x_m - x_n\| < \varepsilon$, $\forall m, n \geq n_0$

4) Let $A \subseteq X$ and $x \in X$. Then x is called limit point of A if $A \cap S_r(x) - \{x\} \neq \phi$, $\forall r > 0$.

5) A point x in a subset A of X is called an interior point of A if $\exists\, r > 0$ such that $S_r(x) \subseteq A$ (i.e. A is neighbourhood of x).

6) A subset A of normed linear space X is said to be bounded if there exists K > 0 such that $\|x\| \leq K$, $\forall x \in A$.

7) Let $\{x_n\}_{n=1}^{\infty}$ be a sequence in a normed space X. The series $\sum_{n=1}^{\infty} x_n$ is said to convergent if the sequence $\{S_n\}_{n=1}^{\infty}$ of the partial sums $S_n = \sum_{j=1}^{n} x_j$, $(n = 1, 2, ...)$ is convergent.

If $S_n \to x$ i.e. $\|S_n - x\| \to 0$ then $\sum_{n=1}^{\infty} x_n = x$.

Further the series $\sum_{n=1}^{\infty} x_n$ is said to absolutely convergent if $\sum_{n=1}^{\infty} \|x_n\|$ is convergent.

1.3.2 Theorem : In a normed linear space every convergent sequence is a Cauchy sequence.

Proof : Proof is similar as in metric space.

1.3.3 Theorem : In a normed space $(N, \|\cdot\|)$,

$$\big|\, \|x\| - \|y\| \,\big| \leq \|x - y\|, \text{ for all } x, y \in N.$$

Proof : Let $(N, \|\cdot\|)$ be a normed space and let any $x, y \in \mathbb{N}$. Then we have

$$\|x\| = \|(x - y) + y\| \leq \|x - y\| + \|y\|$$

This implies,

$$\|x\| - \|y\| \leq \|x - y\| \qquad ..... (1)$$

Interchanging role of x and y we get

$$\|y\| - \|x\| \le \|y - x\|$$

$$= \|-(x - y)\|$$

$$= |(-1)| \|x - y\|$$

$$= \|x - y\|$$

$$\Rightarrow -\|x - y\| \le \|x\| - \|y\| \qquad \text{..... (2)}$$

From (1) and (2), we have

$$-\|x - y\| \le \|x\| - \|y\| \le \|x - y\|$$

Therefore,

$$\big|\ \|x\| - \|y\|\ \big| \le \|x - y\|$$

1.3.4 Theorem : Let $(N, \|\cdot\|)$ be a normed space. Then the mapping $\|\cdot\| : N \to \mathbb{R}$ is continuous i.e. norm is continuous function.

Proof : Let $(N, \|\cdot\|)$ be a normed space.

Let $x_n \to x$ in N. Then

$\big|\|x_n\| - \|x\|\big| \le \|x_n - x\| \to 0$ as $n \to \infty$.

$\Rightarrow \|x_n\| \to \|x\|$ as $n \to \infty$

Therefore $\|.\|$ on N is continuous.

1.3.5 Definition : Let (X, d_X), (Y, d_Y) and (Z, d_Z) are metric spaces. A mapping $f : X \times Y \to Z$ is jointly continuous if and ony if $x_n \to x$ in X and $y_n \to y$ in Y implies $f(x_n, y_n) \to f(x, y)$.

1.3.6 Remark : If f is jointly continuous then it is continuous in each variable separately but the converse is not true.

1.3.7 Theorem : The operations of addition and scalar multiplications in a normed space are jointly continuous.

Proof : Let $(N, \|\cdot\|)$ be a normed space over $\mathbb{K} = \mathbb{R}$ (or $\mathbb{C}$).

Let $x_n \to x$ in N, $y_n \to y$ in N and let $\alpha_n \to \alpha$ in $\mathbb{K}$.

(i) $$\|(x_n + y_n) - (x + y)\| = \|(x_n - x) + (y_n - y)\|$$

$$\leq \|x_n - x\| + \|y_n - y\| \to 0$$

This gives $\|(x_n + y_n) - (x + y)\| \to 0$ as $n \to \infty$ i.e. $x_n + y_n \to x + y$.

Therefore vector addition $+ : N \times N \to N$ is jointly continuous.

(ii) $$\|\alpha_n x_n - \alpha x\| = \|(\alpha_n x_n - \alpha_n x) + (\alpha_n x - \alpha x)\|$$

$$\leq \|\alpha_n (x_n - x)\| + \|(\alpha_n - \alpha) x\|$$

$$= |\alpha_n| \|x_n - x\| + |\alpha_n - \alpha| \|x\|$$

$$\to 0 \text{ as } n \to \infty$$

Thus $\|\alpha_n x_n - \alpha x\| \to 0$ i.e. $\alpha_n x_n \to \alpha x$

This proves scalar multiplication $\bullet : \mathbb{K} \times N \to N$ is jointly continuous.

1.3.8 Definition (Seminorm) : A seminorm on a linear space X over $\mathbb{K} = \mathbb{R}$ (or $\mathbb{C}$) is a function $f : \mathrm{X} \to \mathbb{R}$ satisfying.

(i) $f(x) \geq 0$

(ii) $f(\alpha x) = |\alpha| f(x)$

(iii) $f(x + y) \leq f(x) + f(y)$

for all $x, y \in X$ and all $\alpha \in \mathbb{K}$.

1.3.9 Remark : We observe that

1) $f(0)=0$

2) $|f(x)-f(y)| \le f(x-y)$, $x, y \in X$

3) If $f(x)=0$ implies $x=0$ then f is norm on X.

1.4 BANACH SPACES

1.4.1 Definition : A normed linear space $(N, \|\cdot\|)$ is said to be complete if each Cauchy sequence in N converges to a point in N.

Note : The normed space $(N, \|\cdot\|)$ is complete means it is complete in a metric space (N, d) where $d(x, y) = \|x-y\|$, $x, y \in N$.

1.4.2 Definition : A complete normed linear space $(N, \|\cdot\|)$ is called a Banach Space.

The Banach Space N is called real (or complex) if the underlying field $\mathbb{K}$ is $\mathbb{R}$ (or $\mathbb{C}$).

1.4.3 Example : The real linear space $\mathbb{R}$ is Banach space with norm $\|x\| = |x|$, $x \in \mathbb{R}$.

1.4.4 Example : The complex linear space $\mathbb{C}$ is Banach space with norm $\|z\| = |z| = \sqrt{x^2+y^2}$, $z = x+iy \in \mathbb{C}$.

1.4.5 Cauchy-Schwartz Inequality (for n-tuples)

Let $x=(x_1, x_2, \ldots, x_n)$, $y=(y_1, \mathrm{y}_2, \ldots, \mathrm{y}_n) \in \mathbb{K}^n$, ($\mathbb{K}=\mathbb{R}$ or $\mathbb{C}$), then

$$\sum_{j=1}^{n} |x_j y_j| \le \left[\sum_{j=1}^{n} |x_j|^2\right]^{1/2} \left[\sum_{j=1}^{n} |y_j|^2\right]^{1/2}$$

1.4.6 Problem : Prove that $\mathbb{K}^n$, ($\mathbb{K}=\mathbb{R}$ or $\mathbb{C}$) is Banach space with the norm

$$\|x\| = \left[\sum_{j=1}^{n} |x_j|^2\right]^{1/2},$$

where $x = (x_1, x_2, ..., x_n) \in \mathbb{K}^n$.

Solution :

Part - I : To prove $(\mathbb{K}^n, \|\cdot\|)$ is normed space.

Let any $x = (x_1, x_2, ..., x_n)$, $y = (y_1, y_2, ..., y_n)$ in $\mathbb{K}^n$, ($\mathbb{K}=\mathbb{R}$ or $\mathbb{C}$), and let α be any scalar.

(i) Since $|x_j| \geq 0$, $\forall j$ (j = 1, 2, ..., n), we have

$\sum_{j=1}^{n} |x_j|^2 \geq 0$. This gives $\|x\| \geq 0$.

(ii) $x = 0 \Leftrightarrow (x_1, x_2, ..., x_n) = (0, 0, ..., 0)$

$$\Leftrightarrow x_j = 0, \ \forall j \ (j = 1, 2, ..., n)$$

$$\Leftrightarrow |x_j| = 0, \forall j \ (j = 1, 2, ..., n)$$

$$\Leftrightarrow \sum_{j=1}^{n} |x_j|^2 = 0$$

$$\Leftrightarrow \|x\| = 0$$

(iii) Since $x + y = (x_1 + y_1, x_2 + y_2, ..., x_n + y_n)$ we have

$$\|x+y\|^2 = \sum_{j=1}^{n} |x_j + y_j|^2$$

$$= \sum_{j=1}^{n} |x_j + y_j| |x_j + y_j|$$

$$\leq \sum_{j=1}^{n} |x_j + y_j| \left(|x_j| + |y_j|\right)$$

$$= \sum_{j=1}^{n} |x_j + y_j||x_j| + \sum_{j=1}^{n} |x_j + y_j||y_j|$$

Using Cauchy-Schwartz inequality we get

$$\|x+y\|^2 \le \left[\sum_{j=1}^{n} |x_j + y_j|^2\right]^{1/2} \left[\sum_{j=1}^{n} |x_j|^2\right]^{1/2} + \left[\sum_{j=1}^{n} |x_j + y_j|^2\right]^{1/2} \left[\sum_{j=1}^{n} |y_j|^2\right]^{1/2}$$

$$= \|x+y\|\|x\| + \|x+y\|\|y\|$$

$$= \|x+y\|\left(\|x\| + \|y\|\right)$$

This gives

$$\|x+y\| \le \|x\| + \|y\|$$

(iv) As $\alpha x = (\alpha x_1, \alpha x_2, ..., \alpha x_n)$ we have

$$\|\alpha x\| = \left[\sum_{j=1}^{n} |\alpha x_j|^2\right]^{1/2}$$

$$= \left[|\alpha|^2 \sum_{j=1}^{n} |x_j|^2\right]^{1/2}$$

$$= |\alpha| \left[\sum_{j=1}^{n} |x_j|^2\right]^{1/2}$$

$$= |\alpha| \|x\|$$

We have proved that, $(\mathbb{K}^n, \|\cdot\|)$ is normed space.

Part - II : To prove $(\mathbb{K}^n, \|\cdot\|)$ is complete.

Let $\{x_m\}_{m=1}^{\infty}$ be any Cauchy sequence in $\mathbb{K}^n$, where for each m, $x_m = \left(x_1^{(m)}, x_2^{(m)}, ..., x_n^{(m)}\right)$.

Then for given $\varepsilon > 0$, $\exists\, n_0 \in \mathbb{N}$ such that

$$m, k \ge n_0 \Rightarrow \|x_m - x_k\| < \varepsilon$$

$$\Rightarrow \|x_m - x_k\|^2 < \varepsilon^2$$

$$\Rightarrow \sum_{j=1}^{n} \left|x_j^{(m)} - x_j^{(k)}\right|^2 < \varepsilon^2$$

But $\quad \left|x_j^{(m)} - x_j^{(k)}\right|^2 \le \sum_{j=1}^{n} \left|x_j^{(m)} - x_j^{(k)}\right|^2$, for all j ($j = 1, 2, ..., n$)

Therefore,

$$m, k \ge n_0 \Rightarrow \left|x_j^{(m)} - x_j^{(k)}\right|^2 < \varepsilon^2, \ \forall j$$

$$\Rightarrow \left|x_j^{(m)} - x_j^{(k)}\right| < \varepsilon, \ \forall j$$

This shows that for each j ($j = 1, 2, ..., n$), $\left\{x_j^{(m)}\right\}_{m=1}^{\infty}$ is a Cauchy sequence in $\mathbb{K}$.

Since $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ is complete space, there exists $x_j \in \mathbb{K}$ such that,

$$x_j^{(m)} \to x_j \text{ as } m \to \infty, \text{ for each } j \ (j = 1, 2, ..., n)$$

Define $x = (x_1, x_2, ..., x_n)$ then $x \in \mathbb{K}^n$.

We prove that $x_m \to x$ as $m \to \infty$.

By (1), $\exists n_1 \in \mathbb{N}$ such that,

$$m \ge n_1 \Rightarrow \left|x_j^{(m)} - x_j\right| < \frac{\varepsilon}{\sqrt{n}}, \ \forall j$$

$$\Rightarrow \left|x_j^{(m)} - x_j\right|^2 < \frac{\varepsilon^2}{n}, \ \forall j$$

$$\Rightarrow \sum_{j=1}^{n} \left|x_j^{(m)} - x_j\right|^2 < n\left(\frac{\varepsilon^2}{n}\right) = \varepsilon^2$$

$$\Rightarrow \left[\sum_{j=1}^{n} \left|x_j^{(m)} - x_j\right|^2\right]^{1/2} < \varepsilon$$

$$\Rightarrow \|x_m - x\| < \varepsilon$$

Therefore $x_m \to x$ in $\mathbb{K}^n$.

By part I and II, $\mathbb{K}^n$ is Banach Space.

1.4.7 Minkowski Inequality (for n-tuples)

Let $x = (x_1, x_2, ..., x_n)$ and $y = (y_1, y_2, ..., y_n)$ be the elements of $\mathbb{K}^n$, ($\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$) and let p be a real number such that $1 \le p < \infty$, then

$$\left[\sum_{j=1}^{n} |x_j + y_j|^p\right]^{1/p} \le \left[\sum_{j=1}^{n} |x_j|^p\right]^{1/p} + \left[\sum_{j=1}^{n} |y_j|^p\right]^{1/p}$$

1.4.8 Problem : Let p be a real number such that $1 \le p < \infty$, and denote by ℓ_p^n the space $\mathbb{K}^n$ ($\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$), with the norm

$$\|x\|_p = \left[\sum_{j=1}^{n} |x_j|^p\right]^{1/p}, \text{ where } x = (x_1, x_2, ..., x_n) \in \ell_p^n$$

Prove that ℓ_p^n is a Banach space.

Solution : Part I : To prove ℓ_p^n is normed space.

Let any $x = (x_1, x_2, ..., x_n)$, $y = (y_1, y_2, ..., y_n)$ in ℓ_p^n, and let α be any scalar.

(i) Since $|x_j| \ge 0$, $\forall j$, (j = 1, 2, ..., n), we have

$$\sum_{j=1}^{n} |x_j|^p \ge 0 \Rightarrow \|x\|_p \ge 0$$

(ii) $x = 0 \Leftrightarrow (x_1, x_2, ..., x_n) = 0$

$$\Leftrightarrow x_j = 0, \ \forall j$$

$$\Leftrightarrow |x_j|^p = 0, \ \forall j$$

$$\Leftrightarrow \sum_{j=1}^{n} |x_j|^p = 0 \Leftrightarrow \left(\sum_{j=1}^{n} |x_j|^p \right)^{\frac{1}{p}} = 0$$

$$\Leftrightarrow \|x\|_p = 0$$

(iii) As $x + y = (x_1 + y_1, x_2 + y_2, ..., x_n + y_n)$ we have

$$\|x + y\|_p = \left[\sum_{j=1}^{n} |x_j + y_j|^p \right]^{1/p}$$

By Minkowski inequality we have,

$$\left[\sum_{j=1}^{n} |x_j + y_j|^p \right]^{1/p} \le \left[\sum_{j=1}^{n} |x_j|^p \right]^{1/p} + \left[\sum_{j=1}^{n} |y_j|^p \right]^{1/p}$$

Therefore, $\|x + y\|_p \le \|x\|_p + \|y\|_p$

(iv) As $\alpha x = (\alpha x_1, \alpha x_2, ..., \alpha x_n)$ we have

$$\|\alpha x\|_p = \left[\sum_{j=1}^{n} |\alpha x_j|^p \right]^{1/p}$$

$$= \left[|\alpha|^p \sum_{j=1}^{n} |x_j|^p \right]^{1/p}$$

$$= |\alpha| \left[\sum_{j=1}^{n} |x_j|^p \right]^{1/p}$$

$$= |\alpha| \|x\|_p$$

Therefore, ℓ_p^n is normed space.

Part II : To prove ℓ_p^n is complete.

Let $\{x_m\}_{m=1}^{\infty}$ be any Cauchy sequence in ℓ_p^n, where for each m, $x_m = \left(x_1^{(m)}, x_2^{(m)}, ..., x_n^{(m)}\right)$.

For given $\varepsilon > 0$, $\exists n_0 \in \mathbb{N}$ such that

$$m, k \geq n_0 \Rightarrow \|x_m - x_k\| < \varepsilon$$

$$\Rightarrow \|x_m - x_k\|^p < \varepsilon^p$$

$$\Rightarrow \sum_{j=1}^{n} \left|x_j^{(m)} - x_j^{(k)}\right|^p < \varepsilon^p$$

But for each j $(j = 1, 2, ..., n)$

$$\left|x_j^{(m)} - x_j^{(k)}\right|^p \leq \sum_{j=1}^{n} \left|x_j^{(m)} - x_j^{(k)}\right|^p$$

Therefore,

$$m, k \geq n_0 \Rightarrow \left|x_j^{(m)} - x_j^{(k)}\right|^p < \varepsilon^p \text{ , for each } j$$

$$\Rightarrow \left|x_j^{(m)} - x_j^{(k)}\right| < \varepsilon \text{ , for each } j$$

This shows that for each j $(1 \leq j \leq n)$, $\left\{x_j^{(m)}\right\}_{m=1}^{\infty}$ is Cauchy sequence in complete space $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$.

Hence $\exists x_j \in \mathbb{K}$ such that,

$$x_j^{(m)} \to x_j \text{ as } m \to \infty \text{ for each } j. \qquad .... (1)$$

Define $x = (x_1, x_2, ..., x_n)$; then $x \in \ell_p^n$.

We prove that $x_m \to x$ in ℓ_p^n as $m \to \infty$.

By (1) $\exists n_1 \in \mathbb{N}$ such that,

$$m \geq n_1 \Rightarrow \left|x_j^{(m)} - x_j\right| < \frac{\varepsilon}{n^{1/p}} \text{ , } \forall j$$

$$\Rightarrow \left|x_j^{(m)} - x_j\right|^p < \frac{\varepsilon^p}{n}, \forall j$$

$$\Rightarrow \sum_{j=1}^{n} \left| x_j^{(m)} - x_j \right|^p < n \frac{\varepsilon^p}{n} = \varepsilon^p$$

$$\Rightarrow \left\| x_m - x \right\|_p < \varepsilon$$

This proves $x_m \to x$ in ℓ_p^n as $m \to \infty$.

By Part I and II, ℓ_p^n is a Banach space.

1.4.9 Remark : In above example for $p = 2$ we have two important Banach spaces :

1) **Unitary n-space :** The space $\mathbb{C}^n$ with the norm $\|x\|_2 = \left[\sum_{j=1}^{n} |x_j|^2 \right]^{1/2}$, $x = (x_1, x_2, ..., x_n) \in \mathbb{C}^n$, is a Banach space. The Banach space $(\mathbb{C}^n, \|\cdot\|_2)$ is called unitary n-space.

2) Similarly, the Banach space $(\mathbb{R}^n, \|\cdot\|_2)$ is called Euclidean n-space.

1.4.10 Minkowski Inequality (for sequences) :

Let $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ be any sequences in $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ such that $\sum_{n=1}^{\infty} |x_n|^p < \infty$ and $\sum_{n=1}^{\infty} |y_n|^p < \infty$, and p be a real number such that $1 \le p < \infty$. Then,

$$\left[\sum_{n=1}^{\infty} |x_n + y_n|^p \right]^{1/p} \le \left[\sum_{n=1}^{\infty} |x_n|^p \right]^{1/p} + \left[\sum_{n=1}^{\infty} |y_n|^p \right]^{1/p}$$

1.4.11 Problem : Let p be a real number such that $1 \le p < \infty$. Denote by ℓ_p the space of all sequences $x = \{x_n\}_{n=1}^{\infty}$ in $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ such that $\sum_{n=1}^{\infty} |x_n|^p < \infty$, with the norm

$$\|x\|_p = \left[\sum_{n=1}^{\infty} |x_n|^p\right]^{1/p}$$

Prove that ℓ_p is Banach space.

Solution : Let any $x = \{x_n\}_{n=1}^{\infty}$ and $y = \{y_n\}_{n=1}^{\infty}$ in ℓ_p and α be any scalar. We know ℓ_p is linear space with vector addition and scalar multiplication given by,

$$x + y = \{x_n + y_n\}_{n=1}^{\infty},$$

and $$\alpha x = \{\alpha x_n\}_{n=1}^{\infty}$$

Part - I : To Prove ℓ_p is normed space :

(i) Since $|x_n| \geq 0 \ \forall n$, $\sum_{n=1}^{\infty} |x_n|^p \geq 0$

This gives $\|x\|_p \geq 0$.

(ii) $x = 0 \Leftrightarrow \{x_n\}_{n=1}^{\infty} = 0$

$$\Leftrightarrow x_n = 0, \ \forall n$$

$$\Leftrightarrow |x_n|^p = 0, \ \forall n$$

$$\Leftrightarrow \sum_{n=1}^{\infty} |x_n|^p = 0$$

$$\Leftrightarrow \|x\|_p = 0$$

(iii) Since $x + y = \{x_n + y_n\}_{n=1}^{\infty}$, we have

$$\|x + y\|_p = \left[\sum_{n=1}^{\infty} |x_n + y_n|^p\right]^{1/p}$$

By Minkowski inequality,

$$\left[\sum_{n=1}^{\infty} |x_n + y_n|^p\right]^{1/p} \leq \left[\sum_{n=1}^{\infty} |x_n|^p\right]^{1/p} + \left[\sum_{n=1}^{\infty} |y_n|^p\right]^{1/p}$$

Thus, $\|x+y\|_p \le \|x\|_p + \|y\|_p$.

(iv) As $\alpha x = \{\alpha x_n\}_{n=1}^{\infty}$, we have

$$\|\alpha x\|_p = \left[\sum_{n=1}^{\infty} |\alpha x_n|^p\right]^{1/p} = |\alpha| \left[\sum_{n=1}^{\infty} |x_n|^p\right]^{1/p} = |\alpha| \|x\|_p$$

Therefore, ℓ_p is a normed space.

Part - II : To prove ℓ_p is complete.

Let $\{x_m\}_{m=1}^{\infty}$ be any Cauchy sequence in ℓ_p, where for each m, $x_m = \left\{x_n^{(m)}\right\}_{n=1}^{\infty}$ such that $\sum_{n=1}^{\infty} \left|x_n^{(m)}\right| < \infty$.

Then for given $\varepsilon > 0$, $\exists n_0 \in \mathbb{N}$ such that

$$m, k \ge n_0 \Rightarrow \|x_m - x_k\|_p < \varepsilon$$

$$\Rightarrow \sum_{n=1}^{\infty} \left|x_n^{(m)} - x_n^{(k)}\right|^p < \varepsilon^p \qquad \text{.... (1)}$$

Since $\left|x_n^{(m)} - x_n^{(k)}\right|^p \le \sum_{n=1}^{\infty} \left|x_n^{(m)} - x_n^{(k)}\right|^p$ for each n, we have

$$m, k \ge n_0 \Rightarrow \left|x_n^{(m)} - x_n^{(k)}\right|^p < \varepsilon^p, \ \forall n$$

$$\Rightarrow \left|x_n^{(m)} - x_n^{(k)}\right| < \varepsilon, \ \forall n$$

Thus for each n, $\left\{x_n^{(m)}\right\}_{m=1}^{\infty}$ is Cauchy sequence in complete space $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$.

Hence $\exists x \in \mathbb{K}$ such that for each n

$$x_n^{(m)} \to x_n \text{ as } m \to \infty. \qquad \text{...... (2)}$$

Define $x = \{x_n\}_{n=1}^{\infty}$. We show that $x \in \ell_p$ and $x_m \to x$ as $m \to \infty$.

From (1), we have,

$$m, k \geq n_0 \Rightarrow \sum_{n=1}^{r} \left| x_n^{(m)} - x_n^{(k)} \right|^p < \varepsilon^p \text{, for each } r, (r = 1, 2, 3, ...) \qquad ..... (3)$$

Letting $k \to \infty$ in (3), and using (2), we get

$$m \geq n_0 \Rightarrow \sum_{n=1}^{r} \left| x_n^{(m)} - x_n \right|^p < \varepsilon^p \text{, for each } r, (r = 1, 2, 3, ...)$$

This on tending $r \to \infty$, we obtain

$$m \geq n_0 \Rightarrow \sum_{n=1}^{\infty} \left| x_n^{(m)} - x_n \right|^p < \varepsilon^p \qquad ...... (4)$$

This shows that $x_m - x \in \ell_p$.

By Minkowski inequality and using (4), we obtain

$$\left[\sum_{n=1}^{\infty} |x_n|^p \right]^{1/p} = \left[\sum_{n=1}^{\infty} \left| \left(x_n - x_n^{(m)} \right) + x_n^{(m)} \right|^p \right]^{1/p}$$

$$\leq \left[\sum_{n=1}^{\infty} \left| x_n^{(m)} - x_n \right|^p \right]^{1/p} + \left[\sum_{n=1}^{\infty} \left| x_n^{(m)} \right|^p \right]^{1/p}$$

$$= \|x_m - x\|_p + \|x_m\|_p$$

$$< \varepsilon + \|x_m\|_p$$

$$\Rightarrow \sum_{n=1}^{\infty} |x_n|^p < \left[\varepsilon + \|x_m\|_p \right]^p < \infty$$

$$\Rightarrow x = \{x_n\}_{n=1}^{\infty} \in \ell_p.$$

Finally, from (4) we have

$$m \geq n_0 \Rightarrow \|x_m - x\|_p < \varepsilon$$

This gives $x_m \to x$ in ℓ_p as $m \to \infty$.

Therefore ℓ_p is complete.

By Part I and II, ℓ_p is Banach space.

1.4.12 Remark : In above example, for $p = 2$ we have following two important Banach spaces.

1. The set of all sequences $x = \{x_n\}_{n=1}^{\infty}$ in $\mathbb{C}$ such that $\sum_{n=1}^{\infty} |x_n|^p < \infty$, with the norm $\|x\|_2 = \left[\sum_{n=1}^{\infty} |x_n|^2\right]^{1/2}$ is denoted by $\mathbb{C}^{\infty}$. Then $\mathbb{C}^{\infty}$ is Banach space and it is called **infinite dimensional unitary space.**

2. The **infinite dimensional Euclidean space** is defined similarly as above.

1.4.13 Problem : Denote by ℓ_{∞}^{n} the space $\mathbb{K}^n$, ($\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$) with the norm defined by

$\|x\|_{\infty} = \max_{1 \le j \le n} |x_j|$, where $x = (x_1, x_2, ..., x_n)$. Prove that ℓ_{∞}^{n} is Banach space.

Solution : Part - I : To prove ℓ_{∞}^{n} is normed space

Let any $x = (x_1, x_2, ..., x_n)$, $y = (y_1, y_2, ..., y_n)$ in ℓ_{∞}^{n} and α be any scalar.

(i) Since $|x_j| \ge 0 \ \forall j$, (j = 1, 2, ..., n) $\Rightarrow \max_{1 \le j \le n} |x_j| \ge 0$

$$\Rightarrow \|x\|_{\infty} \ge 0$$

(ii) $x = 0 \Leftrightarrow (x_1, x_2, ..., x_n) = (0, ..., 0)$

$$\Leftrightarrow x_j = 0, \quad \forall j \quad (j = 1, 2, ..., n)$$

$$\Leftrightarrow |x_j| = 0, \ \forall j \quad (j = 1, 2, ..., n)$$

$$\Leftrightarrow \max_{1 \le j \le n} |x_j| = 0$$

$$\Leftrightarrow \|x\|_{\infty} = 0$$

(iii) For each j, (j = 1, 2, ..., n)

$|x_j + y_j| \le |x_j| + |y_j|$

$$\le \max_{1 \le j \le n} |x_j| + \max_{1 \le j \le n} |y_j|$$

$$= \|x\|_\infty + \|y\|_\infty$$

Thus $|x_j + y_j| \le \|x\|_\infty + \|y\|_\infty \quad \forall j \quad (1 \le j \le n)$

$\Rightarrow \max\limits_{1\le j\le n} |x_j + y_j| \le \|x\|_\infty + \|y\|_\infty$

Therefore $\|x + y\|_\infty \le \|x\|_\infty + \|y\|_\infty$.

(iv) As $\alpha x = (\alpha x_1, \alpha x_2, ..., \alpha x_n)$ we have ,

$\|\alpha x\| = \sup\limits_{1\le j\le n} |\alpha x_j| = |\alpha| \sup\limits_{1\le j\le n} |x_j|$

$$= |\alpha| \|x\|$$

We have proved that, ℓ_p^n is normed space.

Part II : To prove ℓ_∞^n is complete.

Let $\{x_m\}_{m=1}^\infty$ be any Cauchy sequence in ℓ_∞^n, where for each m, $x_m = \left(x_1^{(m)}, ..., x_2^{(m)}\right)$.

Thus for given $\varepsilon > 0$, $\exists\ n_0 \in \mathbb{N}$ such that

$m, k \ge n_0 \Rightarrow \|x_m - x_k\|_\infty < \varepsilon$

$$\Rightarrow \max_{1\le j\le n} \left|x_j^{(m)} - x_j^{(k)}\right| < \varepsilon$$

But for each j,

$$\left|x_j^{(m)} - x_j^{(k)}\right| \le \max_{1\le j\le n} \left|x_j^{(m)} - x_j^{(k)}\right|$$

Therefore for each j, (j = 1, 2, ..., n)

$$m, k \ge n_0 \Rightarrow \left|x_j^{(m)} - x_j^{(k)}\right| < \varepsilon \qquad ..... (1)$$

This implies for each j, $\left\{x_j^{(m)}\right\}_{m=1}^\infty$ is Cauchy sequence in complete space $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. Therefore $\exists\ x_j \in \mathbb{K}$ such that,

$$x_j^{(m)} \to x_j \text{ as } m \to \infty.$$

Define $x = (x_1, x_2, ..., x_n)$. Then $x \in \ell_\infty^n$.

We prove that $x_m \to x$ as $m \to \infty$.

Let $k \to \infty$ in (1) we have,

$$m \geq n_0 \Rightarrow \left| x_j^{(m)} - x_j \right| < \varepsilon, \ \forall j, (j = 1, 2, ..., n)$$

$$\Rightarrow \max_{1 \leq j \leq n} \left| x_j^{(m)} - x_j \right| < \varepsilon$$

$$\Rightarrow \|x^m - x\| < \varepsilon \cdot$$

This proves $x_m \to x$ in ℓ_∞^n as $m \to \infty$.

By part I and II, ℓ_∞^n is Banach space.

1.4.14 Problem : Denote by ℓ_∞ the space of all bounded sequences $x = \{x_n\}_{n=1}^\infty$ in $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ with the norm

$$\|x\|_\infty = \sup_{n \in \mathbb{N}} |x_n|$$

Prove that ℓ_∞ is Banach space.

Solution : Part I : ℓ_∞ is normed space.

We leave it for students, as it can be completed looking toward the part I of solution of problem 1.4.13.

Part II : ℓ_∞ is complete.

Let $\{x_m\}_{m=1}^\infty$ is Cauchy sequence in ℓ_∞, where for each *m*, $x_m = \left\{x_n^{(m)}\right\}_{n=1}^\infty$ is a bounded sequence.

Then for given $\varepsilon > 0$, $\exists \ n_0 \in \mathbb{N}$ such that,

$$m, k \geq n_0 \Rightarrow \|x_m - x_k\|_\infty < \varepsilon$$

$$\Rightarrow \sup_{n \in \mathbb{N}} \left| x_n^{(m)} - x_n^{(k)} \right| < \varepsilon$$

But $\left|x_n^{(m)} - x_n^{(k)}\right| \leq \sup_{n\in\mathbb{N}}\left|x_n^{(m)} - x_n^{(k)}\right|$ for all $n \in \mathbb{N}$.

Therefore for each $n \in \mathbb{N}$, we have

$$m, k \geq n_0 \Rightarrow \left|x_n^{(m)} - x_n^{(k)}\right| < \varepsilon \qquad \text{..... (1)}$$

This implies for each $n \in \mathbb{N}$, $\left\{x_n^{(m)}\right\}_{m=1}^{\infty}$ is Cauchy sequence in complete space $\mathbb{K}$.

Hence $\exists\, x_n \in \mathbb{K}$ such that $x_n^{(m)} \to x_n$ as $m \to \infty$.

Define $x = \left\{x_n\right\}_{n=1}^{\infty}$. We prove that $x \in \ell_\infty$ and $x_m \to x$ as $m \to \infty$.

Taking limit as $k \to \infty$ in (1) we get,

$$m \geq n_0 \Rightarrow \left|x_n^{(m)} - x_n\right| < \varepsilon,\ \forall n \in \mathbb{N} \qquad \text{..... (2)}$$

$$\Rightarrow \sup_{n\in\mathbb{N}}\left|x_n^{(m)} - x_n\right| < \varepsilon$$

$$\Rightarrow \left\|x_m - x\right\|_\infty < \varepsilon$$

This proves $x_m \to x$ as $m \to \infty$.

It remains to prove $x \in \ell_\infty$.

Since for each m, $x_m = \left\{x_n^{(m)}\right\}_{n=1}^{\infty}$ is bounded sequence, $\exists\, L_m > 0$ such that

$$\left|x_n^{(m)}\right| \leq L_m,\ \forall n \qquad \text{.... (3)}$$

Since $\left|x_n\right| = \left|x_n - x_n^{(m)} + x_n^{(m)}\right| \leq \left|x_n - x_n^{(m)}\right| + \left|x_n^{(m)}\right|$,

From (2) and (3), we have,

$$\left|x_n\right| \leq \varepsilon + L_m,\ \forall n.$$

This prove $x = \left\{x_n\right\}_{n=1}^{\infty}$ is bounded, and hence $x \in \ell_\infty$.

By part I and II ℓ_∞ is Banach space.

1.4.15 Problem : We denote by C the space of all convergent sequences $x = \{x_n\}_{n=1}^{\infty}$ in $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ with the norm.

$$\|x\|_{\infty} \Rightarrow \sup_{n \in \mathbb{N}} |x_n|,$$

Prove that C is Banach space.

Solution : We know every convergent sequence in $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ is bounded. Hence we have $C \subseteq \ell_{\infty}$.

Clearly C is subspace of complete normed space ℓ_{∞}. Thus to prove C is complete it is sufficient to prove that C is closed in ℓ_{∞}.

As $C \subseteq \overline{C}$ always, to prove C is closed we show that $\overline{C} \subseteq C$.

Let any $x \in \overline{C}$. Then there exists sequence $\{x_n\}_{n=1}^{\infty}$ in C such that,

$$x_n \to x \text{ as } n \to \infty,$$

Here for each n, $x_n = \{x_m^{(n)}\}_{m=1}^{\infty}$ is convergent sequence in $\mathbb{K}$ and hence it is bounded, and $x = \{x_m\}_{m=1}^{\infty} \in \ell_{\infty}$.

As $x_n \to x$, for given $\varepsilon > 0$, $\exists\, N \in \mathbb{N}$ such that,

$$n \geq N \Rightarrow \|x_n - x\|_{\infty} < \frac{\varepsilon}{3}$$

$$\Rightarrow \sup_{m \in \mathbb{N}} \left\| x_m^{(n)} - x_m \right\| < \frac{\varepsilon}{3}$$

But for any $m \in \mathbb{N}$,

$$\left| x_m^{(n)} - x_m \right| \leq \sup_{m \in \mathbb{N}} \left| x_m^{(n)} - x_m \right|$$

Therefore,

$$n \geq N \Rightarrow \left| x_m^{(n)} - x_m \right| < \frac{\varepsilon}{3}, \ \forall m$$

In particular for $n = N$,

$$\left|x_m^{(N)} - x_m\right| < \frac{\varepsilon}{3}, \ \forall m. \qquad .... (1)$$

Now $x_N = \left\{x_m^{(N)}\right\}_{m=1}^{\infty} \in C$, and hence it is Cauchy sequence.

This implies $\exists\, M \in \mathbb{N}$ such that,

$$m, k \geq M \Rightarrow \left|x_m^{(N)} - x_k^{(N)}\right| < \frac{\varepsilon}{3} \qquad ..... (2)$$

By triangle inequality,

$$\left|x_m - x_k\right| \leq \left|x_m - x_m^{(N)}\right| + \left|x_m^{(N)} - x_k^{(N)}\right| + \left|x_k^{(N)} - x_k\right| \qquad ..... (3)$$

From (1), (2) and (3) we have

$$\left|x_m - x_k\right| < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon$$

This proves $x = \left\{x_m\right\}_{m=1}^{\infty}$ is Cauchy sequence in complete space $\mathbb{K} = \mathbb{C}$ or $\mathbb{R}$, and hence it is convergent. Thus $x \in C$. We have proved that $\overline{C} \subseteq C$.

This gives C is closed subspace of complete space ℓ_∞, and hence C is also complete.

We have proved that C is Banach space.

1.4.16 Theorem : Let $E \subseteq \mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ and $f_n : E \to \mathbb{K}$, $(n = 1, 2, ...)$. Suppose $\lim\limits_{n\to\infty} f_n(x) = f(x)$, $(x \in E)$

Define $M_n = \sup\limits_{x\in E} \left|f_n(x) - f(x)\right|$

Then $f_n \to f$ uniformly on E iff $M_n \to 0$.

1.4.17 Problem : Consider the space C (X) of all bounded continuous scalar valued function defined on topological space X with the norm,

$\|f\| = \sup_{x\in X}|f(x)|$, where $f \in C(X)$.

Prove that C (X) is Banach space.

Solution :

Part I : To prove C (X) is normed space.

Let any $f, g \in C(X)$ and α be any scalar.

(i) Since $|f(x)| \geq 0$, $\forall x \in X$, we have

$\sup_{x\in X}|f(x)| \geq 0$, that is, $\|f\| \geq 0$.

(ii) $f = 0 \Leftrightarrow f(x) = 0$, $\forall x \in X$

$$\Leftrightarrow |f(x)| = 0, \ \forall x \in X$$

$$\Leftrightarrow \sup_{x\in X}|f(x)| = 0$$

$$\Leftrightarrow \|f\| = 0$$

(iii) For each $x \in X$,

$$|(f+g)(x)| \leq |f(x)| + |g(x)|$$

$$\leq \sup_{x\in X}|f(x)| + \sup_{x\in X}|g(x)|$$

$$= \|f\| + \|g\|$$

i.e. $|(f+g)(x)| \leq \|f\| + \|g\|$, $\forall x \in X$

$$\Rightarrow \sup_{x\in X}|(f+g)(x)| \leq \|f\| + \|g\|$$

$$\therefore \|f+g\| \leq \|f\| + \|g\|.$$

(iv) $\|\alpha f\| = \sup_{x\in X}|(\alpha f)(x)| = \sup_{x\in X}|\alpha \cdot f(x)|$

$$= |\alpha| \sup_{x\in X}|f(x)|$$

$$= |\alpha| \|f\|$$

We have proved that, C (X) is normed space.

Part II : To prove C (X) is complete.

Let $\{f_n\}_{n=1}^{\infty}$ be any Cauchy sequence in C (X). Then for given $\varepsilon > 0$, $\exists\, N \in \mathbb{N}$ such that

$$m, n \geq N \Rightarrow \|f_m - f_n\| < \varepsilon$$

$$\Rightarrow \sup_{x \in X} |f_m(x) - f_n(x)| < \varepsilon$$

But, $|f_m(x) - f_n(x)| \leq \sup_{x \in X} |f_m(x) - f_n(x)|$, $\forall x \in X$

Thus, $m, n \geq N \Rightarrow |f_m(x) - f_n(x)| < \varepsilon$, $\forall x \in X$ .... (1)

This implies for each $x \in X$, $\{f_n(x)\}_{n=1}^{\infty}$ is Cauchy sequence in complete space $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, and hence convergent.

Let $\lim_{n \to \infty} f_n(x) = f(x)$, $x \in X$

Letting $m \to \infty$ in (1) we obtain

$$n \geq N \Rightarrow |f_n(x) - f(x)| < \varepsilon, \ \forall x \in X$$

$$\Rightarrow \sup_{x \in X} |f_n(x) - f(x)| < \varepsilon \qquad \text{.... (2)}$$

Define $M_n = \sup_{x \in X} |f_n(x) - f(x)|$. From (2) it follows that $M_n \to 0$ as $n \to \infty$.

Hence $f_n \to f$ uniformly on X.

Since for each n, f_n is bounded continuous scalar valued function defined on X, f is bounded continuous scalar valued function on X, that is, $f \in C(X)$.

Thus $f_n \to f$ in C (X). This proves C (X) is complete space.

By part (I) and (II), C (X) is Banach space.

EXERCISE

1. Denote by C_0 the space of all sequences $x = \{x_n\}_{n=1}^{\infty}$ in $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ converging to zero with the norm $\|x\|_\infty = \sup_{n \in \mathbb{N}} |x_n|$. Prove that C_0 is a Banach space.

2. Prove that the space of all sequences $x = \{x_n\}_{n=1}^{\infty}$ in $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ such that $\sum_{n=1}^{\infty} x_n$ is convergent, is a Banach space with the norm,

$$\|x\|_\infty = \sup_{n \in \mathbb{N}} \left| \sum_{j=1}^{n} x_j \right|$$

3. Prove that the space C^n [a, b] of all n times continously differentiable scalar valued functions x on [a, b] with the norm $\|x\| = \sum_{j=0}^{n} \|x^{(j)}\|_\infty$ is a Banach space.

1.5 INCOMPLETE NORMED LINEAR SPACES

We have seen many examples of complete normed linear spaces (Banach spaces). But every normed linear space need not be complete. Here we provide some examples of normed linear spaces which are not complete.

Note : If there exists a Cauchy sequence in normed space $(X, \|\cdot\|)$ which is not convergent in X then $(X, \|\cdot\|)$ is incomplete normed space.

1.5.1 Problem : Let X = C [–1, 1] be the linear space of all real valued functions defined on closed interval [–1, 1]. Define the norm on X by

$$\|f\| = \int_{-1}^{1} |f(t)| \, dt, \quad f \in X,$$

where integral is taken in the sense of Riemann. Prove that $(X, \|\cdot\|)$ is incomplete normed linear space.

Solution : Let X = C [–1, 1] be the linear space of all real valued functions defined on closed interval [–1, 1].

Let any $f, g \in X$ and α be any scalar. Then sum $f + g$ and scalar multiplication αf is defined by

$$(f+g)(x) = f(x) + g(x)$$

and $(\alpha f)(x) = \alpha f(x)$, for all $x \in [-1,1]$.

Part I : To prove $(X, \|\cdot\|)$ is normed space.

(i) Since $|f(x)| \geq 0$, $\forall x \in X$

$$\|f\| = \int_{-1}^{1} |f(x)| dx \geq 0$$

(ii) $f = 0 \Leftrightarrow f(x) = 0$, $\forall x \in [-1,1]$

$$\Leftrightarrow |f(x)| = 0, \ \forall x \in [-1,1]$$

$$\Leftrightarrow \int_{-1}^{1} |f(x)| dx = 0$$

$$\Leftrightarrow \|f\| = 0 .$$

(iii) $\|f+g\| = \int_{-1}^{1} |(f+g)(x)| dx$

$$\leq \int_{-1}^{1} (|f(x)| + |g(x)|) dx$$

$$= \int_{-1}^{1} |f(x)| dx + \int_{-1}^{1} |g(x)| dx$$

$$= \|f\| + \|g\| .$$

(iv) $\|\alpha f\| = \int_{-1}^{1} |(\alpha f)(x)|\,dx = \int_{-1}^{1} |\alpha f(x)|\,dx$

$$= |\alpha| \int_{-1}^{1} |f(x)|\,dx$$

$$= |\alpha|\|f\|.$$

Thus, $(X, \|\cdot\|)$ is normed linear space.

Part II : We show that $(X, \|\cdot\|)$ is incomplete normed space.

Consider the sequence $\{f_n\}_{n=1}^{\infty}$ in X = C [–1, 1] defined by,

$$f_n(x) = \begin{cases} 1 & \text{if} \quad -1 \le x \le 0 \\ 1 - nx & \text{if} \quad 0 < x < \dfrac{1}{n} \\ 0 & \text{if} \quad \dfrac{1}{n} < x \le 1 \end{cases}$$

Let $m > n \ge 1$ then $\frac{1}{m} < \frac{1}{n}$. Thus we have,

$$\|f_m - f_n\| = \int_{-1}^{1} |f_m(x) - f_n(x)|\,dx$$

$$= \int_{-1}^{0} (0)\,dx + \int_{0}^{1} |f_m(x) - f_n(x)|\,dx$$

$$\le \int_{0}^{1} |f_m(x)|\,dx + \int_{0}^{1} |f_n(x)|\,dx \qquad \text{..... (1)}$$

Consider,

$$\int_{0}^{1} |f_n(x)|\,dx = \int_{0}^{1/n} |1 - nx|\,dx + \int_{1/n}^{1} (0)\,dx$$

$$= \int_{0}^{1/n} (1 - nx)\,dx$$

$$= \left[x - n\frac{x^2}{2} \right]_0^{1/n}$$

$$= \frac{1}{n} - \frac{1}{2n} = \frac{1}{2n}$$

Similarly,

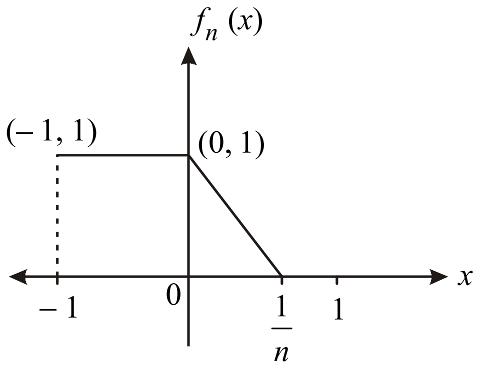
$$\int_0^1 |f_m(x)|\, dx = \frac{1}{2m}$$

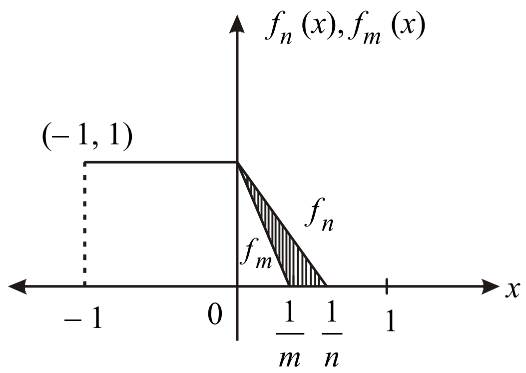
Thus from inequality (1) we have,

$$\|f_m - f_n\| \le \frac{1}{2}\left(\frac{1}{m} + \frac{1}{n}\right) \to 0 \text{ as } m, n \to \infty .$$

This proves $\{f_n\}_{n=1}^{\infty}$ is Cauchy sequence in X = C [–1, 1].

The continuity of each f_n and $\{f_n\}_{n=1}^{\infty}$ is Cauchy sequence also follows from following figures.





Let $f_n \to f$ as $n \to \infty$. Then $\|f_n \to f\| \to 0$ as $n \to \infty$.

But

$$\|f_n - f\| = \int_{-1}^{1} |f_n(x) - f(x)|\, dx$$

$$= \int_{-1}^{0} |1 - f(x)|\,dx + \int_{0}^{1/n} |f_n(x) - f(x)|\,dx + \int_{1/n}^{1} |f(x)|\,dx \qquad \text{..... (2)}$$

On the right hand side we observe that all the integrands are non-negative and hence each integral is non-negative.

Hence from (2), $\|f_n - f\| \to 0$ as $n \to \infty$ imply that

$$\int_{-1}^{0} |1 - f(x)|\,dx = 0, \quad \int_{0}^{1/n} |f_n(x) - f(x)|\,dx \to 0 \text{ and } \int_{1/n}^{1} |f(x)|\,dx \to 0 \text{ as } n \to \infty.$$

This implies,

$$f(x) = \begin{cases} 1 & \text{if } -1 \le x \le 0 \\ 0 & \text{if } \quad 0 < x \le 1 \end{cases}$$

But we see that

$$1 = f(0) \ne \lim_{x \to 0^+} f(x) = 0$$

Therefore f is not continuous on [–1, 1], and hence $f \notin C[-1,1]$.

Thus the Cauchy sequence $\{f_n\}_{n=1}^{\infty}$ in C [–1, 1] defined above is not convergent in C [–1, 1].

We have proved that X = C [–1, 1] is incomplete w.r.t. the norm $\|.\|$ defined above.

1.5.2 Problem : Consider the real linear space X = C^1 [0, 1] of all continuously differentiable functions defined on [0, 1] with the norm,

$$\|f\| = \sup_{x \in [0,1]} |f(x)|.$$

Prove that $(X, \|\cdot\|)$ is incomplete normed space.

Solution : Part I : $(X, \|\cdot\|)$ is normed space.

We omit the proof as it is similar to the part I of problem 1.4.17.

Part II : $(X,\|\cdot\|)$ is incomplete normed space.

Consider the sequence $\{f_n\}_{n=1}^{\infty}$ in X = C^1 [0, 1] defined by $f_n(x)=\sqrt{x^2+\frac{1}{n}}$; $x\in[0,1]$.

Then for any $m>n\geq 1$,

$$\|f_m-f_n\|=\sup_{x\in[0,1]}\left|\sqrt{x^2+\frac{1}{m}}-\sqrt{x^2+\frac{1}{n}}\right| \leq\sqrt{\frac{1}{m}}+\sqrt{\frac{1}{n}}\to 0 \text{ as } m,n\to\infty .$$

This proves $\{f_n\}_{n=1}^{\infty}$ is Cauchy sequence in X = C^1 [0, 1].

Note that for each $x\in[0,1]$,

$$\lim_{n\to\infty} f_n(x)=\lim_{n\to\infty}\sqrt{x^2+\frac{1}{n}}=|x| .$$

Thus $f:[0,1]\to\mathbb{R}$, defined by $f(x)=|x|$, $x\in[0,1]$ is pointwise limit of $\{f_n\}_{n=1}^{\infty}$.

