SHIVAJI UNIVERSITY, KOLHAPUR

CENTRE FOR DISTANCE AND ONLINE EDUCATION

Ordinary Differential Equations

(Mathematics)

For

M. Sc. Part-I : Sem.-I

(In accordance with National Education Policy 2020)
(Academic Year 2023-24 onwards)

First Edition 2008
Second Edition 2010
Revised Edition 2023

Prescribed for **M. Sc. Part-I**

Copies : 500

Published by:
Dr. V. N. Shinde
Registrar,
Shivaji University,
Kolhapur-416 004

Printed by :

Shri. A. S. Mane,
I/c. Superintendent,
Shivaji University Press,
Kolhapur-416 004

ISBN- 978-81-8486-012-2

★ Further information about the Centre for Distance and Online Education & Shivaji University may be obtained from the University Office at Vidyanagar, Kolhapur-416 004, India.

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Preface

Large numbers of students appear for M.A./M. Sc. Examinations externally every year. In view of this, Shivaji University has introduced the Distance Education Mode for external students from the year 2007-2008, and entrusted the task to us to prepare the Self Instructional Material (SIM) for aspirants.

It is hoped that students must learn Mathematics not only to become competent mathematicians but also skilled users of Mathematics in the solution of problems in the real world. They must learn how to use their Mathematical knowledge in solving the problems of the real world. Differential equations usually are description of physical systems. This book on Ordinary Differential Equations consists of four chapters. Chapter one contains the complete discussion of linear equations with constant coefficients, including the uniqueness theorem. In chapter two linear equations with variable coefficients are trea. Equations with analytic coefficients are introduced and series solutions are obtained by a simple formal process. A detailed treatment of linear equations with regular singular points is discussed in chapter four. Classification of regular singular points and regular singular points at infinity is studied. In chapter five existence and uniqueness of solutions of first order initial value problem are established. The innumerable examples and exercises are given at the end of each unit.

The book introduces the students to some of the abstract topics that pervade modern analysis. The first chapter deals with the Riemann Stieltjes integration. The problems in Physics and Chemistry which involve mass distribution that are partly discrete and partly continuous can be solved by using Riemann Stietjes integrations. The Chapter 2 deals with convergence and uniform convergence of sequences of functions and series where as the Chapter 3 consists of multidimensional calculus. The Chapter 4 deals with implicit functions and extremum problems which have wide applications in optimization theory. Line integrals, surface integrals and Volume integrals are the subject matter of Chapter 5. This provides sufficient background to study the Gauss divergence Theorem and Stokes Theorem.

We owe a deep sense of gratitude to the Vice-Chancellor who has given impetus to go ahead with ambitious projects like the present one. Dr. Sarita Thakar, Professor, Department of Mathematics, Shivaji University has to be profusely thanked for the ovations he has poured to prepare the SIM on Differential Equations. We also thank Prof. M. S. Chaudhary, Former Head, Department of Mathematics, Shivaji University, Director of Centre for Distance and Online Education for their help and keen interest in completion of the SIM.

Prof. Dr. S. H. Thakar
Head Department of Mathematics &
I/c Dean, Faculty of Science & Technology,
Shivaji University, Kolhapur

Centre for Distance and Online Education
Shivaji University,
Kolhapur.

Ordinary Differential Equations

Writing Team	Unit No.
Dr. (Mrs.) S. H. Thakar Head Department of Mathematics & I/c Dean, Faculty of Science & Technology Shivaji University, Kolhapur. Maharashtra	**All**

■ **Editor** ■

Prof. Dr. S. H. Thakar
Head Department of Mathematics &
I/c Dean, Faculty of Science & Technology,
Shivaji University, Kolhapur

M. Sc. (Mathematics)

Ordinary Differential Equations

Contents

Each Unit begins with the section Objectives -

Objectives are directive and indicative of :

1. What has been presented in the Unit and
2. What is expected from you
3. What you are expected to know pertaining to the specific Unit once you have completed working on the Unit.

The self check exercises with possible answers will help you to understand the Unit in the right perspective. Go through the possible answers only after you write your answers. These exercises are not to be submitted to us for evaluation. They have been provided to you as Study Tools to help keep you in the right track as you study the Unit.

M. Sc. (Mathematics)

Paper III

Differential Equations

❁ Dr. Sarita Thakar ❁

Department of Mathematics

Shivaji University, Kolhapur (M.S.)

Chapter 1

Linear Equations with Constant Coefficients

Contents :

Introduction :

We live in a world of interrelated changing entities. The position of the earth changes with time, the velocity of falling body changes with distance, the bending of a beam changes with the weight of the load placed on it, the area of circle changes with the size of the radius, the path of projectile changes with the velocity and angle at which it is fired.

In the language of mathematics changing entities are called variables and the rate of change of one variable with respect to another is called derivative. Equations which express a relation among these variables and their derivatives are called differential equations.

A Linear differential equation of order n with constant coefficients is an equation of the form

$$a_0 y^{(n)} + a_1 y^{(n-1)} + a_2 y^{(n-2)} + \cdots + a_n y = b(x),$$

where, $a_0 \neq 0,\ a_1,\ a_2, \cdots, a_n$ are complex constants

and b is complex valued function on an interval $I\ :\ a < x < b$.

The operator L defined by

$$L(\phi)(x) = \phi^{(n)}(x) + a_1 \phi^{(n-1)}(x) + a_2 \phi^{(n-2)}(x) + ... + a_n \phi(x)$$ is called as differential operator of order n with constant coefficients.

The equation $L(y) = b(x)$ is called non-homogenous equation. If $b(x) = 0$ for all x in I the corresponding equation $L(y) = 0$ is called a homogenous equation.

Unit 1 : Initial Value Problems for Second Order Equations

Here, we are concerned with the equation

$$L(y) = y'' + a_1 y' + a_2 y = 0$$

where a_1 and a_2 are constants.

Theorem 1.1.1

Let, a_1, a_2 be constants and consider the equation $L(y) = y'' + a_1 y' + a_2 y = 0$

1. If r_1, r_2 are distinct roots of the characteristic polynomial

$$p(r) = r^2 + a_1 r + a_2$$

then the functions $\phi_1(x) = e^{r_1 x}$ and $\phi_2(x) = e^{r_2 x}$ are solutions of $L(y) = 0$.

2. If r_1 is a repeated root of the characteristic polynomial $p(r)$, then the functions $\phi_1(x) = e^{r_1 x}$ and $\phi_2(x) = xe^{r_1 x}$ are solutions of $L(y) = 0$.

Proof : Let $\phi(x) = e^{rx}$ be a solutions of $L(y) = 0$.

$$L(e^{rx}) = (e^{rx})'' + a_1 (e^{rx})' + a_2 e^{rx}$$

$$= (r^2 + a_1 r + a_2) e^{rx}$$

$L(e^{rx}) = 0$ if and only if $p(r) = r^2 + a_1 r + a_2 = 0$.

1. If r_1 and r_2 are distinct roots of $p(r)$ then $L(e^{r_1 x}) = L(e^{r_2 x}) = 0$ *and* $\phi_1(x) = e^{r_1 x}$ *and* $\phi_2(x) = e^{r_2 x}$ are solutions of $L(y) = 0$.

2. If r_1 is a repeated root of $p(r)$ then

$$P(r) = (r - r_1)^2 \quad and \quad p'(r) = 2(r - r_1)$$

$$L(e^{rx}) = P(r)e^{rx} \text{ for all } r \,\&\, x.$$

$$\frac{\partial}{\partial r} L(e^{rx}) = \frac{\partial}{\partial r}\left[P(r)e^{rx}\right]$$

$$\Rightarrow \quad L(xe^{rx}) = \left[P'(r) + xP(r)\right]e^{rx}.$$

At $r = r_1$, $P(r_1) = P'(r_1) = 0$.

i.e. $L(xe^{r_1 x}) = 0$ thus, showing that $xe^{r_1 x}$ is a solution of $L(y) = 0$.

Thus if r_1 is a repeated root of the characteristic polynomial $P(r)$, then $\phi_1(x) = e^{r_1 x}$ and $\phi_2(x) = xe^{r_1 x}$ are solutions of $L(y) = 0$.

Theorem 1.1.2 :

If ϕ_1 and ϕ_2 are two solutions of $L(y) = 0$ then $C_1\,\phi_1 + C\phi_2$ is also a solution of $L(y) = 0$. Where, C_1 and C_2 are any two constants.

Proof : Let ϕ_1 and ϕ_2 be two solutions of $L(y) = 0$

$$L(\phi_1) = \phi_1'' + a_1 \phi_1' + a_2\, \phi_1 = 0$$

$$L(\phi_2) = \phi_2'' + a_1\phi_2' + a_2\ \phi_1 = 0$$

Suppose C_1 and C_2 are any two constants then the function ϕ defined by $\phi = C_1\ \phi_1 + C_2\ \phi_2$ is also a solution of $L(y) = 0$.

$$L(\phi) = (a\phi_1 + c_2\phi_2)'' + a_1(a\phi_1 + c_2\phi_2) + a_2(a\phi_1 + c_2\phi_2)$$

$$= c_1(\phi_1'' + a_1\phi_1' + a_2\phi_1) + c_2(\phi_2'' + a_1\phi_2' + a_2\phi_2)$$

$$= c_1L(\phi_1) + c_2L(\phi_2)$$

$$= 0$$

The function ϕ which is zero for all x is also a solution called the trivial solution of $L(y) = 0$.

The results of above two theorems allow us to solve all homogeneous linear second order differential equations with constant coefficients.

Definition 1.1 :

An initial value problem $L(y) = 0$ is a problem of finding a solution ϕ satisfying $\phi(x_0) = \alpha_0$ and $\phi'(x_0) = \beta_0$ where, x_0 is some real number and α_0, β_0 are given constants.

Theorem 1.1.3 : (Existence Theorem)

For any real x_0 and constants α, β, there exists a solution ϕ of the initial value problem

$$L(y) = y'' + a_1y' + a_2y = 0, \quad y(x_0) = \alpha,\ y'(x_0)\beta, \quad -\infty < x < \infty.$$

Proof : By theorem 1.1.1 there exist two solutions ϕ_1 and ϕ_2 that satisfy $L(\phi_1) = L(\phi_2) = 0$. From theorem 1.1.2 we know that $c_1\ \phi_1 + c_2\ \phi_2$ is a solution of $L(y) = 0$. We show that there are unique constants c_1, c_2 such that $\phi = c_1\phi_1 + c_2\phi_2$ satisfies $\phi(x_0) = \alpha$ and $\phi'(x_0) = \beta$.

$$\phi(x_0) = c_1\phi_1(x_0) + c_2\phi_2(x_0) = \alpha$$

$$\phi'(x_0) = c_1\phi_1'(x_0) + c_2\phi_2'(x_0) = \beta$$

Above system of equations will have a unique solution c_1, c_2 if the determinant

$$\Delta = \begin{vmatrix} \phi_1(x_0) & \phi_2(x_0) \\ \phi_1'(x_0) & \phi_2'(x_0) \end{vmatrix} = \phi_1(x_0)\phi_2'(x_0) - \phi_2(x_0)\phi_1'(x_0) \neq 0.$$

By theorem 1.1.1 (1), $\phi_1(x)e^{r_1x}$ and $\phi_2(x)e^{r_2,x}$ are two solution of $L(y) = 0$ for $r_1 \neq r_2$ and

$$\Delta = e^{r_1x_0}r_2\,e^{r_2x_0} - e^{r_2x_0}\,r_1\,e^{r_1x_0}$$

$$= (r_2 - r_1)e^{(r_1+r_2)x_0} \neq 0.$$

By theorem 1.1.1 (2), $\phi_1(x) = e^{r_1x}$ and $\phi_2(x)\,xe^{r_1,x}$ are solutions of $L(y) = 0$ and

$$\Delta = e^{r_1x_0}\left[e^{r_1x_0} + x_0r_1e^{r_1x_0}\right] - x_0e^{r_1x_0}r_1e^{r_1x_0}$$

$$= e^{2r_1x_0} \neq 0$$

Thus, the determinant condition is satisfied in both the cases. Therefore, c_1, c_2 are uniquely determined. The function $\phi = c_1\phi_1 + c_2\phi_2$ is a desired solution of the initial value problems.

Defination 1.2 :

A solution of a differential equation will be called a particular solution if it satisfies the equation and does not contain arbitrary constants.

Theorem 1.1.4 :

Let, ϕ be any solution of

$$L(y) = y'' + a_1 y' + a_2 y = 0$$

on an interval I containing a point x_0, Then for all x in I.

$$\|\phi(x_0)\| e^{-k|x-x_0|} \le \|\phi(x)\| \le \|\phi(x_0)\| e^{k|x-x_0|}$$

Where,

$$\|\phi(x)\| = \left[|\phi(x)|^2 + |\phi'(x)|^2\right]^{1/2} \text{ and } k = 1 + |a_1| + |a_2|.$$

Proof : Let,

$$u(x) = \|\phi(x)\|^2$$

$$= |\phi(x)|^2 + |\phi'(x)|^2$$

$$= \phi(x)\bar{\phi}(x) + \phi'(x)\bar{\phi}'(x)$$

Then, $$u'(x) = \phi'(x)\bar{\phi}(x) + \phi(x)\bar{\phi}'(x) + \phi''(x)\bar{\phi}'(x) + \phi'(x)\bar{\phi}''(x)$$

and $$|u'(x)| \le 2|\phi(x)||\phi'(x)| + 2|\phi'(x)||\phi''(x)|$$

as $|\phi(x)| = |\bar{\phi}(x)|$

Since ϕ is a solution of $L(y) = 0$, $L(\phi) = \phi'' + a_1\phi' + a_2\phi = 0$

i.e. $\phi''(x) = -a_1\phi'(x) - a_2\phi(x)$ and the above inequality becomes

$$|u'(x)| \le 2|\phi(x)||\phi'(x)| + 2|\phi'(x)|\left[|a_1||\phi'(x)| + |a_2||\phi(x)|\right]$$

$$\le 2[1 + |a_2|]|\phi(x)||\phi'(x)| + 2|a_1||\phi'(x)|^2$$

But, $$2|\phi(x)||\phi'(x)| \le |\phi(x)|^2 + |\phi'(x)|^2$$

Therefore,

$$|u'(x)| \le 2(1 + |a_1| + |a_2|)|\phi'(x)|^2 + 2(1 + |a_2|)|\phi(x)|^2$$

$$\le 2(1 + |a_1| + |a_2|)\left[|\phi'(x)|^2 + |\phi(x)|^2\right]$$

$$\le 2k\,u(x)$$

Thus, we get

$$-2u(x) \le u'(x) \le 2ku(x)$$

$u'(x) \le 2ku(x)$ is equivalent to $u'(x) - 2k\,u(x) \le 0$ since exponential functions are positive on multiplying above inequality by e^{-2kx} we get

$$e^{-2kx}\left(u'(x)-2ku(x)\right)=\left(e^{-2kx}u(x)\right)'\le 0.$$

Integrating above inequality between the limits x_0 to x for $x > x_0$ yields.

$$e^{-2kx}u(x)-e^{-2kx_0}\,u(x_0)\le 0$$

$$u(x)\le e^{2k(x-x_0)}\,u(x_0)$$

Thus, $$\|\phi(x)\|^2\le e^{2k(x-x_0)}\,\|\phi(x_0)\|^2$$

Similarly, for $x > x_0$ the inequality $-2\,k\,u\,(x)\le u'(x)$ implies

$$\|\phi(x_0)\|^2\,e^{-2k(x-x_0)}\le\|\phi(x)\|^2$$

Therefore for $x > x_0$ we get

$$\|\phi(x_0)\|^2\,e^{-2k(x-x_0)}\le\|\phi(x)\|^2\le e^{2k(x-x_0)}\,\|\phi(x_0)\|^2 \qquad \text{.......... (1.1.1)}$$

For $x < x_0$, the sign of above inequality will get changed

$$\|\phi(x_0)\|^2\,e^{-2k(x-x_0)}\ge\|\phi(x)\|^2\ge e^{2k(x-x_0)}\,\|\phi(x_0)\|^2$$

This inequality can be written as

$$e^{2k(x-x_0)}\,\|\phi(x_0)\|^2\le\,\|\phi(x_0)\|^2\le\|\phi(x_0)\|^2\,e^{-2k(x-x_0)}$$

since $x < x_0$, $\;x_0 - x > 0$.

$$e^{-2k(x_0-x)}\,\|\phi(x_0)\|^2\le\,\|\phi(x)\|^2\le\|\phi(x_0)\|^2\,e^{2k(x_0-x)} \qquad \text{.......... (1.1.2)}$$

Equation (1.1.1) and (1.1.2) together can be put in the form

$$e^{-2k|x_0-x|}\,\|\phi(x_0)\|^2\le\,\|\phi(x)\|^2\le\|\phi(x_0)\|^2\,e^{2k|x_0-x|}$$

Since all the terms in above inequality are positive the square root of each term results into the required inequality.

Theorem 1.1.5 (Uniqueness Theorem)

Let α, β be any two constants and let x_0 be any real number. On any interval I containing x_0 there exists at most one solution ϕ of the initial value problem

$$L(y)=y''+a_1y'+a_2y=0,\quad y(x_0)=\alpha,\;\; y'(x_0)=\beta$$

Proof : Suppose ϕ and ψ are two solutions.