Further,

$$\|f_n-f\|=\sup_{x\in[0,1]}|f_n(x)-f(x)|$$

$$=\sup_{x\in[0,1]}\left|\sqrt{x^2+\frac{1}{n}}-|x|\right|$$

$$=\frac{1}{\sqrt{n}}\to 0 \text{ as } n\to\infty .$$

Hence $f_n\to f$ uniformly on [0, 1].

But $f\notin C^1[0,1]$, because f is not differentiable at $x=0$.

Hence the Cauchy sequence $\{f_n\}_{n=1}^{\infty}$ defined above is not convergent in normed space $(X,\|\cdot\|)$.

Hence $X=C^1[0,1]$ is incomplete.

1.5.3 Example : Let $X = (0,1)$ and define $\|x\| = |x|$, $x \in (0,1)$.

Then $(X, \|\cdot\|)$ is incomplete normed space.

Solution : Let $x_n = \frac{1}{n}$. Then $\{x_n\}$ is Cauchy sequence in normed space $(X, \|\cdot\|)$ but $\{x_n\}$ does not converges in X.

1.5.4 Example : Let $X = C[0,2]$ be the space of all real valued functions and define

$$\|f\| = \int_0^2 |f(x)|\,dx \,, \quad f \in X$$

Then $(X, \|\cdot\|)$ is incomplete normed linear space.

Solution : Part-I : The proof of $(X, \|\cdot\|)$ is normed linear space is similar to the solution in part I of problem 1.5.1.

Part - II : To prove $(X, \|\cdot\|)$ is incomplete normed space we must have to prove a Cauchy sequence in X which is not convergent.

Define $f_n : [0,2] \to \mathbb{R}$, $(n = 1, 2, ...)$ by

$$f_n(x) = \begin{cases} x^n & ; \ 0 \le x < 1 \\ 1 & ; \ 1 \le x \le 2 \end{cases}$$

Then clearly $f_n \in X$.

Further for any $m, n \ge 1$,

$$\|f_m - f_n\| = \int_0^2 |f_m(x) - f_n(x)|\,dx$$

$$\le \int_0^2 |f_m(x)|\,dx + \int_0^2 |f_n(x)|\,dx$$

$$= \int_0^1 x^m\,dx + \int_0^1 x^n\,dx$$

$$= \left[\frac{x^{m+1}}{m+1}\right]_0^1 + \left[\frac{x^{n+1}}{n+1}\right]_0^1$$

$$= \frac{1}{m+1} + \frac{1}{n+1} \to 0 \text{ as } m,n \to \infty$$

That is $\|f_m - f_n\| \to 0$ as $m,n \to \infty$.

This implies $\{f_n\}_{n=1}^{\infty}$ is Cauchy sequence in X.

Suppose there exists a function $f:[0,2] \to \mathbb{R}$ such that $\|f_n - f\| \to 0$ as $n \to \infty$.

But

$$\|f_n - f\| = \int_0^2 |f_n(x) - f(x)|\,dx$$

$$= \int_0^1 |f_n(x) - f(x)|\,dx + \int_1^2 |f_n(x) - f(x)|\,dx$$

$$= \int_0^1 |x^n - f(x)|\,dx + \int_1^2 |1 - f(x)|\,dx$$

On the right hand side all the integrands are non-negative and hence all integrands are non-negative.

Hence $\|f_n - f\| \to 0$ must imply,

$$\int_0^1 |x^n - f(x)|\,dx \to 0 \text{ as } n \to \infty$$

and $$\int_1^2 |1 - f(x)|\,dx = 0$$

This is possible only if $f:[0,2] \to \mathbb{R}$ must be of the form,

$$f(x) = \begin{cases} 0 & \text{if } 0 \le x < 1 \\ 1 & \text{if } 1 \le x \le 2 \end{cases}$$

But we see that

$$1 = f(1) \neq \lim_{x \to 1^-} f(x) = 0$$

Hence $f \notin X = C[0,2]$.

Thus the Cauchy sequence $\{f_n\}_{n=1}^{\infty}$ is not convergent in $X = C[0,2]$.

Hence $(X, \|\cdot\|)$ is incomplete normed linear space.

1.6 SUBSPACES OF A NORMED SPACES AND BANACH SPACES

1.6.1 Definition : Let N be a normed linear space. A non-empty subset M of N is said to be a subpace of N if M is a linear subspace of N considered as a linear space and the norm $\|\cdot\|_M$ of M is obtained by restricting the norm $\|.\|$ on N. i.e. $\|y\|_M = \|y\|$, $y \in M$.

The norm $\|.\|_M$ on M is said to be induced by the norm $\|.\|$ on N.

1.6.2 Definition : A subspace M of a Banach space B is a subspace B considered as a normed space.

1.6.3 Remark : A subspace of a Banach space need not be complete.

1.6.4 Definition : A subspace M of a normed space N is called a closed subspace of N, if M is closed in N considered as a metric space.

1.6.5 Theorem : Let M be a subspace of normed space N and $\{x_n\}$ be a sequence in M. If $\{x_n\}$ is Cauchy in N then it is Cauchy in M and conversely.

Proof : Let $(M, \|\cdot\|_M)$ be a subspace of normed space $(N, \|\cdot\|)$. Then $\|x\|_M = \|x\|$, $x \in M$.

Let $\{x_n\}$ be a sequence in M.

Let $\{x_n\}$ be a Cauchy sequence in M.

Then for given $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that,

$$\|x_m - x_n\|_M < \varepsilon \,,\ \forall m, n \ge n_0$$

$$\Rightarrow \|x_m - x_n\| < \varepsilon \,,\ \forall m, n \ge n_0$$

This implies $\{x_n\}$ is Cauchy sequence in N. The proof of converse part follows by replacing role of $\|.\|_M$ and $\|.\|$.

1.6.6 Theorem : If M is a complete subspace of normed linear space N then M is closed.

Proof : Let M is a complete subspace of normed space N.

Let x be a limit point of M.

Then $S_r(x) \cap M - \{x\} \neq \phi$, $\forall r > 0$.

Inparticular for each n (n = 1, 2, 3, ...),

$S_{1/n}(x) \cap M - \{x\} \neq \phi$

Thus for each n, $\exists\, x_n \in N$ such that $x_n \in S_{1/n}(x) \cap M - \{x\}$.

Hence $\{x_n\}$ is a sequence in M such that $x_n \neq x$ and $\|x_n - x\| < \dfrac{1}{n}$, $\forall n$.

$\Rightarrow \lim\limits_{n \to \infty} x_n = x$ in N.

$\Rightarrow \{x_n\}$ is Cauchy in N and hence in M.

But as M is complete, we must have $x \in M$.

This proved M is closed.

1.6.7 Theorem : If M is closed subspace of Banach space B then M is complete.

Proof : Let M be a closed linear subspace of a Banach space B.

Let $\{x_n\}$ be any Cauchy sequence in M.

Hence $\{x_n\}$ is a Cauchy sequence in B.

As B is complete $\exists\, x \in B$ such that $x_n \to x$.

If $x \in M$ then there is nothing to prove otherwise $x_n \to x$ implies each open sphere of *x* contains the point x_n other than *x*. Thus *x* is imit point of M. But M being closed, it follows that $x \in M$.

1.6.8 Corollary : Let M be a subspace of a Banach space B. Then, M is complete iff M is closed.

1.6.9 Problem : Let N be a non-zero normed linear space. Prove that N is a Banach space iff $\{x \in N : \|x\| = 1\}$ is complete.

Proof : Let N be a non-zero normed linear space.

Assume N is a Banach space. To prove $X = \{x \in N : \|x\| = 1\}$ is complete, let $\{x_n\}$ be a Cauchy sequence in X. Then $\|x_n\| = 1$ for all *n*. As $X \subseteq N$, $\{x_n\}$ is Cauchy sequence in complete normed space N, and hence $\exists\, y \in N$ such that $x_n \to y$.

Since norm is continuous function we have $\|x_n\| \to \|y\|$.

Therefore,

$$\|y\| = \lim_{n\to\infty} \|x_n\| = \lim_{n\to\infty} (1) = 1$$

This implies $y \in X$. Hence X is complete.

Conversely let X is complete. To prove that normed space N is complete, let $\{y_n\}$ be a Cauchy sequence in N.

Then $\|y_m - y_n\| \to 0$ as $m, n \to \infty$ .... (1)

For each *n* (*n* = 1, 2, 3, ...) define $x_n = \dfrac{y_n}{\|y_n\|}$.

Then $\|x_n\| = \left\| \frac{y_n}{\|y_n\|} \right\| = \frac{\|y_n\|}{\|y_n\|} = 1$ for all n.

This gives $x_n \in X$ for all n.

We prove that $\{x_n\}$ is Cauchy sequence in X.

For any $m > n$ we have,

$$\|x_m - x_n\| = \left\| \frac{y_m}{\|y_m\|} - \frac{y_n}{\|y_n\|} \right\|$$

$$= \left\| \left(\frac{y_m}{\|y_m\|} - \frac{y_n}{\|y_m\|} \right) + \left(\frac{y_n}{\|y_m\|} - \frac{y_n}{\|y_n\|} \right) \right\|$$

$$\leq \left\| \frac{y_m}{\|y_m\|} - \frac{y_n}{\|y_m\|} \right\| + \left\| \left(\frac{1}{\|y_m\|} - \frac{1}{\|y_n\|} \right) y_n \right\|$$

$$= \frac{\|y_m - y_n\|}{\|y_m\|} + \frac{\left\| \|y_n\| - \|y_m\| \right\|}{\|y_m\|\|y_n\|} \|y_n\|$$

$$\leq \frac{\|y_m - y_n\|}{\|y_m\|} + \frac{\|y_n - y_m\|}{\|y_m\|} \qquad [\because \left| \|x\| - \|y\| \right| \leq \|x - y\|, \text{ for all } x, y \in N\,]$$

Therefore

$$\|x_m - x_n\| \leq \frac{2\|y_m - y_n\|}{\|y_m\|} \to 0 \text{ as } m, n \to \infty \qquad [\because \text{ By}\,(1)]$$

This proves $\{x_n\}$ is Cauchy sequence in complete space X, and hence $\exists\; x \in X$ such that

$$x_n \to x \quad \text{i.e.} \quad \frac{y_n}{\|y_n\|} \to x \qquad \text{.... (2)}$$

Also note that

$$\left| \|y_m\| - \|y_n\| \right| \leq \|y_m - y_n\| \to 0 \text{ as } m, n \to \infty$$

This implies $\{\|y_n\|\}$ is Cauchy sequence in complete space $\mathbb{R}$. Thus $\exists\ \alpha \in \mathbb{R}$ such that

$$\|y_n\| \to \alpha \text{ as } n \to \infty \qquad \text{..... (3)}$$

Using (2) and (3) we have,

$$y_n \to \alpha x \text{ as } n \to \infty$$

Since $x \in X \subseteq N$ and α is a scalar. We have $\alpha x \in N$.

We proved that $\lim\limits_{n\to\infty} y_n = \alpha x \in N$.

Hence N is complete normed space, and hence is a Banach space.

1.6.10 Problem : Let a Banach space B be the direct sum of the linear subspaces M and N, so that $B = M \oplus N$. If $z = x + y$ is the unique expression of a vector z in B as the sum of vectors *x* and *y* in M and N, then a new norm can be defined on the linear space B by $\|z\|' = \|x\| + \|y\|$. Prove that this actually a norm. If B' symbolizes the linear space B equipped with this new norm, prove that B' is a Banach space if M and N are closed in B.

Solution : Let B be a Banach space with the norm $\|.\|$.

Let any $z \in B = M \oplus N$ where M and N are the linear subspaces of B. Then $z = x + y$ is the unique expression, where $x \in M$ and $y \in N$.

Define $\|.\|'$ on $B = M \oplus N$ by

$$\|z\|' = \|x\| + \|y\|$$

We have to prove that :

(I) $B' = (B, \|\cdot\|')$ is normed linear space.

(II) $B' = (B, \|\cdot\|')$ is Banach space if M and N are closed.

Part (I) : Let any $z, w \in B = M \oplus N$ and α be any scalar. Then $z = x + y$ and $w = u + v$ are unique representation where $x, u \in M$ and $y, v \in N$.

(i) Since $\|x\| \geq 0$ and since $\|y\| \geq 0$ we have $\|z\|' = \|x\| + \|y\| \geq 0$.

(ii) $\quad \|z\|' = 0 \Leftrightarrow \|x\| + \|y\| = 0$

$\Leftrightarrow \|x\| = \|y\| = 0$

$\Leftrightarrow x = y = 0$

$\Leftrightarrow z = x + y = 0$

(iii) $\quad \|z + w\|' = \|(x+y) + (u+v)\|'$

$= \|(x+u) + (y+v)\|'$

$= \|x+u\| + \|y+v\|$

$\leq \|x\| + \|u\| + \|y\| + \|v\|$

$= (\|x\| + \|y\|) + (\|u\| + \|v\|)$

$= \|z\|' + \|w\|'$

Thus $\|z + w\|' \leq \|z\|' + \|w\|'$.

(iv) $\quad \|\alpha z\|' = \|\alpha(x+y)\|'$

$= \|\alpha x + \alpha y\|'$

$= \|\alpha x\| + \|\alpha y\|$

$= |\alpha|\|x\| + |\alpha|\|y\|$

$= |\alpha|(\|x\| + \|y\|)$

$= |\alpha|\|z\|'$

We have proved that B' is normed space.

Part -II : To prove B' is complete, let $\{z_n\}$ be any sequence in B'. Then for each $n \in \mathbb{N}$, $z_n = x_n + y_n$ is unique expression, where $x_n \in M$ and $y_n \in N$.

For any $\varepsilon > 0$, $\exists\, n_0 \in \mathbb{N}$ such that

$m, n \geq n_0 \Rightarrow \|z_m - z_n\|' < \varepsilon$

$$\Rightarrow \left\|\left(x_m + y_m\right) - \left(x_n + y_n\right)\right\|' < \varepsilon$$

$$\Rightarrow \left\|\left(x_m - x_n\right) + \left(y_m - y_n\right)\right\|' < \varepsilon$$

$$\Rightarrow \left\|x_m - x_n\right\| + \left\|y_m - y_n\right\| < \varepsilon$$

$$\Rightarrow \left\|x_m - x_n\right\| < \varepsilon \text{ and } \left\|y_m - y_n\right\| < \varepsilon$$

Thus $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences in M and N respectively. But M and N are closed linear subspaces of complete space $(B, \|\cdot\|)$ and hence M and N are complete spaces. Therefore there exists $x \in M$ and $y \in N$ such that $x_n \to x$ and $y_n \to y$.

Define $z = x + y$. Then $z \in B = M \oplus N$ and

$$\|z_n - z\|' = \left\|\left(x_n + y_n\right) - \left(x + y\right)\right\|' = \left\|\left(x_n - x\right) + \left(y_n - y\right)\right\|'$$

$$= \|x_n - x\| + \|y_n - y\| \to 0 \text{ as } n \to \infty.$$

This proves $z_n \to z$ in B'.

We have proved that $(B', \|\cdot\|')$ is complete normed space, and hence Banach space.

1.7 QUOTIENT SPACE

1.7.1 Definition : A partition of a non-empty set X is a disjoint family of non-empty subsets of X whose union is X.

1.7.2 Theorem : Let M be a subspace of a linear space L, and let the coset of M in L generated by $x \in L$ be defined by, $x + M = \{x + m : m \in M\}$

Then the distinct cosets form a partition of L. Let L/M denote the set of all cosets of M in L *i.e.* $L/M = \{x + M : x \in L\}$.

Define addition and scalar multiplication in L/M by

$$(x + M) + (y + M) = (x + y) + M$$

and $$\alpha(x + M) = \alpha x + M,$$

Then L/M is linear space over the same field L. This space is called the quotient space (or factor space) of M in L (or quotient space of L with respect to M).

Note :

(i) If $m \in M$ then m + M = M

(ii) If $x - y \in M$ then $x + M = y + M$. Thus coset of M in L have more than one representation.

(iii) If 0 is zero vector in L, then 0 + M = M is a zero in L/M.

(iv) The negative of x + M is $(-x)$ + M.

1.7.3 Theorem : If a Cauchy sequecne $\{x_n\}$ in a metric space X has convergent subsequence having limit x then the sequence $\{x_n\}$ is convergent with same limit x.

1.7.4 Theorem : Let M be a closed linear subspace of a normed linear space N. If the norm of coset x + M in the quotient space N/M is define by

$$\|x + M\| = \inf\{\|x + m\| : m \in M\},$$

then N/M is a normed linear space. Further, if N is a Banach space, then N/M is a Banach space.

Proof : Let M be a closed linear subspace of a normed linear space N.

Part - I : Firstly we prove that $\|\cdot\| : N / M \to [0, \infty)$ defined by,

$\|x + M\| = \inf\{\|x + M\| : m \in M\}$ defines a norm on N/M.

Let any x + M, y + M in N/M and α be any scalar.

(i) Since $\|x + m\| \geq 0$ for all $m \in M$, we have $\|x + M\| \geq 0$.

(ii) Let $x + M = M$ (a zero vector in N/M).

Then $x \in M$ and we have,

$$\|x + M\| = \inf\{\|x + m\| : m \in M\}$$

$$= \inf\{\|y\| : y \in M\} \quad [\because x, m \in M \Rightarrow y = x + m \in M]$$

$$= 0 \quad [\because \text{ zero vector, } 0 \in M \text{ and } \|0\| = 0]$$

Conversely, let $\|x+M\|=0$.

Then, $\inf\{\|x+M\|: m\in M\}=0$

$\Rightarrow$ there exists a sequence $\{m_n\}_{n=1}^{\infty}$ in M such that $\|x+m_n\|\to 0$ as $n\to\infty$.

$\Rightarrow -m_n \to x$ as $n\to\infty$

Since $\{-m_n\}_{n=1}^{\infty}$ is sequence in M and M is closed, $-m_n\to x$ implies $x\in M$.

Therefore $x+M=M$ (zero vector in N/M)

(iii) $\|(x+M)+(y+M)\| = \|(x+y)+M\|$

$$= \inf\{\|(x+y)+m\| : m\in M\}$$

$$= \inf\{\|(x+m_1)+(y+m_2)\| : m_1,m_2\in M\}$$

$$\le \inf\{\|x+m_1\|+\|y+m_2\| : m_1,m_2\in M\}$$

[$\because$ Triangle inequality of norm in N]

$$\le \inf\{\|x+m_1\| : m_1\in M\}+\inf\{\|x+m_2\| : m_2\in M\}$$

$$= \|x+M\|+\|y+M\|$$

Therefore, $\|(x+M)+(y+M)\|\le\|x+M\|+\|y+M\|$

(iv) For $\alpha\neq 0$, we have

$\|\alpha(x+M)\| = \|\alpha x+M\|$

$$= \inf\{\|\alpha x+m\| : m\in M\}$$

$$= \inf\{\|\alpha x+\alpha m'\| : m'\in M\} \qquad [\because m' = \frac{m}{\alpha}\in M\ ]$$

$$= \inf\{|\alpha|\|x+m'\| : m'\in M\}$$

$$= |\alpha|\inf\{\|x+m'\| : m'\in M\}$$

$$= |\alpha|\|x+M\|$$

For $\alpha = 0$,

$\|0(x+M)\| = \|0x+M\| = \|M\| = 0 = |0|\|x+M\|$

Therefore,

$$\|\alpha(x+M)\| = |\alpha|\|x+M\| \text{ for any scalar } \alpha.$$

Thus N/M is a normed linear space.

Part - II : Let N is complete (Banach) space. We prove thet N/M is complete (Banach) space.

Let $\{x_n + M\}$ be any Cauchy sequence in N/M.

Then it is possible to select a subsequence $\{x_{n_k} + M\}$ of $\{x_n + M\}$ such that

$$\left\|\left(x_{n_2} + M\right) - \left(x_{n_1} + M\right)\right\| < \frac{1}{2}$$

$$\left\|\left(x_{n_3} + M\right) - \left(x_{n_2} + M\right)\right\| < \frac{1}{2^2}$$

$$\vdots$$

$$\left\|\left(x_{n_{k+1}} + M\right) - \left(x_{n_k} + M\right)\right\| < \frac{1}{2^k}$$

$$\vdots$$

Now choose any vector $y_1 \in x_{n_1} + M$ and select $y_2 \in x_{n_2} + M$ such that $\|y_2 - y_1\| < \frac{1}{2}$.

We next select $y_3 \in x_{n_3} + M$ such that $\|y_3 - y_2\| < \frac{1}{2^2}$. Continuing in this way we obtain a sequence $\{y_k\}_{k=1}^{\infty}$ such that

$$\|y_{k+1} - y_k\| < \frac{1}{2^k}, \ (k = 1, 2, 3, ...)$$

and $\quad x_{n_k} + M = y_k + M \qquad [\because y_k \in x_{n_k} + M]$

Let any $\varepsilon > 0$. Choose $n_0 \in \mathbb{N}$ such that $\frac{1}{2^{n_0-1}} < \varepsilon$.

Then for any $k > r \geq n_0$ we have,

$$\|y_k - y_r\| = \|(y_k - y_{k-1}) + (y_{k-1} - y_{k-2}) + \ldots + (y_{r+1} - y_r)\|$$

$$\leq \|y_k - y_{k-1}\| + \|y_{k-1} - y_{k-2}\| + \ldots + \|y_{r+1} - y_r\|$$

$$< \frac{1}{2^{k-1}} + \frac{1}{2^{k-2}} + \ldots. + \frac{1}{2^r}$$

$$= \sum_{j=r}^{k-1} \left(\frac{1}{2}\right)^j = \frac{\left(\frac{1}{2}\right)^r}{1 - \frac{1}{2}} = \frac{1}{2^{r-1}} \leq \frac{1}{2^{n_0-1}} < \varepsilon$$

That is for given $\varepsilon > 0$, $\exists\, n_0 \in \mathbb{N}$ such that $\|y_k - y_r\| < \varepsilon$, $\forall k > r \geq n_0$.

This proves $\{y_k\}_{k=1}^{\infty}$ is Cauchy sequence in complete normed space N. Therefore there exists $y \in N$ such that $y_k \to y$ as $k \to \infty$.

Now,

$$\|(x_{n_k} + M) - (y + M)\| = \|(y_k + M) - (y + M)\|$$

$$= \|(y_k - y) + M\|$$

$$= \inf\left\{\|(y_k - y) + m\| : m \in M\right\}$$

$$\leq \|(y_k - y) + m\|, \ \forall m \in M$$

Inparticular for $m = 0 \in M$, we have

$$\|(x_{n_k} + M) - (y + M)\| \leq \|y_k - y\| \to 0 \text{ as } k \to \infty.$$

We have proved that Cauchy sequence $\{x_n + M\}$ has convergent subsequence $\{x_{n_k} + M\}$ with $\lim_{k\to\infty} x_{n_k} + M = y + M \in \frac{N}{M}$.

We know that if a subsequence of Cauchy sequence converges, the sequence itself converges. Hence the Cauchy sequence $\{x_n + M\}$ converges in N/M, so N/M is complete normed space.

❑❑❑

UNIT - II

BOUNDED LINEAR TRANSFORMATIONS

In this unit we study bounded linear transformation and their properties. Well known theorem which are considered as piller of functional analysis, namely, open mapping theorem, closed graph theorem, uniform bounded principle are proved in this unit.

2.1 LINEAR TRANSFORMATIONS

2.1.1 Definition : Let L and V be linear spaces over the same field $\mathbb{K}$. A function $T : L \to V$ is said to be linear transformation if $T(\alpha x + \beta y) = \alpha T(x) + \beta T(y)$ for all $x, y \in L$ and $\alpha, \beta \in \mathbb{K}$.

2.1.2 Definition : Let $T : L \to V$ be a linear transformation. Then

(a) **Kernel of T** is defined as

$$\ker(T) = \{x \in L : T(x) = 0\},$$

which is also called **null space** of T and some time it is denoted as $\mathcal{N}$ (T).

(b) **Range of T** is defined as

$$\begin{aligned}\mathcal{R}(T) &= \{T(x) : x \in L\} \\ &= \{y \in V : y = T(x) \text{ for some } x \in L\}\end{aligned}$$

2.1.3 Theorem : Let $T : L \to V$ be a linear transformation. Then,

(a) ker (T) is linear subspace of L.

(b) $\mathcal{R}$ (T) is linear subspace of V.

(c) T (0) = 0

(d) $T(-x) = -T(x),\ x \in L$.

(e) T is bijective $\Rightarrow$ T is invertible $\Rightarrow \ker(T)=\{0\}$.

(f) T^{-1} if exists, is a linear transformation.

Remark : For linear transformation $T: L \to V$ we always assume L and V are the linear spaces over same field of scalar $\mathbb{K}=\mathbb{R}$ or $\mathbb{C}$.

2.1.4 Some Important linear Transformations :

(a) Identity Transformation : Let L be a linear space. The function $I: L \to L$ defined by $I(x)=x$, $x \in L$, is a linear transformation, called identity transformation on L.

(b) Zero Transformation : Let L and V be the linear spaces over the same scalar field. The function $O: L \to V$ defined by $O(x)=0$, $x \in L$, is a linear transformation, called zero transformation.

2.2 BOUNDED LINEAR TRANSFORMATION IN A NORMED SPACE

2.2.1 Definition : Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be normed spaces and $T: X \to Y$ a linear transformation. Then T is said to be bounded linear transformation if there is a real number $K \geq 0$ such that,

$$\|Tx\|_Y \leq K\|x\|_X,\ \forall x \in X.$$

If T is not bounded, then it is said to unbounded linear transformation.

2.2.2 Remark : Bounded linear transformations are not same as those of ordinary real (or complex) bounded functions. Bounded function is one whose range is a bounded set.

e.g. consider the identity operator $I: \mathbb{R} \to \mathbb{R}$, $I(x)=x$, $x \in \mathbb{R}$. Then,

(i) $I(\alpha x+\beta y)=\alpha x+\beta y=\alpha I(x)+\beta I(y)$

for all $x, y \in \mathbb{R}$ and $\alpha, \beta \in \mathbb{R}$.

(ii) $|I(x)|=|x|,\ \forall x \in \mathbb{R}$.

If $K \geq 1$ then we have

$|I(x)| \leq K|x|,\ \forall x \in \mathbb{R}$.

Thus identity operator is bounded linear transformation. But there does not exists constant $M \geq 0$ such that $|I(x)| \leq M$, $\forall x \in \mathbb{R}$. Therefore I is not a bounded function.

2.2.3 Examples of bounded linear tranformations :

Example 1 : Identity transformation and zero transformation are bounded linear transformations.

Example 2 : Consider the normed space C [0, 1] of all real (or complex) valued functions with supremum norm.

$$\|x\| = \sup_{t\in[0,1]} |x(t)|$$

Define $T : C[0,1] \to \mathbb{R}$ by

$$T(x) = x(0), \ x \in C[0,1].$$

(i) Let any $x, y \in C[0,1]$ and $\alpha, \beta \in \mathbb{R}$. Then

$$T(\alpha x + \beta y) = (\alpha x + \beta y)(0) = \alpha x(0) + \beta y(0) = \alpha T(x) + \beta T(y)$$

This implies T is a linear transformation.

(ii) For any $x \in C[0,1]$, we have,

$$|T(x)| = |x(0)| \leq \sup_{t\in[0,1]} |x(t)| = \|x\|$$

Therefore,

$$|T(x)| = K\|x\|, \ \forall x, y \in C[0,1].$$

where, K =1.

We have proved that T is bounded linear transformation.

Example 3 : Consider the Banach space B = C [0,1] with the supremum norm.

$$\|x\| = \sup_{t\in[0,1]} |x(t)|, \ x \in B.$$

Let $K:[0,1]\times[0,1]\to\mathbb{R}$ is continuous.

Define $T:B\to B$ by

$$T(x)(t)=\int_0^1 K(t,s)x(s)\,ds\,,\ x\in B\,.$$

(i) Let any $x,y\in B$ and $\alpha\in\mathbb{R}$. Then

$$T(x+y)(t)=\int_0^1 K(t,s)(x+y)(s)\,ds$$

$$=\int_0^1 K(t,s)\big(x(s)+y(s)\big)\,ds$$

$$=\int_0^1 K(t,s)x(s)\,ds+\int_0^1 K(t,s)y(s)\,ds$$

$$=T(x)(t)+T(y)(t)$$

and

$$T(\alpha x)(t)=\int_0^1 K(t,s)(\alpha x)(s)\,ds$$

$$=\alpha\int_0^1 K(t,s)x(s)\,ds$$

$$=\alpha T(x)(t)$$

Thus T is linear transformation.

(ii) Since $K:[0,1]\times[0,1]\to\mathbb{R}$ is continuous on compact set $[0,1]\times[0,1]$, there is constant $\mathrm{M}>0$ such that

$$|K(t,s)|\le M\,,\ \forall(s,t)\in[0,1]\times[0,1]\,.$$

Now for each $t\in[0,1]$ we have,

$$|x(t)|\le\sup_{t\in[0,1]}|x(t)|=\|x\|,\ x\in B\,.$$

Therefore for any $x \in B$, and $t \in [0,1]$ we have,

$$|T(x)(t)| = \left| \int_0^1 K(t,s)x(s)\,ds \right|$$

$$\leq \int_0^1 |K(t,s)||x(s)|\,ds$$

$$\leq M\|x\| \int_0^1 ds$$

$$= M\|x\|$$

Thus $|T(x)(t)| \leq M\|x\|$, $x \in B$, $t \in [0,1]$.

This gives

$$\|Tx\| = \sup_{t\in[0,1]} |Tx(t)| \leq M\|x\|, \ x \in B.$$

By part (i) and (ii), T is bounded linear transformation.

2.2.4 Examples of Unbounded Linear Transformation

1. Let $X = P[0,1]$ - the set of all polynomials with real coefficients defined on [0, 1].

   Then X is normed linear space with the norm,

$$\|x\| = \sup_{t\in[0,1]} |x(t)|, \ x \in X.$$

   Define $T : X \to X$ by

$$T(x)(t) = x'(t), \ t \in [0,1],$$

   where $x'(t) = \dfrac{dx}{dt}$.

(i) Let any $x, y \in X$ and $\alpha, \beta \in \mathbb{R}$,

   Then for any $t \in [0,1]$.

$$T(\alpha x+\beta y)(t)=(\alpha x+\beta y)'(t)$$

$$=\alpha x'(t)+\beta y'(t)$$

$$=\alpha Tx(t)+\beta Ty(t)$$

$$\Rightarrow T(\alpha x+\beta y)=\alpha T(x)+\beta T(y)$$

Thus T is a linear transformation.

(ii) For each n ($n = 1, 2, 3, ...$) define $x_n(t)=t^n$, $t\in[0,1]$.

Then $x_n \in X$ for all n.

Also $T(x_n)(t)=nt^{n-1}$

and $\|x_n\|=\sup\limits_{t\in[0,1]}|x_n(t)|=\sup\limits_{t\in[0,1]}|t^n|=1$, for all n.

Thus

$$\|Tx_n\|=\sup_{t\in[0,1]}|T(x_n)(t)|$$

$$=\sup_{t\in[0,1]}|nt^{n-1}|$$

$$=n\sup_{t\in[0,1]}|t^{n-1}|$$

$$=n(1)=n=n\|x_n\|$$

Thus $\|Tx_n\|=n\|x_n\|$, $\forall n\in\mathbb{N}$.

$$\Rightarrow \frac{\|Tx_n\|}{\|x_n\|}=n,\ \forall n\in\mathbb{N}.$$

$\Rightarrow$ there is no fixed number K > 0 such that,

$$\frac{\|Tx_n\|}{\|x_n\|}\le K,\ \forall n\in\mathbb{N}.$$

Thus we cannot find, K > 0 such that

$$\|Tx_n\| \leq K\|x_n\|$$

Hence T is unbounded linear transformation.

2.2.5 Definition :

Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be normed linear spaces over same field of scalar $\mathbb{K} = \mathbb{C}$ or $\mathbb{R}$. A transformation $T: X \to Y$ (linear or not) is said to be continuous at a point $x_0 \in X$ if for given $\varepsilon > 0$, $\exists \delta > 0$ such that

If $x \in X$, $\|x - x_0\|_X < \delta$ then $\|Tx - Tx_0\|_X < \varepsilon$.

Equivalently, $T: X \to Y$ is continuous at $x_0 \in X$ if and only if.

$$\{x_n\} \subseteq X,\ x_n \to x_0 \text{ in X} \Rightarrow Tx_n \to Tx \text{ in Y.}$$

Further, a transformation $T: X \to Y$ is said to be continuous on X if it is continuous at each $x \in X$.

Notations :

Let X and Y be normed spaces and $T: X \to Y$ a (linear) transformation. Since it is easy to determine which space an element is in and therefore, implicitly, to which norm we referring, we may use the same symbol $\|.\|$ to denote the norm on both normed spaces X and Y, when no confusion will result. When clarification is necessary we may use subsripts to denote different norms.

e.g. Let vector spaces X and Y have norms $\|\cdot\|_X$ and $\|\cdot\|_Y$ respectively.

2.2.6 Theorem : Let N and N' be normed spaces and $T: N \to N'$ a linear transformation. Then, T is continuous at a point (any) in N iff T is continuous on N.

Proof : Fix any $x_0 \in N$, and let the linear transformation $T: N \to N'$ is continuous at x_0.

Let any $x \in N$.

Let $\{x_n\}$ be any sequence in N such that $x_n \to x$. Then $\{x_n - x + x_0\}$ is a sequence in N such that $x_n - x + x_0 \to x_0$.

Therefore, $T(x_n - x + x_0) \to Tx_0$.

Thus $$\lim_{n\to\infty} \|T(x_n - x + x_0) - T(x_0)\| = 0$$

But T is linear, hence we have

$$\lim_{n\to\infty} \|T(x_n) - T(x) + T(x_0) - T(x_0)\| = 0$$

$$\Rightarrow \lim_{n\to\infty} \|T(x_n) - T(x)\| = 0$$

$$\Rightarrow T(x_n) \to T(x).$$

Hence T is continuous at *x*.

We have proved $T : N \to N'$ is continuous.

Conversely if $T : N \to N'$ is continuous, then obviously it is continuous at any point of N.

2.2.7 Theorem : Let N and N' be normed linear spaces and $T : N \to N'$ a linear transformation. Then, T is continuous if and only if T is bounded.

Proof : Assume $T : N \to N'$ is bounded.

Then $\exists K \geq 0$ such that,

$$\|Tx\| \leq K\|x\|, \ \forall x \in N.$$

Let $\{x_n\}$ be any sequence in N such that $x_n \to 0$ in N.

Then $\|Tx_n\| \leq K\|x_n\| \to 0$ as $n \to \infty$.

$\Rightarrow \|Tx_n\| \to 0$ as $n \to \infty$.

$\Rightarrow Tx_n \to 0 = T(0)$ as $n \to \infty$.

Therefore T is continuous at origin in N, and hence it is continuous on N.

Conversely, let T is continuous on N. If possible T is not bounded. Then, for each n, ($n = 1, 2, ...$), $\exists x_n \neq 0$ in N. Such that,

$$\|Tx_n\| > n\|x_n\|$$

$$\Rightarrow \frac{\|Tx_n\|}{n\|x_n\|} > 1$$

$$\Rightarrow \left\|\frac{Tx_n}{n\|x_n\|}\right\| > 1 \qquad [\because \text{ Homogenity of norm}]$$

Therefore, $\left\|T\left(\frac{x_n}{n\|x_n\|}\right)\right\| > 1$, $\forall n$. .... (1) [$\because$ Linerarity of T]

For each n, define $y_n = \frac{x_n}{n\|x_n\|}$.

Then,

$$\|y_n\| = \left\|\frac{x_n}{n\|x_n\|}\right\| = \frac{\|x_n\|}{n\|x_n\|} = \frac{1}{n} \to 0 \text{ as } n \to \infty.$$

This implies $y_n \to 0$ in N.

But from (1), $\|T(y_n)\| > 1$, $\forall n$, and hence $T(y_n) \not\to 0$ as $n \to \infty$.

Therefore T cannot be continuous at origin, which is contradiction to our assumption that $T : N \to N'$ is continuous.

Therefore, T must be bounded.

2.2.8 Theorem : Let N and N' be normed spaces and $T : N \to N'$ a linear transformation.

Then, T is bounded if and only if T maps bounded sets in N into bounded sets in N'.

Proof : Let $T : N \to N'$ is bounded linear transformation.

Then, $\exists k \geq 0$ such that $\|T(x)\| \leq k\|x\|$, $x \in N$. .... (1)

Let $A = \{x \in N : \|x\| \le M\}$ be any bounded set in N. Then for any $x \in A$, we have

$$\|T(x)\| \le K\|x\| \le KM$$

This proves $T(A) = \{T(x) : x \in A\}$ is bounded.

Conversely, let $T : N \to N'$ maps bounded sets in N into bounded sets in N'.

Let $S = \{x \in N : \|x\| \le 1\}$ is closed unit sphere in N. Then by assumption $T(S) = \{T(x) : x \in S\}$ is bounded.

Thus $\exists K \ge 0$ such that,

$$\|T(x)\| \le K, \ \forall x \in S. \qquad \text{..... (2)}$$

We prove that T is bounded linear transformation.

Case 1 : Let $x = 0$ in N. Then $T(x) = 0$.

Hence $\|T(x)\| \le K\|x\|$ is clearly hold for $x = 0 \in N$.

Case 2 : Let any $x \ne 0$ in N. Then $\left\|\frac{x}{\|x\|}\right\| = \frac{\|x\|}{\|x\|} = 1$.

Hence $\frac{x}{\|x\|} \in S$, for any $x \ne 0$ in N.

Therefore by (2), for any $x \ne 0$ in N, we have,

$$\left\|T\left(\frac{x}{\|x\|}\right)\right\| \le K$$

$$\Rightarrow \left\|\frac{1}{\|x\|}T(x)\right\| \le K$$

$$\Rightarrow \frac{\|T(x)\|}{\|x\|} \le K$$

$$\Rightarrow \|T(x)\| \leq K\|x\|.$$

Combining Cases 1 and 2, we have proved that $\exists K \geq 0$ such that

$$\|T(x)\| \leq K\|x\|, \ \forall x \in N.$$

Hence T is bounded linear transformation.

2.2.9 Theorem : Let N and N' be normed linear spaces and $T : N \to N'$ a linear transformation.

Then the following conditions on T are all equivalent to one another :

(a) T is continuous;

(b) T is continuous at origin, in the sense that $x_n \to 0 \Rightarrow T(x_n) \to 0$;

(c) There exists a real number $K \geq 0$ with the property that,

$$\|T(x)\| \leq K\|x\|, \text{ for every } x \in N.$$

(d) If $S = \{x \in N : \|x\| \leq 1\}$ is closed unit sphere in N, then its image T (S) is a bounded set in N'.

Proof :

$(a) \Leftrightarrow (b)$: Please see proof of theorem 2.2.6 with $x_0 = 0$.

$(b) \Leftrightarrow (c)$: Please see proof of theorem 2.2.7.

$(c) \Leftrightarrow (d)$: Please see proof of the theorem 2.2.8 with M = 1.

2.2.10 Corollary : Let N and N' be normed spaces and $T : N \to N'$ a linear transformation. Then following statements are equivalent.

(a) T is continuous.

(b) T is bounded.

(c) T maps bounded sets in N into bounded sets in N'.

2.2.11 Remark : From above corollary, the two adjectives continuous and bounded can be used interchangeably for linear transformations of one normed space to the other normed space.

2.2.12 Theorem : Let N and N' be normed linear spaces and let $T : N \to N'$ be a onto linear transformation. Then T^{-1} exists and is a bounded linear transformation if and only if $\exists$ a constant K > 0 such that,

$$\|T(x)\| \geq K\|x\|, \text{ for all } x \in N.$$

Proof : Let N and N' be normed spaces and let $T : N \to N'$ be onto linear transformation.

Let $\exists K > 0$ such that

$$\|T(x)\| \geq K\|x\|, \ \forall x \in N. \qquad \text{...... (1)}$$

Then from (1), if $Tx = 0$ then $x = 0$.

Therefore T is one-one. Also given that T is onto. Thus T is bijective and hence it follows that T^{-1} exists in algebraic sensense.

By theorem 2.1.3, $T^{-1} : N' \to N$ is a linear transformation.

It remains to prove $T^{-1} : N' \to N$ is bounded.

For each y in the domain of T^{-1}, $\exists$ $x \in N$ such that,

$$T^{-1}(y) = x \Leftrightarrow Tx = y$$

Therefore from (1),

$$K\|T^{-1}(y)\| \leq \|T(T^{-1}(y))\|, \ \forall y \in N'.$$

$$\Rightarrow \|T^{-1}(y)\| \leq \frac{1}{K}\|y\|, \ \forall y \in N'.$$

This implies T^{-1} is bounded.

Hence $T^{-1} : N' \to N$ is bounded linear transformation.

Conversely let $T^{-1} : N' \to N$ is exists and is a bounded linear transformation.

Then $\exists\ M > 0$ such that,

$$\left\|T^{-1}(y)\right\| \le M\left\|y\right\|,\ \forall y \in N'.$$

$$\Rightarrow \left\|T^{-1}(Tx)\right\| \le M\left\|Tx\right\|,\ \forall x \in N.$$

$$\Rightarrow \left\|x\right\| \le M\left\|Tx\right\|,\ \forall x \in X.$$

Therefore $\exists K = \dfrac{1}{M} > 0$ such that,

$$\Rightarrow \left\|T(x)\right\| \ge K\left\|x\right\|,\ \forall x \in N.$$

This proves the theorem.

2.2.13 Definition : Let N and N' be normed spaces and $T : N \to N'$ a bounded linear transformation. The norm of T, is defined as,

$$\left\|T\right\| = \sup\left\{\left\|T(x)\right\| : x \in N, \left\|x\right\| \le 1\right\}$$

This norm is called an operator norm.

Note : From theorem 2.2.9 : $(a) \Leftrightarrow (d)$ and it follows that $\left\|T\right\|$ is well defined.

In the next section we prove that $\left\|T\right\|$ is indeed norm on the space $\mathcal{B}$ (N, N') - the space of all bounded linear transformation of N into N'.

2.2.14 Theorem : Let N and N' be normed spaces and $T : N \to N'$ a bounded linear transformation. Then, $\left\|T\right\|$ can be expressed by any one of the following formulae.

(i) $\left\|T\right\| = \sup\left\{\left\|Tx\right\| : x \in N, \left\|x\right\| \le 1\right\}$

(ii) $\left\|T\right\| = \sup\left\{\left\|Tx\right\| : x \in N, \left\|x\right\| = 1\right\}$

(iii) $\|T\| = \sup\left\{\dfrac{\|Tx\|}{\|x\|} : x \in N, x \neq 0\right\}$

(iv) $\|T\| = \inf\left\{K : K \geq 0 \text{ and } \|Tx\| \leq K\|x\|, \forall x \in N\right\}$

Further, $\|Tx\| \leq \|T\|\|x\|$, $\forall x \in N$.

Proof : Let N and N' be normed spaces and $T : N \to N'$ a bounded linear transformation.

By definition of an operator norm,

$$\|T\| = \sup\left\{\|Tx\| : x \in N, \|x\| \leq 1\right\}$$

Define,

$$a = \sup\left\{\|Tx\| : x \in N, \|x\| = 1\right\}$$

$$b = \sup\left\{\frac{\|Tx\|}{\|x\|} : x \in N, x \neq 0\right\}$$

$$c = \inf\left\{K : K \geq 0 \text{ and } \|Tx\| \leq K\|x\|, \forall x \in N\right\}$$

We prove that,

$$\|T\| = a = b = c$$

Since $\left\{x \in N : \|x\| = 1\right\} \subseteq \left\{x \in N : \|x\| \leq 1\right\}$, we have,

$$\left\{\|Tx\| : x \in N, \|x\| = 1\right\} \subseteq \left\{\|Tx\| : x \in N, \|x\| \leq 1\right\}$$

$$\Rightarrow \sup\left\{\|Tx\| : x \in N, \|x\| = 1\right\} \leq \sup\left\{\|Tx\| : x \in N, \|x\| \leq 1\right\}$$

$$\Rightarrow a \leq \|T\| \qquad \text{.... (1)}$$

By homogenity of norm and linearity of T, for any $x \in N$, $x \neq 0$ we have,

$$\frac{\|Tx\|}{\|x\|} = \left\|\frac{Tx}{\|x\|}\right\| = \left\|T\left(\frac{x}{\|x\|}\right)\right\|$$

Therefore,

$$b = \sup\left\{\frac{\|Tx\|}{\|x\|} : x \in N, x \neq 0\right\}$$

$$= \sup\left\{\left\|T\left(\frac{x}{\|x\|}\right)\right\| : x \in N, x \neq 0\right\}$$

If $y = \dfrac{x}{\|x\|}$, $x \in N$, $x \neq 0$ then $y \in N$ and $\|y\| = 1$, then

$$b = \sup\left\{\|Ty\| : y \in N, \|y\| = 1\right\} = a$$

Thus,

$$b = a \qquad \text{.... (2)}$$

By definition of c, we have

$$\|Tx\| \le c\|x\|, \ \forall x \in N$$

$$\Rightarrow \|Tx\| \le c, \ \forall x \in N \text{ with } \|x\| \le 1$$

$$\Rightarrow \sup\left\{\|Tx\| : x \in N, \|x\| \le 1\right\} \le c$$

$$\Rightarrow \|T\| \le c \qquad \text{.... (3)}$$

Finally

$$b = \sup\left\{\frac{\|Tx\|}{\|x\|} : x \in N, x \neq 0\right\}$$

$$\Rightarrow \frac{\|Tx\|}{\|x\|} \le b, \ \forall x \in N, \ x \neq 0.$$

$$\Rightarrow \|Tx\| \le b\|x\|, \ \forall x \in N, \ x \neq 0.$$

Clearly $\|Tx\| \le b\|x\|$ for $x = 0$ in N.

Therefore $\|Tx\| \le b\|x\|$, for all $x \in N$. .... (4)

$$\Rightarrow b \in \{K : K \geq 0 \text{ and } \|Tx\| \leq K\|x\|, \forall x \in N\}$$

$$\Rightarrow b \geq \inf\{K : K \geq 0 \text{ and } \|Tx\| \leq K\|x\|, \forall x \in N\}$$

$$\Rightarrow b \geq c \qquad \text{.... (5)}$$

Combining (1), (2), (3) and (5) we obtain,

$$\|T\| \leq c \leq b = a \leq \|T\|$$

$$\Rightarrow \quad \|T\| = a = b = c \qquad \text{.... (6)}$$

Further from (4) and (6) we have,

$$\|Tx\| \leq \|T\|\|x\|, \ \forall x \in N.$$

This complete the proof.

2.2.15 Problem : If M is a closed linear subspace of a normed linear space N, and if T is the natural mapping of N onto N/M defined by T $(x) = x +$ M. Show that T is a continuous linear transformation for which $\|T\| \leq 1$.

Solution : We know that "If M is closed linear subspace of a normed space N then N/M is normed space with the norm of coset $x +$ M in N/M defined by

$$\|x + M\| = \inf\{\|x + m\| : m \in M\}$$

Define $T : N \to N/M$ by $T(x) = x + M$, $x \in N$.

(i) **T is linear :** Let any $x, y \in N$ and α, β be any scalar.

Then,

$$\begin{aligned} T(\alpha x + \beta y) &= (\alpha x + \beta y) + M \\ &= (\alpha x + M) + (\beta y + M) \\ &= \alpha(x + M) + \beta(y + M) \\ &= \alpha T(x) + \beta T(y) \end{aligned}$$

(ii) **T is continuous :** For any $x \in N$, we have,

$$\|T(x)\| = \|x + M\|$$

$$= \inf\{\|x+m\| : m \in M\}$$

$$\leq \|x+m\|, \ \forall m \in M.$$

Inparticular for $m = 0$,

$$\|T(x)\| \leq \|x\|, \ \forall x \in N. \qquad \text{..... (1)}$$

This implies T is bounded linear transformation with bound K = 1.

Hence T is continuous linear transformation.

From (1) it follows that

$$\|T\| = \sup\{\|Tx\| : x \in N, \|x\| \leq 1\} \leq 1, \text{ i.e. } \|T\| \leq 1.$$

2.2.16 Problem : Let N and N' be normed spaces and $T : N \to N'$ a continuous linear transformation. Prove that the null space $\mathcal{N}$ (T) (Kernel of T, Ker (T)) is closed.

Solution : The null-space of $T : N \to N'$ is given by,

$$\mathcal{N}(T) = \{x \in N : Tx = 0\}$$

Let any $x \in \overline{\mathcal{N}(T)}$. Then there exists a sequence $\{x_n\}$ in $\mathcal{N}$(T) such that $x_n \to x$.

Since T is continuous, $Tx_n \to Tx$.

But $x_n \in \mathcal{N}(T), \ \forall n \Rightarrow T(x_n) = 0, \ \forall n$.

Therefore,

$$T(x) = \lim_{n\to\infty} T(x_n) = \lim_{n\to\infty}(0) = 0$$

$$\Rightarrow x \in \mathcal{N}(T)$$

Hence $\overline{\mathcal{N}(T)} \subseteq \mathcal{N}(T) \Rightarrow \mathcal{N}(T)$ is closed.

2.2.17 Problem : If T is continuous linear transformation of a normed space N into a normed space N', and if M is its null space, show that T induces a natural linear transformation T' of N/M into N' and $\|T'\| = \|T\|$.

Solution : Let $T : N \to N'$ be a continuous linear transformation of normed space N into normed space N'. Then its null space $M = \mathcal{N}(T)$ is closed linear subspace of N. The N/M is normed space with norm of coset x + M in N/M defined by

$$\|x + M\| = \inf\{\|x + m\| : m \in M\}$$

Define $T' : N/M \to N'$ by

$$T'(x + M) = T(x), \; x \in N.$$

$$\begin{tikzcd} x \in N \arrow[rr, "T"] \arrow[dr] & & T(x) \in N' \\ & x + M \in N/M \arrow[ur, "T'"'] & \end{tikzcd}$$

(i) **T' is well defined :** Let $x + M = y + M$ in N/M.

Then $x - y \in M = \mathcal{N}(T)$

$\Rightarrow T(x - y) = 0$

$\Rightarrow T(x) - T(y) = 0$

$\Rightarrow T(x) = T(y)$

$\Rightarrow T'(x + M) = T'(y + M).$

(ii) **T' is linear :** Let any $x, y \in N$ and α, β be any scalar.

Then

$$T'\big(\alpha(x + M) + \beta(y + M)\big) = T'\big((\alpha x + \beta y) + M\big)$$

$$= T(\alpha x + \beta y)$$

$$= \alpha T(x) + \beta T(y)$$

$$= \alpha T'(x+M) + \beta T'(y+M)$$

Therefore T' is linear.

(iii) To prove $\|T\| = \|T'\|$:

$\|T'\| = \sup\{\|T'(x+M)\| : x \in N, \|x+M\| \le 1\}$

$= \sup\{\|T(x)\| : x \in N, \inf\{\|x+m\| : m \in M\} \le 1\}$

$= \sup\{\|T(x)\| : x \in N, \|x+m\| \le 1 \text{ for some } m \in M\}$

Since $m \in M = \mathcal{N}(T) \Rightarrow T(m) = 0$

Thus $\|T(x)\| = \|T(x) + T(m)\| = \|T(x+m)\|, x \in N$

Therefore

$\|T'\| = \sup\{\|T(x+m)\| : x \in N, \|x+m\| \le 1 \text{ for some } m \in M\}$

For any $m \in M$, $y = x + m \in N$, $\forall x \in N$.

Thus

$\|T'\| = \sup\{\|Ty\| : y \in N, \|y\| \le 1\} = \|T\|$

$\Rightarrow \|T'\| = \|T\|$.

2.3 SPACE OF BOUNDED LINEAR TRANSFORMATIONS

Let N and N' be normed linear spaces. The collection of all bounded (or continuous) linear transformations of N into N' is denoted by $\mathcal{B}$ (N, N'). The letter $\mathcal{B}$ is intended to suggest the adjective "bounded".

Note that :

(i) The zero operator $O : N \to N'$, $O(x) = 0$, $x \in N$, is bounded linear transformation, with $\|O\| = \sup\{\|O(x)\| : x \in N, \|x\| \le 1\} = 0$

(ii) The identity operator $I : N \to N$, $I(x) = x$, $x \in N$, is bounded linear transformation with $\|I\| = \sup\{\|I(x)\| : x \in N, \|x\| \leq 1\} = 1$

Therefore both zero operator O and identity operator I belongs to $\mathscr{B}$ (N, N') and hence $\mathscr{B}$ (N, N') is nonempty.

2.3.1 Theorem : Let N and N' are normed spaces over the same field of scalar $\mathbb{K}$ ($\mathbb{C}$ or $\mathbb{R}$).

Then,

(a) $\mathscr{B}$ (N, N') is a vector space over $\mathbb{K}$ with respect to pointwise operation and scalar multiplication.

(b) The function $\|\cdot\| : \mathscr{B}(N, N') \to \mathbb{R}$, defined by $\|T\| = \sup\{\|Tx\| : x \in N, \|x\| \leq 1\}$, is norm on $\mathscr{B}(N, N')$.

(c) If N' is a Banach space then $\mathscr{B}(N, N')$ is a Banach space.

Proof :

(a) The family $L(N, N')$ of all linear transformations of N into N' is the vector space over $\mathbb{K}$ with addition and scalar multiplication given by

$$(T+U)(x) = T(x) + U(x) \text{ and } (\alpha T)(x) = \alpha T(x)$$

for all $x \in N$ and $\alpha \in \mathbb{K}$, where $T, U \in L(N, N')$.

Clearly $\mathscr{B}(N, N') \subseteq L(N, N')$.

Since zero linear transformation and identity linear transformations are the member of $\mathscr{B}(N, N')$, it is nonempty.

To prove $\mathscr{B}(N, N')$ is vector space over $\mathbb{K}$, we show that it is linear subspace of $L(N, N')$.

Let any $T, U \in B(N, N')$ and $\alpha \in \mathbb{K}$.

Then $\exists K_1, K_2 \geq 0$ such that

$$\|Tx\| \leq K_1 \|x\|, \ \forall x \in N.$$

and $\|Ux\| \le K_2 \|x\|$, $\forall x \in N$.

Therefore,

$$\|(T+U)(x)\| = \|Tx + Ux\|$$
$$\le \|Tx\| + \|Ux\|$$
$$\le K_1 \|x\| + K_2 \|x\|$$
$$= (K_1 + K_2)\|x\|.$$

Therefore,

$$\|(T+U)(x)\| \le (K_1 + K_2)\|x\|, \ \forall x \in N.$$

$$\Rightarrow T + U \in B(N, N')$$

Also,

$$\|(\alpha T)(x)\| = \|\alpha Tx\| = |\alpha| \|Tx\| \le |\alpha| K_1 \|x\|, \ \forall x \in N.$$

$$\Rightarrow \alpha T \in \mathcal{B}(N, N')$$

Thus $\mathcal{B}(N, N')$ is linear subspace of $L(N, N')$ and hence $\mathcal{B}(N, N')$ itself is a vector space.

(b) $\mathcal{B}(N, N')$ is normed space :

Define $\|\cdot\| : \mathcal{B}(N, N') \to \mathbb{R}$, by

$$\|T\| = \sup\{\|Tx\| : x \in N, \|x\| \le 1\}$$

where $T \in \mathcal{B}(N, N')$.

Let any $T, U \in \mathcal{B}(N, N')$ and $\alpha \in \mathbb{K}$.

(i) Since $\|T(x)\| \ge 0$, $\forall x \in N$, $\|x\| \le 1$, we have $\|T\| \ge 0$.

(ii) $$\|T\| = 0 \Leftrightarrow \sup\left\{\frac{\|Tx\|}{\|x\|} : x \in N, x \ne 0\right\} = 0$$

$$\Leftrightarrow \frac{\|Tx\|}{\|x\|} = 0; x \in N, x \ne 0$$

$\Leftrightarrow \|Tx\| = 0, \forall x \in N$

$\Leftrightarrow Tx = 0, \forall x \in N$

$\Leftrightarrow T = 0$

(iii) $\|T+U\| = \sup\{\|(T+U)(x)\| : x \in N, \|x\| \le 1\}$

$= \sup\{\|Tx+Ux\| : x \in N, \|x\| \le 1\}$

$\le \sup\{\|Tx\|+\|Ux\| : x \in N, \|x\| \le 1\}$

$\le \sup\{\|Tx\| : x \in N, \|x\| \le 1\} + \sup\{\|Ux\| : x \in N, \|x\| \le 1\}$

$= \|T\| + \|U\|$.

Thus $\|T+U\| \le \|T\| + \|U\|$.

(iv) $\|\alpha T\| = \sup\{\|(\alpha T)(x)\| : x \in N, \|x\| \le 1\}$

$= \sup\{|\alpha|\|Tx\| : x \in N, \|x\| \le 1\}$

$= |\alpha| \sup\{\|Tx\| : x \in N, \|x\| \le 1\}$

$= |\alpha|\|T\|$

We have proved that $\mathscr{B}(N, N')$ is normed linear space.

(c) To prove $\mathscr{B}(N, N')$ is complete if N' is complete, let $\{T_n\}_{n=1}^{\infty}$ be any Cauchy sequence in $\mathscr{B}(N, N')$.

Then $\|T_m - T_n\| \to 0$ as $m, n \to \infty$.

For each $x \in N$, we have

$\|T_m x - T_n x\| = \|(T_m - T_n)(x)\|$

$\le \|T_m - T_n\|\|x\| \to 0$ as $m, n \to \infty$.

i.e. $\|T_m x - T_n x\| \to 0$ as $m, n \to \infty$.

Thus for each $x \in N$, $\{T_n x\}_{n=1}^{\infty}$ is Cauchy sequence in complete normed space N'. Thus $\exists$ vector $Tx \in N$ such that $T_n x \to Tx$.

Define $T: N \to N'$ by $T(x) = \lim_{n\to\infty} T_n(x)$, $x \in N$.

We prove that $T \in \mathscr{B}(N, N')$ and $T_n \to T$ in $\mathscr{B}(N, N')$.

(i) **T is linear :** Let any $x, y \in N$ and α, β be any scalar. Then,

$$T(\alpha x + \beta y) = \lim_{n\to\infty} T_n(\alpha x + \beta y)$$

$$= \lim_{n\to\infty}\left(\alpha T_n(x) + \beta T_n(y)\right) \qquad [\because T_n \text{ is linear}]$$

$$= \alpha \lim_{n\to\infty} T_n(x) + \beta \lim_{n\to\infty} T_n(y)$$

$$= \alpha T(x) + \beta T_n(y)$$

(ii) **T is bounded :** For any $x \in N$, we have,

$$\|T(x)\| = \lim_{n\to\infty}\|T_n(x)\| \qquad [\|.\| \text{ is continuous}]$$

$$\le \lim_{n\to\infty}\|T_n\|\|x\|$$

$$= \left(\lim_{n\to\infty}\|T_n\|\right)\|x\|$$

$$\le \left(\sup_{n\in\mathbb{N}}\|T_n\|\right)\|x\|$$

Thus $\|Tx\| \le K\|x\|$, $\forall x \in N$, where $K = \sup_{n\in\mathbb{N}}\|T_n\|$.

Hence T is bounded.

We have proved that $T \in \mathscr{B}(N, N')$.

Finally, since $\{T_n\}$ is Cauchy sequence in $\mathscr{B}(N, N')$, for each $\varepsilon > 0$, $\exists$ an integer $n_0 \in \mathbb{N}$ such that,

$$m, n \ge n_0 \Rightarrow \|T_m - T_n\| < \varepsilon$$

Hence for any $x \in N$,

$$m,n \geq n_0 \Rightarrow \|T_m x - T_n x\| = \|(T_m - T_n)(x)\|$$

$$\leq \|T_m - T_n\|\|x\|$$

$$< \varepsilon \|x\|.$$

This gives,

$$m,n \geq n_0 \Rightarrow \|T_m x - T_n x\| < \varepsilon, \ \forall x \in N, \ \|x\| \leq 1.$$

Taking $m \to \infty$, we obtain,

$$n \geq n_0 \Rightarrow \|T_n x - Tx\| < \varepsilon, \ \forall x \in N, \ \|x\| \leq 1$$

Therefore,

$$n \geq n_0 \Rightarrow \sup\left\{\|(T_n - T)(x)\| : x \in N, \|x\| \leq 1\right\} < \varepsilon$$

$$\Rightarrow \|T_n - T\| < \varepsilon$$

Hence $T_n \to T$ is $\mathscr{B}(N, N')$.

This proves $\mathscr{B}(N, N')$ is complete normed space.

Notations : Let N be a normed space. We call continuous linear transformation of N into itself an operator on N. We denote normed space of all operators on N by $\mathscr{B}$ (N) instead of $\mathscr{B}(N, N)$.

2.3.2 Theorem : Let N be a normed space and $\mathscr{B}$ (N) the set of all operators on N. Then :

(a) $\mathscr{B}$ (N) is normed space.

(b) $\mathscr{B}$ (N) is Banach space if N is Banach space.

(c) If $T, T' \in \mathscr{B}(N, N')$ then $T, T' \in \mathscr{B}(N)$ and $\|TT'\| \leq \|T\|\|T'\|$.

(d) Multiplication is jointly continuous in $\mathscr{B}$ (N) :

$$T_n \to T, \ T_n' \to T' \Rightarrow T_n T_n' \to TT'.$$

Proof : Proof of part (a) and (b) follows from the theorem 2.3.1, by taking N = N'.

(c) Let any $T, T' \in \mathscr{B}(N)$.

(i) TT' is linear : Let any $x, y \in N$ and α, β be any scalar.

Using the linearity of T and T', we obtain,

$$(TT')(\alpha x+\beta y)=T\left(T'(\alpha x+\beta y)\right)$$

$$=T\left(\alpha T'x+\beta T'y\right)$$

$$=\alpha T\left(T'x\right)+\beta T\left(T'y\right)$$

$$=\alpha\left(TT'\right)(x)+\beta\left(TT'\right)(y)$$

(ii) TT' is continuous : Let $x_n \to 0$ in N. Then $T'(x_n) \to T(0)$.

Hence $TT'(x_n) \to T\left(T'(x_n)\right) \to T(0)=0$.

This prove TT' is continuous at origin, and hence it is continuous.

The assertitions (i) and (ii) proves $T, T' \in \mathscr{B}(N)$.

(iii) $\|TT'\| = \sup\left\{\|TT'(x)\| : x \in N, \|x\| \leq 1\right\}$

$$= \sup\left\{\left\|T\left(T'(x)\right)\right\| : x \in N, \|x\| \leq 1\right\}$$

$$\leq \sup\left\{\|T\|\|T'x\| : x \in N, \|x\| \leq 1\right\}$$

$$= \|T\| \sup\left\{\|T'x\| : x \in N, \|x\| \leq 1\right\}$$

$$= \|T\|\|T'\|$$

Therefore, $\|TT'\| \leq \|T\|\|T'\|$.

(d) Let $T_n \to T$ and $T_n' \to T'$ in $\mathscr{B}(N)$.

Then,

$$\left\|T_nT_n' - TT'\right\| = \left\|\left(T_nT_n' - T_nT'\right) + \left(T_nT' - TT'\right)\right\|$$

$$\leq \left\|T_n\left(T_n' - T'\right)\right\| + \left\|\left(T_n - T\right)T'\right\|$$

$$\leq \|T_n\| \|T_n' - T'\| + \|T_n - T\| \|T'\| \to 0 \text{ as } n \to \infty.$$

$$\Rightarrow \|T_n T_n' - TT'\| \to 0 \text{ as } n \to \infty.$$

$$\Rightarrow T_n T_n' \to TT' \text{ as } n \to \infty.$$

Thus multiplication is jointly continuous in $\mathscr{B}(N)$.

2.4 BANACH ALGEBRA

2.4.1 Definition : An algebra A over the field $\mathbb{K}$ is a vector space A over $\mathbb{K}$ such that for each ordered pair of elements $x, y \in A$ a unique product $xy \in A$ is defined with the properties :

(i) $(xy)z = x(yz)$

(ii) $x(y+z) = xy + xz$

(iii) $(x+y)z = xz + yz$

(iv) $\alpha(xy) = (\alpha x)y = x(\alpha y)$

for all $x, y, z \in A$ and $\alpha \in \mathbb{K}$.

An algebra is said to be real or complex according as $\mathbb{K}$ is $\mathbb{R}$ or $\mathbb{C}$.

2.4.2 Definition : An algebra A is said to be commutative (or abelian) if the multiplication in A is commutative, that is, if for all $x, y \in A$.

$$xy = yx$$

2.4.3 Definition : An algebra A is called an algebra with identity if A contains an element e such that for all $x \in A$,

$$xe = ex = x$$

This element e is called an identity of A.

2.4.4 Definition : A normed algebra A is a normed space which is an algebra such that for all $x, y \in A$

$$\|xy\| \le \|x\|\|y\|,$$

and if A has an identity e, $\|e\| = 1$.

2.4.5 Definition : A Banach algebra is a normed algebra which is complete, considered as a normed space.

2.4.6 Theorem : If $N \neq \{0\}$ is a Banach space, then $\mathscr{B}$ (N) is a Banach algebra.

Proof : Let $N \neq \{0\}$ is a Banach space. Then $\mathscr{B}$ (N) is a Banach space.

For $T, U \in \mathscr{B}(N)$, define,

$$(TU)(x) = T(Ux), \ x \in N \cdot$$

Then by Theorem 2.3.2, $T, U \in \mathscr{B}(N)$.

Further, for any $S, T, U \in \mathscr{B}(N)$ and a scalar α, we have,

(i) $S(TU) = (ST)U$

(ii) $S(T+U) = ST + SU$

(iii) $(S+T)U = SU + TU$

(iv) $\alpha(ST) = (\alpha S)T = S(\alpha T)$

Further, for any $T, U \in \mathscr{B}(N)$, we have already proved $\|TU\| \le \|T\|\|U\|$.

Also, an identity transformation $I : N \to N$, $I(x) = x \in N$, is an identity element for $\mathscr{B}$ (N) with $\|I\| = 1$.

From above discussion it follows that $\mathscr{B}$ (N) is a Banach algebra.

2.5 THE OPEN MAPPING THEOREM

In this section we prove the open mapping theorem which gives condition under which a bounded linear transformation is open mapping.

We give some basic definitions and theorems which we need subsequently.

2.5.1 Definition : Let X and Y be metric spaces.

A mapping $f : X \to Y$ is said to be open if f (A) is open in Y for every open set A in X. i.e. A mapping which maps open sets into open sets is called open mapping.

2.5.2 Theorem : Let X and Y be metric spaces. Then following conditions are all equivalent,

(a) $f : X \to Y$ is homeomorphism.

(b) $f : X \to Y$ is bijective and bicontinuous.

(c) $f : X \to Y$ is bijective, open and continuous.

(d) $f : X \to Y$ is bijective, closed and continuous.

2.5.3 Theorem : If f is one-to-one mapping of metric space X into metric space Y.

Then, $f : X \to Y$ is homeomorphism if and only if $f(\overline{A}) = \overline{f(A)}$, $\forall A \subseteq X$.

2.5.4 Theorem (Baire Category Theorem) :

If a complete metric space is the union of a sequence of its subsets, then the closure of at least one set in the sequence must have non-empty interior.

2.5.5 Problem : Let N be a normed space, $x_0 \in N$ and $r > 0$. Then :

(i) $S_r(x_0) = x_0 + S_r(0)$

(ii) $S_r(0) = rS_1(0)$

Solution : $S_r(x_0) = \{x \in N : \|x - x_0\| < r\}$

and $S_r(0) = \{x \in N : \|x\| < r\}$

(i) $x \in S_r(x_0) \Leftrightarrow \|x - x_0\| < r$

$$\Leftrightarrow x - x_0 \in S_r(0)$$

$$\Leftrightarrow x_0 + (x - x_0) \in x_0 + S_r(0)$$

$$\Leftrightarrow x \in x_0 + S_r(0)$$

Thus $S_r(x_0) = x_0 + S_r(0)$.

(ii) $x \in S_r(0) \Leftrightarrow \|x\| < r$

$$\Leftrightarrow \left\|\frac{x}{r}\right\| < 1$$

$$\Leftrightarrow \frac{x}{r} \in S_1(0)$$

$$\Leftrightarrow r\left(\frac{x}{r}\right) \in rS_1(0)$$

$$\Leftrightarrow x \in rS_1(0)$$

Therefore $S_r(0) = rS_1(0)$.

Combining (i) and (ii) we have,

$$S_r(x_0) = x_0 + rS_1(0).$$

Firstly we prove the following Lemma which play a key role to prove the open mapping theorem.

2.5.6 Lemma : If B and B' are Banach spaces, and if T is a continuous linear transformation of B on to B', then the image of each open sphere centered on the origin in B contains an open sphere centered on the origin in B'.

Proof : Let $S_r = \{x \in B : \|x\| < r\}$ be the open sphere of radius r centered at origin in B. Then by linearity of T we have,

$$T(S_r) = T(rS_1) = rT(S_1).$$

Therefore to prove the lemma it is sufficient to show that $T(S_1)$ contains an open sphere $S'_\varepsilon = \{x \in B' : \|x\| < \varepsilon\}$, centered at origin in B' for some $\varepsilon > 0$.

To each $x \in B$, choose $n \in \mathbb{N}$ sufficiently large so that $\|x\| < n$. Then $x \in S_n$.

Therefore, $$B = \bigcup_{x \in B} \{x\} \subseteq \bigcup_{n \in \mathbb{N}} S_n \subseteq B$$

$$\Rightarrow B = \bigcup_{n \in \mathbb{N}} S_n$$

Since $T : B \to B'$ is onto, we have

$$B' = T(B) = T\left(\bigcup_{n \in \mathbb{N}} S_n\right) = \bigcup_{n \in \mathbb{N}} T(S_n)$$

As B' is complete, by Baire's category theorem $\exists n_0 \in \mathbb{N}$ such that $\overline{T(S_{n_0})}$ has non-empty interior.

Let y_0 is an interior point of $\overline{T(S_{n_0})}$ such that $y_0 \in T(S_{n_0})$.

Define $f : B' \longrightarrow B'$ by $f(y) = y - y_0$, $y \in B'$.

Claim 1 : f is homeomorphism.

***f* is one-one :** Let $y_1, y_2 \in B'$. Then,

$$f(y_1) = f(y_2) \Rightarrow y_1 - y_0 = y_2 - y_0 \Rightarrow y_1 = y_2$$

***f* is onto :** To each $x \in B'$, $\exists y = x + y_0 \in B'$, such that,

$$f(y) = y - y_0 = (x + y_0) - y_0 = x$$

***f* and f^{-1} are continuous :** Fix any $y \in B'$ and let $\{y_n\} \subseteq B'$ such that $y_n \to y$. Then,

$$f(y_n) = y_n - y_0 \to y - y_0 = f(y)$$

$$f^{-1}(y_n) = y_n + y_0 \to y + y_0 = f^{-1}(y)$$

We have proved that f is bijective and bicontinuous. Hence f is a homeomorphism.

Claim 2 : '0' is an interior point of $\overline{T(S_{n_0})} - y_0$.

Since y_0 is the interior point of $\overline{T(S_{n_0})}$, $\exists$ an open set G such that,

$y_0 \in G \subseteq \overline{T(S_{n_0})}$

$\Rightarrow f(y_0) \in f(G) \subseteq f\left(\overline{T(S_{n_0})}\right)$.

But $f(y_0) = y_0 - y_0 = 0$ and $f\left(\overline{T(S_{n_0})}\right) = \overline{T(S_{n_0})} - y_0$

Therefore, $0 \in f(G) \subseteq \overline{T(S_{n_0})} - y_0$.

Since *f* is homeomorphism, it is an open map and hence *f* (G) is open in B'.

Hence '0' is an interior point of $\overline{T(S_{n_0})} - y_0$.

Claim 3 : $T(S_{n_0}) - y_0 \subseteq T(S_{2n_0})$

Let any $y \in T(S_{n_0}) - y_0$.

Then $y = T(x) - y_0$ for some $x \in S_{n_0}$.

Further $y_0 \in T(S_{n_0}) \Rightarrow y_0 = T(x_0)$ for some $x_0 \in S_{n_0}$.

Therefore $y = T(x) - T(x_0) = T(x - x_0)$, $x, x_0 \in S_{n_0}$.

But $x, x_0 \in S_{n_0} \Rightarrow \|x\| < n_0$, $\|x\| < n_0$

$\Rightarrow \|x - x_0\| \leq \|x\| + \|x_0\| < 2n_0$

$\Rightarrow x - x_0 \in S_{2n_0}$

$\Rightarrow T(x - x_0) \in T(S_{2n_0})$

$$\Rightarrow y \in T\left(S_{2n_0}\right)$$

Therefore, $T\left(S_{n_0}\right) - y_0 \subseteq T\left(S_{2n_0}\right)$ ..... (1)

Claim 4 : '0' is an interior point of $\overline{T\left(S_1\right)}$.

By using (1), we have,

$$\overline{T\left(S_{n_0}\right) - y_0} \subseteq \overline{T\left(S_{2n_0}\right)} = \overline{T\left(2n_0 S_1\right)} = \overline{2n_0 T\left(S_1\right)} \quad \text{...... (2)}$$

Since f is homeomorphism.

$$f\left(\overline{T\left(S_{n_0}\right)}\right) = \overline{f\left(T\left(S_{n_0}\right)\right)}$$

$$\Rightarrow \overline{T\left(S_{n_0}\right)} - y_0 = \overline{T\left(S_{n_0}\right) - y_0} \quad \text{...... (3)}$$

Combining (2) and (3), we obtain,

$$\overline{T\left(S_{n_0}\right)} - y_0 \subseteq \overline{2n_0 T\left(S_1\right)} \quad \text{..... (4)}$$

Note that, the mapping $g : B' \to B'$ defined by $g(x) = 2n_0 x$ is homeomorphism.

Therefore, $g\left(\overline{T\left(S_1\right)}\right) = \overline{g\left(T\left(S_1\right)\right)}$

$$\Rightarrow 2n_0 \overline{T\left(S_1\right)} = \overline{2n_0 T\left(S_1\right)} \quad \text{..... (5)}$$

Using (5) in (4), we have,

$$\overline{T\left(S_{n_0}\right)} - y_0 \subseteq 2n_0 \overline{T\left(S_1\right)}$$

Since 0 is an interior point of $\overline{T\left(S_{n_0}\right)} - y_0$, it follows that 0 is the interior point of $2n_0 \overline{T\left(S_1\right)}$.

This implies, is the interior point of $\overline{T\left(S_1\right)}$.

Therefore $\exists$ an open sphere, $S_\varepsilon^{'} = \{x \in B' : \|x\| < \varepsilon\}$ centered at origin in B' such that,

$$S_\varepsilon^{'} \subseteq \overline{T\left(S_1\right)} \quad \text{..... (6)}$$

We conclude the proof by showing that $S'_{\varepsilon} \subseteq T(S_3)$ which is equivalent to $S'_{\varepsilon/3} \subseteq T(S_1)$.

Let any $y \in S'_{\varepsilon}$.

Then by (6), $y \in \overline{T(S_1)}$.

$\Rightarrow S'_r(y) \cap T(S_1) \neq \phi$, $\forall r > 0$.

In particular for $r = \dfrac{\varepsilon}{2}$, we have,

$$S'_{\varepsilon/2}(y) \cap T(S_1) \neq \phi$$

Let $y_1 \in S'_{\varepsilon/2}(y) \cap T(S_1)$.

Then $\|y - y_1\| < \dfrac{\varepsilon}{2}$ and $y_1 = T(x_1)$ for some $x_1 \in S_1$ so that $\|x_1\| < 1$.

Hence $y - y_1 \in S'_{\varepsilon/2} \subseteq \overline{T(S_{1/2})}$ .... (7) [$\because$ By (6)]

$\Rightarrow S'_{\varepsilon/2^2}(y - y_1) \cap T(S_{1/2}) \neq \phi$

Let $y_2 \in S'_{\varepsilon/2^2}(y - y_1) \cap T(S_{1/2})$.

Then $\|y - y_1 - y_2\| < \dfrac{\varepsilon}{2^2}$ and $y_2 = T(x_2)$ for some $x_2 \in S_{1/2}$ so that $\|x_2\| < \dfrac{1}{2}$.

Again we see that $y - y_1 - y_2 \in S'_{\varepsilon/2^2} \subseteq \overline{T\left(S_{1/2^2}\right)}$.

$\Rightarrow S'_{\varepsilon/2^2}(y - y_1 - y_2) \cap T\left(S_{1/2^2}\right) \neq \phi$

Let $y_3 \in S'_{\varepsilon/2^2}(y - y_1 - y_2) \cap T\left(S_{1/2^2}\right)$.

Then $\|y - y_1 - y_2 - y_3\| < \dfrac{\varepsilon}{2^3}$ and $y_3 = T(x_3)$ for some $x_3 \in S_{1/2^2}$ so that $\|x_3\| < \dfrac{1}{2^2}$.

Continuing in this way we get a sequence $\{x_n\}$ in B such that $\|x_n\| < \dfrac{1}{2^{n-1}}$, and

$$\left\| y - \left(y_1 + y_2 + \ldots.. + y_n \right) \right\| < \frac{\in}{2^n} \qquad \ldots..(8)$$

Where $y_n = T\left(x_n\right)$.

Define $S_n = x_1 + x_2 + \ldots.. + x_n$. Then,

$$\left\| S_n \right\| = \left\| x_1 \right\| + \left\| x_2 \right\| + \ldots.. + \left\| x_n \right\|$$

$$< 1 + \frac{1}{2} + \ldots. + \frac{1}{2^{n-1}}$$

$$= \frac{1 - \left(\frac{1}{2}\right)^n}{1 - \frac{1}{2}} = 2\left(1 - \frac{1}{2^n}\right) < 2, \ \forall n.$$

Thus $\left\| S_n \right\| < 2$, for all n. ..... (9)

For $n > m$ we have,

$$\left\| S_n - S_m \right\| = \left\| x_{m+1} + x_{m+2} + \ldots.. + x_n \right\|$$

$$\leq \left\| x_{m+1} \right\| + \left\| x_{m+2} \right\| + \ldots. + \left\| x_n \right\|$$

$$< \frac{1}{2^m} + \frac{1}{2^{m+1}} + \ldots.. + \frac{1}{2^{n-1}}$$

$$= \frac{1}{2^m}\left(\frac{1 - \frac{1}{2^{n-m}}}{1 - \frac{1}{2}} \right)$$

$$= 2\left(\frac{1}{2^m} - \frac{1}{2^n} \right) \to 0 \text{ as } m, n \to \infty.$$

Therefore $\left\| S_n - S_m \right\| \to 0$ as $m, n \to \infty$.

This implies $\{S_n\}$ is Cauchy sequence in complete space B and hence $\exists x \in B$ such that $S_n \to x$. Using the continuity of norm, we have,

$$\|x\| = \lim_{n\to\infty}\|S_n\| \le 2 < 3 \qquad [\because \text{By}(9)]$$

$$\Rightarrow x \in S_3$$

Further, using continuity of T, we have,

$$T(x) = \lim_{n\to\infty} T(S_n)$$

$$= \lim_{n\to\infty} T(x_1 + ... + x_n)$$

$$= \lim_{n\to\infty}\left(T(x_1) + ... + T(x_n)\right)$$

$$= \lim_{n\to\infty} T(y_1 + ... + y_n)$$

$$= y \qquad [\because \text{By}(8)]$$

But $x \in S_3$ implies $y = T(x) \in T(S_3)$.

We have proved that,

$$y \in S_\varepsilon^{'} \Rightarrow y \in T(S_3)$$

Therefore, $S_\varepsilon^{'} \subseteq T(S_3) \Rightarrow S_{\varepsilon/3}^{'} \subseteq T(S_1)$.

This complete the proof.

2.5.7 Theorem : If B and B' are Banach spaces, and T is a continuous linear transformation of B onto B', then T is an open mapping.

Proof : Let B and B' are Banach spaces and $T : B \to B'$ is onto, continuous linear transformation.

Let G be any open set in B.

We prove that T (G) is open set in B'.

Case 1 : If $T(G) = \phi$, the T (G) is open in B'.

Case 2 : Let $T(G) \neq \phi$.

Let $y \in T(G)$. Then $y = T(x)$ for some $x \in G$.

Since G is open in B $\exists$ an open sphere $S_r(x)$ in B such that $S_r(x) \subseteq G$.

But $S_r(x) = x + S_r(0)$.

Also $S_r(0)$ is open sphere contered at origin in B, thus by lemma 2.5.6 $\exists$ an open sphere $S_\varepsilon^{'}(0)$ centered at origin in B' such that

$$S_\varepsilon^{'}(0) \subseteq T\left(S_r(0)\right)$$

$$\Rightarrow y + S_\varepsilon^{'}(0) \subseteq y + T\left(S_r(0)\right)$$

$$\Rightarrow S_\varepsilon^{'}(y) \subseteq T(x) + T\left(S_r(0)\right)$$

$$\Rightarrow S_\varepsilon^{'}(y) \subseteq T\left(x + S_r(0)\right) = T\left(S_r(x)\right)$$

But $S_r(x) \subseteq G \Rightarrow T\left(S_r(x)\right) \subseteq T(G)$

Therefore $S_\varepsilon^{'}(y) \subseteq T(G)$.

This implies T (G) is an open set in B'.

2.5.8 Theorem : A one-to-one continuous linear transformation of one Banach space onto another is a homeomorphism. In particular, if a one-to-one linear transformation T of a Banach space onto itself is continuous, then its inverse T^{-1} is automatically continuous.

Proof : Let B and B' are Banach spaces and $T : B \to B'$ is bijective, continuous linear transformation.

To prove T is homeomorphism it remains to prove T^{-1} is continuous.

Since $T : B \to B'$ is bijective, $T^{-1} : B' \to B$ exists and it is linear.

Let G be any open set in B, then by open mapping theorem, T (G) is open in B'.

But $\left(T^{-1}\right)^{-1}(G) = T(G)$ implies $\left(T^{-1}\right)^{-1}(G)$ is open in B'.

This implies, inverse image under T of an open set G in B is open in B'.

Therefore T^{-1} is continuous.

2.6 PROJECTIONS ON BANACH SPACES

2.6.1 Projection on Linear Space

A projection P on a linear space L is an idempotent ($P^2 = P$) linear transformation of L into itself.

The projection on linear space described geometrically as follows :

(a) A projection P determines a pair of linear subspaces M and N scuh that $L = M \oplus N$, where $M = \{P(x) : x \in L\}$ is the range of P.

and $L = \{x \in L : P(x) = 0\}$ is null space of P.

(b) A pair of linear subspaces M and N such that $L = M \oplus N$ determines a projection P whose range and null space are M and N.

Indeed, if $z = x + y$ is unique expression of vector in $L = M \oplus N$ then P is defined by $P(z) = x$.

These facts shows that the study of projections on L is equivalent to study of pairs of linear subspaces which are disjoint and span L.

2.6.2 Projection on Banach Space :

A projection on a Banach space B is an idempotent operator on B in the algebraic sense which is also continuous.

In other words P is projection on Banach space B if :

(i) $P^2 = P$ (P is projection on B in algebraic sense).

(ii) $P : B \to B$ is continuous (bounded).

2.6.3 Theorem :

If P is projection on a Banach space B, and if M and N are its range and null space, then M and N are closed linear subspaces of B such that $B = M \oplus N$.

Proof : Let P is projection on a Banach space B.

Then,

(i) P is projection on B in algebraic sense i.e. $P^2 = P$.

(ii) $P : B \to B$ is continuous (bounded).

Thus (ii) implies that $B = M \oplus N$, where $M = \{P(x) : x \in B\}$ is the range of P.

and $N = \{x \in B : P(x) = 0\}$ is null space of P.

Note that,

$$M = \{P(x) : x \in B\} = \{x \in B : P(x) = x\}$$

$$= \{x \in B : (I - P)(x) = 0\}$$

$\Rightarrow M$ is the null space of the continuous linear transformation I – P on B.

We know the null space of any continuous linear transformation is closed (please see problem 2.2.16). Therefore both M and N are closed linear subspaces of B.

2.6.4 Theorem :

Let B be a Banach space, and let M and N be closed linear subspaces of B such that $B = M \oplus N$. If $z = x + y$ is the unique representation of vector in B as a sum of vectors in M and N, then the mapping P defined by $P(z) = x$ is projection on B, whose range and null space are M and N.

Proof : Let M and N are closed linear subspaces of Banach space B such that $B = M \oplus N$. Then the pair M and N determines a projection P on linear space B whose range and nullspaces are M and N respectively.

Thus to prove $P : B \to B$ is projection on Banach space B it remains to prove P is continuous.

Let $z = x + y$ is unique expression of vector in $B = M \oplus N$. Let B' is the linear space B equipped with new norm $\|.\|'$ defined by,

$$\|z\|' = \|x\| + \|y\|$$

Then $B' = (B, \|\cdot\|')$ is Banach space [$\because$ please refer problem 1.6.10].

Note that,

$$\|P(z)\| = \|x\| \le \|x\| + \|y\| = \|z\|'$$

$\Rightarrow P : B' \to B$ is bounded linear transformation and hence it is continuous.

It is therefore sufficient to prove that B' and B have same topology.

Let $T : B' \to B$ be a identity map. Then T is bijective and

$$\|T(z)\| = \|z\| = \|x + y\| \leq \|x\| + \|y\| = \|z\|'$$

$\Rightarrow T$ is bounded linear transformation and hence continuous.

By Theorem 2.5.8, $T : B' \to B$ is homeomorphism. Hence B' and B have same topology. This completes the proof.

2.7 CLOSED GRAPH THEOREM

In this section we give the proof of closed graph theorem which states the sufficient condition under which a closed linear operator on a Banach space is bounded (continuous).

We know given linear spaces X and Y over same scalar field $\mathbb{K}$ ($\mathbb{C}$ or $\mathbb{R}$), the cartesian product X $\times$ Y is again linear space over $\mathbb{K}$ under the algebraic operations given by,

$$(x, y) + (u, v) = (x + u, y + v) \text{ and } \alpha(x, y) = (\alpha x, \alpha y)$$

where $(x, y), (u, v) \in X \times Y$ and $\alpha \in \mathbb{K}$.

Problem 2.7.1 : Let $(X, \|\cdot\|)$ and $(Y, \|\cdot\|_Y)$ be normed spaces. Prove that each one of the following defines norm of $X \times Y$.

(a) $\|(x, y)\| = \max\{\|x\|_X, \|y\|_Y\}$, $(x, y) \in X \times Y$.

(b) $\|(x, y)\| = \|x\|_X + \|y\|_Y$, $(x, y) \in X \times Y$.

(c) $\|(x, y)\| = \|x\|_X^P + \|y\|_Y^P$, $(1 \leq P < \infty)$, $(x, y) \in X \times Y$.

Solution : We have already discussed (a) in the first unit. Remaining we leave for students.

Problem 2.7.2 : Let X and Y are Banach spaces with norm $\|\cdot\|_X$ and $\|\cdot\|_Y$ respectively. Prove that $X \times Y$ is Banach space with the norm $\|.\|$ defined by,

$$\|(x, y)\| = \max\{\|x\|_X, \|y\|_Y\}$$

Solution : Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ are Banach spaces.

Then $X \times Y$ is normed space with the norm $\|.\|$ defined by,

$$\|(x, y)\| = \max\{\|x\|_X, \|y\|_Y\}$$

To prove $X \times Y$ is complete, let $\{z_n\}$ be any Cauchy sequence in $X \times Y$, where for each n, $z_n = (x_n, y_n)$.

Then for given $\varepsilon > 0$, $\exists N \in \mathbb{N}$ such that

$$m, n \geq N \Rightarrow \|z_m - z_n\| < \varepsilon$$

$$\Rightarrow \max\{\|x_m - x_n\|_X, \|y_m - y_n\|_Y\} < \varepsilon$$

$$\Rightarrow \|x_m - x_n\|_X < \varepsilon \text{ and } \|y_m - y_n\|_Y < \varepsilon$$

This implies $\{x_n\}$ and $\{y_n\}$ are Cauchy sequence in complete normed linear space X and Y respectively. Therefore $\exists x \in X$, $y \in Y$ such that $x_n \to x$ and $y_n \to y$.

Define $z = (x, y)$ then clearly $z \in X \times Y$.

We prove that $z_n \to z$.

Note that,

$$\|z_n - z\| = \max\{\|x_n - x\|_X, \|y_n - y\|_Y\} \to 0 \text{ as } n \to \infty.$$

Thus $z_n \to z$ in $X \times Y$.

Therefore $X \times Y$ is complete normed linear space and hence Banach space.

2.7.3 Definition : Let X and Y are linear space over the same system of scalar and $T : X \to Y$ be a linear transformation. The set given by $G(T) = \{(x, Tx) : x \in X\}$ is called graph of T.

2.7.4 Remark : (i) If X and Y are linear spaces then G (T) is linear subspace of $X \times Y$.

(ii) Graph of T is also denoted by T_G or G_T.

2.7.5 Definition : Let X and Y be normed spaces and $T : X \to Y$ a linear transformation. Then T is called closed linear transformation if its graph $G(T) = \{(x, Tx) : x \in X\}$ is closed in the normed space $X \times Y$.

2.7.6 Theorem : Let X and Y be normed linear spaces over the same system of scalar $\mathbb{K}$ ($\mathbb{C}$ or $\mathbb{R}$), then the linear transformation $T : X \to Y$ is closed iff for every sequence $\{x_n\}$ in X with $x_n \to x$ and $T(x_n) \to y$ we have $x \in X$ and $T(x) = y$.

Proof : Let X and Y be normed linear spaces with the norm $\|\cdot\|_X$ and $\|\cdot\|_Y$ respectively.

Then $X \times Y$ is normed linear space with the norm given by,

$$\|(x, y)\| = \max\{\|x\|_X, \|y\|_Y\}, \ (x, y) \in X \times Y.$$

Let the linear transformation $T : X \to Y$ is closed.

Then by definition its graph $G(T) = \{(x, Tx) : x \in X\}$ is closed.

Let $\{x_n\}$ be any sequence in X such that,

$$x_n \to x \text{ and } T(x_n) \to y$$

Then $\{(x_n, T(x_n))\}$ is sequence in G (T) such that

$$\|(x_n, T(x_n)) - (x, y)\| = \max\{\|x_n - x\|_X, \|T(x_n) - y\|_Y\} \to 0$$

This implies $\{(x_n, T(x_n))\}$ is the sequence in G (T) such that $(x_n, T(x_n)) \to (x, y)$.

But G (T) is closed. Thus, we must have, $(x, y) \in G(T)$.

Therefore $x \in X$ and $y = T(x)$.

Conversely, let for every sequence $\{x_n\}$ in X with $x_n \to x$ and $T(x_n) \to y$ we have, $x \in X$ and $T(x) = y$. We have to prove that T is closed i.e. its graph G (T) is closed.

Let $(x_n, T(x_n))$ be any sequence in G (T) such that $(x_n, T(x_n)) \to (x, y)$.

Then, $\|(x_n, T(x_n)) - (x, y)\| \to 0$ as $n \to \infty$.

$$\max\left\{\|(x_n - x)\|_X, \|(T(x_n) - y)\|_Y\right\} \to 0$$

But $\|x_n - x\|_X$, $\|T(x_n) - y\|_Y \le \max\left\{\|x_n - x\|_X, \|T(x_n) - y\|_Y\right\}$.

$\Rightarrow \|x_n - x\|_X \to 0$ and $\|T(x_n) - y\|_Y \to 0$

$\Rightarrow x_n \to x$ and $T(x_n) \to y$

But by hypothesis, we must have $x \in X$ and $T(x) = y$.

Therefore, $(x, y) = (x, T(x)) \in G(T)$.

This implies G (T) is closed.

2.7.7 Theorem (Closed Graph Theorem) : If B and B' are the Banach spaces and if T is linear transformation of B into B', then T is continuous iff its graph is closed (T is closed).

Proof : Let B and B' are Banach spaces w.r.t. norm $\|.\|$ and $\|.\|'$ respectively and $T : B \to B'$ be a linear transformation.

Let T is continuous. We prove that its graph $G(T) = \{(x, T(x)) : x \in B\}$ is closed.

Let $\{(x_n, T(x_n))\}$ be any sequence in G (T) such that $(x_n, T(x_n)) \to (x, y)$.

Then $x_n \to x$ and $T(x_n) \to y$.

But continuity of T gives that

$$x_n \to x \Rightarrow T(x_n) \to T(x)$$

Therefore we must have $y = T(x)$.

Thus $(x, y) = (x, T(x)) \in G(T)$.

This proves G (T) is closed, that is T is closed.

Conversely, let G (T) is closed.

We denote by B_1 the linear space B with the norm $\|x\|_1 = \|x\| + \|T(x)\|'$, $x \in B$.

Then $B_1 = (B, \|\cdot\|_1)$ is normed linear space.

Moreover $\|T(x)\|' \le \|x\| + \|T(x)\|' = \|x\|$, $x \in B$.

This implies, $T : B_1 \to B'$ is bounded linear transformation, hence continuous.

To prove $T : B \to B'$ is continuous.

We must show that B and B_1 have same topology that is they are homeomorphic.

Consider the identity map.

$$I : B_1 \to B,\ I(x) = x,\ x \in B.$$

Then I is clearly bijective linear transformation.

Further,

$$\|I(x)\| = \|x\| \le \|x\| + \|T(x)\|' = \|x\|_1,\ x \in B.$$

implies I is bounded.

Thus, we have proved that I is bijective, continuous linear transformation.

Therefore, by the theorem "A one to one continuous linear transformation from one Banach space onto other is homeomorphism", $I : B_1 \to B$ will be homeomorphism if B_1 is complete.

Thus to conclude the proof we show that B_1 is complete.

Let $\{x_n\}$ be any Cauchy sequence in B_1, then for given $\varepsilon > 0$, $\exists N \in \mathbb{N}$ such that.

$$m, n \ge N \Rightarrow \|x_m - x_n\|_1 < \varepsilon$$

$$\Rightarrow \|x_m - x_n\| + \|T(x_m - x_n)\|' < \varepsilon$$

$$\Rightarrow \|x_m - x_n\| < \varepsilon \text{ and } \|T(x_m - x_n)\|' < \varepsilon$$

This implies $\{x_n\}$ and $\{T(x_n)\}$ are Cauchy sequences in complete normed linear spaces B and B' respectively. Hence $\exists$ vector $x \in B$ and $y \in B'$ such that,

$$\|x_n \to x\| \to 0 \text{ and } \|T(x_n) - y\| \to 0 \qquad \text{.... (1)}$$

$$\Rightarrow (x_n, T(x_n)) \to (x, y)$$

Note that, $\{(x_n, T(x_n))\}$ is sequence in G (T) such that $(x_n, T(x_n)) \to (x, y)$.

But by assumption G (T) is closed and hence $(x, y) \in G(T)$, so $y = T(x)$.

Now,

$$\|x_n - x\|_1 = \|x_n - x\| + \|T(x_n) - T(x)\|'$$

$$= \|x_n - x\| + \|T(x_n) - y\|' \to 0 \text{ as } n \to \infty. \qquad [\because \text{ by } (1)]$$

This proves B_1 is complete.

This complete the proof of the theorem.

2.8 UNIFORM BOUNDEDNESS PRINCIPLE

2.8.1 Definition : Let X and Y are norm linear spaces and $\mathcal{F} \subseteq B(X)$. Then $\mathcal{F}$ is said to be :

(a) Pointwise bounded : If for each $x \in X$, the set $\{T(x) : T \in \mathcal{F}\}$ is bounded in Y.

(b) Uniformly bounded : If $\mathcal{F}$ is bounded set in the normed linear space $\mathscr{B}$ (X, Y), that is $\exists K \geq 0$ such that $\|T\| \leq K$, $\forall T \in \mathcal{F}$.

2.8.2 Remark : If $\mathcal{F}$ is uniformly bounded set then $\mathcal{F}$ is pointwise bounded but converse need not be true.

The uniform boundedness principle which is also known as Banach-Steinhaus theorem is one of the fundamental results in functional analysis which has significant applications in the field of analysis. It asserts that for a family of continuous linear transformations of Banach spaces to normed spaces, pointwise boundedness is equivalent uniform boundedness.

2.8.3 Theorem (Uniform Boundedness Principle)

Let B be a Banach space and N a normed linear space. If { T_i } is a nonempty set of continuous linear transformation of B into N with the property that { $T_i(x)$] is bounded subset of N for each x in B, then $\{\|Ti\|\}$ is bounded as a subset of numbers, that is, { T_i } is bounded as a subset of $\mathscr{B}$ (B, N).

OR

Let B be a Banach space, N a normed linear space and $\{T_i\} \subseteq \mathscr{B}(B,N)$. If $\{T_i\}$ is pointwise bounded than $\{T_i\}$ is uniformly bounded.

Proof : Let B be a Banach space, N a normed linear space and $\{T_i\} \subseteq \mathscr{B}(B,N)$.

Assume that, $\{T_i(x)\}$ is bounded subset of N for each $x \in B$.

We have to prove that $\{T_i\}$ is bounded subset of $\mathscr{B}$ (B, N).

For each $n \in \mathbb{N}$, define,

$$F_n = \{x \in B : \|T_i(x)\| \leq n, \forall i\} \qquad \text{...... (1)}$$

Claim : F_n is closed set :

Let $\{x_k\}$ be any sequence in F_n such that $x_k \to x$.

Then $\|T_i(x_k)\| \leq n$, for all *i* and all *k*. ...... (2)

Now, T_i is continuous for each *i*, we have

$$T_i(x_k) \to T_i(x), \text{ for each } i.$$

Further using continuity of norm we have,

$$\|T_i(x_k)\| \to \|T_i(x)\|, \text{ for each } i.$$

Therefore, by (2) we get

$$\|T_i(x)\| \leq n, \forall i$$

This implies, $x \in F_n$. Hence F_n is closed.

Now, as $F_n \subseteq B$, $\forall n \in \mathbb{N}$, we have $\bigcup_{n=1}^{\infty} F_n \subseteq B$.

We prove that $\bigcup_{n=1}^{\infty} F_n = B$.

If possible $\bigcup_{n=1}^{\infty} F_n \neq B$ then $\bigcup_{n=1}^{\infty} F_n \subset B$ and $\exists x \in B$ such that $x \notin \bigcup_{n=1}^{\infty} F_n$.

$\Rightarrow x \notin F_n$, for each *n*.

$\Rightarrow \exists$ *i* such that $\|T_i(x)\| > n$, for each *n*.

Which is contradiction to the fact that $\{T_i(x)\}$ is bounded subset of N for each $x \in B$.

Therefore we must have,

$$B = \bigcup_{n=1}^{\infty} F_n$$

But B being complete by Baire's category theorem $\exists n_0 \in \mathbb{N}$ such that $\overline{F_{n_0}}$ has nonempty interior.

Since F_{n_0} is closed, we have,

$$\overline{F_{n_0}} = F_{n_0}$$

$\Rightarrow F_{n_0}$ has nonempty interior.

Let x_0 is the interior point of F_{n_0}.

Then $S_{r_0}(x_0) \subseteq F_{n_0}$, for some $r_0 > 0$.

$\Rightarrow \|T_i(x)\| \le n_0$, $\forall x \in S_{r_0}(x_0)$ and $\forall i$.

$\Rightarrow$ Each vector in $T_i(S_{r_0}(x_0))$ has norm less than or equal to n_0.

For the sake of brevity we express this fact by writting.