Let $\quad \theta=\phi-\psi.\;$ Since $\;L(\phi)=L(\psi)=0,$

$$L(\theta)=L(\theta-\psi)=L(\theta)-L(\psi)=0$$

Since $\;\phi(x_0)=\psi(x_0)=\alpha\;$ and $\;\phi'(x_0)=\psi'(x_0)=\beta,$

$$\theta(x_0)=\phi(x_0)-\psi(x_0)=0 \;\text{ and }\; \theta'(x_0)=\phi(x_0)-\psi'(x_0)=0$$

Thus, $\;L(\theta)=0,\;\; \theta(x_0)=0\;$ and $\;\theta'(x_0)=0.$

$$\|\theta(x_0)\|^2=\left[|\,\theta(x_0)|^2+|\,\theta'(x_0)^2\,|\right]=0$$

By theorem (1.1.4) we see that

$$||\theta(x)|| = \left[|\theta(x)|^2 + |\theta'(x)^2|\right] = 0 \quad \text{for all } x \text{ in } I$$

This implies $\theta(x) = 0$ for all x in I.

But $\theta(x) = \theta(x) - \psi(x) = 0$ i.e. $\phi(x) \equiv \phi(x)$.

Theorem 1.1.6 :

Let ϕ_1, ϕ_2 be two solutions of $L(y) = 0$ given by theorem 1.1.1. If c_1, c_2 are any two constants the function $\phi = c_1 \phi_1 + c_2 \phi_2$ is a solution of $L(y) = 0$ on $-\infty < x < \infty$.

Conversely, if ϕ is any solution of $L(y) = 0$ on $-\infty < x < \infty$, then there are unique constants C_1 and C_2 such that $\phi = C_1 \phi_1 + C_2 \phi_2$.

Proof : First part of the theorem follows from theorem 1.1.2.

Conversely suppose ϕ is any solution of $L(y) = 0$. Let $\phi(x_0) = \alpha$ and $\phi'(x_0) = \beta$ for some constants α and β. In the proof of existence theorem 1.1.3 we showed that there is a solution ψ of $L(y) = 0$ satisfying.

$$\psi(x_0) = \alpha, \quad \psi'(x_0) = \beta \quad \text{of the form}$$

$\psi(x) = c_1\phi_1(x) + c_2\phi_2(x)_1$ where c_1 and c_2 are uniquely determined by α and β. By uniqueness theorem 1.1.5 $\phi = \psi$, for all x.

Examples :

1. Find all solutions of the following equations.

(a) $y'' - 4y = 0$ (b) $y'' + 2iy' + y = 0$ (c) $y'' - 4y' + 5y = 0$

Answer :

(a) The characteristic polynomial is $p(r) = r^2 - 4$. $r_1 = 2$ and $r_2 = -2$ are two distinct roots of $p(r) = 0$.

Therefore $\phi_1(s) = e^{2x}$ and $\phi_2(x) = e^{-2x}$ are two solutions. For any constants c_1 and c_2, $c_1e^{2x} + c_2e^{-2x}$ is a solution. Thus the general solution is $\phi_1(x) = c_1e^{2x} + c_2e^{-2x}$.

(b) The characteristic polynomial $p(r) = r^2 + 2ir + 1$

$$p(r) = 0 \Rightarrow \quad r = \frac{1}{2}\left[-2i \pm \sqrt{(2i)^2 - 4}\right]$$

$$= \frac{1}{2}\left[-2i \pm \sqrt{-8}\right]$$

$$= -i \pm \sqrt{2}\, i$$

$$= \left(-1 \pm \sqrt{2}\right) i$$

Thus $r_1 = \left(-1+\sqrt{2}\right)i$ and $r_2 = \left(-1-\sqrt{2}\right)i$ are two district roots of $p(r) = 0$.

Therefore $\phi_1(x) = e^{\left(-1+\sqrt{2}\right)ix}$ and $\phi_2(x) = e^{\left(-1-\sqrt{2}\right)ix}$ are two solutions. Thus, for any constants c_1 and c_2, $\phi(x) = c_1 e^{\left(-1+\sqrt{2}\right)ix} + c_2\, e^{\left(-1-\sqrt{2}\right)ix}$ is a general solution.

(c) The characteristic polynomial $p(r) = r^2 - 4r + 5$. $p(r) = 0$ gives $r_1 = 2 + i$ and $r_2 = 2 - i$ as two distinct roots. $\phi_1(x) = e^{(2+i)x}$ and $\phi_2(x) = e^{(2-i)x}$ are two solutions of the differential equation. For any constants c_1 and c_2, $\phi(x) = c_1 e^{(2-i)x} + c_2 e^{(2+i)x}$ is a general solution. In particular for $c_1 = c_2 = \dfrac{1}{2}$ we get,

$$\phi(x) = e^{2x}\left(\frac{e^{-ix}+e^{ix}}{2}\right) = e^{2x}\cos x. \text{ and for}$$

$$c_1 = \frac{-1}{2i} \text{ and } c_2 = \frac{1}{2i} \text{ we get}$$

$$\phi(x) = e^{2x}\left(\frac{e^{ix}-e^{-ix}}{2i}\right) = e^{2x}\sin x$$

Thus, $\phi(x) = A\, e^{2x}\cos x + B\, e^{2x}\sin x$ is a solution of the differential equation for any constants A & B.

2. Find the solutions ϕ of the following initial value problems.

(a) $\phi'' + \phi' - 6\phi = 0, \quad \phi(0) = 1,\ \phi'(0) = 0$

(b) $\phi'' + \phi = 0, \quad \phi(0) = 1, \quad \phi\left(\dfrac{\pi}{2}\right) = 0$

(c) $\phi'' + k\phi = 0, \quad k$ is any constant, $\phi(0) = 0, \quad \phi(\pi) = 0$

(d) $\phi'' - 2\phi' - 3\phi = 0, \quad \phi(0) = 0, \quad \phi'(0) = 1$

Answer :

(a) The characteristic polynomial $p(r) = r^2 + r - 6$. $r_1 = 2$ and $r_2 = -3$ are distinct roots $\phi(x) = c_1 e^{2x} + c_2 e^{-3x}$ is a general solution.

$\phi(0) = 1 \Rightarrow c_1 + c_2 = 1$ .... (1)

$\phi'(0) = 0 \Rightarrow \phi'(x) = 2c_1 e^{2x} - 3c_2 e^{-3x}$ at $x = 0$, gives $\phi'(0) = 2c_1 - 3c_2 = 0$ .... (2)

solving equation (1) and (2) for c_1 and c_2 we get $c_1 = 3/5$ and $c_2 = +2/5$.

Thus, the required solution is $\phi(x) = \dfrac{3e^{2x}}{5} + \dfrac{2e^{3x}}{5}$.

(b) The characteristic polynomial is $p(r) = r^2 + 1$. $r_1 = i$ and $r_2 = -i$ are distinct roots

$\phi(x) = c_1 \cos x + c_2 \sin x$ is a general solution.

$$\phi(0) = 1 \Rightarrow c_1 \cos 0 + c_2 \sin = 1 \text{ gives } c_1 = 1$$

$$\phi(\pi/2) = 2 \Rightarrow c_1 \cos \pi/2 + c_2 \sin \pi/2 = 2 \text{. gives } c_2 = 2.$$

Thus, $\phi(x) = \cos x + 2 \sin x$ is the required solution.

(c) The characteristic polynomial is $p(r) = r^2 + k$ since k is any constants, k can be positive, negative or zero.

<u>Case 1.</u> $\boldsymbol{k > 0}$

Then $r_1 = \sqrt{k}\, i$ and $r_2 = -\sqrt{k}\, i$; are distinct roots.

$\therefore \ \phi(x) = c_1 e^{\sqrt{k}\, ix} + c_2 e^{-\sqrt{k}\, ix}$ is a general solution

In general $\phi(x) = A \cos \sqrt{k}\ x + B \sin \sqrt{k}\ x$ is a solution.

$$\phi(0) = 0 \Rightarrow A \cos 0 + B \sin 0 = 0 \quad \text{i.e. } A = 0$$

$$\phi(\pi) = 0 \Rightarrow A \cos \pi + B \sin \pi = 0 \quad \text{i.e. } A = 0$$

Thus, $\phi(x) = B \sin \sqrt{k}\ x$ is a solution where B is any constant.

<u>Case 2.</u> $\boldsymbol{k = 0}$

$p(r) = r^2 = 0 \Rightarrow r = 0$ a repeated root.

$\therefore \ \phi(x) = c_1 e^0 + c_2 x e^0 = c_1 + c_2 x$ is a solution

$$\phi(0) = 0 \Rightarrow c_1 = 0$$

$$\phi(\pi) = 0 \Rightarrow c_1 + c_2 \pi = 0 \Rightarrow c_2 = 0$$

Therefore there is no nontrivial solution corresponding to $k = 0$.

<u>Case 3.</u> $\boldsymbol{k < 0}$

for $k = 0$, $p(r) = r^2 + k$ has distinct roots

$$r_1 = \sqrt{-k} \quad \& \quad r_2 = -\sqrt{-k} \quad (\text{ Since } k < 0,\ -k > 0)$$

$$\phi(x) = c_1 e^{\sqrt{-k}\ x} + c_2 e^{-\sqrt{-k}\ x}$$

$$\phi(0) = c_1 + c_2 = 0$$

$$\phi(\pi) = c_1 e^{\sqrt{-k}\ \pi} + c_2 e^{-\sqrt{-k}\ \pi} = 0$$

Simultaneous evaluation of above two equations give $c_1 = c_2 = 0$.

Thus, there is no non-trival solution corresponding to $k < 0$.

The only non-trivial solution for the given equation is $\phi(x) = B \sin \sqrt{k}\ x$.

(d) The characteristic polynomial $p(r) = r^2 - 2r - 3$

$r_1 = 3$, $r_2 = 1$ are two distinct roots.

$\therefore \qquad \phi(x) = c_1 e^{3x} + c_2 e^{-x}$ is a general solution

$$\phi(0) = 0 \;\Rightarrow\; \phi(0) = c_1 + c_2 = 0$$

$$\phi'(x) = 3c_1 e^{3x} - c_2 e^{-x}$$

$$\phi'(0) = 1 \;\Rightarrow\; \phi'(0) = 1 = 3c_1 - c_2$$

Thus, $c_1 + c_2 = 0$ and $3c_1 - c_2 = 1$ gives

$$c_1 = \frac{1}{4} \text{ and } c_2 = -\frac{1}{4}$$

Therefore $\phi(x) = \frac{1}{4} e^{3x} - \frac{1}{4} e^{-x}$ is the required solution.

EXERCISES

1. Fill in the blanks.

(i) If r_1, r_2 are distinct roots of characteristic polynomial $p(r) = r^2 + a_1 r + a_2$ then $\phi_1(x) =$ .... and $\phi_2(x) =$ .... are solutions of the differential equation $y'' + a_1 y' + a_2 y = 0$

(ii) If $p(r) = (r - r_1)^2$ is a characteristic polynomial then $\phi_1(x) =$ .... and $\phi_2(x) =$ .... are two solutions of the differential equation $y'' - 2r_1 y' + {r_1}^2 y = 0$.

(iii) Uniqueness theorem states that ....

(iv) Solution of $y'' - 2y' + 4y = 0$ are $\phi_1(x) =$ .... and $\phi_2(x) =$ ....

(v) The general solution of $y'' - 3y' + 2y = 0$ is.....

2. Find the gental solution of each of the following equation.

(i) $y'' + 4y' = 0$ (ii) $y'' - y = 0$ (iii) $y'' + y' - 6y = 0$

(iv) $y'' + 4ky' - 12k^2 y = 0$ (v) $y'' - 2ay' + a^2 y = 0$ (vi) $y'' - 4y' + 20y = 0$

3. Find the solution of the following initial value problems :

(i) $y'' = 0, \quad y(1) = 2, \quad y'(1) = -1$

(ii) $y'' + 4y' + 4y = 0, \quad y(0) = 1, \quad y'(0) = 1$

(iii) $y'' - 2y' + 5y = 0, \quad y(0) = 2, \quad y'(0) = 4$

(iv) $y'' - 4y' + 20y = 0, \quad y(\pi/2) = 0, \quad y'(\pi/2) = 1$

Answers :

1. (i) $\phi_1(x) = e^{r_1 x}$, $\phi_2(x) = e^{r_2 x}$ (ii) $\phi_1(x) = e^{r_1 x}$, $\phi_2(x) = x e^{r_1 x}$

(iii) theorem 1.1.5 (iv) $\phi_1(x) = e^{2x}$, $\phi_2(x) = x e^{2x}$

(v) $c_1 e^{2x} + c_2 e^{x}$

2. (i) $c_1 + c_2 e^{-4x}$ (ii) $c_1 e^{x} + c_2 e^{-x}$

(iii) $c_1 e^{2x} + c_2 e^{-3x}$ (iv) $c_1 e^{-6kx} + c_2 e^{2kx}$

(v) $(c_1 + c_2 x)e^{ax}$ (vi) $e^{2x}(c_1 \cos 4x + c_2 \sin 4x)$

3. (i) $3 - x$ (ii) $(1 + 3x)\, e^{-2x}$

(iii) $e^{x}(2\cos 2x + \sin 2x)$ (iv) $\frac{1}{4} e^{2x-\pi} \sin 4x$

Unit 2 : Linear Dependence and Independence

Every solution of the equation $L(y) = 0$ is a linear combination of two solutions obtained in theorem 1.1.1. Therefore these two solutions span the solution space of the differential equation $L(y) = 0$.

Defination 1.3 : A set of n real or complex functions $f_1, f_2, f_3, ..., f_n$ defined on an interval (a, b) is said to be linearly independent when $c_1 f_1(x) + c_2 f_2(x) + c_3 f_3(x) + \cdots + c_n f_n(x) = 0$ for every x in (a, b) implies $c_1 = c_2 = c_3 = \cdots = c_n = 0$.

Defination 1.4 : Given the functions $f_1, f_2, f_3, \cdots, f_n$ if constants $c_1, c_2, c_3, \cdots, c_n$ not all zero exist such $c_1 f_1(x) + c_2 f_2(x) + c_3 f_3(x) + \cdots + c_n f_n(x) = 0$ for every x in (a, b), then these functions are linearly dependent.

A set which is not linearly independent is said to be linearly dependent.

There are two notions of linear independence, according as we allow the coefficients c_k, $k = 1, 2, 3, ..., n$ to assume only real values or also complex values. In the first case, one says that the functions are linearly independent over the field of reals; in the second case, that they are linearly independent over the complex field.

***Lemma* 1.2.1 :** A set of real valued functions on an interval (a, b) is linearly independent over the complex field if and only if it is linearly independent over the real field.

Proof : If the set of real valued functions on an interval (a, b) is linearly independent over the complex field then it is linearly independent over the field of reals.

Conversely suppose the set is linearly independent over the real field. Therefore for $\alpha_j \in R$, $\sum_{j=1}^{n} \alpha_j f_j(x) = \alpha_1 f_1(x) + \alpha_2 f_2(x) + \alpha_3 f_3(x) + \cdots + \alpha_n f_n(x) = 0$ for all x in (a, b) implies $\alpha_j = 0$ for all $j = 1, 2, 3...., n$. Let $\sum_{j=1}^{n} c_j f_j(x) = 0$ for all x in (a, b) and for some $c_j \in C$, $j = 1, 2, 3 \cdots, n$. Since the function f_j are real valued and $\sum c_j f_j(x) = 0$, $\left[\sum c_j f_j(x)\right]^* = 0$. implies $\sum_{j=1}^{n} c_j^* f_j(x) = 0$. Thus, $\sum_{j=1}^{n} \left(\frac{c_j - c_j^*}{i}\right) f_j(x) = 0$. But $\left(c_j - c_j^*\right)/i$ are all real and the set is linearly independent over the real field therefore $c_j = c_j^*$. But then c_j's are all real therefore $\sum_{j=1}^{n} c_j \, f_j(x) = 0$ implies $c_j = 0$ for $j = 1, 2, \ldots, n$.

A set of functions which is linearly dependent on a given domain may become linearly independent when the functions are extended to a larger domain. However, a linearly independent set of functions clearly remain linearly independent on the restricted domain.

Illustration 1 : The functions ϕ_1 and ϕ_2 define by $\phi_1(x)$ = Cos x and $\phi_2(x)$ = Sin x are linearly independet on the real line IR and therefore are linearly independent on $(0, 2\pi)$.

Illustration 2 : The functions ϕ_1 and ϕ_2 define by $\phi_1(x) = x$, $\phi_2(x) = |x|$ are linearly indepent on the interval (–1, 1) but is not linearly independent on the interval (0, 1) as on the interval (0, 1), $\phi_1(x) = \phi_2(x)$.

Theorem 1.2.1 :

Let a_1, a_2 be constants and consider the equation $L(y) = y'' + a_1 y' + a_2 y = 0$. The two solutions of $L(y) = 0$ given in the theorem 1.1.1 are linearly independent on any interval I.

Proof : Let r_1, r_2 be the roots of characteristic polynomial $p(r) = r^2 + a_1 r + a_2$.

<u>**Case 1.**</u>

If $r_1 \neq r_2$, then $\phi_1(x) = e^{r_1 x}$ and $\phi_2(x) = e^{r_2 x}$ are two solutions of the equation $L(y) = 0$ on an interval I.

Suppose $c_1 e^{r_1, x} + c_2 e^{r_2 x} = 0$ for all x in I.

Then $c_1 + c_2 e^{(r_2 - r_1)x} = 0$ for all x in I.

Differentiation of above equation with respect to x gives $c_2 (r_2 - r_1) e^{(r_2 - r_1)x} = 0$ for all x in I.

Since, $r_2 \neq r_1$ and exponential function in non-zero, c_2 is zero. But if c_2 is zero then $c_1 + c_2 \, e^{(r_2 - r_1)x} = 0$ implies c_1 is zero. Thus, $c_1 e^{r_1 x} + c_2 e^{r_2 x} = 0$ implies $c_1 = c_2 = 0$.

Therefore $\phi_1(x) = e^{r_1 x}$ and $\phi_2(x) = e^{r_2 x}$ are linearly independent.

<u>**Case 2.**</u>

If $r_1 = r_2$, then $\phi_1(x) = e^{r_1 x}$ and $\phi_2(x) = x e^{r_1 x}$ are two solutions of the equation $L(y) = 0$ on an interval I.

Suppose $c_1 e^{e_1 x} + c_2 x e^{r_2 x} = 0$ then $c_1 + c_2 x = 0$ for all x in I. Therefore $c_1 = c_2 = 0$. Thus, ϕ_1 and ϕ_2 are linearly independent

Thus, in both cases the two solutions ϕ_1 and ϕ_2 of $L(y) = 0$ are linearly independent.

Defination 1.5 : Assume that each of the functions $f_1(x), f_2(x), f_3(x), \cdots, f_n(x)$ are differentiable atleast $(n - 1)$ times in the interval (a, b). Then the determinant

$$\begin{vmatrix} f_1(x) & f_2(x) & f_3(x) & \cdots & f_n(x) \\ f_1'(x) & f_2'(x) & f_3'(x) & \cdots & f_n'(x) \\ f_1''(x) & f_2''(x) & f_3''(x) & \cdots & f_n''(x) \\ \vdots & \vdots & \vdots & & \vdots \\ f_1^{(n-1)}(x) & f_2^{(n-1)}(x) & f_3^{(n-1)}(x) & \cdots & f_n^{(n-1)}(x) \end{vmatrix}$$

denoted by $W(f_1, f_2, f_3, \ldots, f_n)(x)$ is called the wronskian of the n functions $f_1, f_2, f_3, \ldots, f_n$.

Theorem 1.2.2 :

Two solutions ϕ_1, ϕ_2 of $L(y) = 0$ are linearly independent on an interval I if and only if $W(\phi_1, \phi_2)(x) \neq 0$ for all x in I.

Proof : Suppose $W(\phi_1, \phi_2)(x) \neq 0$ for all x in I

Let c_1, c_2 be constants such that

$c_1 \phi_1(x) + c_2 \phi_2(x) = 0$ for all x in I. Then

$c_1 \phi_1'(x) + c_2 \phi_2'(x) = 0$ for all x in I.

Above two equations can be written as

$$\begin{bmatrix} \phi_1(x) & \phi_2(x) \\ \phi_1'(x) & \phi_2'(x) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Since, $W(\phi_1, \phi_2)(x) \neq 0$ for all x in I, the coefficient matrix is invertible. On premultiplying the inverse of the coefficient matrix results in $c_1 = c_2 = 0$. This proves that ϕ_1 and ϕ_2 are linearly independent on I.

Conversely, assume that ϕ_1, ϕ_2 are linearly independent on I. Suppose that there is a point x_0 in I such that $W(\phi_1, \phi_1)(x_0) = 0$. Then the system of equations

$$\begin{bmatrix} \phi_1(x_0) & \phi_2(x_0) \\ \phi_1'(x_0) & \phi_2'(x_0) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

has a solution c_1, c_2 where at least one of these numbers is not zero. Let c_1, c_2, be such a solution and consider the function $\psi(x) = c_1 \phi_1(x) + c_2 \phi_2(x)$. Now $L(\psi) = 0$ and $\psi(x_0) = 0, \psi'(x_0) = 0$. Therefore $\|\psi(x_0)\| = \left[|\psi(x_0)|^2 + |\psi'(x_0)|^2\right]^{\frac{1}{2}} = 0$. By theorem 1.1.4 $\|\psi(x)\| = 0$. But $\|\psi(x)\| = \left[|\psi(x)|^2 + |\psi'(x)|^2\right] = 0$. Therefore $\psi(x) = 0$ for all x in I and thus $c_1 \phi_1(x) + c_2 \phi_2(x) = 0$ for all x in I. But then ϕ_1 and ϕ_2 are linearly dependent. Thus, the supposition $W(\phi_1, \phi_2)(x_0) = 0$ must be false and therefore $W(\phi_1, \phi_2)(x) \neq 0$ for all x in I.

In the next theorem we will prove that we need to compute $W(\phi_1, \phi_2)$ at only one point to test the linear independence of the solutions ϕ_1 and ϕ_2.

Theorem 1.2.3 :

Let ϕ_1, ϕ_2 be two solution of $L(y) = 0$ on an interval I and let x_0 be any point in I. Then two solutions ϕ_1 and ϕ_2 are linearly independent on I if and only if $W(\phi_1, \phi_2)(x_0) \neq 0$.

Proof : If ϕ_1 and ϕ_2 are linearly independent on I then by theorem 1.2.2, $W(\phi_1, \phi_2)(x) \neq 0$ for all x in I. In particular $W(\phi_1, \phi_2)(x_0) \neq 0$ conversely, suppose $W(\phi_1, \phi_2)(x_0) \neq 0$ and suppose c_1, c_2 are constants such that $c_1\phi_1(x) + c_2\phi_2(x) = 0$ for all x in I. Then $c_1\phi_1(x_0) + c_2\phi_2(x_0) = 0$ and $c_1\phi_1'(x_0) + c_2\phi_2'(x_0) = 0$.

i.e.
$$\begin{bmatrix} \phi_1(x_0) & \phi_2(x_0) \\ \phi_1'(x_0) & \phi_2'(x_0) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

But since the determinant of the coefficient is $W(\phi_1, \phi_2)(x_0) \neq 0$ we obtain $c_1 = c_2 = 0$. Thus ϕ_1, ϕ_2 are linearly independent on I.

In the next theorem we show that the knowledge of two linearly independent solutions of $L(y) = 0$ is sufficient to generate all solutions of $L(y) = 0$.

Theorem 1.2.4 :

Let ϕ_1, ϕ_2 be any two linearly independent solutions of $L(y) = 0$ on an interval I. Every solution ϕ of $L(y) = 0$ can be written uniquely as

$$\phi = c_1\phi_1 + c_2\phi_2 \text{ where } c_1, c_2 \text{ are constants.}$$

Proof : Let x_0 be a point in I. Let $\phi(x_0) = \alpha, \phi'(x_0) = \beta$. Since ϕ_1, ϕ_2 are linearly independent on I we know that $W(\phi_1, \phi_2)(x_0) \neq 0$. Consider the two equations.

$$\begin{bmatrix} \phi_1(x_0) & \phi_2(x_0) \\ \phi_1'(x_0) & \phi_2'(x_0) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$$

Since $W(\phi_1, \phi_2)(x_0) \neq 0$, above system of equations has a unique solution c_1, c_2. For this choice of c_1, c_2 the function $\psi(x) = c_1\phi_1(x) + c_2\phi_2(x)$ satisfies $\psi(x_0) = c_1\phi_1(x_0) + c_2\phi_2(x_0)$ $= \alpha = \phi(x_0)$ i.e. $\psi(x_0) = \phi(x_0)$ similarly $\psi'(x_0) = \phi'(x_0)$ and $L(\psi) = 0$. From the uniqueness theorem 1.1.5 it follows that $\psi = \phi$ on I i.e. $\phi = c_1\phi_1 + c_1\phi_2$.

Examples :

Q1. Show that the functions e^x, e^{2x}, e^{3x} are linearly independent.

Ans. :

<u>Method 1 :</u>

Let $c_1e^x + c_2e^{2x} + c_3e^{3x} = 0$

then $c_1 + c_2e^x + c_3e^{2x} = 0$ .... (1)

Differentiate above equation (1) with respect to x then $c_2e^x + 2\,c_3\,e^{2x} = 0$ implies

$$c_2 + 2c_3e^x = 0 \qquad \text{.......... (2)}$$

By differentiating equation (2) with respect to x we get $2c_3e^x = 0$ therefore $c_3 = 0$.

But then by equation (2) $c_2 = 0$ and by equation (1) we get $c_1 = 0$. Thus $c_1 = c_2 = c_3 = 0$. Therefore the functions e^x, e^{2x}, e^{3x} are linearly independent.

Method 2 :

Let $\phi_1(x) = e^x,\ \phi_2(x) = e^{2x},\ \phi_3(x) = e^{3x}$

$$W(\phi_1, \phi_2, \phi_3) = \begin{vmatrix} e^x & e^{2x} & e^{3x} \\ e^x & 2e^{2x} & 3e^{3x} \\ e^x & 4e^{2x} & 9e^{3x} \end{vmatrix} = e^x e^{2x} e^{3x} \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{vmatrix}$$

$$= e^{6x}\,[1(18–12) – 1\,(9–3) + 1\,(4–2)]$$

$$= 2\,e^{6x} \neq 0.$$

by theorem 1.2.2 ϕ_1, ϕ_2, ϕ_3 are linearly independent.

Q2. : The functions ϕ_1, ϕ_2 are defined on $-\infty < x < \infty$. Determine whether they are linearly dependent or independent there.

(i) $\phi_1(x) = x,\ \ \phi_2(x) = e^{rx}$, r is a complex constant

(ii) $\phi_1(x) = x^2,\ \ \phi_2(x) = 5x^2$

(iii) $\phi_1(x) = x,\ \ \phi_2(x) = |x|$

(iv) $\phi_1(x) = \cos x,\ \ \phi_2(x) = \sin x$

Ans. (i) :

Method 1 :

Let $c_1\,\phi_1(x) + c_2\,\phi_2(x) = 0$

i.e. $c_1\,x + c_2\,e^{rx} = 0$ .... (1)

if $r = 0$, $c_1x + c_2 = 0$ for all $x \in R$ implies

$c_1 = 0$ and $c_2 = 0$. $\therefore$ ϕ_1, ϕ_2 are linearly independent if $r \neq 0$, differentiate equation (1) with respect to x then $c_1 + rc_2e^{rx} = 0$

Again differentiate above equation with respect to x then $r^2c_2e^{rx} = 0$. But $r \neq 0$ and $e^{rx} \neq 0$ therefore $c_2 = 0$ and from equation (1) we get $c_1 = 0$. Thus ϕ_1, ϕ_2 are linearly independent.

Method 2 :

$$W(\phi_1, \phi_2) = \begin{vmatrix} x & e^{rx} \\ 1 & re^{rx} \end{vmatrix} = e^{rx} \begin{vmatrix} x & 1 \\ 1 & r \end{vmatrix}$$

$$= e^{rx}(r\,x-1) \neq 0 \text{ for } x \in IR$$

$\therefore$ ϕ_1, ϕ_2 are linearly independent

Method 3 :

$W(\phi_1, \phi_2)\ (0) = \begin{vmatrix} 0 & 1 \\ 1 & r \end{vmatrix} = 1 \neq 0$ therefore by theorem 1.2.3 ϕ_1, ϕ_2 are linearly independent.

Ans. (ii) :

Let $c_1\phi_1 + c_2\phi_2 = 0$

i.e. $c_1 x^2 + c_2\, 5x^2 = 0$

if $(c_1 + 5c_2)x^2 = 0$

If we choose $c_1 = -\,5c_2 \neq 0$ then the linear combination $c_1\phi_1 + c_2\phi_2 = 0$ therefore by definition 1.4, ϕ_1, ϕ_2 are linearly dependent.

Ans. (iii) :

For $x > 0$ $c_1\phi_1 + c_2\phi_2 = (c_1 + c_2)x$ as $|x| = x$

and for $x < 0$ $c_1\phi_1 + c_2\phi_2 = (c_1 - c_2)x$ as $|x| = -x$

Thus, $c_1\phi_1 + c_2\phi_2 = 0$ for $x \in R$

$\Rightarrow (c_1 + c_2)x = 0$ and $(c_1 - c_2)x = 0$

for every $x \in R$ above two equations hold true if and only if $c_1 = c_2 = 0$. Thus ϕ_1, ϕ_2 defined by $\phi_1(x) = x$ and $\phi_2(x) = |x|$ are linearly independent.

Ans. (iv) :

$$\phi_1(x) = \cos x; \quad \phi_2(x) = \sin x$$

$$W(\phi_1, \phi_2)\ (x) = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = 1$$

$\because W(\phi_1, \phi_2)(x) = 1 \neq 0$, ϕ_1, ϕ_2 are linearly independent.

Q3. : Let ϕ_n be any function satisfying the boundary value problem

$$y'' + n^2 y = 0, \qquad y(0) = y(2\pi), \qquad y'(0) = y'(2\pi), \qquad n = 0, 1, 2, 3, \ldots..$$

show that $\int_0^{2\pi} \phi_n(x)\phi_m(x)\,dx = 0$ if $n \neq m$.

Ans. :

The characteristic polynomial $p(r) = r^2 + n^2$ has roots $r_1 = in$, $r_2 = -in$ and therefore the general solution $\phi_n(x) = c_n \cos nx + d_n \sin nx$

From the given boundary conditions.

$$\phi_n(0) = c_n \text{ and } \phi_n(2\pi) = c_n \Rightarrow \phi_n(0) = \phi_n(2\pi)$$

and $$\phi_n'(0) = nd_n \text{ and } \phi_n'(2\pi) = nd_n \Rightarrow \phi_n'(0) = \phi_n'(2\pi)$$

Thus, $\phi_n(x) = c_n \cos nx + d_n \sin nx$ satisfies the given boundary conditions.

The solution ϕ_n satisfies $\phi_n''(x) + n^2\phi_n(x) = 0$ where as $\phi_m''(x) + m^2\phi_m(x) = 0$ holds.

Thus, $$(n^2 - m^2)\ \phi_n(x)\phi_m(x) = \phi_n''(x)\phi_m(x) - \phi_n(x)\phi_m''(x)$$

$$= \left[\phi_n'(x)\phi_m(x) - \phi_n(x)\phi_m'(x)\right]'$$

Integrating above equation from 0 to 2π

We get,

$$\left(n^2 - m^2\right)\int_0^{2\pi} \phi_n(x)\phi_m(x)\,dx = \int_0^{2\pi}\left[\phi_n'(x)\phi_m(x) - \phi_n(x)\phi_m'(x)\right]' dx$$

$$= \left[\phi_n'(x)\phi_m(x) - \phi_n(x)\phi_m'(x)\right]_0^{2\pi}$$

But $$\phi_n(0) = c_{n,}\ \phi_n(2\pi) = c_n;\ \ \phi_n'(0) = nd_{n,}\ \phi_n'(2\pi) = nd_n$$

Similarly, $$\phi_m(0) = c_{m,}\ \phi_m(2\pi) = c_m;\ \ \phi_m'(0) = md_m = \phi_m'(2\pi)$$

Thus, $$\left(n^2 - m^2\right)\int_0^{2\pi} \phi_n(x)\phi_m(x)\,dx = [nd_n\, c_m - c_n\, md_m] - [nd_n\, c_m - c_n\, md_m]$$

$$= 0$$

Since, $$n \neq m, \quad \int_0^{2\pi} \phi_n(x)\phi_m(x)\,dx = 0.$$

Q4. (a) : Show that $\phi_n(x)$ = Sin nx satisfies the boundary value problem $y'' + n^2 y = 0$, $y(0) = 0$, $y(\pi) = 0$, $n = 1, 2.....$

(b) : Using (a) show that

$$\int_0^{\pi} \sin nx\ \sin mx\ dx = 0 \text{ if } n \neq m$$

Ans. 4(a) :

<u>Method 1 :</u>

The characteristic polynomial $p(r) = r^2 + n^2$ has roots $r = \pm$ in and therefore the general solution

$$\phi_n(x) = c_n \cos nx + d_n \sin nx$$

$$y(0) = \phi_n(0) = 0 \quad \Rightarrow \quad \phi_n(0) = c_n = 0$$

$$y(\pi) = \phi_n(\pi) = 0 \quad \Rightarrow \quad \phi_n(\pi) = c_n(-1)^n = 0.$$

Thus, $\phi_n(x) = \sin nx$ is a solution for n = 1, 2, 3,....

Method 2 :

$$\phi_n(x) = \sin nx, \ \phi_n'(x) = n\cos nx$$

$$\phi_n''(x) = -n^2 \sin nx$$

Thus, $\phi_n''(x) + n^2\phi_n(x) = -n^2 \sin nx + n^2 \sin nx = 0$.

Since, $\phi_n(x) = \sin nx$ satisfies $\phi_n''(x) + n^2\phi_n(x) = 0$

and $\phi_n(0) = 0, \quad \phi_n(\pi) = 0$

$\phi_n(x) = \sin nx$ is a solution of $y'' + n^2 y = 0, \ y(0) = y(\pi) = 0.$

Ans. 4(b) :

Working on the similar line as in example 2 we get,

$$(n^2 - m^2)\int_0^\pi \phi_n(x)\phi_m(x)\,dx = (n^2 - m^2)\int_0^\pi \sin nx \sin mx \ dx$$

$$= \left[\sin nx(-m\cos mx) - \sin mx(-n\cos nx)\right]_0^\pi$$

$$= 0 \ \text{(as } \sin 0 = \sin n\pi = 0)$$

Since $n \neq m, \ \int_0^\pi \phi_n(x)\phi_m(x)\,dx = 0.$

Q5 : Suppose ϕ_1, ϕ_2 are linearly independent solutions of the constant coefficient equation $y'' + a_1 y' + a_2 y = 0$, Let W (ϕ_1, ϕ_2) be abbreviated to W. Show that W is constant if and only if a_1 = 0.