$\|T_i(S_{r_0}(x))\| \le n_0$, for all *i*.

Note that,

$$S_{r_0}(x_0) = x_0 + r_0 S_1(0)$$

Therefore, for each *i*, we have

$$\|T_i S_1(0)\| = \left\| T_i\left(\frac{S_{r_0}(x_0) - x_0}{r_0}\right)\right\|$$

$$= \frac{1}{r_0} \left\| T_i\left(S_{r_0}(x_0)\right) - T_i(x_0) \right\|$$

$$\leq \frac{1}{r_0} \left[\left\| T_i\left(S_{r_0}(x_0)\right) \right\| + \left\| T_i(x_0) \right\| \right]$$

$$\leq \frac{1}{r_0} [n_0 + n_0]$$

$$= \frac{2n_0}{r_0}$$

$\Rightarrow \|T_i(x)\| \leq \dfrac{2n_0}{r_0}$, $\forall x \in S_1(0)$ and $\forall i$.

$\Rightarrow \|T_i(x)\| \leq \dfrac{2n_0}{r_0}$, $x \in B$, $\|x\| \leq 1$ and $\forall i$.

$\Rightarrow \sup\left\{ \|T_i(x)\| : x \in B, \|x\| \leq 1 \right\} \leq \dfrac{2n_0}{r_0}$, $\forall i$.

$\Rightarrow \|T_i\| \leq \dfrac{2n_0}{r_0}$, $\forall i$.

$\Rightarrow \|T_i\|$ is bounded subset of normed space $\mathcal{B}$ (B, N).

Hence, the proof.

❑❑❑

UNIT - III

BOUNDED LINEAR FUNCTIONALS

This unit deals with bounded linear functional, conjugate spaces, Hahn Banach Theorem and its consequences.

3.1 DEFINITION AND PROPERTIES OF FUNCTIONALS

3.1.1 Definition :

A bounded (or continuous) linear functional is bounded linear transformation of with domain is normed space N and range in the scalar field $\mathbb{K}$ ($\mathbb{R}$ or $\mathbb{C}$) of N.

More precisely, if N be a normed space over field $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ then bounded linear transformation $f : N \to \mathbb{K}$ is called bounded (or continuous) linear functional or more briefly functional, where $\mathbb{K} = \mathbb{R}$ if N is real normed space and $\mathbb{K} = \mathbb{C}$ if N is complex normed space.

3.1.2 Remark :

As a bounded linear functional is a special case of bounded linear transformation, all general theorems and properties studied in Unit 2 for bounded linear transformations are true for bounded linear functionals.

We mention here few important definitions and theorems in the form of functionals.

3.1.3 Definition :

Let N be a normed space over field $\mathbb{K}$ ($\mathbb{R}$ or $\mathbb{C}$). A function $f : N \to \mathbb{K}$ is said to be bounded if $\exists k > 0$ such that $|f(x)| \le k\|x\|$, $\forall x \in N$.

$$|f(x)| \le k\|x\|, \ \forall x \in N \ .$$

3.1.4 Theorem : Let f be a functional on normed space N. Then

(i) f is continuous iff f is continuous at a point (any) in N.

(ii) f is continuous iff f is bounded.

3.1.5 Theorem

Let f be a functional on normed space N. Then, norm of-of f can be expressed by any one of the following formulae.

(a) $\|f\| = \sup\{|f(x)| : x \in N, \|x\| \le 1\}$

(b) $\|f\| = \sup\{|f(x)| : x \in N, \|x\| = 1\}$

(c) $\|f\| = \sup\left\{\frac{|f(x)|}{\|x\|} : x \in N, x \ne 0\right\}$

(d) $\|f\| = \sup\{k : k > 0 \text{ and } |f(x)| \le k\|x\|, x \in X\}$

Further $|f(x)| \le \|f\|\|x\|$, $\forall x \in N$.

Equivalently, $\|f\| \ge \frac{|f(x)|}{\|x\|}$, $x \in N$, $x \ne 0$.

3.1.6 Examples of Functions

Example 1 : Let $\mathbb{R}^n$ be the real normed space with the norm

$$\|x\| = \left[\sum_{j=1}^{n} |x_j|^2\right]^{1/2}, \quad x = (x_1, ..., x_n) \in \mathbb{R}^n.$$

Fix any non zero vector $a = (a_1, ..., a_n)$ in $\mathbb{R}^n$ consider the dot product defined by function

$$f : \mathbb{R}^n \to \mathbb{R}, \; f(x) = x \cdot a = x_1 a_1 + ... + x_n a_n.$$

Then *f* is functional on $\mathbb{R}^n$, with $\|f\| = \|a\|$.

***f* is linear :** Let any $x = (x_1,...,x_n)$, $y = (y_1,...,y_n)$ in $\mathbb{R}^n$ and $\alpha, \beta \in \mathbb{R}$.

Then $$f(\alpha x + \beta y) = (\alpha x_1 + \beta y_1, ..., \alpha x_n + \beta y_n) \cdot (a_1, ..., a_n)$$

$$= (\alpha x_1 + \beta y_1) a_1 + ... + (\alpha x_n + \beta y_n) a_n$$

$$= \alpha (x_1 a_1 + ... + x_n a_n) + \beta (y_1 a_1 + ... + y_n a_n)$$

$$= \alpha (x \cdot a) + \beta (y \cdot a)$$

$$= \alpha f(x) + \beta f(y)$$

***f* is bounded :** By Cauchy Schwartz inequality, we have

$$|f(x)| = |x \cdot a| \le \|x\| \cdot \|a\|, \ \forall x \in \mathbb{R}^n. \qquad ......(1)$$

We have proved that *f* is functional on $\mathbb{R}^n$.

Claim $\|f\| = \|a\|$ **:**

By definition, $$\|f\| = \sup\{|f(x)| : x \in \mathbb{R}^n, \|x\| \le 1\}$$

$$\le \sup\{|x \cdot a| : x \in \mathbb{R}^n, \|x\| \le 1\}$$

Using (1) we have,

$$\|f\| \le \sup\{\|x\| \cdot \|a\| : x \in \mathbb{R}^n, \|x\| \le 1\}$$

$$\le \|a\|$$

Thus, $$\|f\| \le \|a\| \qquad .....(2)$$

Further, $$|f(x)| \le \|f\| \cdot \|x\|, \ \forall x \in \mathbb{R}^n.$$

$$\Rightarrow \|f\| \ge \frac{|f(x)|}{\|x\|}, \ x \in \mathbb{R}^n, \ x \ne 0.$$

Inparticular for $x = a$ we have,

$$\|f\| \ge \frac{|f(a)|}{\|a\|} = \frac{|a \cdot a|}{\|a\|} = \frac{a_1^2 + ... + a_n^2}{\|a\|}$$

$$= \frac{\|a\|^2}{\|a\|} = \|a\|$$

Hence $\|f\| \geq \|a\|$ .... (3)

By (2) and (3), we have $\|f\| = \|a\|$.

Example 2 : Consider the Banach space $B = C([a,b],\mathbb{R})$ with the supremum norm

$$\|x\| = \sup_{t\in[a,b]} |x(t)|, \ x \in B.$$

Define, $f : B \longrightarrow \mathbb{R}$, by

$$f(x) = \int_b^a x(t)\,dt, \ x \in B.$$

Then *f* is functional on B and $\|f\| = b - a$.

***f* is linear :** Let any $x, y \in B$ and $\alpha, \beta \in \mathbb{R}$.

Then $$f(\alpha x + \beta y) = \int_a^b (\alpha x + \beta y)(t)\,dt = \int_a^b (\alpha x(t) + \beta y(t))\,dt$$

$$= \alpha \int_a^b x(t)\,dt + \beta \int_a^b y(t)\,dt$$

$$= \alpha f(x) + \beta f(y)$$

***f* is bounded :** For any $x \in B$, we have,

$$|f(x)| = \left| \int_a^b x(t)\,dt \right|$$

$$\leq \int_a^b |x(t)|\,dt \leq \int_a^b \|x\|\,dt$$

$$\leq \|x\| \int_a^b dt = (b-a)\|x\|$$

Therefore, $\quad |f(x)| \leq (b-a)\|x\|, \; x \in B$. .... (1)

This implies f is bounded.

We have proved that f is functional on B.

Claim : $\|f\| = b-a$.

By definition,

$$\|f\| = \sup\{|f(x)| : x \in B, \|x\| \leq 1\}$$

Using (1), we have

$$\|f\| \leq \sup\{(b-a)\|x\| : x \in B, \|x\| \leq 1\}$$

$$\leq b-a$$

Thus $\quad \|f\| \leq b-a$ ..... (2)

Consider the function $x_0 : [a,b] \longrightarrow \mathbb{R}$ defined by $x_0(t) = 1$, $\forall t \in [a,b]$. Then $x_0 \in B$.

We know for bounded linear functional

$$|f(x)| \leq \|f\| \cdot \|x\|, \; \forall x \in B$$

$$\Rightarrow \|f\| \geq \frac{|f(x)|}{\|x\|}, \; x \in B, \; x \neq 0$$

Inparticular for $x = x_0$ we have

$$\|f\| \geq \frac{|f(x_0)|}{\|x_0\|}$$

$$= \frac{\left|\int_a^b x_0(t)\,dt\right|}{\|x_0\|}$$

$$= \int_a^b dt$$

$$= b - a$$

i.e. $\|f\| \geq b - a$ .... (3)

From (2) and (3), we have

$$\|f\| = b - a .$$

EXERCISE :

1. Consider the Banach space $B = C([a,b],\mathbb{R})$ with the supremum norm

$$\|x\| = \sup_{t\in[a,b]} |x(t)|, \; x \in B .$$

Fix any $x_0 \in B$, and define $f : B \longrightarrow \mathbb{R}$ by

$$f(x) = \int_a^b x(t)|x_0(t)|dt, \; x \in B .$$

Prove that *f* is functional on B with

$$\|f\| = \int_a^b |x_0(t)|dt .$$

2. Let $B = C([a,b],\mathbb{R})$ be a Banach space with the supremum norm

$$\|x\| = \sup_{t\in[a,b]} |x(t)|, \; x \in B .$$

Fix a point $t \in (a,b)$. Define $f_t : B \longrightarrow \mathbb{R}$ by

$$f_t(x) = x(t), \; x \in B .$$

Prove that f_t is a functional on B and $\|f_t\| = 1$.

3.1.7 Theorem : Let N be a normed space. A linear transformation *f* on N is bounded (continuous) if and only if ker (*f*) is closed.

Proof : Let $f : N \to \mathbb{K}$ ($\mathbb{R}$ or $\mathbb{C}$) is bounded linear functional, and hence it is continuous.

Then $$\ker(f) = \{x \in N : f(x) = 0\} = f^{-1}(\{0\}).$$

Since $\{0\}$ is closed subset of $\mathbb{K}$ ($\mathbb{R}$ or $\mathbb{C}$), and $f : N \to \mathbb{K}$ is continuous, it follows that $f^{-1}(\{0\})$ is closed in N.

This proves ker (*f*) is closed in N.

Conversely, let ker (*f*) is closed set in N.

We have to prove that the linear transformation $f : N \to \mathbb{K}$ is bounded.

If *f* = 0 then *f* is clearly continuous and hence bounded.

Let $f \neq 0$ linear transformation.

Then $N - \ker(f) \neq \phi$.

Since ker (*f*) is closed, $N - \ker(f)$ is open set in N.

Fix any $x_0 \in N - \ker(f)$. Then $x_0 \in N$ and $x_0 \notin \ker(f)$. Hence $f(x_0) \neq 0$.

Define $y_0 = \dfrac{x_0}{f(x_0)}$.

Then $y_0 \in N$ and $f(y_0) = f\left(\dfrac{x_0}{f(x_0)}\right) = \dfrac{f(x_0)}{f(x_0)} = 1$.

Therefore $y_0 \in N - \ker(f)$.

Since $N - \ker(f)$ is an open set, $\exists\, r > 0$.

Such that $S_r(y_0) \subseteq N - \ker(f)$.

Claim : $\quad |f(x)| < 1,\ \forall x \in S_r(0)$ ..... (1)

If possible, $\exists\, x_1 \in S_r(0)$ s such that $|f(x)| \geq 1$.

Define, $y_1 = \frac{-x_1}{f(x_1)}$.

Then $y_1 \in N$ and,

$$\|y_1\| = \left\|\frac{-x_1}{f(x_1)}\right\| = \frac{\|x_1\|}{|f(x_1)|} < r.$$

$$\Rightarrow \|(y_0 + y_1) - y_0\| = \|y_1\| < r$$

$$\Rightarrow y_0 + y_1 \in S_r(y_0)$$

Further, $$f(y_0 + y_1) = f(y_0) + f(y_1)$$

$$= 1 + f\left(\frac{-x_1}{f(x_1)}\right)$$

$$= 1 - \frac{f(x_1)}{f(x_1)}$$

$$= 1 - 1 = 0$$

$$\Rightarrow y_0 + y_1 \in \ker(f).$$

Therefore,

$$y_0 + y_1 \in \ker(f) \cap S_r(y_0)$$

$$\Rightarrow \ker(f) \cap S_r(y_0) \neq \phi$$

This contradicts to the fact that

$$S_r(y_0) \subseteq N - \ker(f)$$

Thus (1) must be true.

This proves (1).

Now for any $x \neq 0$ in N, we have,

$$\left\|\frac{rx}{2\|x\|}\right\| = \frac{r}{2}\frac{\|x\|}{\|x\|} = \frac{r}{2} < r$$

Therefore by (1), we have,

$$\left|f\left(\frac{rx}{2\|x\|}\right)\right|<1$$

$$\Rightarrow\left|\frac{r}{2\|x\|}f(x)\right|<1$$

$$\Rightarrow|f(x)|<\left(\frac{2}{r}\right)\|x\|$$

Further for $x=0$,

$$|f(x)|=0=\|x\|$$

Thus $|f(x)|=\left(\frac{2}{r}\right)\|x\|$ if $x=0$.

We have proved that

$$|f(x)|\leq\left(\frac{2}{r}\right)\|x\|,\ \forall x\in N.$$

$\Rightarrow f$ is bounded linear functional.

3.1.8 Remark : The theorem 3.1.6 need not hold, in general for linear transformations between arbitrary normed spaces.

3.2 CONJUGATE SPACE (DUAL SPACE)

We know if N and N' are normed spaces over the same field of scalar $\mathbb{K}$ ($\mathbb{R}$ or $\mathbb{C}$), then the set $\mathscr{B}$ (N, N') of all continuous linear transformations of N into N' is a normed space over $\mathbb{K}$ (See Theorem 2.3.1).

In particular if N be a normed space and $N'=\mathbb{K}$ the set $\mathscr{B}(N,\mathbb{K})$ of all bounded linear functionals on N is normed space with the norm.

$$\|f\|=\sup\{|f(x)|:x\in N,\|x\|\leq 1\}$$

Since $\mathbb{K}$ is complete space, by theorem 2.3.1, it follows that $\mathscr{B}(N,\mathbb{K})$ Banach space over field $\mathbb{K}$.

3.2.1 Definition

If N is an arbitrary normed space, then the set of all bounded (continuous) linear transformation of N into $\mathbb{R}$ or $\mathbb{C}$, according as N is real or complex normed space, is the set $\mathscr{B}(N,\mathbb{R})$ or $\mathscr{B}(N,\mathbb{C})$ and is called conjugate space (or dual space) of N.

The dual space of N is denoted by N*. Thus $N^* = \mathscr{B}(N,\mathbb{R})$ or $\mathscr{B}(N,\mathbb{C})$ according as N is real or complex normed space.

3.2.2 Definition

A member of $N^* = \mathscr{B}(N,\mathbb{K})$, ($\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$) is called bounded linear function or more briefly it is called function.

3.3 THE HAHN-BANACH THEOREM

The theory of conjugate spaces is completely rests on the Hahn-Banach theorem, which is most important theorem in connection with bounded linear functionals. The Hahn-Banach theorem is an extension theorem for bounded linear functional. It asserts that a bounded linear functional f defined on subspace M of a normed linear space N can be extended from M to the entire space N in a such way that the certain basic properties of f continue to hold good for extended functional.

For proving the Hahn-Banach theorem, firstly we prove the Hahn-Banach Lemma.

3.3.1 Lemma (Hahn-Banach Lemma)

Let M be a linear subspace of a normed linear space N, and let f be a functional defined on M. If x_0 is a vector not in M, and if $M_0 = M + [x_0]$ is the linear subspace spanned by M and x_0, then f can be extended to a functional f_0 defined on M_0 such that $\|f_0\| = \|f\|$.

Proof : Let M be a linear subspace of normed space N. Let $f : M \to \mathbb{K}$ ($\mathbb{R}$ or $\mathbb{C}$) be a bounded linear functional.

Without loss of generality we may assume that $\|f\| = 1$.

We give the proof in two parts :

(I) When N is a real normed space.

(II) When N is a complex normed space.

Case - I : Let N be a real normed space.

Let $f : M \longrightarrow \mathbb{R}$ be a bounded linear functional.

Fix $x_0 \notin M$ and let $M_0 = M + [x_0] = \{x + \alpha x_0 : x \in M, \alpha \in \mathbb{R}\}$

Then M_0 is a linear subspace of N and $M \leq M_0$.

Define $f_0 : M_0 \longrightarrow \mathbb{R}$ by,

$$f_0(x + \alpha x_0) = f(x) + \alpha r_0, \; x \in M, \; \alpha \in \mathbb{R},$$

for any choice of the real number $r_0 = f(x_0)$.

***f_0* is an extension of *f* :**

For any $x \in M$ we have,

$$f_0(x) = f(x + 0(x_0)) = f(x) + (0)r_0 = f(x)$$

$\Rightarrow f_0 = f$ on M.

***f_0* is Linear :**

Let any $y_1, y_2 \in M_0$. Then $y_1 = x_1 + \alpha_1 x_0$ and $y_2 = x_2 + \alpha_2 x_0$ for some $x_1, x_2 \in M$ and $\alpha_1, \alpha_2 \in \mathbb{R}$.

Then, $f_0(y_1 + y_2) = f_0((x_1 + x_2) + (\alpha_1 + \alpha_2)x_0)$

$= f(x_1 + x_2) + (\alpha_1 + \alpha_2)r_0$

$= f(x_1) + f(x_2) + (\alpha_1 + \alpha_2)r_0$ [$\because$ f is linear]

$= (f(x_1) + \alpha_1 r_0) + (f(x_2) + \alpha_2 r_0)$

$$= f(x_1 + \alpha_1 x_0) + f(x_2 + \alpha_2 x_0)$$

$$= f(y_1) + f(y_2) \ .$$

Further for any scalar $a \in \mathbb{R}$,

$$f_0(ay_1) = f_0(ax_1 + a\alpha_1 x_0) = f(ax_1) + a\alpha_1 r_0$$

$$= af(x_1) + a\alpha_1 r_0 = a\left[f(x_1) + \alpha_1 r_0\right]$$

$$= a\left[f(x_1 + \alpha_1 x_0)\right] = af(y_1)$$

f_0 is bounded :

Let $x_1, x_2 \in M$. Then,

$$f(x_2) - f(x_1) = f(x_2 - x_1)$$

$$\leq \left|f(x_2 - x_1)\right|$$

$$\leq \|f\|\|x_2 - x_1\|$$

$$= \|x_2 - x_1\| \qquad (\because \|f\| = 1)$$

$$= \|(x_2 + x_0) - (x_1 + x_0)\|$$

$$\leq \|x_2 + x_0\| + \|x_1 + x_0\|$$

This gives,

$$-f(x_1) - \|x_1 + x_0\| \leq -f(x_2) + \|x_2 + x_0\|, \ \forall x_1, x_2 \in M \ .$$

Therefore,

$$\sup\left\{-f(x) - \|x + x_0\| : x \in M\right\} \leq \sup\left\{-f(x) + \|x + x_0\| : x \in M\right\}$$

Choose r_0 to be any real number such that,

$$\sup\left\{-f(x) - \|x + x_0\| : x \in M\right\} \leq r_0 \leq \sup\left\{-f(x) + \|x + x_0\| : x \in M\right\}$$

$$\Rightarrow -f(x) - \|x + x_0\| \leq r_0 \leq -f(x) + \|x + x_0\|, \ \forall x \in M$$

For any $x \in M$ and $\alpha \neq 0 \in \mathbb{R}$, we have $\frac{x}{\alpha} \in M$.

Thus,

$$-f\left(\frac{x}{\alpha}\right) - \left\|\frac{x}{\alpha} + x_0\right\| \leq r_0 \leq -f\left(\frac{x}{\alpha}\right) + \left\|\frac{x}{\alpha} + x_0\right\|, \ \forall x \in M$$

If $\alpha > 0$ then

$$-f(x) - \|x + \alpha x_0\| \leq \alpha r_0 \leq -f(x) + \|x + \alpha x_0\|, \ \forall x \in M$$

$$\Rightarrow -\|x + \alpha x_0\| \leq f(x) + \alpha r_0 \leq \|x + \alpha x_0\|, \ \forall x \in M$$

$$\Rightarrow |f(x) + \alpha r_0| \leq \|x + \alpha x_0\|, \ \forall x \in M$$

Therefore,

$$|f_0(x + \alpha x_0)| \leq \|x + \alpha x_0\|, \ x \in M, \ \alpha \in \mathbb{R}, \ \alpha > 0.$$

On the same line, one can prove that

$$|f_0(x + \alpha x_0)| \leq \|x + \alpha x_0\|, \ x \in M, \ \alpha \in \mathbb{R}, \ \alpha < 0.$$

Combining we have,

$|f_0(x + \alpha x_0)| \leq \|x + \alpha x_0\|$, for all $x \in M$ and all $\alpha \in \mathbb{R}$.

Thus, $|f_0(y)| \leq \|y\|, \ \forall y \in M_0$ ...... (1)

This implies $f_0 : M \longrightarrow \mathbb{R}$ bounded (continuous).

To prove $\|f_0\| = 1$:

Using (1) and definition of norm of functional we have,

$$\|f_0\| = \sup\left\{\frac{|f_0(y)|}{\|y\|} : y \in M_0, y \neq 0\right\} \leq 1$$

i.e. $\|f_0\| \leq 1$

Further,

$$\|f_0\| = \sup\{|f_0(y)| : y \in M_0, \|y\| \le 1\}$$

$$\ge \sup\{|f_0(y)| : y \in M, \|y\| \le 1\} \qquad (\because M \subseteq M_0)$$

$$= \sup\{|f(y)| : y \in M, \|y\| \le 1\} \qquad (\because f = f_0 \text{ on M})$$

$$= \|f\|$$

Thus, $\|f_0\| \ge \|f\|$.

But $\|f\| = 1$ implies $\|f_0\| \ge 1$.

Therefore, $\|f_0\| = 1$.

We have proved that $f_0 : M_0 \to \mathbb{R}$ is a functional extension of $f : M \longrightarrow \mathbb{R}$ such that, $\|f_0\| = \|f\| = 1$.

Case II : Let N be a complex normed space.

Let $f : M \longrightarrow \mathbb{C}$ bounded linear functional with $\|f\| = 1$.

Let $g = \operatorname{Re}(f)$ and $h = \operatorname{Im}(f)$. Then

$$f(x) = g(x) + ih(x),\ x \in M.$$

Where $g : M \longrightarrow \mathbb{R}$ and $h : M \longrightarrow \mathbb{R}$.

Note that $\|g\| \le \|f\| = 1 \Rightarrow \|g\| \le 1$.

(i) Since f is linear, for any $x, y \in M$ and $\alpha \in \mathbb{R}$, we have,

$$f(x+y) = f(x) + f(y)$$

$$\Rightarrow g(x+y) + ih(x+y) = [g(x) + ih(x)] + [g(y) + ih(y)]$$

$$= [g(x) + g(y)] + i[h(x) + h(y)]$$

$$\Rightarrow g(x+y) = g(x) + g(y),\ h(x+y) = h(x) + h(y)$$

and $\quad f(\alpha x) = \alpha f(x)$

$$\Rightarrow g(\alpha x) + ih(\alpha x) = \alpha\left(g(x) + ih(x)\right)$$

$$= \alpha g(x) + i\alpha h(x)$$

$$\Rightarrow g(\alpha x) = \alpha g(x),\ h(\alpha x) = \alpha h(x).$$

Therefore $g, h : M \longrightarrow \mathbb{R}$ both are linear.

(ii) Since *f* is bounded on M and for all $x \in M$.

$$|g(x)| \leq |f(x)| \leq \|f\|\|x\|,\ |h(x)| \leq |f(x)| \leq \|f\|\|x\|.$$

It follows that *g* and *h* both are bounded on M.

By part (i) and (ii), $g, h : M \longrightarrow \mathbb{R}$ are bounded linear functional.

As $f : M \longrightarrow \mathbb{C}$ is linear, for all $x \in M$,

$$f(ix) = if(x)$$

$$\Rightarrow g(ix) + ih(ix) = i\left(g(x) + ih(x)\right)$$

$$= ig(x) - h(x)$$

$$\Rightarrow h(x) = -g(ix) \text{ and } g(x) = h(ix)$$

So we can write,

$$f(x) = g(x) + ih(x)$$

$$= g(x) - ig(ix),\ x \in M .$$

Since $g : M \longrightarrow \mathbb{R}$ is functional, by Case I, *g* can be extended to a functional $g_0 : M_0 \to \mathbb{R}$ such that $\|g\| = \|g_0\|$.

Define $f_0 : M_0 \to \mathbb{C}$ by

$$f_0(x) = g_0(x) - ig_0(ix), \ x \in M_0 \qquad \text{...... (2)}$$

We prove that f_0 is the required functional with desired property.

f_0 is linear :

Let any $x, y \in M_0$. Then,

$$f_0(x+y) = g_0(x+y) - ig_0(i(x+y))$$

$$= g_0(x) + g_0(y) - i[g_0(ix) + g_0(iy)]$$

$$= [g_0(x) - ig_0(ix)] + [g_0(y) - ig_0(iy)]$$

$$= f_0(x) + f_0(y)$$

Also for any $a \in \mathbb{R}$,

$$f_0(ax) = g_0(ax) - ig_0(iax)$$

$$= ag_0(x) - iag_0(ix)$$

$$= a[g_0(x) - ig_0(ix)]$$

$$= af_0(x)$$

Therefore for any $\alpha = a + ib$, $a, b \in \mathbb{R}$, we have

$$f_0(\alpha x) = f_0(ax + ibx) = af_0(x) + bf_0(ix) \qquad \text{..... (3)}$$

But $\qquad f_0(ix) = g_0(ix) - ig_0(i^2 x)$

$$= g_0(ix) - ig_0(-x)$$

$$= g_0(ix) + ig_0(x)$$

$$= i[g_0(x) - ig_0(ix)]$$

$$= if_0(x)$$

Thus (3) becomes

$$f_0(\alpha x) = af_0(x) + ibf_0(x)$$

$$= (a+ib)\,f_0(x)$$

$$= \alpha\, f_0(x)$$

Hence $f_0 : M_0 \to \mathbb{C}$ is linear.

f_0 is bounded :

Since $g_0 : M_0 \to \mathbb{R}$ is bounded, for any $x \in M$, we have,

$$|f_0(x)| = |g_0(x) - ig_0(ix)|$$

$$\leq |g_0(x)| + |g_0(ix)|$$

$$\leq \|g_0\|\|x\| + \|g_0\|\|ix\|$$

$$= 2\|g_0\|\|x\|$$

$$\therefore |f_0(x)| \leq (2\|g_0\|)\|x\|,\ \forall x \in M_0$$

Therefore, $f_0 : M \longrightarrow \mathbb{C}$ is bounded (continuous)

f_0 is extension of f :

Since $g_0 = g$ on M, from (2) it follows that $f_0 = f$ on M.

To prove $\|f_0\| = 1$:

Let $x \in M_0$ and $\|x\| = 1$.

(a) If $f_0(x)$ is real, then $f_0(x) = g_0(x)$.

Thus $|f_0(x)| = |g_0(x)|$

$\leq \|g_0\| \|x\|$

But $\|g_0\| = \|g\| \leq 1$ and $\|x\| = 1$

$\therefore |f_0(x)| \leq 1$.

(b) If $f_0(x)$ is complex, then

$f_0(x) = |f_0(x)| e^{i\theta}$, where $\theta = \arg(f_0(x))$.

Then,

$$|f_0(x)| = e^{-i\theta} f_0(x) = f_0(e^{-i\theta} x) \qquad \text{..... (4)}$$

[$\because f_0 : M_0 \to \mathbb{C}$ is linear]

$\Rightarrow f_0(e^{-i\theta} x)$ is real.

Moreover,

$e^{-i\theta} x \in M$ and $\|e^{-i\theta} x\| = |e^{-i\theta}| \|x\| = \|x\| = 1$

Therefore, by part (a),

$$|f_0(e^{-i\theta} x)| \leq 1 \qquad \text{..... (5)}$$

From (4) and (5), $|f_0(x)| \leq 1$.

Combining part (a) and (b),

$|f_0(x)| \leq 1$, for $x \in M_0$, $\|x\| = 1$.

$\Rightarrow \|f_0\| = \sup\{|f_0(x)| : x \in M_0, \|x\| = 1\} \leq 1$

i.e. $\|f_0\| \leq 1$

Also $\|f_0\| \geq \|f\|$ (already proved) and $\|f\| = 1$, we have $\|f_0\| \geq 1$.

Thus $\|f_0\| = 1$.

Hence $f_0 : M_0 \to \mathbb{C}$ is functional extension of $f : M \longrightarrow \mathbb{C}$ such that $\|f_0\| = \|f\| = 1$.

This complete the proof.

3.3.2 Definition :

A partially ordered set (or poset) is a pair (P, $\leq$) where P is a set and '$\leq$' is a binary relation on P which satisfies for all *x, y* and *z* in P.

(a) Reflexivity : $x \leq x$

(b) Antisymmetry : If $x \leq y$ and $y \leq x$ then $x = y$

(c) If $x \leq y$ and $y \leq z$ then $x \leq z$.

Let (P, $\leq$) is poset. For $x, y \in P$ if either $x \leq y$ or $y \leq x$, then *x* and *y* are called comparable.

A subset C of poset (P, $\leq$) is called chain if every pair of elements of C are comparable.

An upper bound of subset $A \subseteq P$ is any $x \in P$ such that $a \leq x$, $\forall a \in A$.

An element *x* in poset (P, $\leq$) is called a maximal element if there is no element *y* in P such that $x \leq y$. i.e. if $x \leq y$ then $x = y$.

3.3.3 Lemma (Zorn's Lemma) :

If (P, $\leq$) is a partially ordered set in which every chain has an upper bound, then P has a maximal element.

3.3.4 Theorem (Hahn-Banach Theorem)

Let M be a linear subspace of a normed linear space N, and let *f* be a functional defined on M. Then *f* can be extended to a functional f_0 defined on the whole space N such that $\|f_0\| = \|f\|$.

Proof : Let M be a linear subspace of a normed space N. Let *f* be a functional on M.

Let P is the set of all ordered pair (f_α, M_α) where f_α is functional extension of *f* to the subspace $M_\alpha \supseteq M$ and $\|f_\alpha\| = \|f\|$.

Since $(f, M) \in P$, P is nonempty.

Define the relation '$\leq$' on P by $(f_\alpha, M_\alpha) \leq (f_\beta, M_\beta)$ iff $M_\alpha \subseteq M_\beta$ and $f_\alpha = f_\beta$ on M_α. Then clearly (P, $\leq$) is partially ordered set.

Let $\boldsymbol{c} = \{(f_j, M_j)\}$ be any chain in P.

Define $M' = \bigcup_j \{M_j : (f_j, M_j) \in \boldsymbol{c}\}$.

M' is subspace of N :

Let any $x, y \in M'$. Then $x \in M_i$ and $y \in M_j$ for some *i* and *j*. Since $\boldsymbol{c}$ is chain either $M_i \subseteq M_J$ or $M_j \subseteq M_i$.

Let us suppose $M_i \subseteq M_j$. Then $x, y \in M_j$.

Since M_j is linear subspace of N, we have $\mu x + \lambda y \in M_j$ for any scalar μ and λ. But $M_j \subseteq M'$ implies $\mu x + \lambda y \in M'$.

Define $h' : M' \longrightarrow \mathbb{K}$ ($\mathbb{R}$ or $\mathbb{C}$) by

$$h'(x) = f_j(x) \text{ if } x \in M_j \text{ and } (f_j M_j) \in \boldsymbol{c}$$

Clearly $(h', M') \in P$.

Further, for any $(f_j, M_j) \in \boldsymbol{c}$ we have $M_j \subseteq M'$ and $h' = f_j$ on M_j.

Therefore $(f_j, M_j) \leq (h', M')$, $\forall (f_j, M_j) \in \boldsymbol{c}$.

This implies (h', M') is an upper bound of C.

We have provd that every chain $\boldsymbol{c}$ in P has an upper bound. Therefore by zorns lemma P has maximal element, say (f_0, M_0).

Thus f_0 is functional extension of *f* to the subspace $M_0 \supseteq M$ such that $\|f_0\| = \|f\|$.

We claim that $M_0 = N$.

If possible $M_0 \neq N$, then there exists $x_0 \in N - M_0$.

This implies $x_0 \notin M_0 \left(x_0 \in N\right)$.

Therefore by Hahn-Banach Lemma 3.3.1 f_0 can be extended to a functional h_0 defined on subspace $M_0 + [x_0]$ such that $\|h_0\| = \|f_0\|$.

But then $\left(h_0, M_0 + [x_0]\right) \in P$ and $\left(f_0, M_0\right) \leq \left(h_0, M_0 + [x_0]\right)$, which is contradiction to maximality of $\left(f_0, M_0\right)$.

Hence we must have $M_0 = N$.

We have proved that $\exists$ a functional f_0 on N such that $\|f_0\| = \|f\|$.

This completes the proof.

3.4 CONSEQUENCES OF HAHN BANACH THEOREM

3.4.1 Theorem : If N is a normed linear space and x_0 is a non-zero vector in N, then there exists a functional f_0 in N* such that $f_0\left(x_0\right) = \|x_0\|$ and $\|f_0\| = 1$.

Proof : Let N is a normed linear space over field $\mathbb{K}$ ($\mathbb{R}$ or $\mathbb{C}$) and let $x_0 \neq 0$ in N.

Consider the set,

$$M = \left\{\alpha x_0 : \alpha \in \mathbb{K}\right\}$$

Then M, is clearly linear subspace of N spanned by x_0.

Define $f : M \to \mathbb{K}$ by

$$f\left(\alpha x_0\right) = \alpha \|x_0\|, \ \alpha \in \mathbb{K}.$$

We prove that *f* is functional on M with desired property.

***f* is linear :** Let $y, y' \in M$. Then $y = \alpha x_0$ and $y' = \alpha' x_0$ for some $\alpha, \alpha' \in \mathbb{K}$.

Then, $\quad f\left(y + y'\right) = f\left(\alpha x_0 + \alpha' x_0\right)$

$$= f\left((\alpha+\alpha')x_0\right)$$

$$=(\alpha+\alpha')\|x_0\|$$

$$=\alpha\|x_0\|+\alpha'\|x_0\|$$

$$= f\left(\alpha x_0\right)+ f\left(\alpha' x_0\right)$$

$$= f(y)+ f(y')$$

and for any $a\in\mathbb{K}$ we have,

$$f(ay)= f\left(a\alpha x_0\right)= a\alpha\|x_0\|= af\left(\alpha x_0\right)$$

$$= af(y)$$

Thus *f* is linear.

***f* is bounded (Continuous) :**

Let any $y\in M$. Then $y=\alpha x_0$ for some $\alpha\in\mathbb{K}$. Then

$$\left|f(y)\right|=\left|f\left(\alpha x_0\right)\right|=\left|\alpha\|x_0\|\right|=|\alpha|\|x_0\|$$

$$=\|\alpha x_0\|$$

$$=\|y\|$$

Therefore

$$\left|f(y)\right|=\|y\|,\ \forall y\in M \qquad \text{..... (1)}$$

This implies *f* is bounded (continuous).

Using (1), we have

$$\|f\|=\sup\left\{\frac{\left|f(y)\right|}{\|y\|}: y\in M, y\neq 0\right\}=1$$

By definition of *f*, $f\left(x_0\right)=\|x_0\|$ $\quad (\because$ Take $\alpha=1)$

Thus $f : M \to \mathbb{K}$ is functional on M such that,

$$f(x_0) = \|x_0\| \text{ and } \|f\| = 1 \qquad \text{..... (2)}$$

By Hahn-Banach theorem $\exists$ a functional $f_0 \in N^*$ such that,

$$f = f_0 \text{ on M and } \|f\| = \|f_0\| \qquad \text{...... (3)}$$

From (2) and (3), we have

$$f_0(x_0) = f(x_0) = \|x_0\| \qquad (\because x_0 \in M)$$

and $\|f_0\| = \|f\| = 1$.

This completes the proof.

3.4.2 Corollary :

Let N be a normed space. If x and y are any two distinct vectors in N then there exists a functional $f \in N^*$ such that $f(x) \neq f(y)$.

OR

The conjugate space N* separates the vectors in N.

Proof : Let N be a normed space.

Let any $x \neq y$ in N. Then $x - y \neq 0$ in N. By Theorem 3.4.1, $\exists$ functional $f \in N^*$ such that $f(x-y) = \|x-y\| \neq 0$.

This gives $f(x) - f(y) \neq 0 \Rightarrow f(x) \neq f(y)$.

3.4.3 Corollary :

Let N be a normed space.

If $f(x) = 0$, $\forall f \in N^*$ then $x = 0$.

Proof : Let N be a normed space.

Assume $f(x) = 0$, for all $f \in N^*$.

If possible $x \neq 0$ in N. Then, by Theorem 3.4.1, $\exists f \in N^*$ such that $f(x) = \|x\| \neq 0$, a contradiction to our assumption.

Hence we must have $x = 0$.

3.4.4 Corollary :

Let N be a normed space and $x \in N$. Then,

$$\|x\| = \sup\left\{\frac{|f(x)|}{\|f\|} : f \in N^*, f \neq 0\right\}$$

Proof : Let N be a normed space and $x \in N$.

If $x = 0$ then $\|x\| = 0$ and $f(x) = f(0) = 0$, $\forall f \in N^*$.

Therefore,

$$\sup\left\{\frac{|f(x)|}{\|f\|} : f \in N^*, f \neq 0\right\} = 0 = \|x\|$$

Let any $x \neq 0$ in N. By Theorem 3.4.1 there exists functional $f_0 \in N^*$ such that

$$f_0(x) = \|x\| \text{ and } \|f_0\| = 1.$$

Therefore

$$\|x\| = \frac{|f_0(x)|}{\|f_0\|} \leq \sup\left\{\frac{|f(x)|}{\|f\|} : f \in N^*, f \neq 0\right\} \quad \text{..... (1)}$$

For any $f \in N^*$, we have,

$$|f(x)| \leq \|f\|\|x\|.$$

Thus,

$$\sup\left\{\frac{|f(x)|}{\|f\|} : f \in N^*, f \neq 0\right\} \leq \sup\left\{\frac{\|f\|\|x\|}{\|f\|} : f \in N^*, f \neq 0\right\}$$

$$= \|x\|$$

i.e. $\sup\left\{\dfrac{|f(x)|}{\|f\|} : f \in N^*, f \neq 0\right\} \leq \|x\|$ ..... (2)

From (1) and (2) we have,

$$\|x\| = \sup\left\{\dfrac{|f(x)|}{\|f\|} : f \in N^*, f \neq 0\right\}$$

3.4.5 Theorem :

Let M be a subspace of normed space N and let $x_0 \in N$ be such that $d(x_0, M) = d > 0$. Then there exists a functional $f_0 \in N^*$ such that,

$$f_0(x_0) = 1,\ f_0(M) = 0 \text{ and } \|f_0\| = \frac{1}{d}.$$

Proof : Let M be a linear subspace of normed space N over the field $\mathbb{K}$.

Let $x_0 \in N$ be such that $d(x_0, M) = d > 0$.

Then $x_0 \notin M$.

Consider the set,

$$M_0 = M + [x_0]$$

$$= \{m + \alpha x_0 : m \in M, \alpha \in \mathbb{K}\}$$

Then M_0 is linear subspace of N and each $y \in M_0$ has unique expression

$$y = m + \alpha x_0,\ m \in M \text{ and } \alpha \in \mathbb{K}.$$

Define $f : M_0 \to \mathbb{K}$ by

$$f(m + \alpha x_0) = \alpha,\ m \in M,\ \alpha \in \mathbb{K}.$$

We prove that f is a functional on M_0 with desired property.

***f* is linear :** Let any $y, y' \in M_0$. Then $y = m + \alpha x_0$, $y' = m' + \alpha' x_0$ for some $m, m' \in M$ and $\alpha, \alpha' \in \mathbb{K}$. Then,

$$f(y+y') = f\left((m+\alpha x_0)+(m'+\alpha' x_0)\right)$$
$$= f\left((m+m')+(\alpha+\alpha')x_0\right)$$
$$= \alpha + \alpha'$$
$$= f(m+\alpha x_0) + f(m'+\alpha' x_0)$$
$$= f(y) + f(y')$$

and for any scalar $a \in \mathbb{K}$.

$$f(ay) = f\left(a(m+\alpha x_0)\right)$$
$$= f(am + a\alpha x_0)$$
$$= a\alpha$$
$$= af(m+\alpha x_0)$$
$$= af(y)$$

Thus *f* is linear.

***f* is bounded (Continuous) :**

For all $y = m + \alpha x_0$ in M_0,

$$|f(y)| = |f(m+\alpha x_0)| = |\alpha| \qquad \text{...... (1)}$$

Case 1 : Let $\alpha \neq 0$. Then,

$$\|y\| = \|m + \alpha x_0\|$$
$$= \left\| -\alpha\left(-\frac{m}{\alpha} - x_0 \right) \right\|$$

$$= |\alpha| \left\| \left(-\frac{m}{\alpha} \right) - x_0 \right\|$$

$$\geq |\alpha| \inf \left\{ \|x - x_0\| : x \in M \right\}$$

$$= |\alpha| d$$

$$= |f(y)| d \qquad (\because \text{ By } (1))$$

This gives,

$$|f(y)| \leq \frac{1}{d} \|y\|$$

Case 2 : Let $\alpha = 0$. Then,

$$\|y\| = \|m\| \geq 0 = d \cdot (0) = d\,|f(y)| \qquad (\because \text{ By } (1))$$

$$\Rightarrow |f(y)| \leq \frac{1}{d} \|y\|$$

By Cases 1 and 2, *f* is bounded (continuous).

Thus *f* is functional on M_0.

Further by (1),

$$\|f\| = \sup \left\{ \frac{|f(y)|}{\|y\|} : y \in M_0, y \neq 0 \right\} \leq \frac{1}{d}$$

i.e. $$\|f\| \leq \frac{1}{d} \qquad \text{..... (2)}$$

Since $d = \inf \left\{ \|m - x_0\| : m \in M \right\}$ there exists a sequence $\{m_n\}$ in M such that,

$$\|m_n - x_0\| \longrightarrow d \text{ as } n \to \infty.$$

Now, $-1 = f\left(m_n + (-1) x_0\right) = f\left(m_n - x_0\right)$

$$\Rightarrow 1 = \left| f\left(m_n - x_0\right) \right|$$

$$\leq \|f\| \|m_n - x_0\| \text{ for all } n.$$

Taking limit as $n \to \infty$, we obtain

$$1 \leq \|f\| d$$

$$\Rightarrow \frac{1}{d} \leq \|f\| \qquad \text{...... (3)}$$

From (2) and (3) we obtain,

$$\|f\| = \frac{1}{d}.$$

Thus $f : M_0 \to \mathbb{K}$ is functional on M_0 such that,

$$f(x_0) = f(0 + (1)x_0) = 1$$

$$f(m) = f(m + (0)x_0) = 0, \ \forall m \in M \Rightarrow f(M) = 0$$

and $\|f\| = \dfrac{1}{d}$.

By Hahn-Banach Theorem there exists a functional $f_0 \in N^*$ such that $f_0 = f$ on M_0 and $\|f\| = \|f_0\|$.

But then,

$$f_0(x_0) = f(x_0) = 1 \qquad (\because x_0 \in M_0)$$

$$f_0(M) = f(M) = 0 \qquad (\because M \subseteq M_0)$$

and $\|f_0\| = \|f\| = \dfrac{1}{d}$.

This complete the proof of the Theorem.

3.4.6 Corollary : Let M be a subspace of normed space N and let $x_0 \in N$ be such that $d(x_0, M) = d > 0$. Then there exists a functional $f_0 \in N^*$ such that $f_0(x_0) = d$, $f_0(M) = 0$ and $\|f_0\| = 1$.

Proof : Let M be a linear subspace of normed space N. Let $x_0 \in N$ be such that $d(x_0, M) = d > 0$. Then by Theorem 3.4.5 $\exists$ a functional $g_0 \in N^*$ such that

$$g_0(x_0) = 1,\ g_0(M) = 0 \text{ and } \|g_0\| = \frac{1}{d}.$$

Define $f_0 = dg_0$. Then $f_0 \in N^*$ such that,

$$f_0(x_0) = dg_0(x_0) = d(1) = d$$

$$f_0(M) = dg_0(M) = d(0) = 0$$

$$\|f_0\| = \|dg_0\| = d\|g_0\| = d \cdot \frac{1}{d} = 1.$$

3.4.7 Problem : Let M be a closed linear subspace of a normed linear space N, and let x_0 be a vector not in M. If d is the distance from x_0 to M, show that there exists a functional $f_0 \in N^*$ such that

$$f_0(x_0) = 1,\ f_0(M) = 0 \text{ and } \|f_0\| = \frac{1}{d}.$$

Solution : Let M be a closed linear subspace of a normed linear space N.

Let $x_0 \in N$ such that $x_0 \notin M$ and let $d = d(x_0, M)$.

Note that,

$$x_0 \in \overline{M} = M \Leftrightarrow d(x_0, M) = 0$$

Therefore,

$$x_0 \notin M \Leftrightarrow d = d(x_0, M) > 0$$

By Theorem 3.4.5, $\exists f_0 \in N^*$ such that

$$f_0(x_0) = 1,\ f_0(M) = 0 \text{ and } \|f_0\| = \frac{1}{d}.$$

[For complete proof proceed as in the proof of Theorem 3.4.5]

3.4.8 Theorem : If M is closed linear subspace of a normed linear space N and x_0 is a vector not in M, then there exists a functional $f_0 \in N^*$ such that $f_0(M) = 0$ and $f_0(x_0) \neq 0$.

Proof 1 : Follows from Theorem 3.4.5.

Proof 2 : Let M be a closed linear subspace of a normed linear space N. Then N/M is a normed linear space with the norm of coset.

$$\|x + M\| = \inf\{\|x + m\| : m \in M\}$$

Further a natural mapping $T : N \longrightarrow N/M$ defined by

$$T(x) = x + M, \; x \in N. \qquad \text{....... (1)}$$

is continuous linear transformation such that $\|T\| \leq 1$.

By (1), we have

$$T(m) = m + M = M, \; \forall m \in M \qquad \text{..... (2)}$$

Since $x_0 \notin M$, $x_0 + M \neq M$ that is $x_0 + M$ is non-zero vector in N/M.

[Note that M is zero vector in N/M]

By Theorem 3.4.1, $\exists$ a function $g \in (N/M)^*$ such that

$$g(x_0 + M) = \|x_0 + M\| \text{ and } \|g\| = 1.$$

Since $x_0 + M$ is non-zero vector in N/M,

$$g(x_0 + M) \neq 0 \qquad \text{....... (3)}$$

Define $f_0 : M \longrightarrow \mathbb{K}$ by $f_0(x) = g(T(x))$, $x \in N$.

$$\begin{tikzcd} \text{N} \quad x \arrow[rr, "T"] \arrow[dr, "f_0"'] & & \text{N/M} \quad \text{T}(x) \arrow[dl, "g"] \\ & g(\text{T}(x)) \quad \mathbb{K} & \end{tikzcd}$$

We prove that f_0 is functional on M with desired property.

f_0 is linear : Let any $x, y \in N$. Then,

$$f_0(x+y) = g(T(x+y))$$

$$= g(T(x)+T(y))$$

$$= g(T(x))+g(T(y))$$

$$= f_0(x)+f_0(y)$$

and for any scalar $\alpha \in \mathbb{K}$ we have,

$$f_0(\alpha x) = g(T(\alpha x))$$

$$= g(\alpha T(x))$$

$$= \alpha g(T(x))$$

$$= \alpha f_0(x)$$

Thus f_0 is linear.

f_0 is bounded (continuous) :

For any $x \in N$,

$$|f_0(x)| = |g(T(x))|$$

$$\leq \|g\|\|T(x)\|$$

$$\leq \|g\|\|T\|\|x\|$$

$$\leq \|g\|\|x\| \qquad (\because \|T\| \leq 1)$$

Since g is bounded, it follows that f_0 is bounded (continuous).

We have proved that $f_0 \in N^*$.

Further,

$$f_0(m) = g(T(m)) = g(M) = 0$$

($\because$ g is linear and hence preserves the zero vector)

and $f_0(x_0) = g(T(x_0)) = g(x_0 + M) \neq 0$ ($\because$ By (3))

This completes the proof.

3.4.9 Definition : Let (x, d) be a metric space and $A \subseteq X$. Then, A is said to be dense in X (or every where dense) if $\overline{A} = X$.

3.4.10 Definition : A metric space (X, d) is said to be separable if it has a countable subset which is dense in X.

3.4.11 Problem : Prove that a normed linear space N is separable if its conjugate space N* is.

Solution : Let N be a normed linear space.

Let conjugate space N* of N is separable. Then there exists a countable set.

$$A = \{f_n \in \mathbb{N}^* : n = 1, 2, 3, ...\}$$

Which is dense in N*, that is, $\overline{A} = N^*$.

Now for each n (n = 1, 2, 3, ...)

$$\|f_n\| = \sup\{|f_n(x)| : x \in N, \|x\| = 1\}$$

Therefore $\frac{1}{2}\|f_n\|$ is not an upperbound of the set,

$$\{|f_n(x)| : x \in N, \|x\| = 1\}$$

Hence $\exists\, x_n \in N$ such with $\|x_n\| = 1$ such that,

$$\frac{1}{2}\|f_n\| < |f_n(x_n)| \qquad ...... (1)$$

Let $M = \overline{\text{span}\{x_n\}}$

Then M is closed linear subspace of N generated by the sequence $\{x_n\}$.

We claim that M = N (i.e. $\overline{M} = N$).

If possible $M \neq N$. Then $\exists\, x_0 \in N - M$.

By Theorem 3.4.8, $\exists$ a functional $f_0 \in N^*$ such that,

$$f_0(x_0) \neq 0 \text{ and } f_0(M) = 0.$$

Since $\{x_n\} \subseteq M$, $f_0(x_n) = 0 \ \ \forall n$.

Thus from inequality (1), for each *n*, we have,

$$\frac{1}{2}\|f_n\| < |f_n(x_n) - f_0(x_n)|$$

$$= |(f_n - f_0)(x_n)|$$

$$= \|f_n - f_0\|\|x_n\|$$

$$= \|f_n - f_0\| \qquad [\because \|x_n\| = 1]$$

Thus,

$$\|f_n\| < 2\|f_n - f_0\|,\ \forall n \qquad \text{..... (2)}$$

Now,

$$\|f_0\| = \|(f_0 - f_n) + f_n\|$$

$$\leq \|f_0 - f_n\| + \|f_n\|$$

$$< \|f_0 - f_n\| + 2\|f_n - f_0\| \qquad [\because \text{ By (2)}]$$

$$= 3\|f_n - f_0\|$$

i.e. $\|f_0\| < 3\|f_n - f_0\| \quad \forall n$ ..... (3)

On the other hand,

$$f_0 \in N^* \text{ and } \overline{A} = \overline{N^*} \Rightarrow f_0 \in \overline{A}$$

Therefore there exists a sequence $\{f_{n_k}\} \subseteq A$ such that $f_{n_k} \to f_0$ as $k \to \infty$.

But from (3),

$$\|f_0\| < 3\|f_{n_k} - f_0\|$$

Taking limit as $k \to \infty$, we obtain,

$$\|f_0\| \le 0$$

$$\Rightarrow f_0 = 0$$

$$\Rightarrow f_0(x_0) = 0 \qquad (\because x_0 \in N)$$

This is contradiction to the fact $f_0(x_0) \neq 0$.

Hence we must have M = N (i.e. $\overline{M} = N$), which implies N is separable.

❑❑❑

UNIT - IV

SECOND CONJUGATE SPACE, EQUIVALENT NORMS

In this unit we study second conjugate spaces, conjugate of an operator, equivalent norms and finite dimensional spaces.

4.1 SECOND CONJUGATE SPACE

Let N be a normed space over field $\mathbb{K}$ ($\mathbb{R}$ or $\mathbb{C}$). Then $N^* = \mathscr{B}(N, \mathbb{R})$ or $\mathscr{B}(N, \mathbb{C})$ is called conjugate (dual) space of N. Since the conjugate space N* of N itself is normed space with the norm $\|\bullet\| : N^* \to [0, \infty)$ defined by,

$$\|f\| = \sup\{|f(x)| : x \in N, \|x\| \leq 1\}, \ f \in N^*,$$

from Theorem 3.4.1, it follows that if $N \neq \{0\}$ then $N^* \neq \{0\}$. Further N* is always complete (see Theorem 2.3.1) and hence is a Banach space. Therefore it is possible to form conjugate space (N*)* of N*, which is denoted by N** and is called the second conjugate (dual) space of N.

Note that $N^{**} = \mathscr{B}(N^*, \mathbb{C})$ or $\mathscr{B}(N^*, \mathbb{R})$ is again Banach space. (see Theorem 2.3.1) with the norm of $\phi \in N^{**}$, given by

$$\|\phi\| = \sup\{|\phi(f)| : f \in N^*, \|f\| \leq 1\}.$$

4.1.1 Definition :

Let N and N' be normed spaces. An isometric isomorphism of N into N' is a one-to-one linear transformation $T : N \to N'$ such that $\|Tx\| = \|x\|$ for every $x \in N$.

N is said to be isometrically isomorphic to N' if there exists an isometric isomorphism of N to N'.

The importance of second conjugate space N** of normed space N lies in the fact that to each $x \in N$ there is a unique function $F_x \in N^{**}$ having the same norm i.e. $\|F_x\| = \|x\|$. This fact is proved in the following theorem.

4.1.2 Theorem : Let N be a normed space. Then each vector x in N induces a functional F_x on N* defined by,

$$F_x(f) = f(x), \ f \in N^*$$

Such that $\|F_x\| = \|x\|$.

Further the mapping $T : N \longrightarrow N^{**}$ defined by $T(x) = F_x$, $x \in N$, is an isometric isomorphism of N into N**.

Proof : Let N be a normed space over field $\mathbb{K}$ ($\mathbb{R}$ or $\mathbb{C}$).

Part I : Fix any $x \in N$. Define $F_x : N^* \longrightarrow \mathbb{K}$ by

$$F_x(f) = f(x), \ f \in N^*$$

We prove that F_x is functional on N*.

F$_x$ is linear :

Let any $f, g \in N^*$ and α, β be any scalar.

Then, $F_x(\alpha f + \beta g) = (\alpha f + \beta g)(x)$

$$= \alpha f(x) + \beta g(x)$$

$$= \alpha F_x(f) + \beta F_x(g)$$

F$_x$ is bounded (continuous) :

Let any $f \in N^*$. Then,

$$|F_x(f)| = |f(x)| \le \|f\|\|x\|$$

i.e. $|F_x(f)| \le (\|x\|)\|f\|, \ \forall f \in N^*$

$\Rightarrow F_x$ is bounded (continuous) with bound $K = \|x\|$.

We have proved that F_x is functional on N* i.e. $F_x \in N^{**}$.

We prove that $\|F_x\| = \|x\|$:

If $x = 0$ then $F_0(f) = f(0) = 0$, $\forall f \in N^*$. Hence

$$\|F_0\| = \sup\{|F_0(f)| : f \in N^*, \|f\| \le 1\} = 0 = \|0\|$$

Thus, $\|F_x\| = \|x\|$ if $x = 0$

Let $x \neq 0$. Then $\exists f_0 \in N^*$ such that,

$$f_0(x) = \|x\| \text{ and } \|f_0\| = 1$$

But then,

$$\|x\| = |f_0(x)| \le \sup\{|f(x)| : f \in N^*, \|f\| = 1\}$$

$$= \sup\{|F_x(f)| : f \in N^*, \|f\| = 1\}$$

$$= \|F_x\|$$

Thus,

$$\|x\| \le \|F_x\| \qquad \text{....... (1)}$$

On the other hand,

$$\|F_x\| = \sup\{|F_x(f)| : f \in N^*, \|f\| \le 1\}$$

$$= \sup\{|f(x)| : f \in N^*, \|f\| \le 1\}$$

$$\le \sup\{\|f\|\|x\| : f \in N^*, \|f\| \le 1\}$$

$$\le \|x\|$$

This gives,

$$\|F_x\| \le \|x\| \qquad \text{...... (2)}$$

From (1) and (2),

$$\|F_x\| = \|x\|.$$

By Cases 1-2,

$$\|F_x\| = \|x\|,\ \forall x \in N. \qquad \text{...... (3)}$$

Part II : Define $T : N \longrightarrow N^{**}$ by $T(x) = F_x$, $x \in N$.

We prove that T is an isometric isomorphism.

T is Linear :

Let any $x, y \in N$ and any $\alpha \in \mathbb{K}$.

We prove that

$$T(x+y) = T(x) + T(y) \text{ and } T(\alpha x) = \alpha T(x)$$

i.e. $F_{x+y} = F_x + F_y$ and $F_{\alpha x} = \alpha F_x$

Let any $f \in N^*$. Then,

(i) $$\begin{aligned} F_{x+y}(f) &= f(x+y) \\ &= f(x) + f(y) \\ &= F_x(f) + F_y(f) \\ &= (F_x + F_y)(f) \end{aligned}$$

This implies $F_{x+y} = F_x + F_y$.

(ii) $$\begin{aligned} F_{\alpha x}(f) &= f(\alpha x) \\ &= \alpha f(x) \\ &= \alpha F_x(f) \\ &= (\alpha F_x)(f) \end{aligned}$$

Therefore $F_{\alpha x} = \alpha F_x$.

We proved that T is linear.

T Preserves Norm :

Using definition of T and equation (3) we have

$$\|T(x)\| = \|F_x\| = \|x\|, \ \forall x \in N$$

$\Rightarrow T$ preserves the norm.

T is one-to-one :

Let $x, y \in N$. Then,

$$T(x) = T(y) \Rightarrow T(x) - T(y) = 0$$

$$\Rightarrow T(x-y) = 0 \qquad [\because \text{ T is linear}]$$

$$\Rightarrow F_{x-y} = 0$$

$$\Rightarrow \|F_{x-y}\| = 0$$

$$\Rightarrow \|x-y\| = 0 \qquad [\because (3)]$$

$$\Rightarrow x = y$$

Therefore T is one-to-one.

We have proved that T is an isometric isomorphism of N into N**.

This completes the proof.

4.1.3 Theorem : A non-empty subset X of a normed space N is bounded if and only if $f(x)$ is bounded set of numbers for each $f \in N^*$.

Proof : Let N be a normed space over field $\mathbb{K}$.

Let X be a nonempty bounded subset of N.

Then $\exists\ K > 0$ such that,

$$\|x\| \le K, \ \forall x \in X. \qquad \text{.... (1)}$$

Let any $f \in N^*$. Then $f : N \longrightarrow \mathbb{K}$ is bounded linear functional. Therefore $\exists L > 0$ such that

$$|f(x)| \le L\|x\|, \ \forall x \in N \qquad ..... (2)$$

By (1) and (2),

$$|f(x)| \le LK, \ \forall x \in X.$$

$\Rightarrow f(X) = \{f(x) : x \in X\}$ is bounded set of numbers.

Conversely, let $f(X)$ is bounded set of numbers for each $f \in N^*$.

For convenience we write $X = \{x_i\}$.

We know to each $x \in N$ there exists $F_x \in N^{**} = \mathscr{B}(N^*, \mathbb{K})$ defined by,

$$F_x(f) = f(x), \ \forall f \in N^* \qquad ...... (3)$$

Such that,

$$\|F_x\| = \|x\|, \ \forall x \in N \qquad .... (4)$$

By assumption $f(X) = \{f(x_i)\}$ bounded for each $f \in N^*$. This in combination with (3) gives that $\{F_{x_i}(f)\}$ is bounded subset of $\mathbb{K}$ for each $f \in N^*$, where $\{F_{x_i}\} \subseteq \mathscr{B}(N^*, \mathbb{K})$ and N* is Banach space. Therefore by uniform boundedness principle $\{F_{x_i}\}$ is bounded subset of $\mathscr{B}(N^*, \mathbb{K})$, that is, $\exists M > 0$ such that,

$$\|F_{x_i}\| \le M \ \forall i \qquad ..... (5)$$

But (4) and (5) gives,

$$\|x_i\| \le M \ \forall i$$

$\Rightarrow X = \{x_i\}$ is bounded subset of N.

This completes the proof.

4.2 THE NATURAL IMBEDDING OF N IN N**

4.2.1 Definition : Let N be a normed linear space and N** is second conjugate space of N. The isometric isomorphism $T: N \longrightarrow N^{**}$ defined by $T(x) = F_x$, $x \in N^{**}$ is called natural imbedding (or Canonical mapping) of N into N**.

The functional $F_x \in N^{**}$ is called the functional induced by the vector $x \in N$. Such a functional often refered as induced functional.

4.3 THE CONJUGATE OF AN OPERATOR

4.3.1 Definition : Let N be a normed linear space over field $\mathbb{K}$ ($\mathbb{R}$ or $\mathbb{C}$). Let $T: N \longrightarrow N$ be a continuous linear transformation (i.e. T is an operator on N).

The mapping $T^*: N^* \longrightarrow N^*$ defined by

$$T^*(f) = f0T,\ f \in N^*.$$

That is

$$\left[T^*(f)\right](x) = f(T(x)),\ f \in N^*,\ x \in N,$$

is called the conjugate of operator T.

4.3.2 Theorem :

Let N be a normed linear space, T is an opeerator on N (i.e. $T \in \mathscr{B}(N)$) and T* is conjugate of T. Then :

(a) T* is operator on N (i.e. $T^* \in \mathscr{B}(N^*)$)

(b) $\|T^*\| = \|T\|$

(c) The mapping $T \longrightarrow T^*$ is an isometric isomorphism of $\mathscr{B}$ (N) into $\mathscr{B}$ (N*) which reverses products and preserves the identity transformation.

Proof : Let N be a normed space over the field $\mathbb{K}$ ($\mathbb{R}$ or $\mathbb{C}$).

Let $T: N \longrightarrow N$ be an operator (i.e. $T \in \mathscr{B}(N)$)

Then conjugate of T is a mapping $T^*: N^* \longrightarrow N^*$ defined by,

$$[T^*(f)](x) = f(T(x)),\ f \in N^*,\ x \in N \qquad \text{..... (1)}$$

(a) T* is linear :

Let any $f, g \in N^*$ and $\alpha, \beta \in \mathbb{K}$.

Then for all $x \in N$ we have,

$$\begin{aligned}[T^*(\alpha f + \beta g)](x) &= (\alpha f + \beta g)(T(x)) \\ &= \alpha f(T(x)) + \beta g(T(x)) \\ &= \alpha[T^*(f)](x) + \beta[T^*(g)](x) \\ &= [\alpha T^*(f) + \beta T^*(g)](x)\end{aligned}$$

This implies,

$$T^*(\alpha f + \beta g) = \alpha T^*(f) + \beta T^*(g)$$

Thus T* is linear.

T* is bounded :

Let any $f \in N^*$. Then,

$$\begin{aligned}\|T^* f\| &= \sup\{|T^*(f)(x)| : x \in N, \|x\| \le 1\} \\ &= \sup\{|f(T(x))| : x \in N, \|x\| \le 1\} \\ &\le \sup\{\|f\|\|Tx\| : x \in N, \|x\| \le 1\} \\ &= \|f\|\{\|Tx\| : x \in N, \|x\| \le 1\} \\ &= \|f\|\|T\|\end{aligned}$$

Thus

$$\|T^* f\| \le (\|T\|)\|f\|, \ \forall f \in N^* \qquad ..... (2)$$

This proves T* is bounded (continuous).

(b) $\|T^*\| = \sup\{\|T^*(f)\| : f \in N^*, \|f\| \le 1\}$

$\le \sup\{\|T\|\|f\| : f \in N^*, \|f\| \le 1\}$ [By (2)]

$\le \|T\| \sup\{\|f\| : f \in N^*, \|f\| \le 1\}$

$= \|T\|$

Thus $\|T^*\| \le \|T\|$ . ... (3)

By a consequence of Hahn-Banach theorem for any non-zero vector $Tx \in N$, $\exists f \in N^*$ such that

$$f(T(x)) = \|Tx\| \text{ and } \|f\| = 1 \qquad ...... (4)$$

Therefore, $\|Tx\| = |f(T(x))|$

$= |[T^*(f)](x)|$

$\le \|T^*(f)\|\|x\|$

$\le \|T^*\|\|f\|\|x\|$

$= \|T^*\|\|x\|$ [$\because$ By (4)]

Hence, $\|T\| = \sup\left\{\frac{\|Tx\|}{\|x\|} : x \in N, x \ne 0\right\}$

$\le \sup\left\{\frac{\|T^*\|\|x\|}{\|x\|} : x \in N, x \ne 0\right\}$

$= \|T^*\|$

i.e. $\|T\| \le \|T^*\|$ ..... (5)

From (3) and (5),

$$\|T\| = \|T^*\| \qquad \text{...... (6)}$$

(c) Define the mapping $\phi : \mathscr{B}(N) \longrightarrow \mathscr{B}(N^*)$ by

$$\phi(T) = T^*,\ T \in \mathscr{B}(N).$$

Part I : Firstly we prove that ϕ is an isometric isomorphism.

ϕ **is linear :** Let any $T \in \mathscr{B}(N)$ and $\alpha, \beta \in \mathbb{K}$.

We have to prove that,

$$\phi(\alpha T_1 + \beta T_2) = \alpha\phi(T_1) + \beta\phi(T_2)$$

i.e. $(\alpha T_1 + \beta T_2)^* = \alpha T_1^* + \beta T_2^*$

For any $f \in N^*$ and any $x \in N$ we have,

$$\begin{aligned}
\left[(\alpha T_1 + \beta T_2)^*(f)\right](x) &= f(\alpha T_1 + \beta T_2)(x) \\
&= f(\alpha T_1(x) + \beta T_2(x)) \\
&= \alpha f(T_1(x)) + \beta f(T_2(x)) \\
&= \alpha\left[T_1^*(f)\right](x) + \beta\left[T_2^*(f)\right](x) \\
&= \left[\alpha T_1^*(f) + \beta T_2^*(f)\right](x)
\end{aligned}$$

Therefore,

$$\begin{aligned}
(\alpha_1 T_1 + \beta_2 T_2)^*(f) &= \alpha T_1^*(f) + \beta T_2^*(f) \\
&= (\alpha T_1^*)(f) + (\beta T_2^*)(f) \\
&= (\alpha T_1^* + \beta T_2^*)(f)
\end{aligned}$$

$$\Rightarrow (\alpha T_1 + \beta T_2)^* = \alpha T_1^* + \beta T_2^* \qquad \text{...... (7)}$$

This prove ϕ is linear.

ϕ **is one-to-one :** Let any $T_1, T_2 \in \mathscr{B}(N)$.

Then

$$\phi(T_1) = \phi(T_2) \Rightarrow T_1{}^* = T_2{}^*$$

$$\Rightarrow T_1{}^* - T_2{}^* = 0$$

$$= (T_1 - T_2)^* = 0 \qquad [\because \text{By}\,(7)]$$

$$\Rightarrow \|(T_1 - T_2)^*\| = 0$$

$$\Rightarrow \|T_1 - T_2\| = 0 \qquad [\because \text{By}\,(7)]$$

$$\Rightarrow T_1 = T_2$$

Therefore ϕ is one-to-one.

ϕ **Preserves the norm :**

For any $T \in \mathscr{B}(N)$.

$$\|\phi(T)\| = \|T^*\| = \|T\| \qquad [\because \text{By}\,(7)]$$

We have proved that $\phi : \mathscr{B}(N) \longrightarrow \mathscr{B}(N)$ is linear, one-to-one and norm preserving mapping, hence it is an isometric isomorphism.

Part II : It remains to prove ϕ reverses poroducts and preserves the identity transformation means,

$\phi(T_1T_2) = \phi(T_2)\phi(T_1)$ and $\phi(I) = I$

i.e. $(T_1T_2)^* = T_2{}^* T_1{}^*$ and $I^* = I$,

for any $T_1, T_2 \in \mathscr{B}(N)$ and the identity transformation $I \in \mathscr{B}(N)$.

Let any $f \in N^*$ and any $x \in N$.

Then,

$$\left[(T_1T_2)^*(f)\right](x) = f\left((T_1T_2)(x)\right)$$

$$= f\left(T_1\left(T_2(x)\right)\right)$$

$$= \left[T_1^*(f)\right]\left(T_2(x)\right)$$

$$= T_2^*\left(\left[T_1^*(f)\right]\right)(x)$$

$$\Rightarrow (T_1T_2)^*(f) = T_2^*\left(T_1^*(f)\right) = \left(T_2^*T_1^*\right)(f)$$

$$\Rightarrow (T_1T_2)^* = T_2^*T_1^*$$

and

$$\left[I^*(f)\right](x) = f\left(I(x)\right)$$

$$= f(x) \qquad (\because \text{ Identity transformation } I \in \mathscr{B}(N))$$

$$= \left[I(f)\right](x) \qquad (\because \text{ Identity transformation } I \in \mathscr{B}(N^*))$$

$$\Rightarrow I^*(f) = I(f)$$

$$\Rightarrow I^* = I$$

This completes the proof.

4.3.3 Problem : Let T be an operator on a Banach space B. Show that T has an inverse T^{-1} if and only if T* has an inverse $(T^*)^{-1}$ and that in this case $(T^*)^{-1} = (T^{-1})^*$.

Proof : Let T be an operator on a Banach space B.

Let T has an inverse T^{-1}. Then $TT^{-1} = T^{-1}T = I$, where I is an identity operator on B. Therefore,

$$(TT^{-1})^* = (T^{-1}T)^* = I^* \qquad \text{..... (1)}$$

On the other hand,

$$(TT^{-1})^* = (T^{-1})^*T^* \qquad \text{..... (2)}$$

$$\left(T^{-1}T\right)* = T*\left(T^{-1}\right)* \qquad \text{..... (3)}$$

From (1), (2) and (3)

$$\left(T^{-1}\right)*T* = T*\left(T^{-1}\right)* = I*$$

This implies T* has an inverse $(T*)^{-1}$ and $(T*)^{-1} = \left(T^{-1}\right)*$.

Conversely let T* has an inverse $(T*)^{-1}$ and $(T*)^{-1} = \left(T^{-1}\right)*$.

Then,

$$T*(T*)^{-1} = (T*)^{-1}T* = I*$$

$$\Rightarrow T*\left(T^{-1}\right)* = \left(T^{-1}\right)*T* = I*$$

$$\Rightarrow \left(T^{-1}T\right)* = \left(TT^{-1}\right)* = I* \qquad \text{..... (4)}$$

We know that the mapping $T \longrightarrow T*$ is a one-to-one mapping and I* = I.

Therefore from (4), we have,

$$T^{-1}T = TT^{-1} = I$$

$\Rightarrow$ T has inverse T^{-1}.

This completes the proof.

4.4 EQUIVALENT NORMS

Let $\left(X, \|\cdot\|\right)$ be a normed linear space. We know norm $\|.\|$ on X induces a metric on X and metric induces a topology on X. Hence norm on X induces topology on X. We call this topology a norm topology on X.

If different norms on the same linear space induces a same topology on X, we say that they are equivalent norms. More precisely we have the following definition.

4.4.1 Definition : Let $\|.\|$ and $\|.\|'$ be two norms on a linear space X. Then these norms are said to be equivalent, written $\|\bullet\| \sim \|\bullet\|'$, if and only if they generate the same topology on X.

4.4.2 Theorem : Two norms $\|\bullet\|$ and $\|\bullet\|'$ on the linear space X is said to be equivalent if and only if there exists two positive real numbers K_1 and K_2 such that,

$$K_1\|x\| \le \|x\|' \le K_2\|x\|, \ \forall x \in X.$$

Proof : Let $\|\bullet\|$ and $\|\bullet\|'$ be two norms on a linear space X. Then $N = (X, \|\bullet\|)$ and $N' = (X, \|\bullet\|')$ are two normed linear spaces.

Consider the identity transformation $T : N \longrightarrow N'$ defined by $T(x) = x$, $x \in N(= X)$.

(Note that N and N' are the same linear spaces X with two different norms).

Then T is bijective, continuous linear transformation. Thus $T^{-1} : N' \longrightarrow N$ exists, and it is also continuous linear transformation such that,

$$T(x) = x \Leftrightarrow T^{-1}(x) = x, \ x \in X.$$

Note that, T is continuous $\Leftrightarrow$ T is bounded

$\Leftrightarrow \exists\, K_2 > 0$ such that $\|T(x)\|' \le K_2\|x\|, \ \forall x \in X$

$\Leftrightarrow \exists\, K_2 > 0$ such that

$\|x\|' \le K_2\|x\|, \ \forall x \in X \ \ (\because T(x) = x, \ \forall x \in X)$ ..... (1)

Also, T^{-1} is continuous $\Leftrightarrow$ T^{-1} is bounded

$\Leftrightarrow \exists\, M > 0$ such that $\|T^{-1}(x)\| \le M\|x\|', \ \forall x \in X$

$\Leftrightarrow \exists\, K_1$ such that $K_1\|x\| \le \|x\|', \ \forall x \in X$ ..... (2)

$\left(\because K_1 = \dfrac{1}{M}\right.$ and $\left. T^{-1}(x) = x, \ \forall\, x \in X\right)$

Now,

T and T^{-1} continuous

$\Leftrightarrow$ Inverse images of open sets in N' and N under T and T^{-1} respectively are open in N and N'.

$\Leftrightarrow$ Open sets in N and N' are same ($\because$ T and T^{-1} both are identity transformations)

$\Leftrightarrow$ $\|\bullet\|$ and $\|\bullet\|'$ induces the same topology on X.

$\Leftrightarrow$ $\|\bullet\|$ and $\|\bullet\|'$ are equivalent on linear space X. ...... (3)

Combining statements (1), (2) and (3) we obtain,

Two norms $\|\bullet\|$ and $\|\bullet\|'$ on a linear space X are equivalent iff $\exists K_1, K_2 > 0$ such that

$$K_1 \|x\| \le \|x\|' \le K_2 \|x\|, \ \forall x \in X.$$

This completes the proof.

4.4.3 Remark : The relation norm equivalence '~' is an equivalence relation among the norms on X.

Cauchy-Schwartz Inequality :

Let any $x = (x_1, ..., x_n)$, $y = (y_1, ..., y_n) \in \mathbb{K}^n$ where $\mathbb{K}$ (= $\mathbb{R}$ or $\mathbb{C}$). Then,

$$\sum_{j=1}^{n} |x_j y_j| \le \left(\sum_{j=1}^{n} |x_j|^2\right)^{1/2} \left(\sum_{j=1}^{n} |y_j|^2\right)^{1/2}$$

4.4.4 Problem : Prove that the norms $\|\bullet\|_1$, $\|\bullet\|_2$ and $\|\bullet\|_\infty$ are equivalent norms on $\mathbb{K}^n$ ($\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$).

Solution : Let any $x = (x_1, ..., x_n) \in \mathbb{K}^n$ ($\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$). We know

$$\|x\|_p = \left(\sum_{j=1}^{n} |x_j|^p\right)^{1/p}, (1 \le p < \infty).$$

and $\|x\|_\infty = \max_{1\le j\le n} |x_j|$.

Inparticular for p = 1, 2, we have

$$\|x\|_1 = \sum_{j=1}^{n} |x_j| \text{ and } \|x\|_2 = \left(\sum_{j=1}^{n} |x_j|^2\right)^{1/2}$$

Since $|x_j| \le \sum_{j=1}^{n} |x_j|$, $\forall j$ $(j = 1, 2, \ldots\ldots, n)$ we have,

$$\begin{aligned} \|x\|_\infty &= \max_{1\le j\le n} |x_j| \\ &\le \sum_{j=1}^{n} |x_j| \\ &= |x_1| + \ldots\ldots + |x_n| \\ &\le \max_{1\le j\le n} |x_j| + \ldots + \max_{1\le j\le n} |x_j| \quad (n \text{ times}) \\ &= n \max_{1\le j\le n} |x_j| \\ &= n\|x\|_\infty \end{aligned}$$

Therefore,

$$\|x\|_\infty \le \sum_{j=1}^{n} |x_j| \le n\|x\|_\infty$$

$$\Rightarrow \|x\|_\infty \le \|x\|_1 \le n\|x\|_\infty$$

This implies $\|\bullet\|_\infty$ is equivalent $\|\bullet\|_1$ on $\mathbb{K}^n$, that is, $\|\bullet\|_\infty \sim \|\bullet\|_1$ ..... (1)

Further,

$$\|x\|_2 = \left(\sum_{j=1}^{n} |x_j|^2\right)^{1/2}$$

$$\le \sum_{j=1}^{n} |x_j|(1) \qquad \left[\because \left(\sum_{j=1}^{n} a_j\right)^2 \ge \sum_{j=1}^{n} a_j^{\,2},\ \ a_j \ge 0\right]$$

$$\le \left(\sum_{j=1}^{n} |x_j|^2\right)^{1/2} \left(\sum_{j=1}^{n} (1)^2\right)^{1/2} \qquad [\because \text{ Cauchy-Schwartz inequality}]$$

$$= \|x\|_2 \left(\sqrt{n}\right)$$

Thus,

$$\|x\|_2 \le \sum_{j=1}^{n} |x_j| \le \sqrt{n}\, \|x\|_2$$

$$\Rightarrow \|x\|_2 \le \|x\|_1 \le \sqrt{n}\, \|x\|_2$$

$\Rightarrow \|\bullet\|_1$ equivalent to $\|\bullet\|_2$.

$$\Rightarrow \|\bullet\|_1 \sim \|\bullet\|_2 \qquad \text{...... (2)}$$

From (1) and (2) $\|\bullet\|_\infty \sim \|\bullet\|_2$

Since norm equivalence is an equivalence relation among the norms on a linear space, $\|\bullet\|_1$, $\|\bullet\|_2$ and $\|\bullet\|_\infty$ all are equivalent norm on $\mathbb{K}^n$.

4.4.5 Problem : Let $\|\bullet\|$ and $\|\bullet\|'$ be equivalent norms on a linear space X. Prove that :

(a) The Cauchy sequence in $(X, \|\bullet\|)$ and $(X, \|\bullet\|')$ are the same.

(b) The convergent sequences in $(X, \|\bullet\|)$ and $(X, \|\bullet\|')$ are same.

(c) $(X, \|\bullet\|)$ is Banach space if and only if $(X, \|\bullet\|')$ is Banach space.

(d) A set is bounded in $(X, \|\bullet\|)$ if and only if it is bounded in $(X, \|\bullet\|')$.

Solution : Proof of part (a) and (b) is omitted as it can be given on the same line to the proof of (c).

Let $\|\bullet\|$ and $\|\bullet\|'$ are two equivalent on a linear space X. Then $\exists\, k_1, k_2 > 0$ such that,

$$k_1 \|x\| \le \|x\|' \le k_2 \|x\|,\ \forall x \in X \qquad \text{..... (1)}$$

(c) Let $(X, \|\bullet\|)$ is a Banach space.

Let $\{x_n\}$ be any Cauchy sequence in $(X, \|\bullet\|')$. Then

$$\|x_m - x_n\|' \to 0 \text{ as } m, n \to \infty \qquad \text{..... (2)}$$

But by (1),

$$k_1 \|x_m - x_n\| < \|x_m - x_n\|' \text{, for all } m, n \qquad \text{..... (3)}$$

Combining (2) and (3), $\|x_m - x_n\| \to 0$ as $m, n \to \infty$.

$\Rightarrow \{x_n\}$ is Cauchy sequence in complete space $(X, \|\cdot\|)$.

$\Rightarrow \exists\, x \in X$ such that,

$$\|x_n - x\| \to 0 \text{ as } n \to \infty. \qquad \text{....... (4)}$$

Again by (1),

$$\|x_n - x\|' \le k_2 \|x_n - x\| \qquad \text{....... (5)}$$

From (4) and (5), we obtain

$$\|x_n - x\|' \to 0 \text{ as } n \to \infty.$$

This implies $(X, \|\bullet\|')$ is a Banach space.

The converse follows similarly by interchanging the role of $\|\bullet\|$ and $\|\bullet\|'$.

(d) Let A be a bounded subset of $(X, \|\bullet\|)$.

Then $\exists\, M > 0$ such that,

$$\|x\| \le M \text{ , } \forall x \in A \cdot$$

By (1), we have

$$\|x\|' \le k_2 \|x\| \le k_2 M \text{ , } \forall x \in M \cdot$$

$\Rightarrow$ A is bounded subset of $(X, \|\bullet\|')$.

Converse follows by interchanging the role of $\|\bullet\|$ and $\|\bullet\|'$.

4.5 FINITE DIMENSIONAL NORMED SPACES

4.5.1 Definition : Let N and N' be normed spaces over the same system of scalar $\mathbb{K}$ ($\mathbb{R}$ or $\mathbb{C}$).

A mapping is said to be :

(a) **Homeomorphism** if T is bijective and bicontinuous (T and T^{-1} both are continuous).

(b) **Topological isomorphism** if T is linear and homeomorphism.

4.5.2 Remark : The relation of "being topologically isomorphic to" is an equivalence relation on the set of all normed spaces over the field $\mathbb{K}$.

4.5.3 Theorem (Borel-Lebesgue) : A non-empty subset of the normed space $\left(\mathbb{K}^n, \|\bullet\|_2\right)$ is compact if and only if it is closed and bounded.

Here $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ and for $x = \left(x_1, ..., x_n\right) \in \mathbb{K}^n$, $\|x\|_2 = \left(\sum_{j=1}^{n} \left|x_j\right|^2\right)^{1/2}$.

4.5.4 Theorem : Any n-dimensional normed space over the scalar field $\mathbb{K}$ ($\mathbb{R}$ or $\mathbb{C}$) is topologically isomorphic to $\left(\mathbb{K}^n, \|\bullet\|_2\right)$.

Proof : Let $(X, \|\bullet\|)$ be normed linear space over the field $\mathbb{K}$ ($\mathbb{R}$ or $\mathbb{C}$) and dimX = n.

Let $\{e_1, e_2, ..., e_n\}$ be a basis for X. Then for any $x \in X$ $\exists$ unique $\left(\alpha_1, ..., \alpha_n\right) \in \mathbb{K}^n$ such that,

$$x = \sum_{j=1}^{n} \alpha_j e_j$$

Consider the mapping $T : \mathbb{K}^n \longrightarrow X$ defined by,

$$T\left(\alpha_1, ..., \alpha_n\right) = \sum_{j=1}^{n} \alpha_j e_j \text{ , } \left(\alpha_1, ..., \alpha_n\right) \in \mathbb{K}^n .$$

T is well defined :

Let $(\alpha_1,...,\alpha_n)=(\beta_1,...,\beta_n)$ in $\mathbb{K}^n$.

Then $\sum_{j=1}^{n}\alpha_j e_j=\sum_{j=1}^{n}\beta_j e_j \Rightarrow T(\alpha_1,...,\alpha_n)=T(\beta_1,...,\beta_n)$.

T is linear :

Let any $(\alpha_1,...,\alpha_n),(\beta_1,...,\beta_n)\in\mathbb{K}^n$ and any $a,b\in\mathbb{K}$.

Then, $T\left(a(\alpha_1,...,\alpha_n)+b(\beta_1,...,\beta_n)\right)$

$$=T(a\alpha_1+b\beta_1,...,a\alpha_n+b\beta_n)$$

$$=\sum_{j=1}^{n}(a\alpha_j+b\beta_j)e_j$$

$$=\sum_{j=1}^{n}a(\alpha_j e_j)+b(\beta_j e_j)$$

$$=a\left(\sum_{j=1}^{n}\alpha_j e_j\right)+b\left(\sum_{j=1}^{n}\beta_j e_j\right)$$

$$=aT(\alpha_1,...,\alpha_n)+bT(\beta_1,...,\beta_n)$$

Thus T is linear.

T is one-to-one :

Let $(\alpha_1,...,\alpha_n),(\beta_1,...,\beta_n)\in\mathbb{K}^n$

Then, $T(\alpha_1,...,\alpha_n)=T(\beta_1,...,\beta_n)$

$$\Rightarrow\sum_{j=1}^{n}\alpha_j e_j=\sum_{j=1}^{n}\beta_j e_j$$

$$\Rightarrow\sum_{j=1}^{n}(\alpha_j-\beta_j)e_j=0$$

$\Rightarrow \alpha_j - \beta_j = 0$, $\forall j$ $(j = 1, 2, ..., n)$ ($\because \{e_1,...,e_n\}$ is linearly independent)

$\Rightarrow \alpha_j = \beta_j$, $\forall j$

$\Rightarrow (\alpha_1,...,\alpha_n) = (\beta_1,...,\beta_n)$

Therefore T is one-to-one.

T is onto :

For any $x \in X$, $x = \sum_{j=1}^{n} \alpha_j e_j$ is the unique expression, where $(\alpha_1,...,\alpha_n) \in \mathbb{K}^n$.

By definition of T we have,

$$T(\alpha_1,...,\alpha_n) = \sum_{j=1}^{n} \alpha_j e_j = x.$$

This proves T is onto.

T is continuous (bounded) :

For any $(\alpha_1,...,\alpha_n) \in \mathbb{K}^n$,

$$\begin{aligned} \|T(\alpha_1,...,\alpha_n)\| &= \left\| \sum_{j=1}^{n} \alpha_j e_j \right\| \\ &\leq \sum_{j=1}^{n} \|\alpha_j e_j\| \\ &= \sum_{j=1}^{n} |\alpha_j| \|e_j\| \end{aligned}$$

Note that for each j,

$$|\alpha_j| = \left(|\alpha_j|^2\right)^{1/2} \leq \left(\sum_{j=1}^{n} |\alpha_j|^2\right)^{1/2} = \|(\alpha_1,...,\alpha_n)\|_2$$

Therefore,

$$\left\|T\left(\alpha_1,\ldots,\alpha_n\right)\right\| \le \sum_{j=1}^{n}\left\|\left(\alpha_1,\ldots,\alpha_n\right)\right\|_2 \left\|e_j\right\|$$

This gives,

$$\left\|T\left(\alpha_1,\ldots,\alpha_n\right)\right\| \le k\left\|\left(\alpha_1,\ldots,\alpha_n\right)\right\|_2$$

Where, $k = \sum_{j=1}^{n}\left\|e_j\right\| > 0$.

Therefore T is bounded and hence continuous.

T^{-1} is continuous :

Since $T : \mathbb{K}^n \longrightarrow X$ is bijective linear transformation $T^{-1} : X \longrightarrow \mathbb{K}^n$ exists and it is linear transformation.

We prove that $T^{-1} : X \longrightarrow \mathbb{K}^n$ is bounded. Consider the mapping $f : \mathbb{K}^n \longrightarrow \mathbb{R}$ defined by $f(x) = \|T(x)\|$, $x \in \mathbb{K}^n$.

$$\begin{tikzcd}
x \arrow[rr, "T"] \arrow[dr, "f"'] & & T(x) \arrow[dl, "\|\bullet\| \text{ on } X"] \\
& \| T(x) \| &
\end{tikzcd}$$

Then $f = \|\bullet\|$ of (f is composition of $\|\bullet\|$ and T). Since $\|\bullet\|$ and T both are continuous, the function f is also continuous.

Consider the closed unit sphere,

$$S = \left\{x \in \mathbb{K}^n : \|x\|_2 = 1\right\}$$

in $\left(\mathbb{K}^n, \|\bullet\|_2\right)$. Then S is closed and bounded subset of $\mathbb{K}^n$ and hence by Borel-

Lebesgue theorem it is compact subset of $\mathbb{K}^n$.

Since continuous image of compact set is compact, f(S) is compact subset $\mathbb{R}$.

Therefore f(S) is closed and bounded.

Let, $m = \inf\{f(x) : x \in S\}$

$= \inf\{\|Tx\| : x \in S\}$

Then $0 \le m \le \|Tx\|$, $\forall x \in S$.

Since f is continuous compact set,

$m \in \{f(x) : x \in S\}$ i.e. $\exists\, x_0 \in S$

Such that $f(x_0) = m \Rightarrow \|Tx_0\| = m$.

If $m = 0$ then $\|Tx_0\| = 0$.

This gives $T(x_0) = 0 = T(0) \Rightarrow x_0 = 0$ ($\because$ T is one-to-one)

Therefore $\|x_0\| = 0$ a contradiction to the fact $x_0 \in S \Rightarrow \|x_0\| = 1$.

Hence we must have $m \ne 0$.

Therefore,

$0 < m \le \|Tx\|$, $\forall x \in S$.

Now for any $x \in \mathbb{K}^n$, $\left\| \dfrac{x}{\|x\|_2} \right\|_2 = 1$.

Therefore $\dfrac{x}{\|x\|_2} \in S$, $\forall x \in \mathbb{K}^n$.

Hence,

$$m \le \left\| T\left(\frac{x}{\|x\|_2} \right) \right\| = \left\| \frac{Tx}{\|x\|_2} \right\| = \frac{\|Tx\|}{\|x\|_2}$$

$$\Rightarrow m\|x\|_2 \le \|Tx\|, \quad \forall x \in \mathbb{K}^n$$

$$\Rightarrow m\left\|T^{-1}y\right\|_2 \le \left\|T\left(T^{-1}y\right)\right\| = \|y\|, \ \forall y \in X$$

$$\Rightarrow \left\|T^{-1}y\right\|_2 \le \frac{1}{m}\|y\| \ , \ \forall y \in X$$

$\Rightarrow T^{-1}$ is bounded and hence continuous.

We have prove that $T : \mathbb{K}^n \longrightarrow X$ is linear, bijective and bicontinuous.

Therefore $T : \left(\mathbb{K}^n, \|\bullet\|_2\right) \longrightarrow \left(X, \|\cdot\|\right)$ is a topological isomorphism and hence $\left(X, \|\cdot\|\right)$ is topologically isomorphic to $\left(\mathbb{K}^n, \|\bullet\|_2\right)$. This completes the proof.

4.5.5 Theorem : On a finite dimensional space all norms are equivalent.

Proof : Let X be any finite dimensional space with dim X $= n$.

Let $\|\bullet\|$ and $\|\bullet\|'$ be any norm on X..

Then by Theorem 4.5.4 $\left(X, \|\cdot\|\right)$ and $\left(X, \|\cdot\|'\right)$ both are topologically isomorphic to $\left(\mathbb{K}^n, \|\bullet\|_2\right)$. Therefore $\left(X, \|\cdot\|\right)$ is topologically isomorphic to $\left(X, \|\cdot\|'\right)$.

This implies $\|\bullet\|$ and $\|\bullet\|'$ induce the same topology on X.

Therefore $\|\bullet\|$ and $\|\bullet\|'$ are equivalent on X.

4.5.6 Remark : The convergence or divergence of a sequence or a series in a finite dimensional space does not depend on the particular choice of a norm on that space.

4.6 EQUIVALENT NORMS AND FINITE DIMENSIONAL SPACE

4.6.1 Theorem : A finite dimensional normed linear space is Banach space.

Proof : Let N be a normed space over the field $\mathbb{K}$ ($\mathbb{R}$ or $\mathbb{C}$) equipped with the norm $\|\bullet\|$.

Let $\{e_1, e_2, ..., e_n\}$ be a basis for N.

Then, for any $x \in N$, $\exists$ unique $\left(\alpha_1, ..., \alpha_n\right) \in \mathbb{K}^n$, $\alpha_1, ..., \alpha_n$ such that,

$$x = \sum_{i=1}^{n} \alpha_i e_i$$

Define $\|\bullet\|_0 : N \longrightarrow \mathbb{R}$ by,

$$\|x\|_0 = \max_{1 \le j \le n} |\alpha_j|, \ x \in N$$

Then $(N, \|\bullet\|_0)$ is a normed space (verify). Since N is finite dimensional space, $\|\bullet\|$ and $\|\bullet\|_0$ are equivalent norm on N.

Therefore, to prove $(N, \|\bullet\|)$ is complete it will sufficient to show that $(N, \|\bullet\|_0)$ is complete.

Let $\{y_m\}_{m=1}^{\infty}$ be any Cauchy sequence in $(N, \|\bullet\|_0)$.

Then for each m, (m = 1, 2, ...), $\exists$ unique $\left(\alpha_1^{(m)}, ..., \alpha_n^{(m)}\right) \in \mathbb{K}^n$ such that,

$$y_m = \sum_{j=1}^{n} \alpha_j^{(m)} e_j .$$

Since $\{y_m\}$ is Cauchy sequence in $(N, \|\bullet\|_0)$.

$$\|y_m - y_r\| \longrightarrow 0 \text{ as } m, r \to \infty \qquad ...... (1)$$

For each j (j = 1, 2, ..., n)

$$\left|\alpha_j^{(m)} - \alpha_j^{(r)}\right| \le \max_{1 \le j \le n} \left|\alpha_j^{(m)} - \alpha_j^{(r)}\right| = \|y_m - y_r\|_0 \qquad ...... (2)$$

From (1) and (2), for each j (j = 1, 2, ..., n)

$$\left|\alpha_j^{(m)} - \alpha_j^{(r)}\right| \to 0 \text{ as } m, r \to \infty$$

This implies for each j (j = 1, 2, ..., n), $\left\{\alpha_j^{(m)}\right\}_{m=1}^{\infty}$ is Cauchy sequence in complete space $\mathbb{K}$ ($\mathbb{R}$ or $\mathbb{C}$). Therefore $\exists\ \delta_j \in \mathbb{K}$ such that,

$$\alpha_j^{(m)} \longrightarrow \delta_j \text{ as } m \to \infty \qquad (j = 1, 2, ..., n) \qquad .... (3)$$

Define $y = \sum_{j=1}^{n} \delta_j e_j$.

Then $y_m - y = \sum_{j=1}^{n}\left(\alpha_j^{(m)} - \delta_j\right)e_j$.

Therefore by (3), we have,

$$\|y_m - y\|_0 = \max_{1\le j\le n}\left|\alpha_j^{(m)} - \delta_j\right| \to 0 \text{ as } m \to \infty$$

This implies $y_m \to y$ in $\left(N, \|\bullet\|_0\right)$.

We have proved that $\left(N, \|\bullet\|_0\right)$ is complete normed space (Banach space).

4.6.2 Corollary : A finite dimensional subspace M of normed space N is closed subset of N.

Proof : Let M be a finite dimensional subspace of a normed space N.

Then M is finite normed space with the norm induced by N.

But finite dimensional normed spaces are Banach spaces. Therefore M is itself a Banach space. Hence M is closed subset of N.

4.6.3 Corollary : In a finite dimensional normed space every non-empty closed bounded set is compact.

4.6.4 Theorem : A linear transformation on a finite dimensional space is continuous.

Proof : Let $(N, \|\bullet\|)$ be a n-dimensional normed space over the field $\mathbb{K}$ ($\mathbb{R}$ or $\mathbb{C}$) and $(N', \|\bullet\|')$ be any normed linear space.

Let $T : N \longrightarrow N'$ be a linear transformation. We have to prove that T is continuous.

Let $\{e_1, e_2, ..., e_n\}$ be a basis for N. Then to each x $\exists$ unique $\left(\alpha_1, ..., \alpha_n\right) \in \mathbb{K}^n$ such that,

$$x = \sum_{j=1}^{n}\alpha_j e_j .$$

Define a new norm $\|\bullet\|_0 : N \longrightarrow \mathbb{R}$ by,

$$\|x\|_0 = \max_{1 \le j \le n} |\alpha_j|$$

Then $(N, \|\bullet\|_0)$ is a normed space.

Since on a finite dimensional space all norms are equivalent, $\|\bullet\|$ and $\|\bullet\|_0$ are equivalent on N. Thus $\exists\ k_1, k_2 > 0$ such that,

$$k_1 \|x\| \le \|x\|_0 \le k_2 \|x\|,\ \forall x \in N \qquad \text{.... (1)}$$

Now by linearity of T, we have,

$$T(x) = T\left(\sum_{j=1}^{n} \alpha_j e_j\right) = \sum_{j=1}^{n} \alpha_j T(e_j) \text{ for any } x = \sum_{j=1}^{n} \alpha_j e_j \text{ in N.}$$

By triangle inequality we have,

$$\|T(x)\|' \le \sum_{j=1}^{n} \|\alpha_j T(e_j)\|$$

$$= \sum_{j=1}^{n} |\alpha_j| \|T(e_j)\|' \qquad \text{...... (2)}$$

Note that for each j ($j = 1, 2, ... n$)

$$|\alpha_j| \le \max_{1 \le j \le n} |\alpha_j| = \|x\|_0$$

Therefore from (2) we have,

$$\|T(x)\|' \le \sum_{j=1}^{n} \|x\|_0 \|T(e_j)\|'$$

This implies,

$$\|T(x)\|' \le K \|x\|_0,\ \forall x \in N, \qquad \text{.... (3)}$$

where, $K = \sum_{j=1}^{n} \|T(e_j)\|' > 0$.

From (1) and (3) we have,

$$\|T(x)\|' \leq KK_2 \|x\|, \ \forall x \in N$$

Therefore $T : N \longrightarrow N'$ is bounded linear transformation and hence continuous.

4.6.5 Theorem : Let $(X, \|\cdot\|)$ and $(X, \|\cdot\|')$ be two Banach spaces with same underlying linear space X. Suppose $\exists\, k > 0$ such that $\|x\| \leq K\|x\|'$, $\forall x \in X$.

Then $\|\bullet\|$ and $\|\bullet\|'$ are equivalent norms.

Proof : Let $(X, \|\cdot\|)$ and $(X, \|\cdot\|')$ be two Banach spaces with the same underlying space X.

Let $\exists$ a constant $K > 0$ such that,

$$\|x\| \leq K\|x\|', \ \forall x \in X. \qquad \text{..... (1)}$$

Consider the identity transformation.

$$I : (X, \|\cdot\|') \longrightarrow (X, \|\cdot\|), \ I(x) = x, \ x \in X.$$

Then I is bijective and linear. Further by (1),

$$\|I(x)\| = \|x\| \leq K\|x\|', \ \forall x \in X.$$

$\Rightarrow I : (X, \|\cdot\|') \longrightarrow (X, \|\cdot\|)$ is bounded linear transformation.

By theorem 2.5.8, $I^{-1} : (X, \|\cdot\|) \longrightarrow (X, \|\cdot\|')$ is continuous linear transformation and hence bounded.

Therefore $\exists\, L > 0$ such that,

$$\|I^{-1}(x)\|' \leq L\|x\|, \ \forall x \in X$$

$$\Rightarrow \|x\|' \leq L\|x\|, \ \forall x \in X \qquad \text{..... (2)}$$

Combing (1) and (2) we obtain,

$$\frac{1}{K}\|x\| \leq \|x\|' \leq L\|x\|, \ \forall x \in X$$

Hence $\|.\|$ and $\|.\|'$ are equivalent norm on X.