Ans. :

$$W = W(\phi_1, \phi_2) = \begin{vmatrix} \phi_1 & \phi_2 \\ \phi_1' & \phi_2' \end{vmatrix} = \left(\phi_1\phi_2' - \phi_2\phi_1'\right)$$

Then $W' = \left(\phi_1\phi_2' - \phi_2\phi_1'\right)'$

$$= \phi_1\phi_2'' + \phi_1'\phi_2' - \phi_2'\phi_1' - \phi_2\phi_1''$$

$$= \phi_1\phi_2'' - \phi_2\phi_1''$$

But ϕ_1 and ϕ_2 are solutions of $y'' + a_1 y' + a_2 y = 0.$

Therefore $\phi_1'' + a_1\phi_1' + a_2\phi_1 = 0 \Rightarrow \phi_1'' = -a_1\phi_1' - a_2\phi_1$

Similarly, $\phi_2'' = -a_1\phi_2' - a_2\phi_2$

Thus, $W' = \phi_1(-a_1\phi_2' - a_2\phi_2) - \phi_2(-a_1\phi_1' - a_2\phi_1)$

$$= -a_1(\phi_1\phi_2' - \phi_2\phi_1')$$

$$= -a_1 W$$

Thus, $W' = 0$ iff $a_1 = 0$

Therefore W = constant if and only if $a_1 = 0$

Q6 : Let ϕ_1, ϕ_2 be two different function on an interval I, which are not necessarily solutions of an equation $L(y) = 0$. Prove the following

(a) If ϕ_1, ϕ_2 are linearly dependent on I then $W(\phi_1, \phi_2)(x) = 0$ for all x in I

(b) If $W(\phi_1, \phi_2)(x_0) \neq 0$ for some x_0 in I, then ϕ_1, ϕ_2 are linearly independent on I.

(c) $W(\phi_1, \phi_2)(x) = 0$ for all x in I does not imply that ϕ_1, ϕ_2 are linearly dependent on I.

(d) $W(\phi_1, \phi_2)(x) = 0$ for all x in I and $\phi_2(x) \neq 0$ on I, imply that are ϕ_1, ϕ_2 linearly dependent.

Ans. 6(a) :

Suppose ϕ_1, ϕ_2 are linearly dependent on I then $c_1\phi_1(x) + c_2\phi_2(x) = 0$ for some non-zero c_1 and c_2.

i.e. $$\phi_1(x) = -\frac{c_2}{c_1}\phi_2(x).$$

$$W(\phi_1, \phi_2)(x) = \begin{vmatrix} \phi_1 & \phi_2 \\ \phi_1' & \phi_2' \end{vmatrix} = \phi_1(x)\phi_2'(x) - \phi_2(x)\phi_1'(x)$$

$$\therefore\ W(\phi_1, \phi_2)(x) = \left(-\frac{c_2}{c_1}\phi_2(x)\right)\phi_2'(x) - \phi_2(x)\left(-\frac{c_2}{c_1}\phi_2'(x)\right) = 0$$

$$\therefore\ W(\phi_1, \phi_2)(x) = 0 \text{ for all } x \in \text{I}.$$

Ans. 6(b) :

Suppose $c_1\phi_1(x) + c_2\phi_2(x) = 0$ then

$$c_1\phi_1'(x) + c_2\phi_2'(x) = 0$$

Thus we have a system of equation

$$\begin{bmatrix} \phi_1(x) & \phi_2(x) \\ \phi_1'(x) & \phi_2'(x) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Therefore at $x = x_0$

$$\begin{bmatrix} \phi_1(x_0) & \phi_2(x_0) \\ \phi_1'(x_0) & \phi_2'(x_0) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Thus, $c_1 = c_2 = 0$ if and only if the coefficient matrix is invertible i.e. the determinant of coefficient matrix is non-zero

But $$\begin{bmatrix} \phi_1(x_0) & \phi_2(x_0) \\ \phi_1'(x_0) & \phi_2'(x_0) \end{bmatrix} = W(\phi_1, \phi_2)(x_0) \neq 0$$

Since, $W(\phi_1, \phi_2)(x_0) \neq 0 \ \Rightarrow \ c_1 = c_2 = 0$

$\therefore \ c_1 \phi_1(x) + c_2 \phi_2(x) = 0 \ \Rightarrow \ c_1 = c_2 = 0.$

Hence ϕ_1 and ϕ_2 are linearly independent on I.

Ans. 6(c) :

Define $\phi_1(x) = x^2, \quad \phi_2(x) = x\,|\,x\,|$

for $x > 0, \ |\,x\,| = x \quad \therefore \ \phi_1(x) = x^2, \quad \phi_2(x) = x^2$

$$\therefore \ W(\phi_1, \phi_2) = \begin{vmatrix} x^2 & x^2 \\ 2x & 2x \end{vmatrix} = 0.$$

for $x = 0, \quad \phi_1(x) = \phi_2(x) = 0 \quad \therefore \ W(\phi_1, \phi_2) = 0$

for $x < 0, \ |\,x\,| = -x \ \Rightarrow \ \phi_1(x) = x^2$ and $\phi_2(x) = -x^2$

$$\therefore \ W(\phi_1, \phi_2) = \begin{vmatrix} x^2 & -x^2 \\ 2x & -2x \end{vmatrix} = 0.$$

Thus $W(\phi_1, \phi_2)(x) = 0$ for $-\infty < x < \infty$

Let $c_1 \phi_1(x) + c_2 \phi_2(x) = 0$

for $x > 0, \quad c_1 \phi_1(x) + c_2 \phi_2(x) = (c_1 + c_2) x^2 = 0.$

$$\Rightarrow \ c_1 + c_2 = 0 \qquad \text{.......... (i)}$$

for $x < 0, \quad c_1 \phi_1(x) + c_2 \phi_2(x) = c_1 x^2 - c_2 x^2 = 0.$

$$\Rightarrow \ c_1 - c_2 = 0 \qquad \text{.......... (ii)}$$

But $c_1 + c_2 = 0$ and $c_1 - c_2 = 0 \ \Rightarrow \ c_1 = c_2 = 0$

Thus, $c_1 \phi_1 + c_2 \phi_2 = 0 \ \Rightarrow \ c_1 = c_2 = 0$

Therefore ϕ_1, ϕ_2 are linearly independent.

Ans. 6(d) :

$$W(\phi_1, \phi_2)(x) = 0 \Rightarrow W(\phi_1, \phi_2)(x) = \begin{vmatrix} \phi_1(x) & \phi_2(x) \\ \phi_1'(x) & \phi_2'(x) \end{vmatrix} = 0$$

$$\Rightarrow \quad \phi_1(x)\phi_2'(x) - \phi_2(x)\phi_1'(x) = 0$$

$$\Rightarrow \quad \phi_2(x)\phi_1'(x) - \phi_1(x)\phi_2'(x) = 0$$

Since $\quad \phi_2(x) \neq 0 \quad \forall x \in I$

$$\therefore \quad \frac{\phi_2(x)\phi_1'(x) - \phi_1(x)\phi_2'(x)}{\phi_2^2(x)} = 0$$

$$\Rightarrow \left(\frac{\phi_1}{\phi_2}\right)' = 0 \quad \Rightarrow \quad \frac{\phi_1}{\phi_2} = \text{constant} \ = k \text{ (say)}$$

Therefore $\phi_1(x) = k\phi_2(x)$ and hence ϕ_1, ϕ_2 are linearly dependent.

Q7 : If ϕ_1, ϕ_2 are two solution of $L(y) = 0$ on an interval I containing a point x_0, then $W(\phi_1, \phi_2)(x) = e^{-a_1(x-x_0)} W(\phi_1, \phi_2)(x_0)$.

Ans. :

Since ϕ_1, ϕ_2 are solution of $L(y) = 0$,

$$\phi_1'' + a_1\phi_1' + a_2\phi_1 = 0$$

$$\phi_2'' + a_1\phi_2' + a_2\phi_2 = 0$$

On multiplying the first equation by $-\phi_2$, second equation by ϕ_1 and adding we obtain

$$\phi_1\phi_2'' - \phi_2\phi_1'' + a_1(\phi_1\phi_2' - \phi_2\phi_1') + a_2(\phi_1\phi_2 - \phi_2\phi_1) = 0$$

$$(\phi_1\phi_2'' - \phi_2\phi_1'') + a_1(\phi_1\phi_2' - \phi_2\phi_1') = 0 \qquad \text{.......... (i)}$$

Let $\quad W = W(\phi_1, \phi_2)(x) = \begin{vmatrix} \phi_1(x) & \phi_2(x) \\ \phi_1'(x) & \phi_2'(x) \end{vmatrix}$

Then $\quad W = \phi_1(x)\phi_2'(x) - \phi_2(x)\phi_1'(x)$

and $\quad W' = \phi_1(x)\phi_2''(x) + \phi_1'(x)\phi_2'(x) - \phi_2'(x)\phi_1'(x) - \phi_2(x)\phi_1'(x)$

$$= \phi_1(x)\phi_2''(x) - \phi_2(x)\phi_1''(x)$$

Thus, equation (i) becomes

$$W' + a_1 W = 0.$$

Thus W satisfies the first order differential equation

$$W' + a_1 W = 0$$

Hence, $W(x) = c \cdot e^{-a_1 x}$ where c is constant of integration. At $x = x_0$ we get

$$W(x_0) = c \cdot e^{-a_1, x_0} \quad \text{i.e.} \quad c = e^{a_1 x_0} W(x_0)$$

Thus, $$W(x) = e^{a_1 x_0} W(x_0) e^{-a_1 x}$$

$$= e^{-a_1 (x - x_0)} W(x_0)$$

Therefore $W(\phi_{1,} \phi_2)(x) = e^{-a_1(x-x_0)} W(\phi_1, \phi_2)(x_0)$

EXERCISES

1. The functions ϕ_1, ϕ_2 are defined on $-\infty < x < \infty$

Determine whether they are linearly dependent or independent there.

(i) $\phi_1(x) = \cos x, \quad \phi_2(x) = \sin x$

(ii) $\phi_1(x) = \sin x, \quad \phi_2(x) = e^{ix}$

(iii) $\phi_1(x) = \sin nx, \quad \phi_2(x) = \cos nx$

(iv) $\phi_1(x) = 1, \quad \phi_2(x) = \cos x$

(v) $\phi_1(x) = \sin^2 x, \quad \phi_2(x) = \cos^2 x$

(vi) $\phi_1(x) = 1, \quad \phi_2(x) = \sin^2 x, \quad \phi_3(x) = \cos^2 x$

(vii) $\phi_1(x) = \cos x, \quad \phi_2(x) = e^{ix} + e^{-ix}$

2. State whether the following statements are true or false.

(a) If ϕ_1, ϕ_2 are linearly independent functions on an interval I, they are linearly independent on any interval J contained inside I.

(b) If ϕ_1, ϕ_2 are linearly dependent on an internal I, they are linearly dependent on any internal J contained inside I.

(c) If ϕ_1, ϕ_2 are linearly independent solutions of $L(y) = 0$ on an internal I, they are linearly independent an any internal J contained inside I.

(d) If ϕ_1, ϕ_2 are linearly dependent solutions of $L(y) = 0$ on an interval I, they are linearly dependent on any internal J contained inside I.

Ans. : 1.

(i) independent (ii) independent (iii) independent

(iv) independent (v) independent (vi) dependent

(vii) dependent.

Ans. : 2.

(a) false (b) true (c) true (d) true

Unit 3 : The Homogeneous Equation of Order n

Everything we have done for the second order equation can be carried over to the case of the equation of order n. Here, we are concerned with the equation

$$L(y) = y^{(n)} + a_1 y^{(n-1)} + a_2 y^{(n-2)} + \cdots + a_n y = 0,$$

where, $a_1, a_1, a_3, ..., a_n$ are constants.

Theorem 1.3.1 :

Let $r_1,\ r_2,\ r_3, ..., r_s$ be the distinct roots of the characteristic polynomial $p(r) = r^n + a_1 r^{n-1} + a_2 r^{n-2} + \cdots + a_n$ and suppose r_i has multiplicity $m_i (m_1 + m_2 + m_3 + \cdots + m_s = n).$ Then n functions

$$e^{r_1 x},\ xe^{r_1 x}, ..., x^{m_1 - 1} e^{r_1 x};\quad e^{r_2 x}, xe^{r_2 x}, ..., \ x^{m_2 - 1} e^{r_2 x}; .....;$$

$$e^{r_s x},\ xe^{r_s, x},\ x^2 e^{r_s x}, ..., x^{m_s - 1} e^{r_s x}$$

are solutions of $L(y) = y^{(n)} + a_1 y^{(n-1)} + a_2 y^{(n-2)} + \cdots + a_n y = 0$

Proof : Suppose r_i is a root of $p(r)$ of multiplicity m_i. Then $p(r) = (r - r_i)^{m_i} q(r)$ where q is a polynomial of degree $n - m_i$. On differentiating $p(r)$, $(m_i - 1)$ times we get,

$$p'(r) = (r - r_i)^{m_i} q'(r) + m_i (r - r_i)^{m_i - 1} q(r)$$

$$= (r - r_i)^{m_i - 1} \left[q'(r)(r - r_i) + m_i\, q(r) \right]$$

$$p''(r) = (r - r_i)^{m_i} q''(r) + 2m_i (r - r_i)^{m_i - 1} q'(r) + m_i (m_i - 1)(r - r_i)^{m_i - 2} q(r)$$

$$= (r - r_i)^{m_i - 2} \left[(r - r_i)^2 q''(r) + 2m_i (r - r_i) q'(r) + m_i (m_i - 1) q(r) \right]$$

$$= (r - r_i)^{m_i - 2} \left[\text{Polynomial of order } n - m_i \right]$$

and so on

$$p^{(m_i - 1)}(r) = (r - r_i)^{m_i - (m_i - 1)} \left[\text{Polynomial of order } n - m_i \right]$$

$$= (r - r_i) \left[\text{Polynomial of order } n - m_i \right]$$

Therefore, $p(r_i) = p'(r_i) = p''(r_i) = \cdots = p^{(m_i - 1)}(r_i) = 0.$

Let e^{rx} be a solution of $L(y) = 0$. We see that $L(e^{rx}) = p(r) e^{rx}$ where $p(r) = r^n + a_1 r^{n-1} + a_2 r^{n-2} + \cdots + a_n.$

Therefore $L(e^{r_i x}) = p(r_i) e^{r_i x} = 0.$ Thus $e^{r_i x}$ is a solution of $L(y) = 0$.

If we differentiate $L(e^{rx}) = p(r) e^{rx}$ k times with respect to r we obtain

$$\frac{\partial^k}{\partial r^k} L(e^{rx}) = L\left(\frac{\partial^k}{\partial r^k} e^{rx} \right) = L\left(x^k e^{rx} \right)$$

$$=\left[p^{(k)}(r)+kp^{(k-1)}(r)\,x+\frac{k(k-1)}{2!}\,p^{(k-2)}(r)x^2+\cdots+p(r)x^k\right]e^{rx}$$

Thus for $r = r_i$ and $k = 0, 1, 2,\ldots..m_i - 1$ we get $L(x^k e^{r_i x})=0$. Therefore $x^k e^{r_i x},\ k=0,1,2,\ldots\ldots m_i-1$, are solutions of $L(y) = 0$. This is true for every characteristic root r_i with multiplicity m_i. i.e. $x^k e^{r_i x},\ k=0,1,2,\ldots.m_i-1,\ \ i=1,2,3,\ldots.s$ are solutions of $L(y) = 0$ and the result follows.

Theorem 1.3.2 :

The n solutions of $L(y) = 0$ given in theorem 1.3.1 are linearly independent on any interval I.

Proof : We prove that functions given in theorem 1.3.1 satisfy the condition given in defination 1.3.

Suppose we have n constants $c_{ij},\ i=1,2\ldots.s,\ j=0,\ldots..m_i-1$

Such that

$$\left(c_{10}e^{r_1x}+c_{11}\,x\,e^{r_1x}+c_{12}x^2e^{r_1x}+\ldots.+c_{1(m_1-1)}x^{m_1-1}e^{r_1x}\right)$$

$$+\left(c_{20}e^{r_2x}+c_{21}\,x\,e^{r_2x}+c_{22}x^2e^{r_2x}+\ldots.+c_{2(m_2-1)}x^{m_2-1}e^{r_2x}\right)$$

$$+\ldots.+\left(c_{s_0}e^{r_sx}+c_{s_1}\,x\,e^{r_sx}+c_{s_2}x^2e^{r_sx}+\ldots.+c_{s(m_s-1)}x^{m_s-1}e^{r_sx}\right)=0.$$

Define $\quad p_i(x)=c_{i0}+c_{i1}x+c_{i2}x^2+\ldots.+c_{i(m_i-1)}x^{m_i-1}$

Then $\quad p_1(x)e^{r_1x}+p_2(x)e^{r_2x}+p_3(x)e^{r_2x}+\ldots..+p_s(x)e^{r_sx}=0.$

Assume that not all constants c_{ij} are zero. Then there will be at least one of the polynomials p_i which is not identically zero on I. Suppose $p_s(x)$ is not identically zero on I. On dividing above equation by e^{r_1x} we get

$$p_1(x)+p_2(x)e^{(r_2-r_1)x}+p_3(x)e^{(r_3-r_1)x}+\ldots.+p_s(x)e^{(r_s-r_1)x}=0.$$

Upon differentiating above equation sufficiently many (at most m_i) times, we obtain the expression of the form

$$Q_2(x)e^{(r_2-r_1)x}+Q_3(x)e^{(r_3-r_1)x}+\ldots.+Q_s(x)e^{(r_s-r_1)x}=0$$

i.e. $\quad Q_2(x)+Q_3(x)e^{(r_3-r_2)x}+\ldots.+Q_s(x)e^{(r_s-r_2)x}=0$

where the Q_i's are polynomials, degree of Q_i is equal to degree of P_i and Q_s does not vanish identically.