4.6.6 Lemma (Riesz Lemma) : Let M be a closed proper subspace of a normed linear space N, the for every $0 < \varepsilon < 1$ there exists a vector $x_\varepsilon \in N$ such that,

$$\|x_\varepsilon\| = 1 \text{ and } d(x_\varepsilon, M) \geq \varepsilon$$

Proof : Let M be a closed proper subspace of a normed linear space N.

Then $\exists\, x \in N$ such that $x \notin M$.

Let $d = d(x, M)$

$$= \inf\{\|x - m\| : m \in M\}$$

We know

$$x \in \overline{M} \Leftrightarrow d(x, M) = 0$$

Since M is closed, $\overline{M} = M$

Therefore, $x \notin M \Leftrightarrow d(x, M) > 0$. This gives $d > 0$.

Fix any ε such that $0 < \varepsilon < 1$.

Then $\frac{d}{\varepsilon} > d$.

By definition of infimum there exists $m_0 \in M$ such that,

$$d < \|x_0 - m_0\| \leq \frac{d}{\varepsilon} \qquad \text{...... (1)}$$

Define $$x_\varepsilon = \frac{x - m_0}{\|x - m_0\|}.$$

Then $$\|x_\varepsilon\| = \frac{\|x - m_0\|}{\|x - m_0\|} = 1.$$

For any $m \in M$, we have,

$$\|x_\varepsilon - m\| = \left\| \frac{x - m_0}{\|x - m_0\|} - m \right\|$$

$$= \left\| \frac{x - m_0 - m\|x - m_0\|}{\|x - m_0\|} \right\|$$

$$= \frac{\left\| x - \left(m_0 + m\|x - m_0\| \right) \right\|}{\|x - m_0\|}$$

$$= \frac{\|x - m'\|}{\|x - m_0\|},$$

where $m' = m_0 + m\|x - m_0\| \in M$.

By definition of d and (1) we have,

$$\|x_\varepsilon - m\| \geq \|x - m'\| \frac{\varepsilon}{d} \geq d\left(\frac{\varepsilon}{d}\right) = \varepsilon, \ \forall m \in M$$

i.e. $\|x_\varepsilon - m\| \geq \varepsilon, \ \forall m \in M$

$$\Rightarrow d\left(x_\varepsilon m\right) \geq \varepsilon$$

This completes the proof.

4.6.7 Theorem (Riesz) :

A normed space N is finite dimensional if and only if the closed unit sphere in N is compact.

Proof : Let N be a finite dimensional normed space.

Let $S_1[0] = \{x \in N : \|x\| \leq 1\}$ be a closed unit sphere in N.

Then $S_1[0]$ is closed bounded set in N.

But in a finite dimensional normed space every non-empty closed bounded set is compact (Corollary 4.6.3)

Therefore $S_1[0]$ is compact.

Conversely suppose the closed unit sphere $S_1[0]$ in a normed space N is compact.

We have to prove that N is finite dimensional.

If possible N is not finite dimensional.

Choose $x_1 \in S$ and let M_1 be the subspace spanned by $\{x_1\}$.

Then M_1 is proper subspace of N.

Further M_1 is finite dimensional, hence it is closed. Therefore by Riesz Lemma $\exists$ a vector $x_2 \in N$ such that

$$\|x_2\| = 1 \text{ and } \|x_1 - x_2\| \geq \frac{1}{2}.$$

Let M_2 be the closed subspace spanned by $\{x_1, x_2\}$. Then as discussed above $\exists$ a vector $x_3 \in N$ such that,

$$\|x_3\| = 1 \text{ and } \|x_2 - x_3\| \geq \frac{1}{2}$$

Continuing in this way, we obtain the sequence $\{x_n\}$ in $S_1[0]$ such that,

$$\|x_k - x_{k+1}\| \geq \frac{1}{2}, \ \forall\, k \in \mathbb{N}.$$

Therefore, the sequence $\{x_n\} \subseteq S_1[0]$ has no convergent subsequence.

But this contradicts to the assumption $S_1[0]$ is compact in N.

Hence N must be finite dimensional space.

❑❑❑

UNIT - V

HILBERT SPACES

This unit aims at providing a geometric structure to a linear space. The basic concept of an inner product is introduced in section 5.1. An inner product induces a norm on the linear space. If such a space is complete, then it is known as Hibert Space. A Hilbert Space is a special type of Banach Space which possesses additional structure enabling us to tell when two vectors are orthogonal. Just as we were able to embed any normed linear space in a complete normed linear space, we shall be able to embed any inner product space in a complete inner product space or Hibert space. Some examples and simple properties of Hibert spaces are discussed in section 5.2. Some theorems about orthogonal complements are proved in section 5.3. In section 5.4 we discuss Bessel's inequality, Gram Schmidt orthogonalization process, some examples and properties of orthonormal sets. The natural correspondance between the vectors in H and conjugate space H* is established in section 5.5.

5.1 INNER PRODUCT SPACES

Suppose X is a real or complex vector space; i.e. suppose the underlying scalar field is either real or complex numbers $\mathbb{R}$ or $\mathbb{C}$. We now make the following definition.

Definition 5.1.1 :

An inner product on X is a mapping from $X \times X$, the Cartesian product space, into the scalar field, which we shall denote generally by F.

$$X \times X \to F$$

$$\langle x, y \rangle \to (x, y)$$

[$\langle x, y \rangle$ represents only the ordered pair whereas (x, y) denotes the inner product of two vectors]

With the following properties :

1) Let $x, y \in X$ then $(x, y) = \overline{(y, x)}$ where the bar denotes complex conjugation.

2) If α, β are scalars and *z, y, z* are vectors then

$$(\alpha x + \beta y, z) = \alpha(x, z) + \beta(y, z)$$

3) $(x, x) \geq 0$, $\forall x \in X$ and equal to zero iff x is the zero vector.

Definition 5.1.2 :

A real or a complex vector space with an inner product defined on it will be called an inner product space or pre-Hilbert space.

Proposition 5.1.1 :

If a vector y has the property that $(x, y) = 0$, $\forall x \in X$ then $y = 0$.

Proof :

Suppose $(x, y) = 0$, $\forall x \in X$ then $(y, y) = 0$ but then by property (3) of inner product space definition 5.1.1, $y = 0$.

Example 5.1.1 :

Let $X = \mathbb{C}^n$. An inner product of two vectors $x = (\alpha_1, \alpha_2, \alpha_3, ..., \alpha_n)$ and $y = (\beta_1, \beta_2, \beta_3, ..., \beta_n)$ where α_i, β_i, $i = 1, 2, 3, ..., n$ are complex numbers, define

$$(x, y) = \sum_{i=1}^{n} \alpha_i \overline{\beta_i}$$

$\mathbb{C}^n$ with this inner product is referred to as complex Euclidean n-space. With $\mathbb{R}$ in place of $\mathbb{C}$ we get real Euclidean n-space.

Example 5.1.2:

Let $X = C[a,b]$ be a set of complex valued continuous functions on [*a, b*]. For continuous functions *f* and *g*, $(f+g)(x) = f(x)+g(x)$ and for $\alpha \in C$, $(\alpha f)(x) = \alpha f(x)$. An inner product of *f* and *g* is defined as,

$$(f,g) = \int_a^b f(x)\overline{g(x)}dx$$

If *f* and *g* are real valued functions

$$(f,g) = \int_a^b f(x)g(x)\,dx.$$

Example 5.1.3 :

Let $X = \ell_2$ set at sequences of complex / real numbers $(a_1, a_2, a_3, \ldots\ldots, a_n, \ldots.)$ with $\sum |a_i|^2 < \infty$, the inner product of two vectors

$$x = (\alpha_1, \alpha_2, \alpha_3, \ldots\ldots, \alpha_n, \ldots.) \text{ and } y = (\beta_1, \beta_2, \beta_3, \ldots\ldots, \beta_n, \ldots.)$$

We shall define $(x,y) = \sum_{i=1}^{\infty} \alpha_i \overline{\beta}_i$.

Example 5.1.4 :

Let Y = [*a, b*] and S be the set of Lebesgue measurable sets in Y, μ be Lebsesgue measure. For the equivalence classes of square integrable functions on [*a, b*] define inner product between two classes [*f*] and [*g*] as

$$([f],[g]) = \int_a^b f(x)\overline{g(x)}dx$$

Where the integral is Lebesgue integral. This space is usually referred to as L_2 (*a, b*).

Theorem 5.1.1 : (Cauchy Schwarz inequality or Schwarz inequality)

Let X be an inner product space and let $x, y \in X$. Then $|(x,y)| \le \|x\|\|y\|$.

Proof-1 :

When $y = 0$, $(x,y) = (x,0) = (x, y_1 - y_1) = (x, y_1) - (x, y_1) = 0$

Thus $(x,y) = 0$ and $\|y\| = 0$. Since both sides vanish the result is true when $y = 0$.

When $y \ne 0$, $\|y\| \ne 0$ the inequality.

$$|(x,y)| \le \|x\|\|y\| \Rightarrow \frac{1}{\|y\|}|(x,y)| \le \|x\|$$

$$\Rightarrow \left|\left(x, \frac{y}{\|y\|}\right)\right| \le \|x\| \text{ and } \left\|\frac{y}{\|y\|}\right\| = \frac{\|y\|}{\|y\|} = 1$$

Thus it is sufficient to show that if $\|y\| = 1$,

$$|(x,y)| \le \|x\| \qquad \forall x \in X$$

We know that $\|x - (x,y)y\| \ge 0$.

Consider $\|x - (x,y)y\|^2 = (x - (x,y)y, x - (x,y)y)$

$$= (x,x) - \overline{(x,y)}(x,y) - (x,y)(y,x) + (x,y)\overline{(x,y)}(y,y)$$

$$= (x,x) - \overline{(x,y)}(x,y) - (x,y)\overline{(x,y)} + (x,y)\overline{(x,y)} \quad (\because (y,y) = \|y\|^2 = 1)$$

$$= \|x\|^2 - |(x,y)|^2$$

Since $\|x - (x,y)y\|^2 \ge 0$, $\|x\|^2 - |(x,y)|^2 \ge 0$.

Proof 2 : Let $x, y \in X$ and consider $z = (y,y)x - (x,y)y$.

Then $0 \le (z,z) = ((y,y)x - (x,y)y, (y,y)x - (x,y)y)$

$$= (y,y)\overline{(y,y)}(x,x) - (y,y)\overline{(x,y)}(x,y) - (x,y)\overline{(y,y)}(y,x) + (x,y)\overline{(x,y)}(y,y)$$

$=|(y,y)|^2(x,x)-(x,y)\overline{(y,y)}(y,x)$

$=\overline{(y,y)}\left[(y,y)(x,x)-(x,y)\overline{(x,y)}\right]$

$=(y,y)\left[(y,y)(x,x)-|(x,y)|^2\right]$

If $(y,y)>0$, then it follows that $(y,y)(x,x)-|(x,y)|^2\geq 0 \Rightarrow |(x,y)|^2 \leq \|x\|\cdot\|y\|$.

If $(y,y)=0$ then $y=0$ and hence $(x,y)=0$. So $(x,x)(y,y)=|(x,y)|^2$.

Let $|(x,y)|^2=(x,x)(y,y)$ then we have $(z,z)=0$ so that z = 0, that is $(y,y)x=(x,y)y$. Hence the set $\{x,y\}$ is linearly dependent. Conversely if $\{x,y\}$ is linearly dependent then either $x=ky$ or $y=kx$ for some scalar *k*. In this case

$$|(x,y)|^2=|(x,kx)|^2=(x,kx)\overline{(x,kx)}=\bar{k}(x,x)\overline{\bar{k}(x,x)}$$

$$=k\cdot\bar{k}(x,x)(x,x)=(x,x)(kx,kx)=(x,x)(y,y).$$

Corollary 5.1.1 :

The inner product is jointly continuous function [i.e. given $\varepsilon>0$, $\exists\delta>0$ such that $|(x_1,y_1)-(x_2,y_2)|<\varepsilon$ whenever $\|y_1-y_2\|<\delta$, $\|x_1-x_2\|<\delta$]

Proof :

Let $x_3=x_1-x_2$ and $y_3=y_1-y_2$. Consider

$$|(x_1,y_1)-(x_2,y_2)|=|(x_2+x_3,y_2+y_3)-(x_2,y_2)|$$

$$=|(x_2,y_2)+(x_2,y_3)+(x_3,y_2)+(x_3,y_3)-(x_2,y_2)|$$

$$=|(x_2,y_3)+(x_3,y_2)+(x_3,y_3)|$$

$$\leq|(x_2,y_3)|+|(x_3,y_2)|+|(x_3,y_3)|$$

$$\leq \|x_2\|\cdot\|y_3\|+\|x_3\|\cdot\|y_2\|+\|x_3\|\cdot\|y_3\| \qquad \text{(by Schwarz inequality)}$$

$$\leq \|x_2\|\|y_1-y_2\|+\|y_2\|\cdot\|x_1-x_2\|+\|x_1-x_2\|\cdot\|y_1-y_2\|$$

and continuity of the mapping is evident.

Theorem 5.1.2 : (Polarization identity)

Let X be a real inner product space and let $x, y \in X$. Then

$$(x,y)=\frac{1}{4}\|x+y\|^2-\frac{1}{4}\|x-y\|^2$$

Proof : $\frac{1}{4}\|x+y\|^2-\frac{1}{4}\|x-y\|^2$

$$=\frac{1}{4}(x+y,x+y)-\frac{1}{4}(x-y,x-y)$$

$$=\frac{1}{4}\left[(x,x)+(x,y)+(y,x)+(y,y)\right]-\frac{1}{4}\left[(x,x)-(x,y)-(y,x)+(y,y)\right]$$

$$=\frac{1}{4}\left[2(x,y)\right]-\frac{1}{4}\left[-2(x,y)\right] \qquad (\overline{(x,y)}=(y,x)=(x,y) \text{ since X is real})$$

$$=(x,y)$$

Theorem 5.1.3 : (Polarization Identity)

Let X be a complex inner product space and let $x, y \in X$. Then

$$(x,y)=\frac{1}{4}\|x+y\|^2-\frac{1}{4}\|x-y\|^2+\frac{i}{4}\|x+iy\|^2-\frac{i}{4}\|x-iy\|^2$$

Proof : $\frac{1}{4}\|x+y\|^2-\frac{1}{4}\|x-y\|^2+\frac{i}{4}\|x+iy\|^2-\frac{i}{4}\|x-iy\|^2$

$$=\frac{1}{4}\left\{(x+y,x+y)-(x-y,x-y)+i(x+iy,x+iy)-i(x-iy,x-iy)\right\}$$

$$= \frac{1}{4}\{(x,x)+(x,y)+(y,x)+(y,y)-[(x,x)-(x,y)-(y,x)+(y,y)]$$

$$+i[(x,x)+\bar{i}(x,y)+i(y,x)+i\bar{i}(y,y)]-i[(x,x)-\bar{i}(x,y)-i(y,x)+i\bar{i}(y,y)]\}$$

$$= \frac{1}{4}\{2(x,y)+2(y,x)+i[2\bar{i}(x,y)+2i(y,x)]\}$$

$$= \frac{1}{4}\{2(x,y)+2(y,x)+2(x,y)-2(y,x)\}$$

$$=(x,y)$$

Theorem 5.4 : (Parallelogram Law)

Law X be an inner product space and $x, y \in X$. Then,

$$\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2$$

Proof : $\|x+y\|^2 + \|x-y\|^2 = (x+y, x+y) + (x-y, x-y)$

$$=(x,x)+(x,y)+(y,x)+(y,y)-(x,x)-(x,y)-(y,x)+(y,y)$$

$$=2(x,x)+2(y,y)=2\|x\|^2+2\|y\|^2$$

Theorem 5.1.5 :

The mapping of X into F defined by $f(x)=(x,x)^{1/2}$ is a norm on X and will be denoted by $\|x\|$.

Proof : Since $(x,x)\geq 0$, $\|x\|\geq 0$.

$(x,x)=0$ iff $x=0 \Rightarrow \|x\|=0$ iff $x=0$.

$$\|\alpha x\| = (\alpha x, \alpha x)^{1/2} = [\alpha\bar{\alpha}(x,x)]^{1/2} = [|\alpha|^2 (x,x)]^{1/2}$$

$$= |\alpha| (x,x)^{1/2} = |\alpha| \|x\|$$

$$\|x+y\|^2 = (x+y, x+y)$$

$$= (x,x)+(x,y)+(y,x)+(y,y)$$

$$= \|x\|^2 + 2\,\mathrm{Re}(x,y) + \|y\|^2$$

$$\leq \|x\|^2 + 2|(x,y)| + \|y\|^2$$

$$\leq \|x\|^2 + 2\|x\|\|y\| + \|y\|^2 \qquad \text{(Schwarz inequality)}$$

$$\leq (\|x\| + \|y\|)^2$$

Thus, $\|x+y\| \leq \|x\| + \|y\|$

5.2 The Definition and Some Simple Properties

The Banach spaces are little more than linear spaces provided with a resonable notion of the length of a vector. The theory of Hilbert spaces talks about the orthogonality of vectors.

Definition 5.2.1:

An inner product space which is complete in the norm induced by the inner product is called a Hilbert space.

Definition 5.2.2 :

Hibert space is a complete normed linear space in which the norm satisfies the parallelogram law.

Definition 5.2.3 :

A Hilbert space is a complex Banach space whose norm arises from an inner product i.e. in which there is defined a complex function (x, y) of vectors x and y with the following properties.

1) $(\alpha x + \beta y, z) = \alpha (x,z) + \beta (y,z)$

2) $\overline{(x,y)} = (y,x)$

3) $(x,x) = \|x\|^2$

Observe that $(x, \alpha y + \beta z) = \bar{\alpha}(x,y) + \bar{\beta}(x,z)$ is a direct consequence of properties (1) and (2).

Examples of Hilbert Spaces :

Example 1 : The space $\mathbb{C}^n$ with the inner product of two vectors $x = (x_1, x_2, ..., x_n)$, $y = (y_1, y_2, y_3, ..., y_n)$ defined by $(x, y) = \sum_{i=1}^{n} x_i \overline{y}_i$ is Hilbert space.

Let $x, y \in \mathbb{C}^n$, $(y, x) = \sum_{i=1}^{n} y_i \overline{x}_i$.

$\overline{(y, x)} = \overline{\sum y_i \overline{x}_i} = \sum_{i=1}^{n} \overline{y}_i x_i = (x, y)$

If α, β are scalars and $x, y, z \in \mathbb{C}^n$ then

$$(\alpha x + \beta y, z) = \sum_{i=1}^{n} (\alpha x_i + \beta y_i)\overline{z}_i = \alpha \sum x_i \overline{z}_i + \beta \sum y_i \overline{z}_i$$

$$= \alpha (x, z) + \beta (y, z)$$

$$(x, x) = \sum_{i=1}^{n} x_i \overline{x}_i = \sum_{i=1}^{n} |x_i|^2 \geq 0 \qquad \forall x \in \mathbb{C}^n$$

and $\sum |x_i|^2 = 0$ iff $x_i = 0 \ \forall i$ i.e. $x = 0$.

Thus $\mathbb{C}^n$ with inner product defined by $(x, y) = \sum x_i \overline{y}_i$ is an inner product space.

Let $\{x_m\}$ be a Cauchy sequence in $\mathbb{C}^n$ i.e. given $\varepsilon > 0$ $\exists N \in \mathbb{N}$ such that $\|x_m - x_p\| < \varepsilon$, $\forall p, m > N$.

Let $x_m = (x_{m1}, x_{m2}, ..., x_{mn})$ and $x_p = (x_{p1}, x_{p2}, ..., x_{pn})$.

$\|x_m - x_p\| < \varepsilon$

$$\Rightarrow (x_m - x_p, x_m - x_p) < \varepsilon^2 \qquad \forall p, m > N$$

$$\Rightarrow \sum (x_m - x_p)_i \overline{(x_m - x_p)}_i < \varepsilon^2$$

$$\Rightarrow \sum |x_{mi} - x_{pi}|^2 < \varepsilon^2$$

$\Rightarrow \left|x_{mi} - x_{pi}\right| < \varepsilon \qquad \forall m, p > N \qquad \forall i = 1, 2, 3, ..., n.$

$\Rightarrow \left\{x_{mi}\right\}_{m=1}^{\infty}$ is Cauchy sequence in $\mathbb{C}$.

Since $\mathbb{C}$ is complete $x_{mi} \to x_i$, $1 \le i \le n$.

Define $x = \left(x_1, x_2, x_3, \ldots.., x_n\right) \in \mathbb{C}^n$

Then $x_n \to x \in \mathbb{C}^n$. $\therefore \mathbb{C}^n$ is complete with respect to the norm induced by inner product.

Thus, $\mathbb{C}^n$ is a Hilbert space with inner product defined by

$$(x, y) = \sum_{i=1}^{n} x_i \overline{y}_i \qquad i = 1, 2, 3, \ldots., n.$$

Example 2 :

The space $\ell_2 = \left\{\{x_n\} \mid \Sigma |x_i|^2 < \infty\right\}$, the space of square summable sequences with inner product two vectors $x = \{x_n\}$, $y = \{y_n\}$ defined by $(x, y) = \sum_{i=1}^{\infty} x_i \overline{y}_i$ is Hilbert space.

ℓ_2 is a Banach space with $\|x\| = \sum_{i=1}^{\infty} |x_i|^2$.

(i) $\quad (x, x) = \sum_{i=1}^{\infty} x_i \overline{x}_i = \sum_{i=1}^{\infty} |x_i|^2 > 0$

(ii) $\quad (\alpha x + \beta y, z) = \sum_{i=1}^{\infty} (\alpha x_i + \beta y_i) \overline{z}_i$

$$= \alpha \Sigma x_i \overline{z}_i + \beta \Sigma y_i \overline{z}_i$$

$$= \alpha (x, z) + \beta (y, z)$$

where $x = \{x_n\}$, $y = \{y_n\}$ and $z = \{z_n\}$.

(iii) $(x, y) = \Sigma x_i \overline{y}_i$

$$(y, x) = \Sigma y_i \overline{x}_i \Rightarrow \overline{(y, x)} = \overline{\Sigma y_i \overline{x}_i} = \sum_{i=1}^{\infty} \overline{y}_i x_i = (x, y)$$

Thus $\overline{(y, x)} = (x, y)$.

Thus ℓ_2 with inner product defined by $(x, y) = \sum_{i=1}^{\infty} x_i \overline{y}_i$ is complete and therefore is a Hilbert space.

Example 3 :

The space ℓ_p, $p \neq 2$, $1 < p < \infty$ is **not** a Hilbert space.

Suppose ℓ_p, $p \neq 2$, is a Hilbert space. Then ℓ_p, $p \neq 2$, is an inner product space. Inner product space satisfies parallelogram law $\|x+y\|^2 + \|x-y\|^2 = 2\left(\|x\|^2 + \|y\|^2\right)$.

Let $x = (1, 0, 0, 0, ...)$ and $y = (0, 1, 0, 0, 0, ...)$, then,

$\|x\|_p = 1$, $\|y\|_p = 1$, $\|x+y\|_p = 2^{1/p}$,

$\|x-y\|_p = 0$ if p is odd.

$= 2^{1/p}$ if p is even.

Then $\|x+y\|_p^2 + \|x-y\|_p^2 = 2^{2/p}$ if p is odd.

$= 2^{1+2/p}$ if p is even.

$2\left(\|x\|_p^2 + \|y\|_p^2\right) = 4 \qquad \forall p$

Observe that if p is odd $2^{2/p} \neq 4$, for p even, $2^{1+2/p} = 4$ only when $p = 2$.

Thus ℓ_p, $p \neq 2$ do not satisfy parallelogram law.

$\therefore \ell_p$ is not an inner product space for $p \neq 2$.

$\therefore \ell_p$, $p \neq 2$ is not a Hilbert space.

Theorem 5.2.1 :

A closed convex subset C of a Hilbert space H contains a unique vector of smallest norm.

Proof :

Since C is closed convex subset of a Hilbert space H, for $x, y \in C$, $\lambda x + (1-\lambda) y \in C$ where $0 \le \lambda \le 1$.

In particular $\frac{x+y}{2} \in C$.

Let $d = \inf\{\|x\| : x \in C\}$. By property of infimum there is a sequence x_n in C such that $\|x_n\| \to d$. Since C is convex $\frac{x_n + x_m}{2} \in C$ and $\left\|\frac{x_n + x_m}{2}\right\| \ge d$ i.e. $\|x_n + x_m\| \ge 2d$.

Since H is a Hilbert space, parallelogram holds.

$$\|x_m - x_n\|^2 + \|x_m + x_n\|^2 = 2\|x_m\|^2 + 2\|x_n\|^2$$

$$\therefore \|x_m - x_n\|^2 = 2\|x_m\|^2 + 2\|x_n\|^2 - \|x_n + x_m\|^2$$

$$\le 2\|x_m\|^2 + 2\|x_n\|^2 - 4d^2 \qquad (\|x_m + x_n\| \ge 2d)$$

But $2\|x_m\|^2 + 2\|x_n\|^2 - 4d^2 \to 2d^2 + 2d^2 - 4d^2 = 0$

Thus given $\varepsilon > 0$, $\exists N \in \mathbb{N}$ such that

$\|x_m - x_n\|^2 < \varepsilon$, $\forall n, m > N$.

It follows that $\{x_n\}$ is Cauchy sequence in C.

Since H is Hilbert space, H is complete. Since C is closed subset at H, C is complete. Therefore each Cauchy sequence in C converges in C. But $\{x_n\}$ is Cauchy sequence in C. Therefore $x_n \to x$ in C.

$\|x\| = \|\lim x_n\| = \lim \|x_n\| = d$ $\quad$ ($\because \|.\|$ is continuous function)

It means that x is a vector in C with smallest norm.

To show that x is unique, suppose x' is also in C other than x which also has $\|x'\| = d$.

Since C is convex, $\dfrac{x+x'}{2} \in C$ and applying parallelogram law to x and x' we get,

$$\left\|\frac{x+x'}{2}\right\|^2 = 2\left(\left\|\frac{x}{2}\right\|^2\right) + 2\left(\left\|\frac{x'}{2}\right\|^2\right) - \left\|\frac{x-x'}{2}\right\|^2$$

$$< \frac{1}{2}\|x\|^2 + \frac{1}{2}\|x'\|^2 = \frac{1}{2}d^2 + \frac{1}{2}d^2 = d^2$$

Thus, $\left\|\dfrac{x+x'}{2}\right\| < d$.

But d is infimum of $\|x\|$ for $x \in C$.

$\therefore \left\|\dfrac{x+x'}{2}\right\| \not< d$ a contradiction. Therefore, $x = x'$.

Theorem 5.2.2 :

If B is a complex Banach space whose norm obeys the parallelogram law and if an inner product is defined on B by

$$4(x, y) = \|x+y\|^2 - \|x-y\|^2 + i\|x+iy\|^2 - i\|x-iy\|^2$$

Then B is a Hilbert space.

Proof : Since B is a Banach space, B is complete. Thus if B satisfies three properties for inner product space then B is Hilbert space. We shall prove the following.

(i) $(x+y, z) = (x, z) + (y, z)$

(ii) $(\alpha x, y) = \alpha(x, y)$

(iii) $\overline{(x, y)} = (y, x)$

(iv) $(x, x) = \|x\|^2$

(iv) $4(x,x) = \|x+x\|^2 - \|x-x\|^2 + i\|x+ix\|^2 - i\|x-ix\|^2$

$$= 4\|x\|^2 + i|1+i|^2\|x\|^2 - i|1-i|^2\|x\|^2$$

$$= 4\|x\|^2 + i(1+i)(1-i)\|x\|^2 - i(1-i)(1+i)\|x\|^2$$

$$= 4\|x\|^2$$

Thus $(x,x) = \|x\|^2$ and (iv) holds.

Now $4\overline{(x,y)} = \|x+y\|^2 - \|x-y\|^2 - i\|x+iy\|^2 + i\|x-iy\|^2$ .... (1)

$$4(y,x) = \|y+x\|^2 - \|y-x\|^2 + i\|y+ix\|^2 - i\|y-ix\|^2$$

$$= \|y+x\|^2 - \|-(x-y)\|^2 + i\|i(x-iy)\|^2 - i\|(-i)(x+iy)\|^2$$

$$= \|y+x\|^2 - \|x-y\|^2 + i|i|^2\|x-iy\|^2 - i|-i|^2\|x+iy\|^2$$

$$= \|x+y\|^2 - \|x-y\|^2 + i\|x-iy\|^2 - i\|x+iy\|^2 \quad .... (2)$$

From equation (1) and (2) we have $\overline{(x,y)} = (y,x)$. This proves (iii).

To prove (i) and (ii), let $u, v, w \in B$. Since norm on B obeys parallelogram law,

$$\|(u+v)+w\|^2 + \|u+v-w\|^2 = 2\|u+v\|^2 + 2\|w\|^2 \quad .... (3)$$

$$\|(u-v)+w\|^2 + \|(u-v)-w\|^2 = 2\|u-v\|^2 + 2\|w\|^2 \quad .... (4)$$

Equations $(3)-(4)$ gives,

$$\|(u+w)+v\|^2 - \|(u+w)-v\|^2 + \|(u-w)-v\|^2 - \|(u-w)-v\|^2 = 2\left(\|u+v\|^2 - \|u-v\|^2\right)$$

$\therefore$ Real $\langle u+w,v\rangle + \mathrm{Re}\langle u-w,v\rangle = 2\,\mathrm{Re}\langle u,v\rangle$ .... (6)

Similarly,

$$\|(u+iv)+w\|^2 + \|(u+iv)-w\|^2 = 2\|u+iv\|^2 + 2\|w\|^2 \quad .... (7)$$

$$\|(u-iv)+w\|^2 + \|(u-iv)-w\|^2 = 2\|u-iv\|^2 + 2\|w\|^2 \quad .... (8)$$

Equations (7) – (8) gives

$$\|(u+w)+iv\|^2 - \|(u+w)-iv\|^2 + \|(u-w)+iv\|^2 - \|(u-w)-iv\|^2 = 2\|u+iv\|^2 - 2\|u-iv\|^2$$

$\therefore \operatorname{Im}\langle u+w, v\rangle + \operatorname{Im}\langle u-w, v\rangle = 2\operatorname{Im}\langle u, v\rangle$ ..... (9)

From equation (6) and (9) we have,

$\langle u+w, v\rangle + \langle u-w, v\rangle = 2\langle u, v\rangle$

For $u = w$ we have $\langle 2u, v\rangle = 2\langle u, v\rangle$ ..... (10)

For $u + w = x,\ u - w = y,\ v = z$ we have

$\langle x, z\rangle + \langle y, z\rangle = 2\langle u, z\rangle = \langle 2u, z\rangle = \langle x+y, z\rangle$

Thus, $\langle x, z\rangle + \langle y, z\rangle = \langle x+y, z\rangle$ ..... (11)

Equation (10) and (11) proves (ii) and (i)

Since all the conditions (i), (ii), (iii), (iv) are satisfied, B is a Hilbert space.

5.3 ORTHOGONAL COMPLEMENTS

Existence of parallelogram law provide a geometric insight into the place Hilbert spaces.

Definition 5.3.1 :

Two vectory x, y in a Hilbert space H are said to be orthogonal if $(x, y) = 0$ and written as $x \perp y$.

Note :

(i) Since $\overline{(x,y)} = (y,x)$, if $(x,y) = 0$, $(y,x) = 0$ i.e. $x \perp y \Rightarrow y \perp x$.

(ii) Since $(x,0) = 0$, $x \perp 0$ for every $x \in H$.

(iii) Since $(x,x) = \|x\|^2$, $(x,x) = 0 \Rightarrow x = 0$ i.e. 0 is the only vector orthogonal to itself.

If $x \perp y$ then $\|x+y\|^2 = \|x-y\|^2 = \|x\|^2 + \|y\|^2$.

(iv) $\|x+y\|^2 = (x+y, x+y) = (x,x) + (x,y) + (y,x) + (y,y) = \|x\|^2 + \|y\|^2$

Similarly,

(v) $\|x-y\|^2 = (x-y, x-y) = (x,x)-(x,y)-(y,x)+(y,y) = \|x\|^2 + \|y\|^2$

Thus $\|x+y\|^2 = \|x-y\|^2 = \|x\|^2 + \|y\|^2$ is the parallelogram theorem.

Definition 5.3.2 :

A vector x is said to be orthogonal to a non-empty set S (Notation : $x \perp S$) if $x \perp y$, $\forall y \in S$.

Definition 5.3.3 :

Orthogonal complement of S denoted by $S^{\perp}$ is the set of all vectors perpendicular to S.

Note :

(i) $\{0\}^{\perp} = H$

(ii) $H^{\perp} = \{0\}$

(iii) $S \cap S^{\perp} \subseteq \{0\}$

(iv) $S_1 \subseteq S_2 \Rightarrow S_2^{\perp} \subseteq S_1^{\perp}$

(v) $S^{\perp}$ is a closed linear subspace of H.

Theorem 5.3.1 :

Let M be a closed linear subspace of a Hilbert space H. Let x be a vector not in M and let d be the distance from x to M. Then there exists a unique vector y_0 in M such that $\|x - y_0\| = d$.

Proof : Since M is subspace of a Hilbert space H, for $x, y \in M$, $\alpha x + \beta y \in M$ in particular $\alpha x + (1-\alpha) y \in M$. i.e. M is convex. Thus the set C = x + M is a closed convex set.

Since d is distance from x to M, d is distance from origin to $x+$ M $=$ C. By theorem 5.2.1 there exist a unique vector z_0 in C such that $\|z_0\| = d$.($\because$ d is distance from 0 to $x+$ M).

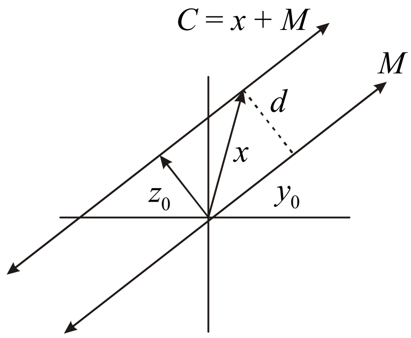
Since $z_0 \in C = x + M$.

$\therefore z_0 = x - y_0$ for some $y_0 \in M$

but then $y_0 = x - z_0$ and $x - z_0 \in M$.

Now $\|x - y_0\| = \|z_0\| = d$.

Thus $\|x - y_0\| = d$.



If y_1 is another vector in M such that $y_1 \neq y_0$ and $\|x - y_1\| = d$ then $z_1 = x - y_1$ is a vector in C such that $z_1 \neq z_0$ and $\|z_1\| = d$.

Which contradicts the uniqueness of z_0 .

Theorem 5.3.2 :

If M is a proper closed linear subspace of a Hilbert space H then there exists a non-zero vector z_0 in H such that $z_0 \perp M$.

Proof : Let $x \notin M$ and let d be the distance from x to M.

By theorem 5.3.1, there exists a vector y_0 in M such that $\|x - y_0\| = d$.

Define $z_0 = x - y_0$.

Since $d > 0,\ z_0 \neq 0$.

Now we shall prove that if $y \in M$ then $y \perp z_0$.

Consider $\|z_0 - \alpha y\| = \|(x - y_0) - \alpha y\| = \|x - (y_0 + \alpha y)\| \geq d = \|z_0\|$

So $\|z_0 - \alpha y\|^2 - \|z_0\|^2 \geq 0$.

$(z_0 - \alpha y, z_0 - \alpha y) - \|z_0\|^2 \geq 0$

$$\Rightarrow (z_0, z_0) - \overline{\alpha}(z_0, y) - \alpha(y, z_0) + |\alpha|^2 (y, y) - \|z_0\|^2 \ge 0$$

$$\Rightarrow -\overline{\alpha}(z_0, y) - \alpha\overline{(z_0, y)} + |\alpha|^2 \|y\|^2 \ge 0$$

Put $\alpha = \beta(z_0, y)$ for an arbitary real number β then

$$-\beta\overline{(z_0, y)}(z_0, y) - \beta(z_0, y)\overline{(z_0, y)} + \beta^2 |(z_0, y)|^2 \|y\|^2 \ge 0$$

$$\therefore -2\beta|(z_0, y)|^2 + \beta^2 |(z_0, y)|^2 \|y\|^2 \ge 0$$

Let $a = |(z_0, y)|^2$ and $b = \|y\|^2$ then we have

$-2\beta a + \beta^2 ab \ge 0 \Rightarrow \beta a(\beta b - 2) \ge 0$ for all real β.

If a is strictly positive, we can chooe β sufficiently small such that $\beta a(\beta b - 2) \le 0$.

Therefore, a cannot be strictly positive but $a \ge 0 \Rightarrow a = 0$. But $a = |(z_0, y)|^2$.

Therefore, $(z_0, y) = 0$ that means $z_0 \perp y$.

Definition 5.3.4 (Orthogonal Sets) :

Two non-empty sets S_1 and S_2 of a Hilbert space H are said to be orthogonal ($S_1 \perp S_2$) if $x \perp y$ for all x in S_1 and y in S_2.

Theorem 5.3.3 :

If M and N are closed linear subspaces of a Hilbert space H such that $M \perp N$ then the linear space M + N is closed in H.

Proof : Let z be a limit point of M + N. Then there is a sequence $\{z_n\}$ in M + N that converges to z. Since $M \perp N$, M and N are disjoint. Therefore each z_n is uniquely written as $z_n = x_n + y_n$ for $x_n \in M$ and $y_n \in N$.

[If $x \perp y$ then $\|x+y\|^2 = \|x-y\|^2 = \|x\|^2 + \|y\|^2$ is called Pythagorean theorem.

$(x+y, x+y) = (x,x) + (x,y) + (y,x) + (y,y)$ Since $x \perp y$, $(x,y) = 0$.

$\therefore \|x+y\|^2 = \|x\|^2 + \|y\|^2$ Similarly $\|x-y\|^2 = \|x\|^2 + \|y\|^2$.]

$$\|z_m - z_n\|^2 = \|(x_m + y_m) - (x_n + y_n)\|^2$$

$$= \|(x_m - x_n) + (y_m - y_n)\|^2$$

Since M and N are subspaces of H,

$x_m, x_n \in M \Rightarrow x_m - x_n \in M$ and $y_n, y_m \in N \Rightarrow y_m - y_n \in N$

But $M \perp N$.

$$\therefore \left\|(x_m - x_n)^2 + (y_m - y_n)^2\right\| = \|x_m - x_n\|^2 + \|y_m - y_n\|^2$$

Since $z_n \to z$, $\{z_n\}$ is a Cauchy sequence. Therefore given $\varepsilon > 0$, $\exists N \in \mathbb{N}$ such that $\|z_n - z_m\| < \varepsilon$, $\forall n, m > N$.

Thus $\|z_m - z_n\|^2 = \|x_m - x_n\|^2 + \|y_n - y_m\|^2 < \varepsilon$.

$\Rightarrow \|x_m - x_n\| < \varepsilon$ and $\|y_n - y_m\| < \varepsilon$

$\Rightarrow \{x_n\}$ and $\{y_n\}$ are cauchy sequences in M and N respectively. But M and N are closed subspaces of H. Therefore $\{x_n\} \to x \in M$ and $y_n \to y \in N$.

Since $x \in M$ and $y \in N$, $x + y \in M + N$.

$$\therefore z = \lim_{n\to\infty} z_n = \lim_{n\to\infty}(x_n + y_n) = x + y \in M + N$$

Thus M + N is closed subspace of a Hilbert space H.

Theorem 5.3.4 :

If M is a closed linear subspace of a Hilbert space H, then $H = M \oplus M^{\perp}$.

Proof : Since M and $M^{\perp}$ are orthogonal closed subspaces of a Hilbert space H, by theorem 5.3.3, $M + M^{\perp}$ is closed subspace of H. Suppose $M + M^{\perp} \neq H$ then $M + M^{\perp} \subset H$ then by theorem 5.3.2, there exists a non-zero vector z_0 in H such that $z_0 \perp (M + M^{\perp})$.

Then $z_0 \in (M + M^{\perp})^{\perp}$ i.e. $z_0 \in M^{\perp} \cap M^{\perp\perp}$.

Since M is closed subspace of H, $M^{\perp\perp} = M$.

Thus $z_0 \in M^{\perp} \cap M$ but $M^{\perp} \cap M = \{0\}$.

Therefore $z_0 = 0$. But $z_0 \neq 0$.

$\therefore M + M^{\perp} = H$.

Since $M \cap M^{\perp} = \{0\}$, $M \oplus M^{\perp} = H$.

5.4 ORTHONORMAL SETS

Definition 5.4.1 :

An orthonormal set in a Hilbert space H is a non-empty subset of H which consists of mutually orthogonal unit vectors.

Non-empty set $\{e_i\}$ of H is said to be orthonormal if

(i) for $i \neq j$, $e_i \perp e_j$ (ii) $\|e_i\| = 1 \ \forall i$.

Example 5.4.1 :

The set $\{e_1, e_2, e_3, ..., e_n\}$ of ℓ_2^n where each e_i is n-tuple with 1 in the i[th] place and 0 elsewhere is an orthonormal set.

Example 5.4.2 :

The set $\{e_1, e_2, e_3, e_4, ..., e_n, ...\}$ where e_n is a sequence have 1 at n^{th} position and zero otherwise is orthonormal set in ℓ_2.

Example 5.4.3 :

Consider a Hilbert space L_2 associated with the measure space [0, 2π], where measure is Lebesgue measure and integrals are Lebesgue integrals.

$$L_2 = \left\{ f \mid \int_0^{2\pi} |f(x)|^2 \, dx < \infty \right\}$$

The norm and inner product are defined by,

$$\|f\| = \left(\int_0^{2\pi} |f(x)|^2 \, dx \right)^{\frac{1}{2}} \text{ and } (f, g) = \int_0^{2\pi} f(x) \overline{g(x)} dx$$

Since $\int_0^{2\pi} e^{imx} e^{-inx} dx = 0$ if $m \neq n$

$= 2\pi$ if $m = n$

Functions $e_n(x)$ defined by $e_n(x) = \dfrac{e^{inx}}{\sqrt{2\pi}}$ form an orthonormal set in L_2.

Theorem 5.4.1 :

Let $\{e_1, e_2, e_3, ..., e_n\}$ be a finite orthonormal set in a Hilbert space H. If x is any vector in H, then $\sum_{i=1}^{n} |(x, e_i)|^2 \leq \|x\|^2$ and $x - \sum_{i=1}^{n} (x, e_i) e_i \perp e_j \forall_j$.

Proof : $\left\| x - \sum (x, e_i) e_i \right\|^2 \geq 0$

$$\Rightarrow \left(x - \sum (x, e_i) e_i, x - \sum (x, e_i) e_i \right) \geq 0$$

$$\Rightarrow (x,x) - \sum_{i=1}^{n} \overline{(x,e_i)}(x,e_i) - \sum_{i=1}^{n} (x,e_i)(e_i,x) + \sum_{i,j=1}^{n} (x,e_i)\overline{(x,e_j)}(e_i,e_j) \geq 0$$

$$\Rightarrow (x,x) - \Sigma|(x,e_i)|^2 - \Sigma(x,e_i)\overline{(x,e_i)} + \Sigma(x,e_i)\overline{(x,e_j)}(e_i,e_j) \geq 0$$

$$\Rightarrow \|x\|^2 - \Sigma|(x,e_i)|^2 - \Sigma|(x,e_i)|^2 + \sum_{i=1}^{0}|(x,e_i)|^2 \geq 0 \qquad [\because (e_i,e_j) = \delta_{ij}]$$

$$\Rightarrow \|x\|^2 - \Sigma|(x,e_i)|^2 \geq 0$$

$$\Rightarrow \Sigma|(x,e_i)|^2 \leq \|x\|^2$$

Observe that,

$$\left(x - \sum_{i=1}^{n}(x,e_i)e_i, e_j\right) = (x,e_j) - \sum_{i=1}^{n}(x,e_i)(e_i,e_j)$$

$$= (x,e_j) - (x,e_j) \qquad [\{e_i\} \text{ is orthonormal set}]$$

$$= 0$$

Since $\left(x - \Sigma(x,e_i)e_i, e_j\right) = 0 \Rightarrow x - \Sigma(x,e_i)e_i \perp e_j$

Theorem 5.4.2 :

If $\{e_i\}$ is an orthonormal set in a Hilbert space H and if x is any vector in H then the set $S = \{e_i \mid (x,e_i) \neq 0\}$ is either empty or countable.

Proof : If $x = 0$, there is nothing to prove. The set $S = \phi$.

Let $x \neq 0$. For j = 1, 2, 3, ... let,

$$E_j = \{e_\alpha : \|x\| \leq j \mid (x,e_\alpha)\}$$

Fix j, suppose E_j contains distinct elements $e_{\alpha 1}, e_{\alpha 2}, ..., e_{\alpha m}$.

Then $0 \leq m\|x\|^2 \subseteq j^2 \sum_{n=1}^{m}|(x,e_{\alpha_n})|^2$.

But by theorem 5.4.1, $\sum_{n=1}^{m}\left|\left(x,e_{\alpha_n}\right)\right|^2 \le \|x\|^2$. This shows that

$m\|x\|^2 \le j^2\|x\|^2 \Rightarrow m \le j^2$

Thus $m \le j^2$ and therefore each E_j contains at most j^2 elements. But $S = \bigcup E_j$ and countable union of countable set is countable. Thus S is countable.

Theorem 5.4.3 (Bessel's inequality) :

If $\{e_i\}$ is an orthonormal set in a Hilbert space H then

$$\Sigma\left|\left(x,e_j\right)\right|^2 \le \|x\|^2 \qquad \forall x \in H$$

Proof : Let $S = \{e_i \mid (x,e_i) \ne 0\}$. If S is empty then $(x,e_i) = 0 \quad \forall i$ and $\Sigma\left|\left(x,e_j\right)\right|^2 = 0$. Therefore result holds.

Suppose $S \ne \phi$ then by theorem 5.4.2, S is countable. If S is finite,

$S = \{e_1, e_2, e_3, ..., e_n\}$ for some integer *n*.

$\Sigma\left|\left(x,e_j\right)\right|^2 = \sum_{j=1}^{n}\left|\left(x,e_j\right)\right|^2$ and by theorem 5.4.1 result holds.

Let $S = \{e_1, e_2, e_3, ..., e_n, ...\}$.

If $\sum_{n=1}^{\infty}\left|\left(x,e_n\right)\right|^2$ converges then every series obtained from rearrangement of terms also converges and all such series have the same sum.

Therefore $\Sigma\left|\left(x,e_i\right)\right|^2 = \sum_{i=1}^{\infty}\left|\left(x,e_i\right)\right|^2$ and $\sum_{i=1}^{\infty}\left|\left(x,e_i\right)\right|^2$ only depends on S and not not the arrangement of $\{e_i\}$. In this case

$$\Sigma\left|\left(x,e_i\right)\right|^2 = \sum_{i=1}^{\infty}\left|\left(x,e_i\right)\right|^2$$

Since $\sum_{i=1}^{n}\left|(x,e_i)\right|^2 \le \|x\|^2$, all partial sums of $\sum_{i=1}^{\infty}\left|(x,e_i)\right|^2$ are bounded by $\|x\|^2$.

$$\therefore \sum_{i=1}^{\infty}\left|(x,e_i)\right|^2 \le \|x\|^2$$

Theorem 5.4.4 :

If $\{e_i\}$ is an orthonormal set in a Hilbert space H and if x is an arbitrary vector in H then

$$x-\sum(x,e_i)e_i \perp e_j \qquad \forall j$$

Proof : Let $S=\{e_i \mid (x,e_i)\neq 0\}$. Then by theorem 5.4.2 either $S=\phi$ or S is countable. If $S=\phi$ then $(x,e_i)=0 \ \forall i$. i.e. x is orthogonal to each e_i.

Suppose $S\neq\phi$. Then S is countable. Let $S=\{e_1,e_2,e_3,....,e_n,...\}$.

Define $S_n=\{e_1,e_2,e_3,....,e_n\}$. In S_n,

$\sum_{i=1}^{\infty}(x,e_i)e_i=\sum_{i=1}^{n}(x,e_i)e_i$. By theorem 5.4.1,

$$x-\sum_{i=1}^{n}(x,e_i)e_i \perp e_j\ ,\ j=1,2,3,....,n$$

Put $s_n=\sum_{i=1}^{n}(x,e_i)e_i$.

Observe that

$$\|s_n-s_m\|=\left\|\sum_{i=m+1}^{n}(x,e_i)e_i\right\|^2=\left\|\sum_{i=m+1}^{n}\left|(x,e_i)\right|^2\|e_i\|\right\|=\sum_{i=m+1}^{n}\left|(x,e_i)\right|^2$$

But by Bessels inequality $\sum_{i=1}^{\infty}\left|(x,e_i)\right|^2\le\|x\|^2$.

$\therefore \sum_{i=1}^{\infty} |(x, e_i)|^2$ is convergent series. Therefore, partial sum form a convergent sequence. Every convergent sequence is Cauchy.

Therefore for $n, m > N$, $\sum_{i=m+1}^{n} |(x, e_i)|^2 < \varepsilon$.

Since $\|s_n - s_m\| = |\Sigma(x, e_i)|^2 < \varepsilon$, $\{s_n\}$ is Cauchy sequence in H. Since H is complete, the sequence $s_n \to s \in H$ i.e. $s = \sum_{i=1}^{\infty} (x, e_i) e_i$.

$$\begin{aligned}
(x - \Sigma(x, e_i) e_i, e_i) = (x - s, e_i) &= (x, e_i) - (s, e_i) \\
&= (x, e_i) - (\lim s_n, e_i) \\
&= (x, e_i) - \lim_{n \to \infty} (s_n, e_i) \\
&= (x, e_i) - \lim_{n \to \infty} \left(\sum_{j=1}^{n} (x, e_j) e_j, e_i \right) \\
&= (x, e_i) - \lim_{n \to \infty} \sum_{j=1}^{n} (x, e_j)(e_j, e_i) \\
&= (x, e_i) - (x, e_i) = 0
\end{aligned}$$

Let the vectors in S be rearranged in any manner,

$$S = \{f_1, f_2, f_3, \ldots., f_n, \ldots\}$$

Put $s_n' = \sum_{i=1}^{n} (x, f_i) f_i$. Then $\{s_n'\}$ is a convergent sequence and

$$s_n' \to s' = \sum_{i=1}^{\infty} (x, f_i) f_i$$

Now we show that $s = s'$.