Continuing this process we finally arrive at a situation where,

$$R_s(x)e^{r_s x}=0,$$

on I and R_s is a polynomial, degree of R_s is equal to degree of P_s, which does not vanish identically on I. But $R_s(x)e^{r_s x}=0$ implies $R_s(x)=0$ is a contradiction. Therefore our supposition that $P_s(x)$ is not identically zero is not true. Thus $P_s(x)=0$ for all x in I.

Thus all constants $C_{ij} = 0$ proving that the n solutions given in theorem 3.1 are linearly independent on an interval I.

*** Initial value problem for n^{th} order equations.**

The problem of finding a solution ϕ of

$$L(y) = y^{(n)} + a_1 y^{(n-1)} + a_2 y^{(n-2)} + ... + a_n y = 0 \text{ satisfying}$$

$\phi(x_0) = \alpha_1,\ \phi'(x_0) = \alpha_2, ..., \phi^{n-1}(x) = \alpha_n$ where $a_1, a_2, a_3, ..., a_n$ and $\alpha_1, \alpha_2, \alpha_3, ..., \alpha_n$ are constants is denoted by

$$L(y) = 0,\quad y(x_0) = \alpha_1,\ \ y'(x_0) = \alpha_2, ..., y^{(n-1)}(x_0) = \alpha_n$$

and is called an initial value problem.

Theorem 1.3.3 :

Let ϕ be any solution of

$$L(y) = y^{(n)} + a_1 y^{(n-1)} + a_2 y^{(n-2)} + ... + a_n y = 0$$

on an interval I containing a point x_0. Then for all x in I

$$\|\phi(x_0)\| e^{-k|x-x_0|} \le \|\phi(x)\| \le \|\phi(x_0)\| e^{k(x-x_0)}$$

where, $\quad k = 1 + |a_1| + |a_2| + |a_3| + ... + |a_n|$

and $\quad \|\phi(x)\| = \left[|\phi(x)|^2 + |\phi'(x)|^2 + ... + |\phi^{(n-1)}(x)|^2\right]^{\frac{1}{2}}$

Proof : This proof is similar to the proof of theorem 1.1.4.

Let $\quad u(x) = \|\phi(x)\|^2$

$$= |\phi|^2 + |\phi'|^2 + ... + |\phi^{(n-1)}|^2$$

$$= \phi\bar{\phi} + \phi'\bar{\phi}' + ... + \phi^{(n-1)}\bar{\phi}^{(n-1)}$$

Hence $\quad u'(x) = \phi'\bar{\phi} + \phi\bar{\phi}' + \phi''\bar{\phi}' + \bar{\phi}'\bar{\phi}'' + ... + \phi^{(n-1)}\bar{\phi}^{(n)} + \phi^{(n)}\bar{\phi}^{(n-1)}$

Therefore $|u'(x)| \le 2|\phi(x)|\ |\phi'(x)| + 2\ |\phi'|\ |\phi''| + ... + 2\ |\phi^{(n-1)}|\ |\phi^{(n)}|$

Since ϕ is solution of $L(y) = 0$, $L(\phi) = 0$ and therefore

$$\phi^{(n)} = -a_1\phi^{(n-1)} - a_2\phi^{(n-2)} - a_3\phi^{(n-3)} - .... - a_n\phi$$

On substituting the expression for $\phi^{(n)}$ we get

$$|u'(x)| \le 2|\phi||\phi'| + 2|\phi'||\phi''| + ... + 2|\phi^{(n-2)}||\phi^{(n-1)}|$$
$$+2|a_1||\phi^{(n-1)}|^2 + 2|a_2||\phi^{(n-1)}||\phi^{(n-2)}| + ... + 2|a_n||\phi^{(n-1)}||\phi|$$

$$\left[(|a| - |b|)^2 \ge 0 \ \Rightarrow\ |a|^2 + |b|^2 \ge 2|a||b|\right]$$

$$|u'(x)| \le (|\phi|^2 + |\phi'|^2) + (|\phi'|^2 + |\phi''|^2) + ... + (|\phi^{(n-2)}|^2 + |\phi^{(n-1)}|^2)$$
$$+|a_1|(|\phi^{(n-1)}|^2 + |\phi^{(n-1)}|^2) + ... + |a_n|(|\phi^{(n-1)}|^2 + |\phi|^2)$$

$$\le (1+|a_n|)|\phi|^2 + (1+1+|a_{n-1}|)|\phi'|^2 + (2+|a_{n-2}|)|\phi''|^2$$
$$+....+(2+|a_2|)|\phi^{(n-2)}|^2 + (1+2|a_1|+|a_2|+....+|a_n|)|\phi^{(n-1)}|^2$$

Since each coefficient on the right hand side is less than $2k$ we have

$$|u'(x)| \le 2k(|\phi|^2 + |\phi'|^2 + ...+|\phi^{(n-1)}|^2)$$
$$= 2k\,\|\phi(x)\|^2 = 2k\,u(x)$$

Therefore $|u'(x)| \le 2ku(x)$

Thus, we get $-2k\,u(x) \le u'(x) \le 2ku(x)$

$$u' - 2\,ku(x) \le 0 \text{ implies } (e^{-2kx}u(x))' \le 0$$

Integrating above inequality between the limits x_0 to x for $x > x_0$ yields

$$e^{-2kx}u(x) - e^{-2kx_0}u(x_0) \le 0$$

i.e. $u(x) \le e^{2k(x-x_0)}\,u(x_0)$

Thus, $\|\phi(x)\| \le e^{k(x-x_0)}\,\|\phi(x_0)\|$

Similarly for $x > x_0$ the inequality

$-2\,ku(x) \le u'(x)$ implies

$$\|\phi(x_0)\|\,e^{-k(x-x_0)} \le \|\phi(x)\|$$

Combining the above two inequalities we get the required result for $x > x_0$.

For $x < x_0$ interchange the role of x and x_0

We get $\|\phi(x_0)\| \le e^{k(x_0-x)}\,\|\phi(x)\| \;\Rightarrow\; \|\phi(x_0)\|\,e^{-k(x_0-x)} \le \|\phi(x)\|$

and $\|\phi(x)\|\,e^{-k(x_0-x)} \le \|\phi(x_0)\| \;\Rightarrow\; \|\phi(x)\| \le e^{k(x_0-x)}\,\|\phi(x_0)\|$

Thus, $\|\phi(x_0)\|\,e^{-k(x_0-x)} \le \|\phi(x)\| \le e^{k(x_0-x)}\,\|\phi(x_0)\|,\ (x < x_0)$

which is the required result for $x < x_0$

Theorem 1.3.4 (Uniqueness theorem)

Let $\alpha_1, \alpha_2, \alpha_3, ..., \alpha_n$ be any n constants and let x_0 be any real number. On any interval I containing x_0 there exists at most one solution ϕ of $L(y) = 0$ satisfying $\phi(x_0) = \alpha_1,\ \phi'(x_0) = \alpha_2$, ,......., $\phi^{(n-1)}(x_0) = \alpha_n$

Proof : Suppose ϕ and ψ were two solutions of $L(y) = 0$ on I satisfying the above conditions at $x = x_0$. i.e.

$$\phi(x_0) = \psi(x_0) = \alpha_1,\ \phi'(x_0) = \psi'(x_0) = \alpha_2,\ ....,\ \phi^{(n-1)}(x_0) = \psi^{(n-1)}(x_0) = \alpha_n$$

Define $\theta = \phi - \psi$. Since ϕ and ψ satisfy $L(\phi) = L(\psi)$ therefore $L(\theta) = 0$ and

$$\theta(x_0) = \phi(x_0) - \psi(x_0) = 0,\ \theta'(x_0) = 0, ..., \theta^{(n-1)}(x_0) = 0.$$

Thus $\|\theta(x_0)\| = \left[|\theta(x_0)|^2 + |\theta'(x_0)|^2 + ...+ |\theta^{(n-1)}(x_0)|^2\right]^{\frac{1}{2}} = 0$

Applying theorem 1.3.3 we obtain $\|\theta(x)\|=0$ for all x in I. This implies $\theta(x)=0$ for all x in I.

i.e. $\phi(x)=\psi(x)$ for all x in I.

Theorem 1.3.5

If $\phi_1, \phi_2, \phi_3, ...\phi_n$, are n solutions of $L(y) = 0$ on an interval I, they are linearly independent if and only if $W(\phi_1, \phi_2, \phi_3, ...\phi_n)(x) \neq 0$ for all x in I. (definition 1.5)

Proof : The proof is entirely similar to the proof of theorem 1.2.2

Suppose $W(\phi_1, \phi_2, \phi_3, ...\phi_n)(x) \neq 0$ for all x in I. Let $c_1, c_2, c_3, ..., c_n$ be constants such that $c_1\phi_1(x)+c_2\phi_2(x)+....+c_n\phi_n(x)=0$ for all x in I.

By differentiating above equation $(n-1)$ times we get a system of equations as follows.

$$\begin{bmatrix} \phi_1(x) & \phi_2(x) & \phi_3(x) & \cdots & \phi_n(x) \\ \phi_1'(x) & \phi_2'(x) & \phi_3'(x) & \cdots & \phi_n'(x) \\ \phi_1''(x) & \phi_2''(x) & \phi_3''(x) & \cdots & \phi_n''(x) \\ \vdots & \vdots & \vdots & & \vdots \\ \phi_1^{(n-1)}(x) & \phi_2^{(n-1)}(x) & \phi_3^{(n-1)}(x) & & \phi_n^{(n-1)}(x) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

The coefficient matrix is invertible because the determinant of coefficient matrix is (definition 1.5) $W(\phi_1, \phi_2, \phi_3, ...\phi_n)(x) \neq 0.$ On premultiplying the inverse of the coefficient matrix we get, $c_1 = c_2 = c_3..... = c_n = 0.$ This proves that $\phi_1, \phi_2, \phi_3, ...\phi_n$ are linearly independent.

Conversely, assume that $\phi_1, \phi_2, ...\phi_n$ are linearly independent on I. Suppose there is a point x_0 in I such that $W(\phi_1, \phi_2, \phi_3, ...\phi_n)(x_0) = 0.$ Then the system of equations

$$\begin{bmatrix} \phi_1(x_0) & \phi_2(x_0) & \phi_3(x_0) & \cdots & \phi_n(x_0) \\ \phi_1'(x_0) & \phi_2'(x_0) & \phi_3'(x_0) & \cdots & \phi_n'(x_0) \\ \phi_1''(x_0) & \phi_2''(x_0) & \phi_3''(x_0) & \cdots & \phi_n''(x_0) \\ \vdots & \vdots & \vdots & & \vdots \\ \phi_1^{(n-1)}(x_0) & \phi_2^{(n-1)}(x_0) & \phi_3^{(n-1)}(x_0) & & \phi_n^{(n-1)}(x_0) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

has a solution $c_1, c_2, c_3, ..., c_n$ where at least one of these numbers is not zero. Let $c_1, c_2, ..., c_n$ be such a solution and consider a function

$$\psi(x) = c_1\phi_1(x) + c_2\phi_2(x) + ... + c_n\phi_n(x).$$

Now $L(\psi)=0$ and $\psi'(x_0)=\psi''(x_0)=....=\psi^{(n-1)}(x_0)=0.$

Therefore $\|\psi(x_0)\|=0$. But then by theorem 1.3.3 , $\|\psi(x)\|=0$, for all x in I. Therefore by defination of $\|\psi(x)\|$, $\psi(x)=0$ for all x in I. But then $\phi_1, \phi_2, \phi_3, ...\phi_n$ are linearly dependent. Thus the supposition $W(\phi_1, \phi_2, \phi_3, ...\phi_n)(x_0)=0$ must be false. Therefore $W(\phi_1, \phi_2, \phi_3, ...\phi_n)(x)\neq 0$ for all x in I.

Theorem 1.3.6 (Existence Theorem)

Let $\alpha_1, \alpha_2, \alpha_3, ..., \alpha_n$ be any n constants and let x_0 be any real number. There exists a solution ϕ of $L(y) = 0$ on $-\infty < x < \infty$ satisfying

$$\phi(x_0)=\alpha_1, \phi'(x_0)=\alpha_2, \phi''(x_0)=\alpha_3, ..., \phi^{(n-1)}(x_0)=\alpha_n$$

Proof : Let $\phi_1, \phi_2, \phi_3, ...\phi_n$ be any set of n linearly independent solutions of $L(y) = 0$ on $-\infty < x < \infty$. We will show that there exist unique constants $c_1, c_2, c_3, ..., c_n$ such that

$$\phi = c_1\phi_1 + c_2\phi_2 + c_3\phi_3 + ... + c_n\phi_n$$

is a solution of $L(y) = 0$ satisfying the given initial conditions $\phi^{(i)}(x_0)=\alpha_i,\ i=0,1,2,....,n-1.$ These constants $c_1, c_2, c_3, ..., c_n$ would have to sartisfy

$$\begin{bmatrix} \phi_1(x_0) & \phi_2(x_0) & \phi_3(x_0) & \cdots & \phi_n(x_0) \\ \phi_1'(x_0) & \phi_2'(x_0) & \phi_3'(x_0) & \cdots & \phi_n'(x_0) \\ \phi_1''(x_0) & \phi_2''(x_0) & \phi_3''(x_0) & \cdots & \phi_n''(x_0) \\ \vdots & \vdots & \vdots & & \vdots \\ \phi_1^{(n-1)}(x_0) & \phi_2^{(n-1)}(x_0) & \phi_3^{(n-1)}(x_0) & & \phi_n^{(n-1)}(x_0) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \vdots \\ \alpha_n \end{bmatrix}$$

Since $\phi_1, \phi_2, \phi_3, ...\phi_n$ are linearly independent, by theorem 1.3.5, the determinant of the coefficients i.e. $W(\phi_1, \phi_2, \phi_3, ...\phi_n)(x_0)\neq 0.$ Thus the coefficient matrix is invertible. Therefore there is a unique set of constants $c_1, c_2, c_3, ..., c_n$ satisfying above system of equations. For this choice of $c_1, c_2, c_3, ..., c_n$ the function

$$\phi(x) = c_1\phi_1(x) + c_2\phi_2(x) + c_3\phi_3(x) + ... + c_n\phi_n(x)$$

will be the desired solution.

Theorem 1.3.7 :

Let $\phi_1, \phi_2, \phi_3, ...\phi_n$ be n linearly independent solutions of $L(y) = 0$ on an interval I. If $c_1, c_2, c_3, ..., c_n$ are any constants

$$\phi(x) = c_1\phi_1(x) + c_2\phi_2(x) + c_3\phi_3(x) + ... + c_n\phi_n(x)$$

is a solution and every solution may be represented in this form.

Proof : Since $\phi_i,\ i=1, 2, 3.....n$ is solution of $L(y) = 0$, $L(\phi_i)=0,\ i=1, 2, 3.....n$.

Therefore $L(\phi)=c_1 L(\phi_1)+c_2 L(\phi_2)+c_3 L(\phi_3)+....+c_n L(\phi_n)=0$ and

$\phi = c_1\phi_1 + c_2\phi_2 + c_3\phi_3 + ... + c_n\phi_n$ is a solution of $L(\phi) = 0$.

Let ϕ be any solution of $L(y) = 0$ and x_0 be in I.

Suppose $\phi(x_0) = \alpha_1, \phi'(x_0) = \alpha_2, \phi''(x_0) = \alpha_3, ..., \phi^{(n-1)}(x_0) = \alpha_n$.

By existence theorem 1.3.6 there exist unique constants $c_1, c_2, c_3, ..., c_n$ such that

$$\psi = c_1\phi_1 + c_2\phi_2 + c_3\phi_3 + ... + c_n\phi_n$$

is a solution of $L(y) = 0$ on I satisfying

$$\psi(x_0) = \alpha_1, \psi'(x_0) = \alpha_2, \psi''(x_0) = \alpha_3, ..., \psi^{(n-1)}(x_0) = \alpha_n$$

The uniqueness theorem 1.3.4 implies that $\phi = \psi$. Thus $\phi = c_1\phi_1 + c_2\phi_2 + c_3\phi_3 + ... + c_n\phi_n$.

Theorem 1.3.8

Let $\phi_1, \phi_2, \phi_3, ...\phi_n$ be n solutions of $L(y) = 0$ on an interval I constaining a point x_0. Then

$$W(\phi_1, \phi_2, \phi_3, ...\phi_n)(x) = e^{-a_1(x-x_0)} W(\phi_1, \phi_2, \phi_3, ...\phi_n)(x_0)$$

Proof :

$$W(\phi_1, \phi_2, \phi_3, ..., \phi_n)(x) = \begin{vmatrix} \phi_1(x) & \phi_2(x) & \phi_3(x) & \cdots & \phi_n(x) \\ \phi_1'(x) & \phi_2'(x) & \phi_3'(x) & \cdots & \phi_n'(x) \\ \phi_1''(x) & \phi_2''(x) & \phi_3''(x) & \cdots & \phi_n''(x) \\ \vdots & \vdots & \vdots & & \vdots \\ \phi_1^{(n-1)}(x) & \phi_2^{(n-1)}(x) & \phi_3^{(n-1)}(x) & & \phi_n^{(n-1)}(x) \end{vmatrix}$$

By differentiating above determinant row-wise we get,

$$W'(\phi_1, \phi_2, \phi_3, ..., \phi_n)(x) = \begin{vmatrix} \phi_1' & \phi_2' & \phi_3' & \cdots & \phi_n' \\ \phi_1' & \phi_2' & \phi_3' & \cdots & \phi_n' \\ \phi_1'' & \phi_2'' & \phi_3'' & \cdots & \phi_n'' \\ \vdots & \vdots & \vdots & & \vdots \\ \phi_1^{(n-1)} & \phi_2^{(n-1)} & \phi_3^{(n-1)} & & \phi_n^{(n-1)} \end{vmatrix}$$

$$+ \begin{vmatrix} \phi_1 & \phi_2 & \phi_3 & \cdots & \phi_n \\ \phi_1'' & \phi_2'' & \phi_3'' & \cdots & \phi_n'' \\ \phi_1'' & \phi_2'' & \phi_3'' & \cdots & \phi_n'' \\ \vdots & \vdots & \vdots & & \vdots \\ \phi_1^{(n-1)} & \phi_2^{(n-1)} & \phi_3^{(n-1)} & \cdots & \phi_n^{(n-1)} \end{vmatrix} + ... + \begin{vmatrix} \phi_1 & \phi_2 & \phi_3 & \cdots & \phi_n \\ \phi_1' & \phi_2' & \phi_3' & \cdots & \phi_n' \\ \phi_1'' & \phi_2'' & \phi_3'' & \cdots & \phi_n'' \\ \vdots & \vdots & \vdots & & \vdots \\ \phi_1^{(n)} & \phi_2^{(n)} & \phi_3^{(n)} & \cdots & \phi_n^{(n)} \end{vmatrix}$$

Since two rows are identical the value of first $(n - 1)$ determinants is zero. Therefore

$$W'(\phi_1, \phi_2, \phi_3, ..., \phi_n)(x) = \begin{vmatrix} \phi_1 & \phi_2 & \phi_3 & \cdots & \phi_n \\ \phi_1' & \phi_2' & \phi_3' & \cdots & \phi_n' \\ \phi_1'' & \phi_2'' & \phi_3'' & \cdots & \phi_n'' \\ \vdots & \vdots & \vdots & & \vdots \\ \phi_1^{(n)} & \phi_2^{(n)} & \phi_3^{(n)} & & \phi_n^{(n)} \end{vmatrix}$$

Since each $\phi_i,\ i=1,2,3,....,n$ is a solution of $L(y) = 0$ $\phi_i^{(n)} = -(a_1\phi_i^{(n-1)} + a_2\phi_i^{(n-2)} + a_3\phi_i^{(n-3)} .... + a_n\phi_i)$. Hence,

$$W'(\phi_1,\ \phi_2, \phi_3, ..., \phi_n)(x) = \begin{vmatrix} \phi_1(x) & \phi_2(x) & \phi_3(x) & \cdots & \phi_n(x) \\ \phi_1'(x) & \phi_2'(x) & \phi_3'(x) & \cdots & \phi_n'(x) \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ \phi_1^{(n-2)}(x) & \phi_2^{(n-2)}(x) & \phi_3^{(n-2)}(x) & & \phi_n^{(n-2)}(x) \\ -a_1\phi_1^{(n-1)}(x) & -a_1\phi_2^{(n-1)}(x) & -a_1\phi_3^{(n-1)}(x) & \cdots & -a_1\phi_n^{(n-1)}(x) \end{vmatrix}$$

Since,
$$\begin{vmatrix} \phi_1(x) & \phi_2(x) & \phi_3(x) & \cdots & \phi_n(x) \\ \phi_1'(x) & \phi_2'(x) & \phi_3'(x) & \cdots & \phi_n'(x) \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ \phi_1^{(n-2)}(x) & \phi_2^{(n-2)}(x) & \phi_3^{(n-2)}(x) & & \phi_n^{(n-2)}(x) \\ -a_k\phi_1^{(n-k)}(x) & -a_k\phi_2^{(n-k)}(x) & -a_k\phi_3^{(n-k)}(x) & \cdots & -a_k\phi_n^{(n-k)}(x) \end{vmatrix} = 0$$

for $k = 2, 3, 4,....n$, as two rows of the determinant are constant multiplies of each other are Thus,

$$W'(\phi_1,\ \phi_2, \phi_3, ..., \phi_n)(x) = -a_1 \begin{vmatrix} \phi_1(x) & \phi_2(x) & \phi_3(x) & \cdots & \phi_n(x) \\ \phi_1'(x) & \phi_2'(x) & \phi_3'(x) & \cdots & \phi_n'(x) \\ \phi_1''(x) & \phi_2''(x) & \phi_3''(x) & \cdots & \phi_n''(x) \\ \vdots & \vdots & \vdots & & \vdots \\ \phi_1^{(n-1)}(x) & \phi_2^{(n-1)}(x) & \phi_3^{(n-1)}(x) & \cdots & \phi_n^{(n-1)}(x) \end{vmatrix}$$

$$= -a_1\ W(\phi_1, \phi_2, \phi_3, ..., \phi_n)(x)$$

Thus $W' + a_1W = 0$. On integrating this equation between the limits x_0 to x we get ,

$$e^{a_1 x}\ W(x) = e^{a_1 x_0}\ W(x_0)$$

or
$$W(x) = e^{-a_1(x-x_0)}\ W(x_0)$$

Thus $W(\phi_1, \phi_2, \phi_3, ..., \phi_n)(x) = e^{-a_1(x-x_0)}\ W(\phi_1, \phi_2, \phi_3, ..., \phi_n)(x_0)$

Theorem 1.3.9

Let $\phi_1,\ \phi_2,\ \phi_3, ...\phi_n$ be n solutions of $L(y) = 0$ on an interval I containing x_0. Then they are linearly independent on I if and only if $W(\phi_1, \phi_2, \phi_3, ..., \phi_n)(x_0) \neq 0$

Proof : By theorem 1.3.5 the solutions $\phi_1, \phi_2, \phi_3, ...\phi_n$ of $L(y) = 0$ are linearly independent on an interval I if and only if $W(\phi_1, \phi_2, \phi_3, ..., \phi_n)(x) \neq 0$ for all x in I.

But $W(\phi_1, \phi_2, \phi_3, ..., \phi_n)(x) = e^{-a_1(x-x_0)} W(\phi_1, \phi_2, \phi_3, ..., \phi_n)(x_0)$ (by theorem 1.3.8.) Therefore $W(\phi_1, \phi_2, \phi_3, ..., \phi_n)(x) \neq 0$ if and only if $W(\phi_1, \phi_2, \phi_3, ..., \phi_n)(x_0) \neq 0$ and the result follows.

EXAMPLES

Q.1. Consider the equation

$$y^{(5)} - y^{(4)} - y' + y = 0$$

(a) Compute five linearly independent solutions.

(b) Compute the wronkian of the solutions found in (a).

(c) Find that solution ϕ satisfying

$$\phi(0) = 1, \ \phi'(0) = \phi''(0) = \phi'''(0) = \phi^{(4)}(0) = 0.$$

Ans (a) :

The characteristic equation

$$\begin{aligned} p(r) &= r^5 - r^4 - r + 1 \\ &= r^4(r-1) - (r-1) \\ &= (r^4 - 1)(r-1) \\ &= (r^2 - 1)(r^2 + 1)(r-1) \\ &= (r+1)(r-1)(r^2+1)(r-1) \end{aligned}$$

Thus the characteristic roots are 1, 1, –1, i, – i

Therefore $\phi_1(x) = e^x$, $\phi_2(x) = x e^x$, $\phi_3(x) = e^{-x}$, $\phi_4(x) = \sin x$ $\phi_5(x) = \cos x$ are solutions of the given differential equation.

Ans (b) :

$$W(\phi_1, \phi_2, \phi_3, \phi_4, \phi_5)(x) = e^{-a(x-x_0)} W(\phi_1, \phi_2, \phi_3, \phi_4, \phi_5)(x_0)$$

For the given equation $a_1 = -1$. Let $x_0 = 0$ then

$$W(\phi_1, \phi_2, \phi_3, \phi_4, \phi_5)(x) = e^x W(\phi_1, \phi_2, \phi_3, \phi_4, \phi_5)(0).$$

$$W(\phi_1, \phi_2, \phi_3, \phi_4, \phi_5)(x) = \begin{vmatrix} e^x & x e^x & e^{-x} & \sin x & \cos x \\ e^x & (1+x)e^x & -e^{-x} & \cos x & -\sin x \\ e^x & (2+x)e^x & e^{-x} & -\sin x & -\cos x \\ e^x & (3+x)e^x & -e^{-x} & -\cos x & \sin x \\ e^x & (4+x)e^x & e^{-x} & \sin x & \cos x \end{vmatrix}$$

$$W(\phi_1,\phi_2,\phi_3,\phi_4,\phi_5)(0)=\begin{vmatrix}1&0&1&0&1\\1&1&-1&1&0\\1&2&1&0&-1\\1&3&-1&-1&0\\1&4&1&0&1\end{vmatrix}$$

The row transformations

$R_2-R_1,\ R_3-R_1,\ R_4-R_1,\ R_5-R_1$ gives

$$W(\phi_1,\phi_2,\phi_3,\phi_4,\phi_5)(0)=\begin{vmatrix}1&0&1&0&1\\0&1&-2&1&-1\\0&2&0&0&-2\\0&3&-2&-1&-1\\0&4&0&0&0\end{vmatrix}$$

$$=\begin{vmatrix}1&-2&1&-1\\2&0&0&-2\\3&-2&-1&-1\\4&0&0&0\end{vmatrix}$$

$$=\begin{vmatrix}0&0&-2\\-2&-1&-1\\0&0&0\end{vmatrix}+2\begin{vmatrix}2&0&-2\\3&-1&-1\\4&0&0\end{vmatrix}+\begin{vmatrix}2&0&0\\3&-2&-1\\4&0&0\end{vmatrix}$$

$$+\begin{vmatrix}2&0&0\\3&-2&-1\\4&0&0\end{vmatrix}=-32$$

Thus, $W(\phi_1,\phi_2,\phi_3,\phi_4,\phi_5)=e^x\,W(\phi_1,\phi_2,\phi_3,\phi_4,\phi_5)(0)=-32e^x$

Ans (c) :

The general solution ϕ is $\phi(x)=c_1e^x+c_2xe^x+c_3e^{-x}+c_4\sin x+c_5\cos x$

The initial conditions $\phi(0)=1,\ \phi'(0)=\phi''(0)=\phi'''(0)=\phi^{(iv)}(0)=0$ gives the following system of equations.

$$\begin{bmatrix}1&0&+1&0&1\\1&1&-1&1&0\\1&2&1&0&-1\\1&3&-1&-1&0\\1&4&1&0&1\end{bmatrix}\begin{bmatrix}c_1\\c_2\\c_3\\c_4\\c_5\end{bmatrix}=\begin{bmatrix}1\\0\\0\\0\\0\end{bmatrix}$$

The row transformation $R_2-R_1,\ R_3-R_1,\ R_4-R_1,\ R_5-R_1$ gives

$$\begin{bmatrix} 1 & 0 & +1 & 0 & 1 \\ 0 & 1 & -2 & 1 & -1 \\ 0 & 2 & 0 & 0 & -2 \\ 0 & 3 & -2 & -1 & -1 \\ 0 & 4 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \\ c_5 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ -1 \\ -1 \\ -1 \end{bmatrix}$$

Solving the above system of equations simultaneously we get the values of c_1, c_2, c_3, c_4, c_5.

From last equation we get $4c_2 = -1$ gives $c_2 = -\frac{1}{4}$

From the third row of the above system we get,

$$2c_2 - 2c_5 = -1 \text{ gives } c_5 = \frac{1}{4}$$

From second and fourth row we get,

$$c_2 - 2c_3 + c_4 - c_5 = -1$$

$$3c_2 - 2c_3 - c_4 - c_5 = -1$$

Substitution of c_2 and c_5 in above equations give

$$-2c_3 + c_4 = -\frac{1}{2}$$

$$-2c_3 - c_4 = 0$$

Thus, $c_3 = \frac{1}{8}$, $c_4 = -\frac{1}{4}$

From first row we get, $c_1 = \frac{5}{8}$

Thus, $\phi(x) = c_1 e^x + c_2 x e^x + c_3 e^{-x} + c_4 \sin x + c_5 \cos x$

$$= \frac{5}{8}e^x - \frac{1}{4}xe^x + \frac{1}{8}e^{-x} - \frac{1}{4}\sin x + \frac{1}{4}\cos x$$

is the required solution.

Q.2. Find all solutions of the following equations.

(a) $y''' - 8y = 0$ (b) $y^{(4)} + 16y = 0$ (c) $y''' - 5y'' + 6y' = 0$

(d) $y^{(iv)} - 16y = 0$ (e) $y''' - 3y' - 2y = 0$ (f) $y^{(4)} + 5y'' + 4y = 0$

Ans. (a) :

The characteristic polynomial is $p(r) = r^3 - 8$ and its roots are $2, -1+\sqrt{3}i, -1-\sqrt{3}i$ Thus, three linearly independent solutions are given by e^{2x}, $e^{(-1+\sqrt{3}i)x}$, $e^{(-1-\sqrt{3}i)x}$ and any solution ϕ has the form $\phi(x) = c_1 e^{2x} + c_2 e^{(-1+\sqrt{3}i)x} + c_3 e^{(-1-\sqrt{3}i)x}$ where c_1, c_2, c_3 are any constants.

Ans. (b) : The characteristic polynomial is $p(r) = r^4 + 16$

$$p(r) = r^4 - (2\sqrt{i})^4 = (r^2 + (2\sqrt{i})^2)\ (r^2 - (2\sqrt{i})^2)$$
$$= (r^2 - i^2(2\sqrt{i})^2)\ (r^2 - (\sqrt{i}\,2)^2)$$
$$= (r + 2i\sqrt{i})\ (r - 2i\sqrt{i})\ (r + 2\sqrt{i})\ (r - 2\sqrt{i})$$

Thus, $p(r) = (r + 2i\sqrt{i})\ (r - 2i\sqrt{i})\ (r + 2\sqrt{i})\ (r - 2\sqrt{i})$

$$i = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2} = e^{i\frac{\pi}{2}}$$

$$\therefore \quad \sqrt{i} = \left(e^{i\frac{\pi}{2}}\right)^{\frac{1}{2}} = e^{i\frac{\pi}{4}} = \cos\frac{\pi}{4} + i\sin\frac{\pi}{4}$$

Therefore $\quad \sqrt{i} = \dfrac{1+i}{\sqrt{2}}, \quad i\sqrt{i} = \dfrac{i(1+i)}{\sqrt{2}} = \dfrac{-1+i}{\sqrt{2}}$

The roots of characteristic polynomial are $-\sqrt{2}(-1+i),\ \sqrt{2}(-1+i),\ \sqrt{2}(1+i),\ -\sqrt{2}(1+i)$

Thus four linearly independent solutions are

$$e^{(\sqrt{2}-i\sqrt{2})x},\quad e^{(-\sqrt{2}+\sqrt{2}\,i)x},\quad e^{(\sqrt{2}+i\sqrt{2})x},\quad e^{(-\sqrt{2}-i\sqrt{2})x}$$

and every solution ϕ has the form

$$\phi(x) = c_1 e^{(\sqrt{2}-i\sqrt{2})x} + c_2\, e^{(-\sqrt{2}+i\sqrt{2})x} + c_3\, e^{(\sqrt{2}+i\sqrt{2})x} + c_4 e^{(-\sqrt{2}-i\sqrt{2})x}$$

Ans. (c) : The characteristic polynomial is $p(r) = r^3 - 5r^2 + 6r$ and its roots are 0, 3, 2. Thus three linearly independent solutions are given by 1, e^{3x}, e^{2x} and any solution ϕ has the form $\phi(x) = c_1 e^{3x} + c_2 e^{2x} + c_3$

Ans. (d) : The characteristic polynomial is $p(r) = r^4 - 16 = (r^2+4)(r^2-4) = (r+2i)(r-2i)$ $(r+2)(r-2)$ and its roots are 2, – 2, 2i, –2i. Thus four linearly independent solutions are given by e^{2x}, e^{-2x}, $\cos 2x$, $\sin 2x$ and every solution ϕ has the form

$$\phi(x) = c_1 e^{2x} + c_2 e^{-2x} + c_3 \cos 2x + c_4 \sin 2x$$

Ans. (e) : The characteristic polynomial is

$$p(r) = r^3 - 3r - 2 = (r+1))(r^2 - r - 2)$$

and its roots are $-1,\ \dfrac{1+\sqrt{5}}{2},\ \dfrac{1-\sqrt{5}}{2}.$

Thus, three linearly independent solutions are e^{-x}, $e^{(1+\sqrt{5})\frac{x}{2}}$, $e^{\left(\frac{1-\sqrt{5}}{2}\right)x}$, and every solution ϕ has the form

$$\phi(x) = c_1 e^{-x} + c_2 e^{(\frac{1+\sqrt{5}}{2})x} + c_3 e^{(\frac{1-\sqrt{5}}{2})x}$$

Ans. (f) : The characteristic polynomial is

$$p(r) = r^4 + 5r^2 + 4 = (r^2 + 4)(r^2 + 1)$$

and its roots are $2i, -2i, i, -i$. Thus four linearly independent solutions are $\cos 2x, \sin 2x, \cos x, \sin x$ and every solution ϕ has the form

$$\phi(x) = c_1 \cos 2x, + c_2 \sin 2x + c_3 \cos x + c_4 \sin x.$$

Q.3. Consider the equation $y''' - 4y' = 0$

(a) Compute three linearly independent solutions.