Let $\varepsilon > 0$ be given and let n_0 be a positive integer so large that if $n \geq n_0$ then $\|s_n - s\| < \varepsilon$ and $\|s_n' - s'\| < \varepsilon$ and $\sum_{i=n_0+1}^{\infty} |(x, e_i)|^2 < \varepsilon^2$. For some positive integer $m_0 > n_0$, all terms of s_{n_0} occur among those of s'_{m_0}, so $s'_{m_0} - s_{n_0}$ is a finite sum of terms of the form $(x, e_i)e_i$ for $i = n_0 + 1, n_0 + 2, \ldots\ldots$. This gives $\left\|s'_{m_0} - s_{n_0}\right\|^2 < \sum_{i=n_0+1}^{\infty} |(x, e_i)|^2 < \varepsilon^2$

So $\left\|s'_{m_0} - s_{n_0}\right\| < \varepsilon$ and $\|s' - s\| \le \left\|s' - s'_{m_0}\right\| + \left\|s'_{m_0} - s_{n_0}\right\| + \left\|s_{n_0} - s\right\| < 3\varepsilon$

Since ε is arbitrary $s' = s$. Thus rearrangement of series gives same limit.

Theorem 5.4.5 :

Every non-zero Hilbert space contains a complete orthonormal set.

Proof : Let H be a non-zero Hilbert space. Let S be the class of all its orthonormal sets. This class is a partially ordered set with respect to set inclusion. An orthonomal set $\{e_i\}$ in H is said to be complete if it is maximal in this partially ordered set. Every chain in S has upper bound (union of the chain). Since every chain in S has upper bound, by Zorn's lemma S has maximal elements. This maximal element of S is complete orthonormal set.

Theorem 5.4.6 :

Let H be a Hilbert space and let $\{e_i\}$ be an orthonormal set in H. Then the following conditions are all equivalent to one another.

1. $\{e_i\}$ is complete
2. $x \perp \{e_i\} \Rightarrow x = 0$
3. If x is an arbitrary vector in H then $x = \sum (x, e_i) e_i$.
4. If x is an arbitrary vector in H then $\|x\|^2 = \sum |(x, e_i)|^2$.

Proof : We prove that (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) $\Rightarrow$ (1)

Suppose (2) is not true. Then there is $x \neq 0$ such that $x \perp \{e_i\}$. Define $e = \frac{x}{\|x\|}$ then $\|e\| = 1$. Observe that $\{e, e_i\}$ is an orthonormal set and $\{e_i\} \subset \{e, e_i\}$. Observe that $\{e, e_i\}$ is orthonormal set which contradicts the completeness of $\{e_i\}$.

(2) $\Rightarrow$ (3) By theorem 5.4.4, $x-\sum(x,e_i)e_i \perp e_j$, $\forall j$.

Therefore by (2) we have $x-\sum(x,e_i)e_i = 0 \Rightarrow x = \sum(x,e_i)e_i$.

(3) $\Rightarrow$ (4) Let $x=\sum(x,e_i)e_i$ then $\|x\|^2 = (x,x)$.

and $(x,x) = \left(\sum_i (x,e_j)e_j, \sum_j (x,e_i)e_i\right) = \sum_{i,j}(x,e_j)\overline{(x,e_i)}(e_j,e_i)$

$$= \sum_{i,j}(x,e_j)\overline{(x,e_i)}\delta_{ij} = \sum\left|(x,e_i)\right|^2$$

Thus $\|x\|^2 = \sum\left|(x,e_i)\right|^2$.

(4) $\Rightarrow$ (1), Suppose (1) is not true. If $\{e_i\}$ is not complete then it is a proper subset of an orthonormal set $\{e_i, e\}$. Since e is orthonormal to all the e_i's, then

$\|e\|^2 = (e,e) = \sum_i \left|(e,e_i)\right|^2 = 0 \Rightarrow e = 0$ a contraction to $\|e\| = 1$.

Thus (4) $\Rightarrow$ (1).

Theorem 5.4.7 : Let $\{e_\alpha\}$ be an orthonormal set in a Hilbert space H. Then the following conditions are equivalent.

(i) $\{e_\alpha\}$ is an orthonormal basis for H.

(ii) (Fourier Expansion) for every $x \in H$ we have,

$$x = \sum(x,e_\alpha)e_\alpha$$

where $\{e_1, e_2, e_3, ...\} = \{e_\alpha \mid (x,e_\alpha) \neq 0\}$ (This set is countable)

(iii) (Parseval formula) for every $x \in H$ we have,

$$\|x\|^2 = \sum_n \left|(x,e_n)\right|^2$$

where $\{e_1, e_2, ...\} = \{e_\alpha \mid (x,e_\alpha) \neq 0\}$

(iv) Span $\{e_\alpha\}$ is dense in H.

(v) If $x \in H$ and $(x, e_\alpha) = 0$, $\forall \alpha$ then $x = 0$.

Proof : (i)$\Rightarrow$(ii) : Let $\{e_\alpha\}$ be a maximal orthonormal set in H.

Consider $x \in H$, By theorem 5.4.2 and theorem 5.4.4,

$\Sigma(x, e_i)e_i$ converges to y for some $y \in H$ and $x - y \perp e_j \quad \forall j$.

If $y \neq x$, let $e = \dfrac{(y-x)}{\|y-x\|}$ then $\|e\| = 1$ and $e \perp e_j \quad \forall j$ so that

$\{e_\alpha\} \cup \{e\} = \{e_\alpha\}$ is an orthonormal set in H but $\{e_\alpha\}$ is a basis.

Therefore, $\{e_\alpha\} \cup \{e\} = \{e_\alpha\}$ i.e. $x = y$ i.e. $x = \Sigma(x, e_j)e_j$

(ii)$\Rightarrow$(iii) follows from theorem 5.4.6.

(ii)$\Rightarrow$(iv) Since $\sum_{j=1}^{m}(x, e_j)e_j \in$ span $\{e_\alpha\}$ for each $m = 1, 2, 3, ...$

and $x = \sum_j (x, e_j)e_j$, $x \in \overline{\text{span } \{e_\alpha\}}$. Thus $\{e_\alpha\}$ is dense in H.

(iv)$\Rightarrow$(v) : Let $x \in H$ be such that $(x, e_\alpha) = 0 \quad \forall \alpha$ and let

$x_m \to x$ where $x_m \in \text{span}\{e_\alpha\}$. Consider (x_m, x).

$= \left(\sum_{j=0}^{m} a_i e_i, x\right) = \Sigma a_i (e_i, x) = 0$. Thus $(x_m, x) = 0$.

$\therefore \operatorname*{Lt}_{m\to\infty} (x_m, x) = \left(\operatorname*{Lt}_{m\to\infty} x_m, x\right) = (x, x) = 0 \Rightarrow x = 0$

(v)$\Rightarrow$(i) Let E be an orthonormal set in H containing $\{e_\alpha\}$.

If $e \in E$ and $e \neq e_\alpha \quad \forall \alpha$ then by orthonormality,

$(e, e_\alpha) = 0 \quad \forall \alpha$ but then by (v) $e = 0$. But $e \in E$ and E is orthonormal.

$\therefore \|e\| = 1$

This contradiction shows that $E = \{e_\alpha\}$.

Thus E is maximal orthonormal set in H i.e. an orthonormal basis for H.

Example 5.4.4 : Consider the Hilbert space L_2 associated with Lebesgue measurable space on $[0, 2\pi]$ and integrals are Lebesgue integrals.

Then $L_2 = \left\{ f \mid \int_0^{2\pi} |f(x)|^2 \, dx < \infty \right\}$

$$\|f\| = \left(\int_0^{2\pi} |f(x)|^2 \, dx \right)^{1/2} \text{ and } (f, g) = \int_0^{2\pi} f(x) \overline{g(x)} dx$$

Let $u_n = e^{inx}$ for $n = 0, \pm 1, \pm 2, \pm 3, \ldots\ldots$

$$\int_0^{2\pi} e^{imx} e^{-inx} dx = 0 \qquad \text{if } m \neq n$$

$$= 2\pi \qquad \text{if } m = n$$

$\therefore$ Define $e_n = \dfrac{u_n}{\sqrt{2\pi}}$ then the set $\{e_n\}$ form an orthonormal set in L_2.

For any $f \in L_2$, $c_n = (f, e_n) = \dfrac{1}{\sqrt{2\pi}} \int_0^{2\pi} f(x) e^{-inx} dx$ are its classical Fourier coefficients and by Bessel's inequality we have,

$$\sum_{n=-\infty}^{\infty} |c_n|^2 \subseteq \int_0^{2\pi} |f(x)|^2 \, dx$$

Theorem 5.4.6 and 5.4.7 proves the importance of orthonormal basis.

Theorem 5.4.8 : (Gram-Schmidt Orthonormalization)

Let $\{x_1, x_2, x_3, x_4, ...\}$ be a linearly independent subset of an inner product space X.

Define $y_1 = x_1$ and $e_1 = \dfrac{y_1}{\|y_1\|}$ and for $n = 2, 3, 4, ...$

$$y_n = x_n - (x_n, e_1)e_1 - (x_n, e_2)e_2 - ..... - (x_n, e_{n-1})e_{n-1} \text{ and } e_n = \frac{y_n}{\|y_n\|}$$

Then $\{e_1, e_2, e_3, ...\}$ is an orthonormal set in X and for $n = 1, 2, 3, ...$

$\text{span}\{e_1, e_2, e_3, ...\} = \text{span}\{x_1, x_2, x_3, ...\}$.

Proof : We shall prove this result by Methematical induction.

$n = 1$. As $\{x_1\}$ is linearly independent set, $y_1 = x_1 \neq 0$ and $\|e_1\| = \dfrac{\|y_1\|}{\|y_1\|} = 1$ and $\text{span}\{x_1\} = \text{span}\{e_1\}$.

For $n \geq 1$, assume that we have defined y_n and e_n as stated in the statement of theorem and $\{e_1, e_2, e_3, ..., e_n\}$ is an orthonormal set satifying $\text{span}\{e_1, e_2, e_3, ..., e_n\}$ = $\text{span}\{x_1, x_2, ..., x_n\}$.

Define $y_{n+1} = x_{n+1} - (x_{n+1}, e_1)e_1 - (x_{n+1}, e_2)e_2 ..... - (x_{n+1}, e_n)e_n$

As $\{x_1, x_2, x_3, ..., x_{n+1}\}$ is a linearly independent set, x_{n+1} does not belong to $\text{span}\{x_1, x_2, x_3, ..., x_n\} = \text{span}\{e_1, e_2, e_3, ..., e_n\}$, therefore $y_{n+1} \neq 0$.

Let $e_{n+1} = \dfrac{y_{n+1}}{\|y_{n+1}\|}$. Then $\|e_{n+1}\| = 1$ and for all $j \leq n$ we have,

$$\begin{aligned}(y_{n+1}, e_j) &= (x_{n+1}, e_j) - \sum_{k=1}^{n} (x_{n+1}, e_k)(e_k, e_j) \\ &= (x_{n+1}, e_j) - (x_{n+1}, e_j) = 0 \text{ as } (e_k, e_j) = \delta_{kj}\end{aligned}$$

Thus $\left(e_{n+1}, e_j\right) = \left(\frac{y_{n+1}}{\|y_{n+1}\|}, e_j\right) = \frac{1}{\|y_{n+1}\|}\left(y_{n+1}, e_j\right) = 0$

Hence $\{e_1, e_2, e_3, ..., e_n, e_{n+1}\}$ is an orthonormal set.

span $\{e_1, e_2, e_3, ..., e_{n+1}\}$ = span $\{x_1, x_2, ..., x_n, y_{n+1}\}$

= span $\{x_1, x_2, x_3, ..., x_n, x_{n+1}\}$

Example 5.4.5 :

Let $X = \ell^2$ For $n = 1,2,3, ...$ let $x_n = (1, 1, 1, ..., 1, 0, 0, ...)$ where 1 occurs only in first n entries. By Gram-Schmidt orthonormalization processes.

$y_1 = x_1 = (1,0,0,0.....)$ and $e_1 = \frac{y_1}{\|y_1\|} = (1,0,0,0.....)$

$x_2 = (1,1,0,0,0.....)$

$$y_2 = x_2 - \left(x_2, e_1\right)e_1 = (1,1,0,0,0.....) - (1.1 + 1.0 + 0)(1,0,0....)$$
$$= (1, 1, 0, 0, 0, ...) - (1, 0, 0, ...) = (0, 1, 0, ...)$$

$x_3 = (1,1,1,0,0,....)$

$$y_3 = x_3 - \left(x_3, e_1\right)e_1 - \left(x_3, e_2\right)e_2$$
$$= (1, 1, 1, 0, 0, 0, ...) - 1\ (1, 0, 0, 0, ...) - 1\ (0, 1, 0, 0,.....)$$
$$= (0, 0, 1, 0, ...)$$

Thus in general Gram Schmidt orthonormalization process yields $e_n = (0,0,...,0,1,0,...)$ where 1 occur only in n^{th} entry.

5.5 CONJUGATE SPACE H*

Fundamental properties of a Hilbert space is that there is a natural correspondence between the vectors in H and the functionals in dual space H*. In this section the features of this correspondence are discussed.

Theorem 5.5.1 :

The map $T: H \to H^*$, $y \to f_y$ defined by $f_y(x) = (x, y)$ is a norm preserving mapping of H into H*.

Proof : Let y be a fixrd vector in H and consider function f_y on H defined by $f_y(x) = (x, y)$. Then,

$$f_y(x_1 + x_2) = (x_1 + x_2, y) = (x_1, y) + (x_2, y) = f_y(x_1) + f_y(x_2)$$

$$f_y(\alpha x) = (\alpha x, y) = \alpha(x, y) = \alpha f_y(x)$$

Thus f_y is linear map. Further f_y is continuous as

$$|f_y(x)| = |(x, y)| \leq \|x\| \cdot \|y\| \qquad \text{(by Schwarz inequality)}$$

Thus f_y is functional defined on H.

Observe that $|f_y(x)| \leq \|f_y\| \cdot \|x\|$. But $|f_y(x)| \leq \|x\| \cdot \|y\|$.

Thus by definition of $\|f_y\|$, $\|f_y\| \leq \|y\|$.

If $y = 0$ then $\|y\| = 0$ and $\|f_y\| = 0$ and $\|f_y\| = \|y\|$.

Suppose $y \neq 0$, then,

$$\|f_y\| = \sup\{|f_y(x)| : \|x\| = 1\} \geq \left| f_y\left(\frac{y}{\|y\|}\right) \right|$$

$$\left| f_y\left(\frac{y}{\|y\|}\right) \right| = \left| \left(\frac{y}{\|y\|}, y\right) \right| = \frac{1}{\|y\|}|(y, y)| = \|y\|$$

Thus, $\|f_y\| \geq \|y\|$. But $\|f_y\| \leq \|y\|$.

Therefore $\|f_y\| = \|y\|$ and hence f_y is norm preserving map.

Theorm 5.5.2 (Riesz Representation Theoem)

Let H be a Hilbert space and let f be an arbitrary functional in H*. Then there exists a unique vector y in H such that $f(x) = (x, y)\ \forall x \in H$.

Proof : Uniquencess

Suppose *f* is an arbitrary functional in H* and *y* and *y'* are two vectors in H such that,

$f(x) = (x, y') = (x, y) \Rightarrow (x, y') - (x, y) = 0 \Rightarrow (x, y' - y) = 0$, $\forall x \in H$

Since $(x, y' - y) = 0$, $\forall x \in H$ therefore $y' - y = 0$ i.e. $y' = y$

Existence :

If *f* = 0 then choose *y* = 0. Suppose $f \neq 0$.

Let $M = \{x \mid f(x) = 0\}$. Then M is proper closed linear subspace of H. By theorem 5.3.2 there exists a non-zero vector y_0 which is orthogonal to M. Now we shall show that if α is suitably chosen, then the vector $y = \alpha y_0$ satisfy $f(x) = (x, y)$, $\forall x \in H$.

If $f(x) = 0$ then $(x, y) = (x, \alpha y_0) = \overline{\alpha}(x, y_0) = 0$. Since $y_0 \perp M$.

Now we choose α in such a way that $f(x) = (x, y)$ holds for $x = y_0$.

$f(y_0) = (y_0, \alpha y_0) = \bar{\alpha}(y_0, y_0) = \bar{\alpha}\|y_0\|^2$. Choose $\alpha = \dfrac{\overline{f(y_0)}}{\|y_0\|^2}$ and thus $f(x) = (x, y)$ is true for every *x* in *M* and for $x = y_0$ also.

Since $y_0 \in M^{\perp}$ observe that each *x* in H can be written as $x = m + \beta y_0$ for some $m \in M$. Now choose β in such a way that $f(x - \beta y_0) = f(m) = 0$.

But $f(x - \beta y_0) = f(x) - \beta f(y_0) = 0 \Rightarrow \beta = \dfrac{f(x)}{f(y_0)}$.

$$f(x) = f(m + \beta y_0) = f(m) + \beta f(y_0) = (m, y) + \beta(y_0, y)$$

$$= (m + \beta y_0, y) = (x, y)$$

Thus norm preserving mapping of H into H* defined by $y \to f_y$ where $f_y(x) = (x, y)$ is a mapping of H onto H*.

The mapping $y \to f_y$ constitutes one-one onto isometric mapping from a Hilbert space H to its conjugate H*.

Theorem 5.5.3 :

Let H be a Hilbert space.

(a) For $f \in H^*$ let y_f be the representer of *f* in H. Then the mapping $T : H^* \to H$ given by $T(f) = y_f$ is onto conjugate linear isometry.

(b) For $f, g \in H^*$, define $(f, g)^* = (T(f), T(g))$.

Then $(f, g)^*$ is an inner product on H*, $(f, f)^* = \|f\|^2$ for all $f \in H^*$ and H* is a Hilbert space.

(c) For $y \in H$ define $j_y : H^* \to K$ by $j_y(f) = f(y)$, $f \in H^*$. Then j_y is a continuous linear functional on H* and the map $J : H \to H^{**}$ defined by $J(y) = j_y$ for $y \in H$ is onto linear isometry i.e. H is reflexive.

Proof :

(a) For $f, g \in H^*$ we have,

$$(f+g)(x) = f(x) + g(x)$$

$$= (x, y_f) + (x, y_g) = (x, y_f + y_g), \ \forall x \in H$$

Hence $y_f + y_g$ is a representer of $f + g \in H^*$ i.e. $T(f+g) = T(f) + T(g)$

Similarly for $f \in H^*$ and $k \in K$

$(kf)(x) = kf(x) = k(x, y_f) = (x, \bar{k}y_f)$, $\forall x \in H$.

Hence $\bar{k}y_f$ is a representer of kf in H*. i.e. $T(kf) = \bar{k}T(f)$

Thus the map $T : H^* \to H$ is conjugate linear.

To show that T is onto consider $y \in H$ and let $f(x) = (x, y)$, $\forall x \in X$.

Then $y = T(f)$. We have proved that $\|y_f\| = \|f\| = \|T(f)\|$ so that T is an isometry.

(b) For all $f \in H^*$, we have $(f,f)^* = (T(f),T(f)) \geq 0$.

and $(f,f)^* = 0$ iff $\|T(f)\| = \|y_f\| = \|f\| = 0$ i.e. $f = 0$.

For $f,g,h \in H^*$ and $k \in K$,

$$(f+g,h)^* = (T(h),T(f+g)) = (T(h),T(f)) + (T(g),T(f))$$

$$= (f,h)^* + (f,g)^*$$

$$(kf,h)^* = (T(h),T(kf)) = (T(h),\bar{k}T(f))$$

$$= k(T(h),T(f)) = k(f,h)^*$$

Similarly $(f,h)^* = \overline{(h,\mathrm{f})}^*$. Thus (,)* is an inner product space on H*.

Since H is complete and $T : H^* \to H$ is onto isometry H* is complete.

Thus H* is Hilbert space.

(c) Let $y \in H$. $j_y : H^* \to K$ is linear and $|j_y(f)| = |f(y)| \leq \|f\| \cdot \|y\|$, $\forall f \in H^*$.

Therefore j_y is continuous and $\|j_y\| \leq \|y\|$. If we define $f \in H^*$ by $f(x) = (x,y)$, $\forall x \in H$ then, $|j_y(f)| = |f(y)| = |(y,y)| = \|y\|^2$ so that $\|j_y\| = \|y\|$.

A map $J : H \to H^{**}$ is linear as, $J(y_1 + y_2) = j_{y_1+y_2}$.

But $j_{y_1+y_2}(f) = f(y_1 + y_2) = f(y_1) + f(y_2) = j_{y_1}(f) + j_{y_2}(f)$

and $J(\alpha y) = j_{\alpha y}$ and $j_{\alpha y}(f) = f(\alpha y) = \alpha f(y) = \alpha j_y(f) = \alpha J(y)$

To show that J is onto, consider $\phi \in H^{**}$ then by theorem 4.5.2 there exist a unique representer $g \in H^*$ of ϕ. Then $\phi(f) = (f,g)^* = (y_g, y_f) = f(y_g) = J(y_g)(f)$ for all $f \in H^*$. Thus $\phi = J(y_g)$. Also J is an isometry,

since $\|J(y)\| = \|j_y\| = \|y\|$.

❑❑❑

UNIT - VI

BOUNDED OPERATORS ON HILBERT SPACES

This chapter presents a detailed study of bounded linear maps from a Hilbert space to itself. The adjoint of such a bounded operator is introduced in section 6.1. It corresponds to the conjugate transpose of a matrix in a finite dimensional situation. Self adjoint operators and their properties are discussed in section 6.2. In section 6.3, normal and unitary operators are discussed. The properties of normal and unitary operator and relation between these operators are proved. Section 6.3 is devoted to Projections. The projections whose range and null spaces are orthogonal are of much use. Since in certain circumstances sum of projections is also a projections.

6.1 ADJOINT OF AN OPERATOR

By an operator T on an inner product space X over K we mean a linear map T from X to X. The map is said tobe bounded if $\|Tx\| \le \alpha \|x\|$, $\forall x \in X$ (In working with operators it is common practice to omit parentheses whenever it seems convenient. $Tx = T(x)$). A bounded operator is uniformly continuous on X, since for all x, y in X, $\|Tx - Ty\| \le \alpha \|x - y\|$. Conversely if linear map T from X to X is continuous at 0 then T is bounded operator on X.

The set of all bounded linear operators on X is denoted by B (X). It can be proved that if $A, B \in B(X)$ and $k \in K$ then A+ B and kA and AB belong to B (X). The operator A is invertible if there is some $B \in B(X)$ such that AB = I = BA, where I is identity operator on X. For $T \in B(X)$, $\|T\| = \sup\{|Tx| : x \in X, \|x\| \le 1\}$. Then $\|.\|$ is a norm on B (X) and $\|Tx\| \le \|T\| \cdot \|x\|$, $\forall x \in X$. If $\{A_n\}$ and $\{B_n\}$ are sequences in B (X) such that $A_n \to A$ and $B_n \to B$ then $A_n + B_n \to A + B$ and $A_n B_n \to AB$,

since $\|(A_n + B_n) - (A + B)\| \le \|A_n - A\| + \|B_n - B\|$ and

$$\|A_nB_n - AB\| \le \|A_nB_n - A_nB + A_nB - AB\|$$

$$\le \|A_n\|\|B_n - B\| + \|B\|\|A_n - A\|$$

Theorem 6.1.1 :

If $\{e_1, e_2, e_3, ...\}$ is an orthonormal basis for a Hilbert space H, then each operator $T \in B(H)$ is defined by a matrix (Te_j, e_i) with respect to this basis.

Proof : Consider the fourier expansion (theorem 6.4.6)

$$x = \sum_j (x, e_j) e_j, \quad x \in H$$

Since the linear operators are continuous, we have,

$$(Tx, e_i) = \left(\sum_j (x, e_j) T(e_j), e_i\right) = \sum_j (x, e_j) T(e_j, e_i) = f_i(x) \quad \text{(say)}$$

Thus we have a Fourier expansion,

$$Tx = \sum_i (Tx, e_i) e_i = \sum_i f_i(x) e_i, \quad x \in H$$

Thus T is defined by a matrix (Te_j, e_i) with respect to basis $\{e_1, e_2, e_3, ...\}$.

Note : If an orthonormal set $\{e_1, e_2, e_3, ...\}$ is not an orthonormal basis for H, then there is some $T \in B(H)$ which is not defined by a matrix with respect to $\{e_1, e_2, e_3, ...\}$.

Theorem 6.1.2 :

Let H be a Hilbert space and $A \in B(H)$. Then there is a unique $B \in B(H)$ such that for all $x, y \in H$, $(Ax, y) = (x, By)$.

Proof : Fix $y \in H$ and consider the map $f_y : H \to K$ defined by $f_y(x) = (Ax, y)$, $x \in H$.

$$f_y(\alpha x_1 + \beta x_2) = (A(\alpha x_1 + \beta x_2), y)$$

$$= (\alpha Ax_1 + \beta Ax_2, y)$$

$$= \alpha(Ax_1, y) + \beta(Ax_2, y)$$
$$= \alpha f_y(x_1) + \beta f_y(x_2)$$

Thus f_y is a linear functional. Also since

$$|f_y(x)| = |(Ax, y)| \le \|Ax\| \cdot \|y\| \le \|A\|\|x\|\|y\|, \ \forall x \in H,$$

each f_y is continuous. By Riesz representation theorem (6.5.2), there is a unique $z \in H$ such that $f_y(x) = (x, z)$, $\forall x \in H$.

Define $By = z$. Then B is an operator on H. Also since

$$\|By\| = \|z\| = \|f_y\| \le \|A\|\|y\|, \ \forall y \in H,$$

the operator B is continuous.

$$(Ax, y) = f_y(x) = (x, z) = (x, By)$$

For each fixed $y \in H$, this condition determines the element $z = By$ of H. Suppose B' is another map that satisfy $(Ax, y) = (x, B'y)$ then,

$$(Ax, y) = (x, B'y) = (x, By) \Rightarrow (x, (B' - B)y) = 0 \Rightarrow B = B'$$

Hence the map B is unique.

Definition 6.1.1 : Let H be a Hilbert space and let $A \in B(H)$. The unique element B of B(H) which satisfies $(Ax, y) = (x, By)$, $\forall x, y \in H$ is called the adjoint of A and is denoted by A*.

Remark : If inner product space X is not complete then for each $A \in B(X)$ there may not exist $B \in B(X)$ such that $(Ax, y) = (x, By)$, $x, y \in X$.

Let X = set of all scalar sequences having only finite number of non-zero entries. For $x = \{x_i\}$, $y = \{y_i\}$ in X define $(x, y) = \sum_i x_i \bar{y}_i$. For $x \in X$ let $Ax = \left\{ \sum_{j=1}^{\infty} \frac{x_j}{j}, 0, 0, ... \right\}$ then $A \in B(X)$.

$\|A\| \le \left(\sum_{j=1}^{\infty} \frac{1}{j^2} \right)^{1/2} = \frac{\pi}{\sqrt{6}}$. For $n = 1, 2, 3, \ldots..,$ let $e_n = (0, 0, \ldots, 1, 0)$ where 1 occurs only in nth entry. If $B \in B(X)$ and $(Ax, y) = (x, By)$, $\forall x, y \in X$ then,

$$\left(\overline{Be_1}\right)_n = (e_n, Be_1) = (Ae_n, e_1) = \frac{1}{n} \ne 0. \; n = 1, 2, 3, \ldots.$$

But then $Be_1 \notin X$ (since all entries Be_1 in are non-zero).

Theorem 6.1.3 : Let T be an operator on a Hilbert space H. Then T* (defined by 6.1.1) is an operator.

Proof : For any $y, z \in H$ and all $x \in H$ we have

$$(x, T^*(y+z)) = (Tx, y+z) = (Tx, y) + (Tx, z)$$

$$= (x, T^* y) + (x, T^* z) = (x, T^* y + T^* z)$$

Thus, $T^*(y+z) = T^* y + T^* z$

$$(x, T^*(\alpha y)) = (Tx, \alpha y) = \bar{\alpha}(Tx, y) = \bar{\alpha}(x, T^* y) = (x, \alpha T^* y)$$

Thus $T^*(\alpha y) = \alpha T^* y$. So T* is linear. Now we shall prove that T* is continuous.

$$\|T^* y\|^2 = (T^* y, T^* y) = (TT^* y, y) \le \|TT^* y\| \|y\| \le \|T\| \|T^* y\| \cdot \|y\|$$

Thus we have $\|T^* y\| \le \|T\| \|y\|$, $\forall y \in H$. So $\|T^*\| \le \|T\|$.

Definition 6.1.2 : The mapping $T \to T^*$ defined on B (H) by $(Tx, y) = (x, T^* y)$, where H is a Hilbert space is called the adjoint operation on B (H).

Theorem 6.1.4 : Let H be a Hilbert space. The adjoint operation $T \to T^*$ on B (H) has the following properties.

1) $(T_1 + T_2)^* = T_1^* + T_2^*$

2) $(\alpha T)^* = \bar{\alpha} T^*$

3) $(T_1T_2)^* = T_2^* T_1^*$

4) $T^{**} = T$

5) $\|T^*\| = \|T\|$

6) $\|T^*T\| = \|T\|^2$

Proof : Let H be a Hilbert space and $x, y \in H$.

1) $(x,(T_1+T_2)^* y) = ((T_1+T_2)x, y) = (T_1x+T_2x, y) = (T_1x, y)+(T_2x, y)$

$$= (x, T_1^* y)+(x, T_2^* y) = (x, T_1^* y+T_2^* y) = (x,(T_1^*+T_2^*)y)$$

2) $(x,(\alpha T)^* y) = (\alpha Tx, y) = \alpha(Tx, y) = \alpha(x, T^* y) = (x, \bar{\alpha}T^* y)$

3) $(x,(T_1T_2)^* y) = (T_1T_2x, y) = (T_2x, T_1^* y) = (x, T_2^* T_1^* y)$

4) $(Tx, y) = (x, T^* y) = (T^{**}x, y)$

5) In theorem 6.1.3 we have proved $\|T^*\| \le \|T\|$. Therefore $\|T^{**}\| \le \|T^*\|$ but by (4) T** = T and we have $\|T\| \le \|T^*\|$.

Thus $\|T\| = \|T^*\|$.

6) $\|T^*T\| \le \|T^*\|\|T\| = \|T\|\cdot\|T\| = \|T\|^2$ (by (5)) .... (i)

$\|Tx\|^2 = (Tx, Tx) = (T^*Tx, x) \le \|T^*Tx\|\|x\| \le \|T^*T\|\|x\|^2$

Taking supremum over all $x \in H$ with $\|x\| \le 1$ we find that

$\|T\|^2 \le \|T^*T\|$ .... (ii)

From (i) and (ii) result follows.

Definition 6.1.3 : Let H be a Hilbert space in T be a bounded operator on H. The subspace,

$$R(T) = \{y \in H \,/\, Tx = y \text{ for some } x \in H\}$$

of H is called the range space of T. The subspace $Z(T) = \{x \in H \,/\, Tx = 0\}$ of H is called zero space of T. If Z (T) = H we write T = 0.

Definition 6.1.4 : A linear map $T : H \to H$ is bounded below if $\beta\|x\| \le \|Tx\|$, $\forall x \in H$ and some $\beta > 0$.

Theorem 6.1.5 : Let H be a Hilbert space and T is bounded linear operator on H.

(a) $Z(T) = R(T^*)^{\perp}$ and $Z(T^*) = R(T)^{\perp}$

(b) $\overline{R(T)} = Z(T^*)^{\perp}$ and $\overline{R(T^*)} = Z(T)^{\perp}$

(c) R(T) = H if and only if T* is bounded below and R(T*) = H if and only if T is bounded below.

Proof : Observe that above results are symmetric in T and T* since T** = T. Therefore it is sufficient to prove one part second part follows immediately.

(a) Let $x \in H$. Then $x \in Z(T)$ i.e. Tx = 0 if and only if $(x, T^*y) = (Tx, y) = 0$, $\forall y \in H$ i.e. $x \in R(T^*)^{\perp}$.

(b) Let $F = \overline{R(T)}$ and note that $F^{\perp} = R(T)^{\perp}$. Since F is closed subspace of H, $F = F^{\perp\perp} = \left[R(T)^{\perp}\right]^{\perp} = Z(T^*)^{\perp}$.

(c) Suppose R(T) = H. Suppose T* is not bounded below. Then there is a sequence $\{x_n\}$ in H such that $\|T^*x_n\| < \frac{\|x_n\|}{n}$ for n = 1, 2, 3, ... Let $y_n = \frac{nx_n}{\|x_n\|}$, so that $\|T^*y_n\| < 1$.

We show that the sequence $\{y_n\}$ is bounded in H. Consider $y \in H$. Since T is onto $\exists x \in H$ such that $Tx = y$. Then,

$$|(y_n, y)| = |(y_n, Tx)| = |(T^*y_n, x)| \le \|T^*y_n\|\|x\| < \|x\|$$

Thus $|(y_n, y)| < \|x\|$ and therefore y_n must be bounded in H. But $\|y_n\| = n \to \infty$. This contradiction proves that T* is bounded below.

6.2 SELF ADJOINT OPERAORS ON HILBERT SPACE H

Definition 6.2.1 : Consider those operators A on a Hilbert space H for which A = A*, such an operator A is called self adjoint operator.

Theorem 6.2.1 : Zero operator and identity operator are self adjoint operators.

Proof : Let Ax = 0, $\forall x \in H$. Then,

$$0 = (x, Ax) = (A^* x, x) \Rightarrow A^* x = 0, \ \forall x \in H \Rightarrow A = A^*$$

Thus if A is zero operator, A = A* i.e. 0* = 0.

Let Ax = x, $\forall x \in H$. Then,

$$\|x\|^2 = (x, x) = (x, Ax) = (A^* x, x) \Rightarrow A^* x = x$$

Thus $Ax = x \Rightarrow A^* x = x$ therefore I* = I where I represents identity map.

Theorem 6.2.2 :

The self adjoint operators in B (H) form a closed real linear subspace of B (H)and therefore a real Banach space which contains identity tranformation.

Proof : Let $S \subset B(H)$ is a set of self adjoint operation. Suppoe A_1 and A_2 are self adjoint operators and α, β are real numbers. Then,

$$(\alpha A_1 + \beta A_2)^* = \bar{\alpha} A_1^* + \bar{\beta} A_2^* = \alpha A_1 + \beta A_2 \qquad (\alpha, \beta \text{ are real})$$

Thus if A_1, A_2 are self adjoint and α, β are real numbers then $\alpha A_1 + \beta A_2$ are self adjoint operator. Thus S is linear subspace of B (H).

Let $\{A_n\}$ be a sequence of self adjoint operators which converges to A.

i.e. $\{A_n\}$ is a sequence in S that converges to A.

Consider,

$$\|A - A^*\| = \|A - A_n + A_n - A_n^* + A_n^* - A^*\|$$

$$\leq \|A - A_n\| + \|A_n - A_n^*\| + \|A_n^* - A^*\|$$

$$\leq \|A - A_n\| + \|(A_n - A)^*\|$$

$$\leq 2\|A - A_n\| \to 0 \text{ as } n \to \infty.$$

Thus $\|A - A^*\| = 0 \Rightarrow A = A^*$ i.e. $A \in S$. Thus S is closed real linear subspace of B(H). By theorem 6.2.1, S contains identity transformation.

Theorem 6.2.3 : If A_1 and A_2 are self adjoint operators on Hilbert space H then their product A_1A_2 is self adjoint if and only if $A_1A_2 = A_2A_1$.

Proof : Suppose A_1, A_2 and A_1A_2 are self adjoint.

Then $(A_1A_2)^* = A_1A_2$ .... (i)

But $(A_1A_2)^* = A_2^* A_1^* = A_2A_1$ ..... (ii)

Thus $(A_1A_2)^* = A_1A_2$ if and only if $A_1A_2 = A_2A_1$, (by equation (i) and (ii))

Theorem 6.2.4 : If T is an operator on H for which $(Tx, x) = 0 \ \forall x \in H$ then $T = 0$.

Proof : We will show that $(Tx, y) = 0$ for any x and any y.

$$(T(\alpha x + \beta y), \alpha x + \beta y) = (\alpha Tx + \beta Ty, \alpha x + \beta y)$$

$$= \alpha\bar{\alpha}(Tx, x) + \beta\bar{\beta}(Ty, y) + \alpha\bar{\beta}(Tx, y) + \beta\bar{\alpha}(Ty, x)$$

$$\Rightarrow (T(\alpha x + \beta y), \alpha x + \beta y) - |\alpha|^2 (Tx, x) - |\beta|^2 (Ty, y) = \alpha\bar{\beta}(Tx, y) + \beta\bar{\alpha}(Ty, x)$$

Since $(Tx, x) = 0$, $\forall x \in H$,

$\alpha\beta(Tx, y) + \beta\bar{\alpha}(Ty, x) = 0$, $\forall \alpha_1 \beta$.

For $\alpha = 1 = \beta$, we have $(Tx, y) + (Ty, x) = 0$ ... (i)

For $\alpha = i$ and $\beta = 1$ we have $i(Tx, y) - i(Ty, x) = 0$ ... (ii)

From equation (i) and (ii) we have $(Tx, \mathrm{y}) = 0$.

Theorem 6.2.5 : An operator T on H is self adjoint if and only if (Tx, x) is real for all $x \in H$.

Proof : Suppoe T i self adjoint then,

$$(\overline{Tx, x}) = (x, Tx) = (x, T^* x) = (Tx, x)$$

Since $(\overline{Tx, x}) = (Tx, x)$, (Tx, x) is real for all x.

Conversely suppoe (Tx, x) is real for all x in H.

$(Tx, x) = \overline{(Tx, x)} = \overline{(x, T^*x)} = (T^*x, x)$ i.e.

$(Tx - T^*x, x) = ((T - T^*)x, x) = 0$ for all x in H.

Thus by theorem 6.2.4 we have T – T* = 0 i.e. T = T*.

Definition 6.2.2 : Suppose A_1 and A_2 are self adjoint operators on a Hilbert pace H. We write $A_1 \le A_2$ if $(A_1x, x) \le (A_2x, x)$, $\forall x \in H$.

Theorem 6.2.6 : The real Banach pace of all self adjoint operators on Hilbert space H is a partially ordered set whose linear structure and order structure are related by the following properties.

1) if $A_1 \le A_2$ then $A_1 + A \le A_2 + A$ for every $A \in B(H)$.

2) if $A_1 \le A_2$ and $\alpha \ge 0$ then $\alpha A_1 \le \alpha A_2$.

Proof : $A_1 \le A_2$ as $(A_1x, x) \le (A_2x, x)$, $\forall x \in H$.

$\therefore \le$ is reflexive.

Suppose $A_1 \le A_2$ and $A_2 \le A_3$, then $(A_1x, x) \le (A_2x, x)$

and $(A_2x, x) \le (A_3x, x) \Rightarrow (A_1x, x) \le (A_3x, x) \Rightarrow A_1 \le A_3$

Thus '$\le$' is transitive.

Let $A_1 \le A_2$ and $A_2 \le A_1$. $(A_1x, x) \le (A_2x, x)$ and $(A_2x, x) \le (A_1x, x)$

$\Rightarrow (A_1x, x) = (A_2x, x) \Rightarrow (A_1x - A_2x, x) = 0$

$\Rightarrow ((A_1 - A_2)x, x) = 0 \Rightarrow A_1 - A_2 = 0 \Rightarrow A_1 = A_2$

Thuss '$\le$' is antisymmetric.

Since '$\le$' is reflective, antisymmetric and transitive, the real Banach space of all self adjoint operators on H with '$\le$' relation is a partially ordered set.

1) Suppose $A_1 \le A_2$ then $(A_1x,x) \le (A_2x,x)$ and therefore

$$(A_1x,x)+(Ax,x) \le (A_2x,x)+(Ax,x)$$

$$\Rightarrow ((A_1+A)x,x) \le ((A_2+A)x,x)$$

$$\Rightarrow A_1 + A \le A_2 + A$$

2) Suppose $A_1 \le A_2$ then $(A_1x,x) \le (A_2x,x)$ and for $\alpha \ge 0$,

$$\alpha(A_1x,x) \le \alpha(A_2x,x) \Rightarrow \alpha A_1 \le \alpha A_2$$

Definition 6.2.3 : A self adjoint operator A is said tobe positive if $A \ge 0$ i.e. $(Ax,x) \ge 0$ $\forall x$.

Note : O, I, T*T, TT* are positive operators for an arbitrary operator T as

$$(Ox,x) = 0 \ge 0,\ (x,x) = \|x\|^2 \ge 0,$$

$$(T^*Tx,x) = (Tx,Tx) = \|Tx\|^2 \ge 0$$

$$(TT^*x,x) = (T^*x,T^*x) = \|T^*x\|^2 \ge 0$$

Theorem 6.2.7 : If A is a positive operator on H, then I + A is non-singular. In particular I + T*T and I + TT* are non-singular for an arbitrary operator T on a Hilbert space H.

Proof : First, we must show that I + A is one to one and onto as a mapping of H into itself.

Suppose $(I+A)x = 0 \Rightarrow Ax = -x \Rightarrow (Ax,x) = (-x,x) = -\|x\|^2$

Since A is positive operator $(Ax,x) \ge 0$ therefore $-\|x\|^2 \ge 0$

But then $x = 0$

Define $M = \{(I+A)x / x \in H\}$.

Let $\{y_n\}$ be a Cauchy sequence in M.

Observe that $\|(I+A)x\|^2 = (x+Ax, x+Ax)$.

$= (x,x) + (x,Ax) + (Ax,x) + (Ax,Ax)$

$= \|x\|^2 + \|Ax\|^2 + 2(Ax,x)$

Since A is positive operator $(Ax,x) \geq 0$ and we have

$\|x\|^2 \leq \|(I+A)x\|^2$ ... (i)

Since $\{y_n\}$ is a Cauchy sequence in M,

$y_n = (I+A)x_n$ for $x_n \in H$ and

$\|x_n - x_m\|^2 \leq \|(I+A)(x_n - x_m)\|^2$ by (i)

$\Rightarrow \|x_n - x_m\|^2 \leq \|y_n - y_m\|^2$

Since $\{y_n\}$ is cauchy in M, $\{x_n\}$ is Cauchy sequence in H. Since H is Hilbert space, H is complete and $x_n \to x \in H$. But then $(I+A)x = y \in M$ and $y_n \to y \in H$. Thus M is complete and therefore closed in H.

Now we will prove that M = H. Suppose not then there would exist a non-zero vector x_0 orthogonal to M.

$(I+A)x_0 \in M$ and $x_0 \perp M$.

Therefore $(x_0, (I+A)x_0) = 0$.

$\Rightarrow (x_0, x_0) + (x_0, \mathrm{A}\,x_0) = 0 \Rightarrow \|x_0\|^2 = -(Ax_0, x_0) \leq 0$

$\Rightarrow x_0 = 0$. Contradiction to $x_0 \neq 0$. Thus M = H.

Thus I + A is one-one and onto and therefore non-singular.

Theorem 6.2.8 : (Generalized Schwarz inequality)

Let $A \in B(H)$ be self adjoint. Then A or – A is positive operator if and only if

$$|(Ax,y)|^2 \leq (Ax,x)(Ay,y),\ \forall x,y \in H.$$

Proof : Suppose A is positive operator i.e. $(Ax,x)\geq 0$ for $x,y\in H$, define $(x,y)_A=(Ax,y)$.

Observe that $(x,x)_A\geq 0,\ \forall x\in H$ and the function $(\ ,\ ):H\times H\to K$ is linear in the first variable and is conjugate symmetric since A is self adjoint.

Consider $z=(y,y)_A x-(x,y)_A y$ where $x,y\in H$.

$$0\leq (z,z)_A=\left((y,y)_A x-(x,y)_A y,(y,y)_A x-(x,y)_A y\right)$$

$$=(y,\mathrm{y})_A^2(x,x)_A-(y,y)_A\left(\overline{x,y}\right)_A(x,y)_A-(x,y)_A(y,y)_A(y,x)_A$$

$$+(x,y)_A\left(\overline{x,y}\right)_A(y,y)_A$$

$$=(y,\mathrm{y})_A\left[(y,y)_A(x,x)_A-(x,y)_A(y,x)_A\right]$$

$$=(y,\mathrm{y})_A\left[(y,y)_A(x,x)_A-\left|(x,y)_A\right|^2\right]$$

Thus if $(y,\mathrm{y})_A>0$ then $\left|(Ax,y)\right|^2\leq (Ay,y)(Ax,x)$ (as $(x,\mathrm{y})_A=(Ax,y)$)

If $(y,\mathrm{y})_A=0$ but $(x,x)_A\neq 0$, then we can interchange *x* and *y* and obtain the result.

Assume that $(x,x)_A=0$ and $(y,y)_A=0$. Then,

$$(x+y,x+y)_A+(x-y,x-y)_A=(x,x)_A+(x,y)_A+(y,x)_A+(y,y)_A$$

$$+(x,x)_A-(x,y)_A-(y,x)_A+(y,y)_A$$

$$=2(x,x)_A+2(y,y)_A=0$$

But since, $(x,x)_A\geq 0\quad x\in H$, $(x+y,x+y)_A\geq 0$, $(x-y,x-y)_A\geq 0$ and

therefore $(x+y,x+y)_A=(x-y,x-y)_A=0$. Similarly

$(x+iy,x+iy)_A=(x-iy,x-iy)_A=0$

Therefore,

$$0=(x+y,x+y)_A-(x-y,x-y)_A+i(x+iy,x+iy)_A-i(x-iy,x-iy)_A$$

$$= 2(x,y)_A + 2(y,x)_A + i\left[2(x,iy)_A + 2(iy,x)_A\right]$$

$$= 2(x,y)_A + 2(y,x)_A + 2(x,y) - 2(y,x)_A$$

$$= 4(x,y)_A = 4(Ax,y)$$

Thus we have $(Ax,y) = 0$ if $(x,x)_A = 0 = (y,y)_A$ and

$$\left|(x,y)_A\right|^2 = \left|(Ax,y)\right| \le (x,x)_A (y,y)_A = (Ax,x)(Ay,y)$$

for all $x, y \in H$, provided A is positive operator.

In case $-$A is positive then,

$$\left|(Ax,y)\right|^2 = \left|(-Ax,y)\right|^2 \le (-Ax,x)(-Ay,y) = (Ax,x)(Ay,y)$$

for all $x, y \in H$.

Conversely assume that $\left|(Ax,y)\right|^2 \le (Ax,x)(Ay,y)$ for all $x, y \in H$.

Then $(Ax,x) \ge 0$, $\forall x \in H$ or $(Ax,x) \le 0$, $\forall x \in H$.

That is A or $-$A is a positive operator.

6.3 NORMAL AND UNITARY OPERATORS ON HILBERT SPACE H

Definition :

An operator N of H is said to be normal if NN* = N*N where N* is adjoint of N.

Theorem 6.3.1 :

The set of all normal operators on H is closed subset of B (H) which contains the set of all self adjoint operators and is closed under scalar multiplication.

Proof : If $A \in B(H)$ is self adjoint then A* = A and AA* = A*A = $\mathbf{A}^2$. Therefore every self adjoint operator is normal operator. If N is normal operator then NN* = N*N. If α is any scalar then

$$|\alpha|^2 NN^* = |\alpha|^2 N^*N \Rightarrow (\alpha N)(\alpha N)^* = (\alpha N)^*(\alpha N)$$

Thus if N is normal operator and α is any scalar then αN is normal. Therefore the set of all normal operators on H is closed under scalar multiplication.

Suppose $\{N_k\}$ is a sequence of normal operators on H that converges to A.

If $N_k \to A$ then $N_k^* \to A^*$ and

$$\|AA^*-A^*A\| = \|AA^*-N_kN_k^* + N_kN_k^* - N_k^*N_k + N_k^*N_k - A^*A\|$$

$$\leq \|AA^*-N_kN_k^*\| + \|N_k^*N_k - A^*A\| \to 0 \text{ as } k \to \infty .$$

$\Rightarrow \|AA^*-A^*A\| = 0 \Rightarrow AA^* = A^*A$

Thus the set of all normal operators on H is closed subset of B (H).

Theorem 6.3.2 :

If N_1 and N_2 are normal operators on Hilbert space H and if $N_1^*N_2 = N_2N_1^*$ or $N_1N_2^* = N_2^*N_1$ then $N_1 + N_2$ and N_1N_2 are normal operators.

Proof :

$$N_1N_2^* = N_2^*N_1 \Leftrightarrow \left(N_1N_2^*\right)^* = \left(N_2^*N_1\right)^*$$

$$\Leftrightarrow N_2^{**}N_1^* = N_1^*N_2^{**} \Leftrightarrow N_2N_1^* = N_1^*N_2$$

So $N_1N_2^* = N_2^*N_1 \Leftrightarrow N_2N_1^* = N_1^*N_2$

Consider,

$$\left(N_1+N_2\right)\left(N_1+N_2\right)^* = \left(N_1+N_2\right)\left(N_1^*+N_2^*\right)$$

$$= N_1N_1^* + N_1N_2^* + N_2N_1^* + N_2N_2^* \quad \text{.... (i)}$$

$$\left(N_1+N_2\right)^*\left(N_1+N_2\right) = \left(N_1^*+N_2^*\right)\left(N_1+N_2\right)$$

$$= N_1^*N_1 + N_1^*N_2 + N_2^*N_1 + N_2^*N_2$$

$$= N_1N_1^* + N_2N_1^* + N_1N_2^* + N_2N_2^* \quad \text{.... (ii)}$$

From (i) and (ii) we have $\left(N_1+N_2\right)$ is normal if

$N_1 N_2^* = N_2^* N_1$ (same as $N_2 N_1^* = N_1^* N_2$)

Similarly,

$$(N_1 N_2)(N_1 N_2)^* = (N_1 N_2)(N_2^* N_1^*) = N_1 (N_2 N_2^*) N_1^*$$

$$= N_1 (N_2^* N_2) N_1^* = N_2^* N_1 N_1^* N_2$$

$$= N_2^* N_1^* N_1 N_2 = (N_1 N_2)^* (N_1 N_2)$$

Thus $N_1 N_2$ is normal operator.