(b) Compute the wronkian of the solutions found in (a).

(c) Find the solution ϕ satisfying

$$\phi(0) = 0, \ \phi'(0) = 1, \ \phi''(0) = 0$$

Ans. (a) : The characteristic polynomial $p(r) = r^3 - 4r$ and its roots are 0, 2, –2. Thus, three linearly independent solution are $e^{\circ} = 1, e^{2x}, e^{-2x}$ and every solution ϕ has the form

$$\phi(x) = c_1 + c_2 e^{2x} + c_3 e^{-2x}$$

Ans. (b) :

$$W(\phi_1, \phi_2, \phi_3)(x) = e^{0(x-0)} W(\phi_1, \phi_2, \phi_3)(0)$$

$$W(\phi_1, \phi_2, \phi_3)(x) = \begin{vmatrix} 1 & e^{2x} & e^{-2x} \\ 0 & 2e^{2x} & -2e^{-2x} \\ 0 & 4e^{2x} & 4e^{-2x} \end{vmatrix}$$

$$W(\phi_1, \phi_2, \phi_3)(0) = \begin{vmatrix} 1 & 1 & 1 \\ 0 & 2 & -2 \\ 0 & 4 & 4 \end{vmatrix}$$

Thus, $W(\phi_1, \phi_2, \phi_3)(x) = 16$.

Ans. (c) :

$$\phi(0) = 0, \phi'(0) = 1, \phi''(0) = 0,$$

$\phi(x) = c_1 + c_2 e^{2x} + c_3 e^{-2x}, \quad \phi(0) = c_1 + c_2 + c_3 = 0$ and so on

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & -2 \\ 0 & 4 & 4 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$R_3 - 2\,R_2$ gives

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & -2 \\ 0 & 0 & 8 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix}$$

Therefore $\quad c_3 = -\frac{1}{4},\ 2c_2 - 2c_3 = 1 \Rightarrow\ c_2 - c_3 = \frac{1}{2}\ \Rightarrow\ c_2 = \frac{1}{4}$

$$c_1 + c_2 + c_3 = 0\ \Rightarrow\ c_1 = 0$$

Thus, $\quad \phi(x) = c_1 + c_2 e^{2x} + c_3 e^{-2x} = \frac{1}{4}\left(e^{2x} - e^{-2x}\right)$ is the required solution.

EXERCISE

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1. Are the following statements true or false ?

(a) If $\phi_1, \phi_2, \phi_3, ..., \phi_n$ are linearly independent functions on an interval I, then any subset of them forms a linearly independent set of functions on I.

(b) If $\phi_1, \phi_2, \phi_3, ..., \phi_n$ are linearly dependent functions on an interval I, then any subset of them forms a linearly dependent set of functions on I.

2. Are the following sets of functions defined on $-\infty < x < \infty$ linearly independent or dependent ? why ?

(a) $\phi_1(x) = 1,\ \phi_2(x) = x,\ \phi_3(x) = x^2$

(b) $\phi_1(x) = e^{ix},\quad \phi_2(x) = \sin x,\quad \phi_3(x) = 2\cos x$

(c) $\phi_1(x) = x,\quad \phi_2(x) = e^{2x},\quad \phi_3(x) = |x|$

3. Find a basis of solutions of the differential equations.

(a) $y'' + 5y' + 4 = 0$ $\qquad$ (b) $y''' + 6y'' + 12y' + 8y = 0$

(c) $y^{(4)} - y = 0$

4. Find the general solution of each of the following equations.

(i) $6y'' - 11y' + 4y = 0$ $\qquad$ (Ans. $y(x) = c_1 e^{\frac{x}{2}} + c_2 e^{\frac{4x}{3}}$)

(ii) $y'' + 2y' - y = 0$ $\qquad$ (Ans. $y(x) = c_1 e^{(-1+\sqrt{2})x} + c_2 e^{(-1-\sqrt{2})x}$)

(iii) $y''' + y'' - 6y' = 0$ $\qquad$ (Ans. $y(x) = c_1 + c_2 e^{2x} + c_3 e^{-3x}$)

(iv) $y^{(4)} - 2y'' = 0$ $\qquad$ (Ans. $y(x) = c_1 + c_2 x + c_3 e^{\sqrt{2}\,x} + c_4 e^{-\sqrt{2}\,x}$)

(v) $y''' + 8y = 0$ $\qquad$ $\left(\text{Ans. } y(x) = c_1 e^{-2x} + c_2 e^{2x} + c_3\, x e^{2x}\right)$

5. For each of the following equations find a particular solution which satisfies the given initial conditions.

(i) $y'' = 0,\ y(1) = 2,\ \ y'(1) = -1$

(ii) $y'' + 4y' + 4y = 0,\ \ y(0) = 1,\ \ y'(0) = 1$

(iii) $y'' - 2y' + 5y = 0,\ \ y(0) = 2,\ \ y'(0) = 4$

(iv) $y'' - 4y' + 20y = 0,\ \ y\left(\pi/2\right) = 0,\ y'\left(\pi/2\right) = 1$

(v) $3y''' + 5y'' + y' - y = 0,\ \ y(0) = 0,\ y'(0) = 1,\ \ y''(0) = -1$

[Ans. : (i) $y(x) = 3 - x,$ **(ii)** $y(x) = (1+3x)e^{-2x}$ **(iii)** $y(x) = e^x(2\cos 2x + \sin 2x)$

(iv) $\frac{1}{4}e^{2x-\pi}\sin 4x$ **(v)** $y = \frac{9}{16}e^{\frac{x}{3}} + \left(\frac{x}{4} - \frac{9}{16}\right)e^{-x}$.**]**

Ans. 1 :

(a) True (b) false

Ans. 2 :

(a) independent (b) dependent (iii) independent

Ans. 3 :

(a) $\phi_1(x) = e^{-4x},\ \phi_2(x) = e^{-x}$

(b) $\phi_1(x) = e^{-2x},\ \phi_2(x) = xe^{-2x},\ \phi_3(x) = x^2e^{-2x}$

(c) $\phi_1(x) = e^x,\ \ \phi_2(x) = e^{-x},\ \phi_3(x) = \cos x,\ \phi_4(x) = \sin x$

Unit 4 : The Non-Homogeneous Equation of Order *n*

We now return to the n^{th} order non-homogeneous linear differential equation with constant coefficients. In the first part we will discuss the method of finding all solutions of the second order non-homogeneous equation.

$$L(y) = y'' + a_1y' + a_2y = b(x),$$

Where b is some continuous function on an interval I. The general solution of the above equation is

$$y(x) = y_c(x) + y_p(x),$$

where, $y_c(x)$, the complementary function is the general solution of the related homogenous equation and $y_p(x)$ is a particular solution of the equation.

Suppose we know that ψ_p is a particular solution of the equation $L(y) = b(x)$ and let ψ be any other solution. Then,

$$L(\psi - \psi_p) = L(\psi) - L(\psi_p) = b(x) - b(x) = 0$$

on I. This shows that $\psi - \psi_p$ is a solution of the homogenous equation $L(y) = 0$. Therefore if ϕ_1, ϕ_2 are linearly independent solutions of $L(y) = 0$, there are unique constants c_1, c_2 such that

$$\psi - \psi_p = c_1\phi_1 + c_2\phi_2$$

In other words every solution ψ of $L(y) = b(x)$ can be written in the form

$$\psi = \psi_p + c_1\phi_1 + c_2\phi_2$$

The problem of finding all solutions of $L(y) = b(x)$ reduces to finding a particular solution ψ_p.

Theorem 1.4.1

Let $b(x)$ be continuous on an interval I. Every solution ψ of $L(y) = b(x)$ on I can be written as $\psi = \psi_p + c_1\phi_1 + c_2\phi_2$.

Where ψ_p is a particular solution, ϕ_1, ϕ_2 are two linearly independent solutions of $L(y) = 0$ and c_1, c_2 are constants. A particular solution ψ_p is given by

$$\psi_p(x) = \int_{x_0}^{x} \frac{[\phi_1(t)\phi_2(x) - \phi_1(x)\phi_2(t)]b(t)}{W(\phi_1, \phi_2)(t)}\, dt.$$

Conversely every such ψ is a solutions of $L(y) = b(x)$

Proof :

Let ψ and ψ_p be two solutions of

$$L(y) = y'' + a_1 y' + a_2 y = b$$

Then $L(\psi - \psi_p) = L(\psi) - L(\psi_p) = 0$

This shows that $\psi - \psi_p$ is a solution of a homogeneous equation $L(y) = 0$. By theorem 1.1.1 there exist two linearly independent solutions ϕ_1, ϕ_2 and every solution of $L(y) = 0$ is of the form $c_1\phi_1 + c_2\phi_2$ where c_1 and c_2 are constants. Such a function $c_1\phi_1 + c_2\phi_2$ cannot be a solution of $L(y) = b(x)$ unless $b(x) = 0$ on I.

Suppose $\phi(x) = u_1(x)\phi_1(x) + u_2(x)\phi_2(x)$ is a solution of $L(y) = b(x)$ on I.

(This procedure is called as the variation of constants.)

Then

$$(u_1\phi_1 + u_2\phi_2)'' + a_1(u_1\phi_1 + u_2\phi_2)' + a_2(u_1\phi_1 + u_2\phi_2) = b(x)$$

i.e. $$a_2(u_1\phi_1 + u_2\phi_2) + a_1(u_1'\phi_1 + u_1\phi_1' + u_2'\phi_2 + u_2\phi_2')$$
$$+(u_1''\phi_1 + 2u_1'\phi_1' + u_1\phi_1'' + u_2''\phi_2 + 2u_1'\phi_1' + u_2\phi_2'') = b(x)$$

Therefore

$$u_1(\phi_1'' + a_1\phi_1' + a_2\phi_1) + u_2(\phi_2'' + a_1\phi_2' + a_2\phi_2)$$
$$+(\phi_1 u_1'' + \phi_1 u_2'') + 2(\phi_1' u_1' + \phi_2' u_2') + a_1(\phi_1 u_1' + \phi_2 u_2') = b(x)$$

i.e. $$(\phi_1 u_1'' + \phi_2 u_2'') + 2(\phi_1' u_1' + \phi_2' u_2') + a_1(\phi_1 u_1' + \phi_2 u_2') = b(x)$$

Observe that if

$$\phi_1 u_1' + \phi_2 u_2' = 0$$

then $(\phi_1 u_1' + \phi_2 u_2')' = (\phi_1' u_1' + \phi_2' u_2') + (\phi_1 u_1'' + \phi_2 u_2'')$

and $\phi_1' u_1' + \phi_2' u_2' = b(x)$

Thus if we can find two functions $u_1(x)$ and $u_2(x)$ such that

$$\phi_1 u_1' + \phi_2 u_2' = 0$$

$$\phi_1' u_1' + \phi_2' u_2' = b(x)$$

Then $u_1\phi_1 + u_2\phi_2$ will satisfy $L(y) = b(x)$.

On solving above two equations for u_1' and u_2'
we get,

$$u_1'(x) = \frac{-\phi_2\, b}{W(\phi_1,\, \phi_2)}, \quad u_2'(x) = \frac{\phi_1\, b}{W(\phi_1,\, \phi_2)},$$

Integration of above equation between the limits x_0 to x provides

$$u_1(x) = -\int_{x_0}^{x} \frac{\phi_2(t)\, b(t)}{W(\phi_1, \phi_2)(t)}\, dt + u_1(x_0)$$

and

$$u_2(x) = \int_{x_0}^{x} \frac{\phi_1(t)\, b(t)}{W(\phi_1, \phi_2)(t)}\, dt + u_2(x_0).$$

The solution $u_1\phi_1 + u_2\phi_2$ takes the form

$$\phi(x) = \phi_1(x)\left[-\int_{x_0}^{x} \frac{\phi_2(t)\, b(t)}{W(\phi_1, \phi_2)(t)}\, dt + u_1(x_0)\right]$$

$$+\phi_2(x)\left[+\int_{x_0}^{x} \frac{\phi_1(t)\, b(t)}{W(\phi_1, \phi_2)(t)}\, dt + u_2(x_0)\right]$$

The term $\phi_1(x)u_1(x_0) + \phi_2(x)u_2(x_0)$ is a complementary function or the solution of corresponding homogeneous equation $L(y) = 0$ and the particular solution takes the form

$$\psi_p(x) = -\phi_1(x)\int_{x_0}^{x} \frac{\phi_2(t)\, b(t)}{W(\phi_1, \phi_2)(t)}\, dt + \phi_2(x)\int_{x_0}^{x} \frac{\phi_1(t)\, b(t)}{W(\phi_1, \phi_2)(t)}\, dt$$

$$\psi_p(x) = \int_{x_0}^{x} \frac{[\phi_1(t)\phi_2(x) - \phi_2(t)\phi_1(x)]\, b(t)}{W(\phi_1, \phi_2)(t)}\, dt$$

The function $\psi_p(x)$ is a solution of $L(y) = b\,(x)$.

Theorem 1.4.1 provides a method to find a solution of second order non-homogeneous differential equation with constant coefficients. The same procedure can be generalized for the non-homogeneous equation of order n.

Theorem 1.4.2

Let b be continuous on an interval I and let $\phi_1, \phi_2, \phi_3, ..., \phi_n$ be n linearly independent solutions of $L(y) = y^{(n)} + a_1 y^{(n-1)} + a_2 y^{(n-2)} + ... + a_n y = 0$ on I. Every solution ψ of $L(y)$ = $b(x)$ can be written as

$$\psi = \psi_p + c_1\phi_1 + c_2\phi_2 + c_3\phi_3 + ... + c_n\phi_n$$

Where ψ_p is a particular solution of $L(y) = b(x)$ and $c_1, c_2, c_3, ..., c_n$ are constants. Every such ψ is a solution of $L(y) = b(x)$. A particular solution ψ_p is given by

$$\psi_p(x) = \sum_{k=1}^{n} \phi_k(x) \int_{x_0}^{x} \frac{W_k(t) b(t)}{W(\phi_1, \phi_2, \phi_3, ..., \phi_n)(t)} dt.$$

Proof : The proof is similar to the proof of theorem 1.4.1 Let b be continuous function on an interval I. Consider the differential equation

$$L(y) = y^{(n)} + a_1 y^{(n-1)} + a_2 y^{(n-2)} + ... + a_n y = b(x)$$

where, a_1, a_2, a_3,..., a_n are constants. If ψ_p is a particular solution of $L(y) = b(x)$ and ψ is any other solution of $L(y) = b(x)$, then

$$L(\psi - \psi_p) = L(\psi) - L(\psi_p) = b(x) - b(x) = 0$$

and $\psi - \psi_p$ is a solution of corresponding homogeneous equation $L(y) = 0$. (is called subtraction principle).

Thus any solution ψ of $L(y) = b(x)$ can be written in the form

$$\psi = \psi_p + c_1 \phi_1 + c_2 \phi_2 + c_3 \phi_3 + ... + c_n \phi_n$$

where, ψ_p is a particular solution of $L(y) = b(x)$, the functions $\phi_1, \phi_2, \phi_3, ..., \phi_n$ are n linearly independent solutions of $L(y) = 0$ (determined in theorem 1.3.1) and $c_1, c_2, c_3, ..., c_n$ are constants.