Theorem 6.3.3 :

An opertor T on H is normal if and only if $\|T^* x\| = \|Tx\|$, $\forall x \in H$.

Proof :

$$\|T^* x\| = \|Tx\| \Leftrightarrow \|T^* x\|^2 = \|Tx\|^2$$

$$\Leftrightarrow (T^* x, T^* x) = (Tx, Tx) \Leftrightarrow (TT^* x, x) = (T^* Tx, x)$$

$$\Leftrightarrow (TT^* x, x) - (T^* Tx, x) = 0$$

$$\Leftrightarrow ((TT^* - T^* T)x, x) = 0$$

$\Leftrightarrow TT^* = T^* T$ i.e. T is normal.

Thus T is normal if and only if $\|Tx\| = \|T^* x\|$, $\forall x \in H$.

Theorem 6.3.4 :

If N is a normal operator on H then $\|N^2\| = \|N\|^2$.

Proof :

$\|N^2 x\| = \|N(Nx)\| = \|N^*(Nx)\|$ (by there 6.3.3)

$= \|N^* Nx\|$, $\forall x \in H$.

Thus $\|N^2\| = \|N^* N\|$. But since $\|N\| = \|N^*\|$, $\|N^* N\| = \|N\|^2$ and we have $\|N^2\| = \|N\|^2$.

For an arbitrary operator T on H, define $A_1 = \frac{T+T^*}{2}$ and $A_2 = \frac{T-T^*}{2i}$. Observe that $A_1^* = \frac{T^*+T^{**}}{2} = \frac{T^*+T}{2} = A_1$.

Similarly $A_2^* = A_2$. Thus A_1 and A_2 are both self adjoint operators. Moreover $T = A_1 + iA_2$ and $T^* = A_1 - iA_2$. The self adjoint operators A_1 and A_2 are called real and imaginary part of T.

Theorem 6.3.5 :

If T is an operator on H then T is normal if and only if its real and imaginary parts commute.

Proof :

Suppose A_1 and A_2 are real and imaginary parts of T then $T = A_1 + iA_2$ and $T^* = A_1 - iA_2$.

$$TT^* = (A_1 + iA_2)(A_1 - iA_2) = A_1^2 + A_2^2 + i(A_2A_1 - A_1A_2) \quad \text{.... (i)}$$

$$T^*T = (A_1 - iA_2)(A_1 + iA_2) = A_1^2 + A_2^2 + i(A_1A_2 - A_2A_1) \quad \text{.... (ii)}$$

From (i) and (ii) we have if $A_2A_1 = A_1A_2$ then $TT^* = T^*T$.

Conversely if $TT^* = T^*T$ then $A_2A_1 - A_1A_2 = A_1A_2 - A_2A_1$,

So $2A_2A_1 = 2A_1A_2$ or $A_1A_2 = A_2A_1$.

Thus T is normal if and only if $A_1A_2 = A_2A_1$.

Definition 6.3.2 :

An operator U on a Hilbert space H is said to be unitary if $UU^* = U^*U = I$.

Theorem 6.3.6 :

A is unitary operator on H if and only if $\|Ax\| = \|x\|$, $\forall x \in H$ and A is onto. In that case $\|A^{-1}x\| = \|x\|$, $\forall x \in H$ and $\|A\| = \|A^{-1}\| = 1$.

Proof :

For $x \in H$, we have

$$\|Ax\|^2 - \|x\|^2 = (Ax, Ax) - (x, x)$$

$$= (A^* Ax, x) - (x, x)$$

$$= ((A^* A - I)x, x)$$

We know that if $(Ax, x) = 0$, $\forall x \in H$ then A = 0.

Therefore $\|Ax\|^2 - \|x\|^2 = 0$ iff $A^* A - I = 0$

i.e. $\|Ax\| = \|x\|$ iff $A^* A = I$.

Thus if $\|Ax\| = \|x\|$, $\forall x \in H$ and A is onto then A*A = I and A is bijective so that

$$AA^* = (AA^*)(AA^{-1}) = A(A^* A)A^{-1} = AA^{-1} = I$$

Thus AA* = A*A = I i.e. A is unitary operator.

Conversely if A is unitary then A*A = I and A^{-1} = A*.

Since A*A = I, A*A – I = 0 i.e. $((A^* A - I)x, x) = 0$

$$\Leftrightarrow (A^* Ax, x) - (x, x) = 0$$

$$\Leftrightarrow (Ax, Ax) - (x, x) = 0 \Leftrightarrow \|Ax\|^2 = \|x\|^2$$

Thus if A is unitary $\|Ax\| = \|x\|$, $\forall x \in H$.

And A is onto. In that case $\|A^{-1}x\| = \|x\|$, $\forall x \in H$.

And we have $\|A\| = \|A^{-1}\| = 1$.

Theorem 6.3.7 :

If T is an operator on H, then the following conditions are all equivalent.

1) T*T = I

2) $(Tx, Ty) = (x, y)$, $\forall x, y \in H$.

3) $\|Tx\| = \|x\|$, $\forall x \in H$.

Proof : (1) $\Rightarrow$ (2)

Suppose T*T = I then $(T^*Tx, y) = (x, y)$ i.e. $(Tx, Ty) = (x, y)$

(2) $\Rightarrow$ (3)

$(Tx, Ty) = (x, y)$, $\forall x, y \in H$ therefore for $x = y$ we have

$(Tx, Tx) = (x, x) \Rightarrow \|Tx\|^2 = \|x\|^2 \Rightarrow \|Tx\| = \|x\| \quad \forall x$

(3) $\Rightarrow$ (1)

$\|Tx\| = \|x\| \Rightarrow \|Tx\|^2 = \|x\|^2 \Rightarrow (Tx, Tx) = (x, x)$

$\Rightarrow (T^*Tx, x) = (x, x) \Rightarrow T^*T = I$

Example : Consider $H = \ell_2$ and $T : \ell_2 \to \ell_2$ is defined by,

$T\{x_1, x_2, x_3, ...\} = \{0, x_1, x_2, x_3, ...\}$

then $\|Tx\|_2 = \|x\|_2$ but T do not have inverse as T is not onto map.

Theorem 6.3.8 : An operator T on H is unitary if and only if it is an isometric isomorphism of H onto itself.

Proof : If T is unitary then T is onto. Since $\|Tx\| = \|x\| \quad \forall x \in H$ (theorem 6.3.7), T is an isometric isomorphism of H onto itself.

Conversely, if T is an isometric isomorphim of H onto itself then T^{-1} exist and by theorem 6.3.7 we have T*T = I.

$(T^*T)T^{-1} = I \cdot T^{-1} \Rightarrow T^* = T^{-1}$ and $TT^* = T^*T = I$

Thus T is unitary operator.

Problem : If T is an arbitrary operator on H and if α, β are scalars such that $|\alpha| = |\beta|$, show that $\alpha T + \beta T^*$ is normal.

Answer : $(\alpha T + \beta T^*)(\alpha T + \beta T^*)^*$

$$= (\alpha T + \beta T^*)(\bar{\alpha} T^* + \bar{\beta} T)$$

$$= |\alpha|^2 TT^* + \alpha\bar{\beta}T^2 + \beta\bar{\alpha}(T^*)^2 + |\beta|^2 T^*T$$

$$= |\alpha|^2 \left(TT^* + T^2 + (T^*)^2 + T^*T\right) \qquad \text{..... (i)}$$

$$(\alpha T + \beta T^*)^*(\alpha T + \beta T^*) = (\bar{\alpha} T^* + \bar{\beta} T)(\alpha T + \beta T^*)$$

$$= |\alpha|^2 T^*T + \bar{\alpha}\beta T^{*2} + \bar{\beta}\alpha T^2 + |\beta|^2 TT^*$$

$$= |\alpha|^2 (T^*T + T^{*2} + T^2 + TT^*) \qquad \text{..... (ii)}$$

From (i) and (ii) we have

$(\alpha T + \beta T^*)(\alpha T + \beta T^*)^* = (\alpha T + \beta T^*)^*(\alpha T + \beta T^*)$

Thus $\alpha T + \beta T^*$ is normal.

6.4 PROJECTIONS ON HILBERT SPACE H

Definition 6.4.1 : Operator P on H with the property that $P^2 = P$ is called projection.

Definition 6.4.2 : A projection on H whose range and null space are orthogonal is called perpendicular projection.

Definition 6.4.3 : Two projections P and Q are said to be orthogonal if PQ = 0.

Theorem 6.4.1 : If P is a projection on a Hilbert space H with range M and null space N then $M \perp N$ if and only if P is self adjoint and in this case $N = M^{\perp}$.

Proof : Since M is range and N is null space $H = M + N$.

Therefore each $z \in H$ can be uniquely written in the form $z = x + y$ with $x \in M$ and $y \in N$. If $M \perp N$ then $x \perp y$. Consider,

$$(P^* z, z) = (z, Pz) = (z, x) = (x + y, x) = (x, x) + (y, x) = (x, x)$$

$$(Pz, z) = (x, z) = (x, x + y) = (x, x) + (x, \mathrm{y}) = (x, x)$$

Thus we have $(P^* z, z) = (Pz, z)$, $\forall z \in H$.

i.e. $((P^* - \mathrm{P}) z, z) = 0$, $\forall z \in H \Rightarrow P^* = P$

Conversely suppose P = P* then for $x \in M$ and $y \in N$

$$(x, y) = (Px, y) = (x, P^* y) = (x, Py) = (x, 0) = 0 \Rightarrow x \perp y$$

Thus for any $x \in M$ and $y \in N$, $x \perp y$ i.e. $M \perp N$.

Now we will show that $N = M^{\perp}$. Observe that $N \subseteq M^{\perp}$. If N is proper subset of $M^{\perp}$ then N is closed linear subspace of $M^{\perp}$ therefore by theorem 5.3.2 there exists a non-zero vector z_0 in $M^{\perp}$ such that $z_0 \perp N$. Since $z_0 \in M^{\perp}$, $z_0 \perp M$ and $z_0 \perp N$ therefore $z_0 \perp M + N = H$ *i.e.* $z_0 \perp H$.

This is impossible therefore $N = M^{\perp}$.

The only projections considered in the theory of Hilbert spaces are those operators which are self adjoint.

Definition 6.4.4 : A projection on a Hilbert space H is an operator P which satisfies the conditions $P^2 = P$ and P* = P.

Let P be a projection on a Hilbert space H. Let $M = \{Px : x \in H\}$ is a closed linear subspace of H.

Conversely to each closed linear space M there corresponds the projection P with range M defined by $P(x + y) = x$ where $x \in M$ and $y \in M^{\perp}$.

Observe that P os projection on M iff I – P is projection on $M^{\perp}$. If P is projection on M then,

$x \in M \Leftrightarrow Px = x \Leftrightarrow \|Px\| = \|x\|$

For every $x \in H$ we have,

$$\|x\|^2 = \|Px + (I-P)x\|^2 = \|Px\|^2 + \|(I-P)x\|^2$$

$\Rightarrow \|Px\| \le \|x\|$, $\forall x \in H$ i.e. $\|P\| \le 1$

$$\|Px\| = \|x\| \Rightarrow \|Px\|^2 = \|x\|^2 \Rightarrow \|Px\|^2 - \|x\|^2 = 0$$

$$\Rightarrow (Px, Px) - (x, x) = 0 \Rightarrow (P^* Px, x) - (x, x) = 0$$

$$\Rightarrow (Px, x) - (x, x) = 0 \Rightarrow (Px - x, x) = 0 \Rightarrow Px = x$$

If $x \in H$ is an arbitrary vector then,

$$(Px, x) = (P^2 x, x) = (Px, P^* x) = (Px, Px) = \|Px\|^2 \ge 0$$

Thus the projection operator is positive operator.

Definition 6.4.5 :

A self adjoint operator A on a Hilbert space H is said to be positive if $(Ax, x) \ge 0$, $\forall x \in H$ and we write $A \ge 0$.

Theorem 6.4.2 :

If P is projection on a Hilbert space H then I–P is also a projection.

Proof : $(I-P)(I-P) = I - P - P + P^2 = I - P - P + P = I - P$

$(I-P)^* = I^* - P^* = I - P$

$\therefore$ I – P is projection.

Since I – P is projection, I – P $\ge$ 0 i.e. I $\ge$ P.

Thus $0 \le P \le I$.

Definition 6.4.6 : Let T be an operator on a Hilbert space H. A closed linear subspace M of H is said to be invariant under T if $T(M) \subseteq M$.

Definition 6.4.7 : If both M and $M^{\perp}$ are invariant under T, we say that M reduces T or **T** is reduced by M.

Theorem 6.4.3 : A closed linear subspace M of H is invariant under an operator T if and only if $M^{\perp}$ is invariant under T*.

Proof : Suppose M is invariant under T. Since M is invariant under T, $Tz \in M \quad \forall z \in M$. Suppose $y \in M^{\perp}$ then $(Tz, y) = 0 \quad \forall z \in M$

i.e. $(z, T^* y) = 0$, $\forall z \in M \Rightarrow T^* y \in M^{\perp}$

Thus if $y \in M^{\perp}$ then $T^* y \in M^{\perp}$ and therefore $M^{\perp}$ is invariant under T*.

Conversely suppose $M^{\perp}$ is invariant under T*. Therefore $T^* z \in M^{\perp}$, $\forall z \in M^{\perp}$. But $M^{\perp\perp} = M$. Let $y \in M$ then $(T^* z, y) = 0$, $\forall y \in M$. i.e. $(z, T^{**}y) = 0$, $\forall y \in M$. i.e. $T^{**}y \in M$. But T** = T therefore $Ty \in M$. Thus $(z, Ty) = 0$, $\forall Ty \in M$. i.e. for $y \in M$, $Ty \in M$ and therefore M is invariant under **T.**

Theorem 6.4.4 : A closed linear subspace M of H reduces an operator T if and only if M is invariant under both T and T*.

Proof : A closed linear subspace M of H reduces an operator T iff both M and $M^{\perp}$ are invariant under T. By theorem 6.4.3 M is invariant under T iff $M^{\perp}$ is invariant under T*. Thus $M^{\perp}$ is invariant under T and T*. Since $M^{\perp}$ is invariant under T then by theorem 6.4.3 $M^{\perp\perp}$ is invariant under T*. But $M^{\perp\perp} = M$. Therefore M is invariant under T and T*.

Conversely if M is invariant under T* then $M^{\perp}$ is invariant under T** = T. Thus M and $M^{\perp}$ are invariant under T. Therefore M reduces T.

Theorem 6.4.5 : If P is the projection on a closed linear subspace M of H then M is invariant under an operator T if and only if TP = PTP.

Proof : Suppose M is invariant under an operator T. Let $x \in H$ then $Px \in M$ and $T(Px) = TPx \in M$.

Since $TPx \in M$ and P is projection on M, $PTPx = TPx$. But $x \in H$ is arbitrary vector. Therefore $PTP = TP$.

Conversely suppose $TP = PTP$. Let $x \in M$. Since P is projection on M, $x = Px$ i.e. $Tx = TPx = PTPx \in M$. Thus for $x \in M$, $Tx \in M$. Therefore M is invariant under T.

Theorem 6.4.6 : If P is the projection on a closed linear subspace M of H then M reduces an operator T if and only if $TP = PT$.

Proof : M reduces T iff M is invariant under T and T* (by theorem 6.4.4) iff $TP = PTP$ (by theorem 6.4.5) and $T^*P = PT^*P$.

$$T^*P = PT^*P \Rightarrow (T^*Px, y) = (PT^*Px, y) \qquad \forall x, y \in H$$

$$\Leftrightarrow (Px, Ty) = (T^*Px, Py) \qquad (\because P^* = P)$$

$$\Leftrightarrow (Px, Ty) = (Px, TPy)$$

$$\Leftrightarrow (x, PTy) = (x, PTPy) \qquad (\because P^* = P)$$

$$\Leftrightarrow PT = PTP$$

Thus we have $TP = PTP$ and $PT = PTP \Leftrightarrow TP = PT$.

Theorem 6.4.7 : If P and Q are the projections on closed linear subspaces M and N of H then $M \perp Q$ iff PQ = 0 iff QP = 0.

Proof : $PQ = 0 \Leftrightarrow (PQ)^* = 0 \Leftrightarrow Q^*P^* = 0 \Leftrightarrow QP = 0$ ($\because$ Q* = Q and P* = P since P and Q are projections). Therefore we shall prove that if P and Q are the projections on closed linear subspaces M and N of H then $M \perp N$ iff PQ = 0. If $M \perp N$ then $N \subset M^{\perp}$. But for every $x \in H$, $Qx \in N$ and $N \perp M$ therefore $PQx = 0$. So $PQ = 0$, $\forall x \in H$.

Conversely if $PQ = 0$ then for every $x \in N$, $Px = PQx = 0$.

Since $Px = 0$, $\forall x \in N$, $N \subset M^{\perp}$ and therefore $M \perp N$.

Definition 6.4.8 : Two projections P and Q are orthogonal if $PQ = 0$.

Theorem 6.4.8 : If $P_1, P_2, P_3, \ldots.., P_n$ are the projections on closed linear subspaces M_1, $M_2, M_3, \ldots., M_n$ of H then $P = P_1 + P_2 + P_3 + \ldots. + P_n$ is a projection iff the Pi's are pairwise orthogonal (i.e. $P_iP_j = 0$, $\forall i \neq j$) and P is projection on $M = M_1 + M_2 + M_3 + \ldots. + M_n$.

Proof : Since P_i is projection for each i, $P_i^* = P_i$ therefore

$$P^* = P_1^* + P_2^* + P_3^* + \ldots\ldots + P_n^* = P_1 + P_2 + P_3 + \ldots.. + P_n = P$$

Thus $P^* = P \Rightarrow P$ is self adjoint. Now P is projection iff $P^2 = P$.

$$P^2 = \left(P_1 + P_2 + P_3 + \ldots.. + P_n\right)\left(P_1 + P_2 + \ldots.. + P_n\right)$$

$$= P_1^2 + P_2^2 + P_3^2 + \ldots.. + P_n^2 + 2\sum_{i\neq j} P_iP_j$$

$$= P_1 + P_2 + P_3 + \ldots.. + P_n + 0 \qquad (\because P_iP_j = 0,\ \forall i \neq j)$$

$= P$

Thus $P^2 = P$ and $P^* = P \Rightarrow P$ is a projection. We have proved that If Pi's are pairwise orthogonal then P is projection.

Conversely assume that P is projection i.e. $P^* = P$ and $P^2 = P$.

Let x be a vector in the range of P_i, so that $x = P_ix$. Then,

$$\|x\|^2 = \|P_ix\|^2 \leq \sum_{j=1}^{n} \|P_jx\|^2 = \sum_{j=1}^{n} \left(P_jx, P_jx\right) = \sum_{j=1}^{n} \left(P_j^* P_jx, x\right) = \sum_{j=1}^{n} \left(P_j^2 x, x\right)$$

$$= \left(\sum_{j=1}^{n} P_jx, x\right) = (Px, x) = \left(P^2x, x\right) = (Px, P^*x) = (Px, Px) \leq \|Px\|^2 \leq \|x\|^2$$

Observe that equality must hold all along the line.

i.e. $\|x\|^2 = \|P_ix\|^2 = \sum_{j=1}^{n} \|P_jx\|^2$

Since $\sum_{j=1}^{n} \|P_jx\|^2 = \|P_ix\|^2 \Rightarrow \|P_jx\| = 0$, $\forall j \neq i$.

Thus range of P_i is contained in the null space of P_j for all $j \neq i$, i.e. $M_i \subset M_j^{\perp}$, $\forall j \neq i$. i.e. $M_i \perp M_j$, $\forall j \neq i$. Therefore by theorem 6.4.7, P_i's are pairwise orthogonal.

Now we shall prove that P is projection on M. Observe that $\|Px\| = \|x\|$, $\forall x \in M_i$, each M_i is contained in the range of P and therefore M is contained in the range of P. If x is a vector in the range of P then $x = Px = P_1x + P_2x + P_3x + ... + P_nx \in M$.

Thus P is projection on M.

Above theorem plays a very important role in the spectral theorem.

❑ ❑ ❑

UNIT - VII

FINITE DIMENSIONAL SPECTRAL THEORY

The aim of this chapter is to prove finite dimensional spectral theorem. In this chapter we assume that the Hilbert space H is finite dimensional i.e. dim H = n.

In section 7.1 we consider linear transformation defined from H to H. The relation between these operators and the corresponding matrices are discussed. Section 6.2 is devoted to define spectrum of an operator and in section 6.3 the spectral theorem is proved for finite dimensional Hilbert spaces.

7.1 LINEAR OPERATORS AND MATRICES

The discussion in this section is independent of the Hilbert space character of H and applies equally well to any non-trival finite dimensional linear space. All the theorems discussed in this section are covered in linear algebra. Here we revise certain correspondence between linear transformation from H to H and A_n, the set of all n $\times$ n matrices.

Let $B = \{e_1, e_2, e_3, ..., e_n\}$ be an ordered basis for H. So that each vector in H is uniquely expressible as linear combination of e_i's. If $T : H \to H$ is an operator then for each $e_j \in B$, $Te_j \in H$, B is basis therefore each Te_j can be expressed as linear combination of vectors from B.

$$Te_j = \sum_{i=1}^{n} \alpha_{ij} e_i$$

The n^2 scalars α_{ij} which are determined in this way by T form a matrix of T relative to the ordered basis B. We denote this matrix by [T] or $[T]_B$.

It is customary to write out a matrix as a square array,

$$[T]=\begin{bmatrix}\alpha_{11} & \alpha_{12} & \alpha_{13} & \cdots & \alpha_{1n}\\ \alpha_{21} & \alpha_{22} & \alpha_{23} & \cdots & \alpha_{2n}\\ \vdots & & & & \\ \alpha_{n1} & \alpha_{n2} & \alpha_{n3} & \cdots & \alpha_{nn}\end{bmatrix}=\left[\alpha_{ij}\right]$$

Thus the construction of [T] is as follows. Write Te_j as a linear combination of $e_1, e_2, e_3, \ldots\ldots, e_n$ and use the resulting coefficients to form j^{th} column of [T].

Theorem 7.1.1 : If $B=\{e_i\}$ is an ordered basis for H, then the mapping $T\rightarrow[T]$, which assigns to each operator T on H its matrix relative to base B, (i.e. [T]) is an isomorphism of the algebra B (H) onto the total matrix algebra A_n where A_n is set of all n $\times$ n matrices.

Proof : Since $Te_j=\sum_{i=1}^{n}\alpha_{ij}e_i$, $\left[\alpha_{ij}\right]$ is the matrix of T.

If $x\in H$ then $x=\sum_{j=1}^{n}\beta_j e_j$ and

$$Tx=T\left(\sum_{j=1}^{n}\beta_j e_j\right)=\sum_{j=1}^{n}\beta_j Te_j=\sum_{j=1}^{n}\beta_j\left(\sum_{i=1}^{n}\alpha_{ij}e_i\right)=\sum_{i=1}^{n}\left(\sum_{j=1}^{n}\alpha_{ij}\beta_j\right)e_i$$

Thus $\left[\alpha_{ij}\right]$ determines Tx for every $x\in H$ and $T\rightarrow[T]$ is one one map.

If $\left[\alpha_{ij}\right]$ is any n $\times$ n matrix then $Te_j=\sum_{i=1}^{n}\alpha_{ij}e_i$ defines T for every vector $e_j\in B$. Since B is basis, every element in H can be expressed as linear combination of vectors in B and therefore T is extended on H.

The resulting operator T has $\left[\alpha_{ij}\right]$ as its matrix. Thus the mapping $T\rightarrow[T]$ is onto.

Now to show that $T\rightarrow[T]$ preserve the algebraic structure.

Let $T_1, T_2\in B(H)$ and let $\left[\alpha_{ij}\right]$ and $\left[\beta_{ij}\right]$ be the matrices of T_1 and T_2 respectively.

$$\left(T_1+T_2\right)e_j = T_1 e_j + T_2 e_j$$

$$= \sum_{i=1}^{n} \alpha_{ij} e_i + \sum_{i=1}^{n} \beta_{ij} e_i$$

$$= \sum_{i=1}^{n} \left(\alpha_{ij} + \beta_{ij}\right) e_i$$

Thus if we define addition of two matrices by

$\left[\alpha_{ij}\right] + \left[\beta_{ij}\right] = \left[\alpha_{ij} + \beta_{ij}\right]$ then $\left[T_1 + T_2\right] = \left[T_1\right] + \left[T_2\right]$

Similarly if $\alpha\left[\alpha_{ij}\right] = \left[\alpha\alpha_{ij}\right]$ then $\left[\alpha T_1\right] = \alpha\left[T_1\right]$

Finally $\left(T_1T_2\right)e_j = T_1\left(T_2 e_j\right) = T_1\left(\sum_{k=1}^{n} \beta_{kj} e_k\right)$

$$= \sum_{k=1}^{n} \beta_{kj} T_1\left(e_k\right)$$

$$= \sum_{k=1}^{n} \beta_{kj} \left(\sum_{j=1}^{n} \alpha_{ik} e_i\right)$$

$$= \sum_{i=1}^{n} \left(\sum_{k=1}^{n} \alpha_{ik} \beta_{kj}\right) e_i$$

Thus if we define multiplication for matrices by,

$$\left[\alpha_{ij}\right]\left[\beta_{ij}\right] = \sum_{k=1}^{n} \alpha_{ik} \beta_{kj} \text{ then } \left[T_1T_2\right] = \left[T_1\right]\left[T_2\right]$$

Thus $T \to [T]$ preserves the algebraic structure.

Note :

(i) The image of the zero operator under the mapping $T \to [T]$ is the zero matrix, all of whose entries are zero.

(ii) The image of the identity operator is identity matrix.

Theorem 7.1.2 : Let B be a basis of H and T an operator whose matrix relative to B is $\left[\alpha_{ij}\right]$. Then T is non-singular iff $\left[\alpha_{ij}\right]$ is non-singular and $\left[\alpha_{ij}\right]^{-1} = [T^{-1}]$.

Proof : A matrix $\left[\alpha_{ij}\right]$ is said to be non-singular if there exists a matrix $\left[\beta_{ij}\right]$ such that,

$$\left[\alpha_{ij}\right]\left[\beta_{ij}\right] = \left[\beta_{ij}\right]\left[\alpha_{ij}\right] = \left[\delta_{ij}\right]$$

where $\left[\delta_{ij}\right]$ is identity matrix. If such matrix exists then it i unique and is denoted by $\left[\alpha_{ij}\right]^{-1}$ and it is called inverse of $\left[\alpha_{ij}\right]$.

Suppose $\left[\alpha_{ij}\right]$ is the matrix of an operator T relative to basis B. Since T is non-singular T^{-1} exists. Moreover $TT^{-1} = T^{-1}T = I$. By theorem 6.1.1 we have

$$[T][T^{-1}] = [T^{-1}][T] = [I]$$

$$\left[\alpha_{ij}\right][T^{-1}] = [T^{-1}]\left[\alpha_{ij}\right] = \left[\delta_{ij}\right]$$

Thus $\left[\alpha_{ij}\right]^{-1} = [T^{-1}]$.

Theorem 7.1.3 : Two matrices in A_n are similar if and only if they are the matrices of a single operator on H relative to different basis.

Proof : If T is a fixed operator on H then its matrix $[T]_B$ relative to basis B depends on the choice of basis.

Let $B = \{e_1, e_2, e_3, ..., e_n\}$ be a basis of H and suppose $B' = \{f_1, f_2, f_3, ..., f_n\}$ is another basis of H.

Suppose $\left[\alpha_{ij}\right]$ and $\left[\beta_{ij}\right]$ are the matrices of T relative to the basis B and B' respectively. Define a non-singular operator A on H by $Ae_i = f_i$, $i = 1, 2, 3, ..., n$.

Let $\left[\gamma_{ij}\right]$ be the matrix of A relative to B so that $Ae_j = \sum_{i=1}^{n} \gamma_{ij} e_i$.

By theorem 6.12 matrix $\left[\gamma_{ij}\right]$ is non-singular.

Consider, $Tf_j = \sum_{k=1}^{n} \beta_{kj} f_k$

$$= \sum_{k=1}^{n} \beta_{kj} A e_k$$

$$= \sum_{k=1}^{n} \beta_{kj} \left(\sum_{i=1}^{n} \gamma_{ik} e_i \right)$$

$$= \sum_{i=1}^{n} \left(\sum_{k=1}^{n} \gamma_{ik} \beta_{kj} \right) e_i \qquad \text{..... (i)}$$

$$Tf_j = TAe_j$$

$$= T\left(\sum_{k=1}^{n} \gamma_{kj} e_k \right)$$

$$= \sum_{k=1}^{n} \gamma_{kj} T e_k$$

$$= \sum_{k=1}^{n} \gamma_{kj} \left(\sum_{i=1}^{n} \alpha_{ik} e_i \right)$$

$$= \sum_{i=1}^{n} \left(\sum_{k=1}^{n} \alpha_{ik} \gamma_{kj} \right) e_i \qquad \text{..... (ii)}$$

From equation (i) and (ii) we have,

$$\sum_{k=1}^{n} \gamma_{ik} \beta_{kj} = \sum_{k=1}^{n} \alpha_{ik} \gamma_{kj} \qquad \forall i, j$$

$$\left[\gamma_{ij}\right]\left[\beta_{ij}\right] = \left[\alpha_{ij}\right]\left[\gamma_{ij}\right]$$

$$\left[\beta_{ij}\right] = \left[\gamma_{ij}\right]^{-1}\left[\alpha_{ij}\right]\left[\gamma_{ij}\right]$$

$$[T]_{B'} = [A]_B^{-1} [T]_B [A]_B$$

Definition 7.1.1 : Two matrices $\left[\alpha_{ij}\right]$ and $\left[\beta_{ij}\right]$ are said to be similar if there exists a non-singular matrix $\left[\gamma_{ij}\right]$ such that $\left[\beta_{ij}\right] = \left[\gamma_{ij}\right]^{-1}\left[\alpha_{ij}\right]\left[\gamma_{ij}\right]$.

We have seen that a given operator on H may have many different matrices relative to different basis. These matrices are related to each other given by definition 7.1.1. Thus working with operators is equivalent to working with square matrices.

7.2 THE SPECTRUM OF AN OPERATOR

Definition 7.2.1 : Let T be an operator on H. Let [T] be the matrix representing T. The scalar λ is said to be an eigen value of an operator T if there exists a non-zero vector $x \in H$ such that $([T]-\lambda I)x = 0$. A non-zero vector $x \in H$ is called eigen vector.

Definition 7.2.2 : Let T be an operator on H. The set of eigen values of T is called spectrum of T and is denoted by $\sigma(T)$.

In definition 7.2.1, a non-zero vector $x \in H$ for which $([T]-\lambda I)x = 0$ exists if det $\det([T]-\lambda I) = 0$. Here we list some properties of determinants.

Let $\left[\alpha_{ij}\right]$ be an n $\times$ n matrix. The determinant of this matrix which we denote by det $\left[\alpha_{ij}\right]$ is a scalar associated with it in such a way that

1) $\det\left(\left[\delta_{ij}\right]\right) = \det(I) = 1$

2) $\det\left(\left[\alpha_{ij}\right]\left[\beta_{ij}\right]\right) = \det\left(\left[\alpha_{ij}\right]\right)\det\left(\left[\beta_{ij}\right]\right)$

3) $\det\left(\left[\alpha_{ij}\right]\right) \neq 0$ if and only if $\left[\alpha_{ij}\right]$ is non-singular.

4) $\det\left(\left[\alpha_{ij}\right]-\lambda\left[\delta_{ij}\right]\right)$ is a polynomial with complex.

Coefficients of degree *n* in the variable λ.

The determinant is a scalar valued function of matrices which has certain properties.

1) $\det(I) = 1$

2) $\det(T_1T_2) = \det(T_1)\det(T_2)$

3) $\det(T) \neq 0$ iff T is non-singular.

4) $\det(T-\lambda I)$ is a polynomials of degree *n* in λ.

Suppose $T: H \to H$ is an operator. Suppose B and B' are basis of H. Suppose $\left[\alpha_{ij}\right]$ and $\left[\beta_{ij}\right]$ are matrices representing the transformation T with respect to basis B and B' respectively. By theorem 7.1.3 there exists a non-singular matrix $\left[\gamma_{ij}\right]$ such that

$$\left[\beta_{ij}\right] = \left[\gamma_{ij}\right]^{-1}\left[\alpha_{ij}\right]\left[\gamma_{ij}\right]$$

$$\det\left[\beta_{ij}\right] = \det\left[\gamma_{ij}\right]^{-1}\det\left[\alpha_{ij}\right]\det\left[\gamma_{ij}\right]$$

$$= \left(\det\left[\gamma_{ij}\right]\right)^{-1}\det\left[\alpha_{ij}\right]\det\left[\gamma_{ij}\right]$$

$$= \det\left[\alpha_{ij}\right]$$

Thus determinant of an operator relative to any basis is same, where we define determinant of an operator T as determinant of its matrix relative to any basis.

Theorem 7.2.1 : If T is an arbitrary operator on H, then the eigenvalues of T constitute a non-empty finite subset of the complex plane. Furthermore the number of points in this set doesnot exceed the dimension *n* of the space H.

Proof : Let T be an operator on H. A scalar λ is an eigenvalue of T if and only if there exists a non-zero vector $x \in H$ such that $(T - \lambda I)x = 0$. Non-zero $x \in H$ satisfy $(T - \lambda I)x = 0$ if and only if $\det(T - \lambda I) = 0$. Thus the eigen values of T are precisely the distinct roots of the equation.

$$\det(T - \lambda I) = 0 \qquad \text{..... (1)}$$

Equation (1) is a polynomial equation of degree *n* in λ. By fundamental theorem of algebra, a polynomial of degree *n* has exactly *n* roots. Some of these roots may be repeated, in which case there may be fewer than *n* distinct roots.

In section 7.1 and 7.2 we briefly state certain properties of operators because operator is a linear transformation from H to H where H is finite dimensional vector space. Therefore the theory of linear transformations on vector spaces holds for operators defined on H. In addition since T is an operator corresponding matrices are square matrices.

7.3 THE SPECTRAL THEOREM

If T is an operator on a finite dimensional Hilbert space H then the scalar λ and non-zero vector $x \in H$ satisfying $(T - \lambda I)x = 0$ are called eigen value and eigen vector of T respectively. Each eigen value has one or more eigen vectors associated with it.

Let λ be an eigen value of T and consider the set M of all its corresponding eigen vectors together with the vector 0. (note that 0 is not an eigen vector). Thus M is the set of all vectors x which satisfy the equation.

$$(T - \lambda I)x = 0$$

The space M is closed subspace of H. We call M the eigen space of T corresponding to λ.

Lemma 7.3.1 : The space M is invariant under T i.e. $T(M) \subseteq M$.

Proof : Let $x \in M$ then $Tx = \lambda x \in M$. Since M is subspace for $x \in M$, $\lambda x \in M$.

Thus $T(M) \subseteq M$.

Let T be an arbitrary operator on H. Let $\lambda_1, \lambda_2, \lambda_3, ..., \lambda_m$ are distinct eigenvalues of T and $M_1, M_2, M_3, ..., M_m$ be their corresponding eigenspaces. Now we shall prove certain results related to subspaces M_i, i = 1, 2, 3, ..., m.

Theorem 7.3.1 : If T is normal operator on a finite dimensional Hilbert space H, then x is an eigen vector of T with eigen values λ if and only if x is an eigenvector of T* with eigen values $\bar{\lambda}$.

Proof : Since T is normal operatir T*T = TT*.

$$(T - \lambda I)^* (T - \lambda I) = (T^* - \bar{\lambda} I)(T - \lambda I) = T^*T - \lambda T^* - \bar{\lambda} T + \lambda \bar{\lambda} I$$

$$= TT^* - \lambda T^* - \bar{\lambda} T + \lambda \bar{\lambda} I = (T - \lambda I)(T^* - \bar{\lambda} I) = (T - \lambda I)(T - \lambda I)^*$$

Therefore if T is normal operator then $T - \lambda I$ is normal. We know that if T is normal operator then $\|Tx\| = \|T^* x\|$ and therefore,

$$\|(T - \lambda I)x\| = \|(T^* - \bar{\lambda} I)x\| = \|T^* x - \bar{\lambda} x\|, \ \forall x \in H$$

This we have, $\|Tx-\lambda x\| = \|T^*x-\bar{\lambda}x\|$

If $\|Tx-\lambda x\| = 0 \Leftrightarrow Tx = \lambda x \Leftrightarrow \|T^*-\bar{\lambda}x\| = 0 \Leftrightarrow T^*x = \bar{\lambda}x$

Thus x is an eigen vector of T with eigen value λ iff x is an eigen vector of T* with eigen value $\bar{\lambda}$.

Theorem 7.3.2: If T is normal operator with eigen values $\lambda_1, \lambda_2, ..., \lambda_m$ and $M_1, M_2, M_3, ..., M_m$ their corresponding eigenspaces, then M_i's are pairwise orthogonal.

Proof : Let x_i, x_j be vectors in M_i and M_j for $i \neq j$.

Then $Tx_i = \lambda_i x_i$ and $Tx_j = \lambda_j x_j$.

$\lambda_i(x_i, x_j) = (\lambda_i x_i, x_j) = (Tx_i, x_j) = (x_i, T^*x_j) = (x_i, \bar{\lambda}_j x_j) = \lambda_j(x_i, x_j)$

Thus $\lambda_i(x_i, x_j) = \lambda_j(x_i, x_j)$ for $\lambda_i \neq \lambda_j$.

Therefore $(x_i, x_j) = 0$.

Hence for $i \neq j$, $M_i \perp M_j$.

Theorem 7.3.3 : If T is normal then each M_i reduces T.

Proof : In lemma 7.3.1 we have seen that each M_i is invariant under T. It is sufficient to show that each M_i is invariant under T*. If $x_i \in M_i$ then $Tx_i = \lambda_i x_i \in M_i$. Since M_i is subspace of H.

But $T^*x_i = \bar{\lambda}_i x_i \in M_i$. Thus $T^*(M_i) \subseteq M_i$.

Since M_i is invariant under T and T*, M_i reduces T.

Theorem 7.3.4 : If T is normal then M_i's span H.

Proof : Let T be a normal operator on Hilbert space. Let $\lambda_1, \lambda_2, \lambda_3, ..., \lambda_m$ be eigenvalues and $M_1, M_2, M_3, ..., M_m$ are corresponding eigen spaces. By theoem 7.3.2, all these eigen spaces are pairwise orthogonal. Let P_i denote the projections on closed linear subspaces M_i. Since $M_i \perp M_j$, $\forall i \neq j$, by theorem 6.4.7, $P_iP_j = 0$, $\forall i \neq j$. Since all M_i's are closed linear subspaces of H, $M = M_1 + M_2 + ... + M_m$ is also a closed linear subspace of H and its associated projection $P = P_1 + P_2 + P_3 + ... + P_m$ (by theorem 6.4.8).

Since each M_i reduces T, and P_i is projection on M_i, by theorem 6.4.6, $TP_i = P_iT$, $\forall$ i = 1, 2, 3, ..., ,m. Therefore TP = PT where $P = P_1 + P_2 + P_3 + ... + P_m$ is projection on $M = M_1 + M_2 + M_3 + ... + M_m$. Since TP = PT where P is projection on M, by theorem 6.4.6, M reduces an operator T. Consequently $M^{\perp}$ is invariant under T. If $M^{\perp} \neq \{0\}$ then since all eigenvectors of T are contained in M, the restriction of T to $M^{\perp}$ is an operator on a non-trival finite dimensional Hilbert space which has no eigenvectors and therefore no eigenvales. But by theorem 7.2.1 eigenvalues of T constitute non-empty finite subset of the complex plane. Therefore T on $M^{\perp}$ donot have any eigenvalue is imposible. Hence $M^{\perp} = \{0\}$. But then M = H and the M_i's span H.

Thus we have seen that if T is normal operator, there are finitely many eigenvalues $\lambda_1, \lambda_2, \lambda_3, ..., \lambda_m$ which are distinct with corresponding eigenspaces $M_1, M_2, M_3, ..., M_m$. There eigen spaces are pairwise orthogonal i.e. $M_i \perp M_j$, $\forall i \neq j$ and these M_i's spane H.

H = $M_1 + M_2 + M_3 + ... + M_m$

Since H = $M_1 + M_2 + M_3 + ... + M_m$, each vectory $x \in H$ can be expressed uniquely in the form

$$x = x_1 + x_2 + x_3 + ... + x_m \quad \text{where } x_i \in M_i \text{ for } i = 1, 2, 3, ..., m$$

Since $M_i \perp M_j$ for $i \neq j$, $x_i \perp x_j$ for $i \neq j$ and

$$Tx = T\left(x_1 + x_2 + x_3 + ... + x_m\right)$$

$$= Tx_1 + Tx_2 + Tx_3 + ... + Tx_m$$

$$= \lambda_1 x_1 + \lambda_2 x_2 + \lambda_3 x_3 + ... + \lambda_m x_m$$

Suppose P_i are projections on H with range M_i. Since $M_i \perp M_j$. P_i's are pairwise orthogonal and $P_i x = x_i$.

Thus $\quad Ix = x = x_1 + x_2 + x_3 + ... + x_m$

$$= P_1 x + P_2 x + P_3 x + ... + P_m x$$

$$= (P_1 + P_2 + P_3 + ... + P_m) x , \ \forall x \in H .$$

Hence $I = P_1 + P_2 + P_3 + ... + P_m$

Since $\quad Tx = \lambda_1 x_1 + \lambda_2 x_2 + \lambda_3 x_3 + ... + \lambda_m x_m$

$$= \lambda_1 P_1 x_1 + \lambda_2 P_2 x_2 + \lambda_3 P_3 x_3 + ... + \lambda_m P_m x$$

$$= (\lambda_1 P_1 + \lambda_2 P_2 + \lambda_3 P_3 + ... + \lambda_m P_m) x , \ \forall x \in H .$$

Thus we have, $T = \lambda_1 P_1 + \lambda_2 P_2 + \lambda_3 P_3 + ... + \lambda_m P_m$ .... (*)

The expression (*) for T is called the spectral resolution of T.

We have shown that if T is normal then it has a spectral resolution.

$$T = \lambda_1 P_1 + \lambda_2 P_2 + \lambda_3 P_3 + ... + \lambda_m P_m$$

Thus we have a spectral theorem.

Theorem 7.3.5 : Spectral Theorem

Let T be an arbitrary operator on Hilbert space H. Then distinct eigenvalues of T form a non-empty finite set of complex numbers $\lambda_1, \lambda_2, \lambda_3, ..., \lambda_m$ with corresponding eigenspaces $M_1, M_2, M_3, ..., M_m$. Let $P_1, P_2, P_3, ..., P_m$ be the projections on $M_1, M_2, M_3, ..., M_m$ respectively. Then following statements are equivalent.

(I) The M_i's are pairwise orthogonal and span H.

(II) The P_i's are pairwise orthogonal, $I = \sum_{i=1}^{m} P_i$ and $T = \sum_{i=1}^{m} \lambda_i P_i$.

(III) T is normal operator.

Theorem 7.3.6 : If T is normal operator on Hilbert space H then the spectral resolution,

$$T = \lambda_1 P_1 + \lambda_2 P_2 + \lambda_3 P_3 + ... + \lambda_m P_m \text{ is unique.}$$

Proof : In theorem 7.3.4 we have seen that if T is normal operator on H then T ha a spectral resolution

$$T = \lambda_1 P_1 + \lambda_2 P_2 + \lambda_3 P_3 + \ldots + \lambda_m P_m \qquad \text{..... (1)}$$

Since P_i's are orthogonal.

$$T^2 = \left(\lambda_1 P_1 + \lambda_2 P_2 + \ldots + \lambda_m P_m\right)^2$$

$$= \lambda_1{}^2 P_1^2 + \lambda_2{}^2 P_2^2 + \ldots + \lambda_m{}^2 P_m{}^2 + 2\left(\lambda_1 \lambda_2 P_1 P_2 + \ldots + \lambda_m \lambda_m P_m P_m\right)$$

$$= \lambda_1{}^2 P_1^2 + \lambda_2{}^2 P_2^2 + \ldots + \lambda_m{}^2 P_m{}^2 \qquad (\because P_i P_j = 0 \text{ as } P_i \perp P_j)$$

In general if n is any positive integer then,

$$T^n = \sum_{i=1}^{m} \lambda_i{}^n P_i \qquad \text{.... (2)}$$

Sinc $I = \sum_{i=1}^{m} P_i$, equation (2) holds for $n = 0$.

Let $p(z)$ be any polynomial with complex coefficients in the complex variable z. Then by equation (2) we have

$$p(T) = \sum_{i=1}^{m} p(\lambda_i) P_i \qquad \text{..... (3)}$$

Define polynomials

$$p_j(z) = \frac{(z-\lambda_1)(z-\lambda_2)\ldots.(z-\lambda_{j-1})(z-\lambda_{j+1})\ldots.(z-\lambda_m)}{(\lambda_j-\lambda_1)(\lambda_j-\lambda_2)\ldots.(\lambda_j-\lambda_{j-1})(\lambda_j-\lambda_{j+1})\ldots.(\lambda_j-\lambda_m)}$$

Since p_j is a polynomial and since $p_j(\lambda_i) = \delta_{ij}$.

From equation (3) we have,

$$p_j(T) = \sum_{i=1}^{m} p_j(\lambda_i) P_i = P_j \qquad \text{..... (4)}$$

The projections $P_j, j = 1, 2, \ldots., m$ are uniquely determined as polynomials in T.

Assume that there is another expression for T similar to (1)

$$T = \alpha_1 Q_1 + \alpha_2 Q_2 + \alpha_3 Q_3 + \ldots. + \alpha_k Q_k \qquad \text{.... (5)}$$

Which is also a spectral resolution of T. i.e. α_i's are distinct complex numbers, Q_i's are non-zero pairwise orthogonal projections and $I = \sum_{i=1}^{k} Q_i$. We shall show that (5) is identical to (1).

First we shall show that α_i's are eigenvalues of T. Since $Q_i \neq 0$ there exists a non-zero vector x in the range of Q_i and for this x, $Q_i x = x$ and for $j \neq i$, $Q_j x = 0$ (Since Q_i's are pairwise orthogonal) from equation (5) we have for x in range of Q_i,

$$Tx = (\alpha_1 Q_1 + \alpha_2 Q_2 + ... + \alpha_m Q_m)x = \alpha_i Q_i x = \alpha_i x$$

Thus $Tx = \alpha_i x$ for some non-zero x. So each α_i is eigenvalue of T. Next we shall prove that if λ is an eigenvalue of T then $\lambda = \alpha_i$ for some i, suppose λ is an eigenvalue of T. So that $Tx = \lambda x$ for some non-zero $x \in H$. Then,

$$Tx = \lambda x = \lambda I x = \lambda \sum_{i=1}^{m} Q_i x = \sum_{i=1}^{m} \lambda Q_i x$$

But since $T = \sum_{i=1}^{m} \alpha_i Q_i$, $Tx = \sum_{i=1}^{m} \alpha_i Q_i x$

Thus $Tx = \sum_{i=1}^{m} \lambda Q_i x = \sum_{i=1}^{m} \alpha_i Q_i x \Rightarrow \sum_{i=1}^{m} (\lambda - \alpha_i) Q_i x = 0$

Since $Q_i x$ are pairwise orthogonal, non-zero vectors among $Q_i x$ are linearly independent. Therefore,

$$\sum (\lambda - \alpha_i) Q_i x = 0 \Rightarrow \lambda - \alpha_i = 0 \Rightarrow \lambda = \alpha_i \text{ for some } i.$$

Thus the set α_i is the set of all eigen values of T.

In particular equation (5) is in the form,

$$T = \lambda_1 Q_1 + \lambda_2 Q_2 + ... + \lambda_m Q_m \qquad ..... (6)$$

But then as discussed earlier,

$$Q_j = p_j(T) \text{ for every } j \qquad .... (7)$$

On comparing (7) with (4) we see that $Q_j = P_j$, $j = 1, 2, ...$ m.

Thus the spectral resolution of T is unique.

In section 7.1 we have seen that the matrix representation of an operator T dependents upon the choice of basis for a Hilbert space H. From spectral theorem we have $H = M_1 + M_2 + ... + M_m$ where M_i's are pairwise orthogonal. If for each M_i we choose orthogonal basis, then we have orthogonal basis for H and relative to this basis the matrix of operator T denoted by [T] has the following form,

$$[T] = \begin{bmatrix} \lambda_1 I & 0 & \cdots & 0 \\ 0 & \lambda_2 I & \cdots & 0 \\ & \vdots & & \\ 0 & 0 & \cdots & \lambda_m I \end{bmatrix}$$

Where each $\lambda_i I$ and 0 are matrices i.e. [T] is partition matrix. Order of $\lambda_i I$ dependents upon the dimension of M_i.

Thus there exist an orthogonal basis for H and relative to this basis, the matrix of T is diagonal.

In this chapter we have proved that a normal operator T on finite dimensional Hilbert space H has spectral resolution i.e. there exist distinct complex numbers $\lambda_1, \lambda_2, \lambda_3, ..., \lambda_m$ and non-zero pairwise orthogonal projections $P_1, P_2, P_3, ..., P_m$, such that $\sum_{i=1}^{m} P_i = I$ and $T = \sum_{i=1}^{m} \lambda_i P_i$. This theorem is generalized for the infinite dimensional case by analytic approach and by algebraic or topological approach.

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