To find a particular solution ψ_p we use the variation of constants method. Suppose

$$\psi_p = u_1(x)\phi_1(x) + u_2(x)\phi_2(x) + u_3(x)\phi_3(x) + ... + u_n(x)\phi_n(x)$$

is a solution of $L(y) = b(x)$. Since ψ_p is a solution it satisfies the equation i.e $L(\psi_p) = b(x)$.

$$\psi_p = u_1\phi_1 + u_2\phi_2 + u_3\phi_3 + ... + u_n\phi_n$$

$$= \sum_{i=1}^{n} u_i \phi_i$$

Then, $$\psi_p' = u_1\phi_1' + u_1'\phi_1 + u_2\phi_2' + u_2'\phi_2 + ... + u_n\phi_n' + u_n'\phi_n$$

$$= (u_1 \phi_1' + u_2 \phi_2' + u_3 \phi_3' + ... + u_n \phi_n') + (u_1'\phi_1 + u_2'\phi_2 + ... + u_n'\phi_n)$$

$$= \sum_{i=1}^{n} u_i \phi_i' + \sum_{i=1}^{n} u_i' \phi_i$$

Let $\quad \sum u_i'\phi_i = 0$ then $\psi_p' = \sum u_i \phi_i'$

We have $\quad \psi_p'' = \sum u_i \phi_i'' + \sum u_i' \phi_i'$

Suppose $\sum u_i'\phi_i' = 0$ then $\psi_p'' = \sum u_i \phi_i''$

Continuing the same assumptions we get,

$$\begin{aligned}
&\sum u_i'\phi_i = 0 \quad ; \quad \psi_p' = \sum u_i \phi_i' \\
&\sum u_i'\phi_i' = 0 \quad ; \quad \psi_p'' = \sum u_i \phi_i'' \\
&\sum u_i'\phi_i'' = 0 \quad ; \quad \psi_p''' = \sum u_i \phi_i''' \\
&\quad \vdots \\
&\sum u_i'\phi_i^{(n-2)} = 0 \quad ; \quad \psi_p^{(n-1)} = \sum u_i \phi_i^{(n-1)} \\
&\psi_p^{(n)} = \sum u_i'\phi_i^{(n-1)} + \sum u_i \phi_i^{(n)}
\end{aligned}$$

If $\sum u_i'\phi_i^{(n-1)} = b(x)$ then $\psi_p^{(n)} = \sum u_i\phi_i^{(n)} + b(x)$ and $L(\psi_p)$ becomes

$$L(\psi_p) = \left[\sum u_i\phi_i^{(n)} + b(x)\right] + a_1 \sum u_i\phi_i^{(n-1)} + a_2 \sum u_i\phi_i^{(n-2)} + ... + a_n \sum u_i\phi_i.$$

$$= b(x) + \sum u_i \left[\phi_i^{(n)} + a_1\phi_i^{(n-1)} + ... + a_n\phi_i\right]$$

Thus, $L(\psi_p) = b(x)$ and therefore ψ_p is a solution of $L(y) = b(x)$. Therefore the problem is now reduced to solving the system given below for the functions $u_1, u_2, u_3, ..., u_n$.

$$\begin{aligned}
&\sum u_i'\phi_i = 0 \\
&\sum u_i'\phi_i' = 0 \\
&\sum u_i'\phi_i'' = 0 \\
&\quad \vdots \\
&\sum u_i'\phi_i^{(n-2)} = 0 \\
&\sum u_i'\phi_i^{(n-1)} = b(x)
\end{aligned}$$

Thus, we have system of equations

$$\begin{bmatrix} \phi_1 & \phi_2 & \phi_3 & \cdots & \phi_n \\ \phi_1' & \phi_2' & \phi_3' & \cdots & \phi_n' \\ \phi_1'' & \phi_2'' & \phi_3'' & \cdots & \phi_n'' \\ \vdots & & & & \\ \phi_1^{(n-1)} & \phi_2^{(n-1)} & \phi_3^{(n-1)} & \cdots & \phi_n^{(n-1)} \end{bmatrix} \begin{bmatrix} u_1' \\ u_2' \\ u_3' \\ \vdots \\ u_n' \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ b(x) \end{bmatrix}$$

By solving above system of equations by Cramer's rule we get,

$$u_k'(x) = \frac{\mathrm{W}_k(x)\, b(x)}{W(\phi_1, \phi_2, \phi_3, ..., \phi_n)(x)}, \quad k = 1, 2, 3, ..., n$$

Where $\mathrm{W_k}(x)$ is the determinant obtained from $\mathrm{W}[\phi_1, \phi_2, \phi_3, ..., \phi_n](x)$ by replacing the k^{th} column i.e. $[\phi_k \ \ \phi_k' \ \ \phi_k'' \ .....\phi_k^{(n-1)}]^T$ by $[0 \quad 0 \quad 0....0 \quad 1]^T$.

If x_0 is any point in I, we can integrate u_k' and the functions u_k can be written as

$$u_k(x) = \int_{x_0}^{x} \frac{W_k(t)\,b(t)\,dt}{W(\phi_1, \phi_2, \phi_3, ..., \phi_n)(t)} \quad k = 1, 2, 3, ..., n.$$

The particular solution ψ_p now takes the form

$$\psi_p(x) = \sum \phi_k(x) \int_{x_0}^{x} \frac{W_k(t)\,b(t)\,dt}{W(\phi_1, \phi_2, \phi_3, ..., \phi_n)(t)}$$

Now we are in a position to find out a solution of the non-homogenous equation of order n.

Observe that a particular solution ψ_p satisfies

$$\psi_p(x_0) = \psi_p{}'(x_0) = \psi_p{}''(x_0) = ... = \psi_p{}^{(n-1)}(x_0) = 0.$$

EXAMPLES

Q.1. Compute the solution ψ of $y''' + y'' + y' + y = 1$ which satisfies $\psi(0) = 0, \psi'(0) = 1,$ $\psi''(0) = 0$.

Ans. : The characteristic polynomial of the corresponding homogeneous equation is $p(r) = r^3 + r^2 + r + 1.$ The characteristic roots are i, $-i$, 1. The basic solutions of the corresponding homogeneous equation are

$$\phi_1(x) = \cos x \quad \phi_2(x) = \sin x \quad \phi_3(x) = e^{-x}$$

To obtain the particular solution of the form

$$\psi_p = u_1\phi_1 + u_2\phi_2 + u_3\phi_3$$

We have to find W (ϕ_1, ϕ_2, ϕ_3) (x) and W_k (t) for k = 1, 2, 3.

$$W(\phi_1, \phi_2, \phi_3)(x) = \begin{vmatrix} \cos x & \sin x & e^{-x} \\ -\sin x & \cos x & -e^{-x} \\ -\cos x & -\sin x & e^{-x} \end{vmatrix}$$

$$W(\phi_1, \phi_2, \phi_3)(x) = e^{-a_1(x-x_0)}\; W(\phi_1, \phi_2, \phi_3)(x_0)$$

$$= e^{-x}\; W(\phi_1, \phi_2, \phi_3)(0)$$

$$W(\phi_1, \phi_2, \phi_3)(0) = \begin{vmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ -1 & 0 & 1 \end{vmatrix}$$

$$= 1[1 - 0] + 1\,[0 + 1] = 2$$

Thus $W(\phi_1, \phi_2, \phi_3)(x) = 2e^{-x}$

$$W_1(x)=\begin{vmatrix} 0 & \sin x & e^{-x} \\ 0 & \cos x & -e^{-x} \\ -1 & -\sin x & e^{-x} \end{vmatrix} = (-1)^{3+1}[-e^{-x}\cos x - e^{-x}\sin x] = -e^{-x}(\cos x+\sin x)$$

$$W_2(x)=\begin{vmatrix} \cos x & 0 & e^{-x} \\ -\sin x & 0 & -e^{-x} \\ -\cos x & 1 & e^{-x} \end{vmatrix} = e^{-x}(\cos x-\sin x)$$

$$W_3(x)=\begin{vmatrix} \cos x & \sin x & 0 \\ -\sin x & \cos x & 0 \\ -\cos x & -\sin x & 1 \end{vmatrix} = 1$$

$$u_1(x)=\int^x \frac{W_1(t)\,b(t)}{W(\phi_1,\phi_2,\phi_3)(t)}\,dt = \int^x -\frac{e^{-t}(\cos t+\sin t)}{2e^{-t}}\,dt$$

$$=-\frac{1}{2}[+\sin x-\cos x]$$

Thus, $u_1(x)=\frac{1}{2}[\cos x-\sin x]$

$$u_2(x)=\int^x \frac{W_2(t)\,b(t)}{W(\phi_1,\phi_2,\phi_3)(t)}\,dt = \int^x \frac{e^{-t}(\cos t-\sin t)}{2e^{-t}}\,dt$$

$$=\frac{1}{2}[+\sin x+\cos x]$$

$$u_3(x)=\int^x \frac{W_3(t)\,b(t)\,dt}{W(\phi_1,\phi_2,\phi_3)(t)} = \int^x \frac{dt}{2e^{-t}} = \frac{1}{2}e^x$$

Therefore a particular solution is given by

$$\psi_p = u_1(x)\phi_1(x)+u_2(x)\phi_2(x)+u_3(x)\phi_3(x)$$

$$=-\frac{1}{2}(\cos x-\sin x)\cos x+\frac{1}{2}(\cos x+\sin x)\sin x+\frac{1}{2}e^x e^{-x} = 1$$

The most general solution is

$$\psi(x)=\psi_p+c_1\phi_1+c_2\phi_2+c_3\phi_3$$

$$=1+c_1\cos x+c_2\sin x+c_3 e^{-x}$$

$$\psi(0)=0 \;\Rightarrow\; 1+c_1+c_3=0$$

$$\psi'(0)=1 \;\Rightarrow\; \psi'(x)=-c_1\sin x+c_2\cos x-c_3 e^{-x}$$

Thus, $\psi'(0)=c_2-c_3=1$

$$\psi''(x)=-c_1\cos x-c_2\sin x+c_3 e^{-x}$$

$$\psi''(0) = -c_1 + c_3 = 0$$

Solving the system of equations

$$1 + c_1 + c_3 = 0$$
$$c_2 - c_3 = 1$$
$$-c_1 + c_3 = 0$$

We get, $c_1 = -\frac{1}{2}, \quad c_2 = \frac{1}{2}, \quad c_3 = -\frac{1}{2}$

Therefore the solution of our problem is given by

$$\psi(x) = 1 + \frac{1}{2}(\sin x - \cos x - e^{-x})$$

Q.2. Find all solutions ψ of the following equations

(a) $y''' - y' = x$

(b) $y'' - 3y' + 2y = \sin e^{-x}$

(c) $y'' + 4y' + 4y = 3xe^{-2x}$

Ans. (a) : The characteristic polynimial $p(r) = r^3 - r$ has roots 0, 1, –1 and the linearly independent solution of the related homogeneous equation are $\phi_1(x) = 1$, $\phi_2(x) = e^x$, $\phi_3(x) = e^{-x}$

Let $\psi_p = u_1(x)\phi_1(x) + u_2(x)\phi_2(x) + u_3(x)\phi_3(x)$

$$u_k(x) = \int^x \frac{W_k(t)\, b(t)\, ds}{W(\phi_1, \phi_2, \phi_3)\ (t)} \quad k = 1, 2, 3.$$

$$W(\phi_1, \phi_2, \phi_3)(x) = \begin{vmatrix} 1 & e^x & e^{-x} \\ 0 & e^x & -e^{-x} \\ 0 & e^x & e^{-x} \end{vmatrix} = 2$$

$$b(x)W_1(x) = \begin{vmatrix} 0 & e^x & e^{-x} \\ 0 & e^x & -e^{-x} \\ x & e^x & e^{-x} \end{vmatrix} = -2x$$

$$b(x)W_2(x) = \begin{vmatrix} 1 & 0 & e^{-x} \\ 0 & 0 & -e^x \\ 0 & x & e^{-x} \end{vmatrix} = xe^{-x}$$

$$b(x)W_3(x) = \begin{vmatrix} 1 & e^x & 0 \\ 0 & e^x & 0 \\ 0 & e^x & x \end{vmatrix} = xe^x$$

$$u_1(x)=\int^x \frac{-2t\,dt}{2}=-\frac{x^2}{2}$$

$$u_2(x)=\int^x \frac{te^{-t}}{2}dt=\frac{1}{2}\int^x t\,e^{-t}dt=-\frac{1}{2}(1+x)e^{-x}$$

$$u_3(x)=\int^x \frac{te^t dt}{2}=\frac{1}{2}\int^x t\,e^t dt=\frac{1}{2}(x-1)e^x$$

$\psi_p=u_1\phi_1+u_2\phi_2+u_3\phi_3=-\frac{x^2}{2}-\frac{1}{2}(1+x)+\frac{1}{2}(x-1)=-\frac{x^2}{2}-1$ is the required particular integral and the solution $\psi=c_1+c_2e^x+c_3e^{-x}+\psi_p$.

(b) : The characteristic polynomial $p(r)=r^2-3r+2$ has roots +2, +1 and therefore the two linearly independent solution of the corresponding homogeneous equation are $\phi_1(x)=e^x$ and $\phi_2(x)=e^{2x}$

Let $\psi_p=u_1(x)\phi_1(x)+u_2(x)\phi_2(x)$ be a particular integral of the given differential equation then by method of separation of parameters we get,

$$u_1(x)=\int\frac{W_1(x)b(x)dx}{W(\phi_1,\phi_2)(x)} \quad \text{and} \quad u_2(x)=\int\frac{W_2(x)b(x)dx}{W(\phi_1,\phi_2)(x)}$$

where, $$W_1(\phi_1,\phi_2)(x)=\begin{vmatrix} e^x & e^{2x} \\ e^x & 2e^{2x}\end{vmatrix}=e^{3x},$$

$$b(x)W_1(x)=\begin{vmatrix} 0 & e^{2x} \\ \sin e^{-x} & 2e^{2x}\end{vmatrix}=-e^{2x}\sin e^{-x},$$

$$b(x)W_2(x)=\begin{vmatrix} e^x & 0 \\ e^x & \sin e^{-x}\end{vmatrix}=e^x\sin e^{-x}.$$

Thus, $$u_1(x)=\int -e^{-x}\sin e^{-x}dx=-\cos e^{-x}$$

and $$u_2(x)=\int +e^{-2x}\sin e^{-x}dx=-\sin e^{-x}+e^{-x}\cos e^{-x}$$

[Integrate above equation with the substitution $t=e^{-x}$].

Then the general solution

$$\begin{aligned}\psi &= c_1\phi_1+c_2\phi_2+\psi_p \\ &= c_1e^x+c_2e^{2x}+(-\cos e^{-x})e^x+(-\sin e^{-x}+e^{-x}\cos e^{-x})e^{2x} \\ &= c_1e^x+c_2e^{2x}-e^{2x}\sin e^{-x}.\end{aligned}$$

(c) : The characteristic polynomial $p(r)=r^2+4r+4$ has roots –2, –2 and therefore the two linearly independent solution of the corresponding homogeneous equation are $\phi_1(x)=e^{-2x}$, $\phi_2(x)=x\,e^{-2x}$

Let $\psi_p = u_1(x)\phi_1(x) + u_2(x)\phi_2(x)$ be a particular integral of the given differential equation then by method of separation of parameters we get,

$$u_1(x) = \int \frac{W_1(x)b(x)\,dx}{W(\phi_1,\phi_2)(x)} \quad , \quad u_2(x) = \int \frac{W_2(x)b(x)\,dx}{W(\phi_1,\phi_2)(x)}$$

where, $\quad b(x) = 3xe^{-2x},\ W_1(\phi_1,\phi_2)(x) = \begin{vmatrix} e^{-2x} & xe^{-2x} \\ -2e^{-2x} & (1-2x)e^{-2x} \end{vmatrix} = e^{-4x},$

$$b(x)W_1(x) = \begin{vmatrix} 0 & xe^{-2x} \\ 3xe^{-2x} & (1-2x)e^{-2x} \end{vmatrix} = -3x^2 e^{-4x},$$

$$b(x)W_2(x) = \begin{vmatrix} -e^{2x} & 0 \\ -2e^{-2x} & 3xe^{-2x} \end{vmatrix} = 3xe^{-4x}.$$

Thus, $\quad u_1(x) = \int -\frac{3x^2e^{-4x}}{e^{-4x}}\,dx = -x^3 \quad$ and

$$u_2(x) = \int +\frac{3x\,e^{-4x}}{e^{-4x}}\,dx = \frac{3}{2}x^2$$

Therefore $\quad \psi_p = u_1\phi_1 + u_2\phi_2 = -x^3e^{-2x} + \frac{3}{2}x^3e^{-2x}$

$$= \frac{1}{2}x^3e^{-2x}$$

The general solution

$$\psi = c_1e^{-2x} + c_2xe^{-2x} + \frac{1}{2}x^3e^{-2x}$$

Q.3. Find the general solution of

$$y'' + y = \tan x \quad , \quad -\frac{\pi}{2} < x < \frac{\pi}{2}.$$

Ans : The characteristic polynimial $p(r) = r^2 + 1$ has roots $+i, -i$ and the two linearly independent solutions are

$$\phi_1(x) = \cos x \quad \text{and} \quad \phi_2(x) = \sin x$$

Let $\psi_p(x) = u_1(x)\phi_1(x) + u_2(x)\phi_2(x)$ be a particular integral of the given differential equation then by method of separation of parameters we get,

$$u_1(x) = \int \frac{W_1(x)b(x)\,dx}{W(\phi_1,\phi_2)(x)} \quad , \quad u_2(x) = \int \frac{W_2(x)b(x)\,dx}{W(\phi_1,\phi_2)(x)}$$

where, $\quad b(x) = \tan x$

$$W(\phi_1,\phi_2)(x)=\begin{vmatrix}\cos x & \sin x\\ -\sin x & \cos x\end{vmatrix}=1,\ b(x)W_1(x)=\begin{vmatrix}0 & \sin x\\ \tan x & \cos x\end{vmatrix}=-\frac{\sin^2 x}{\cos x},$$

$$b(x)W_2(x)=\begin{vmatrix}\cos x & 0\\ -\sin x & \tan x\end{vmatrix}=\sin x.$$

Therefore $$u_1(x)=\int -\frac{\sin^2 x}{\cos x}dx=-\log\ (\sec x+\tan x)+\sin x$$

$$u_2(x)=\int \sin x\ d\ x=-\cos x$$

and $$\psi_p=-\cos x\ \log(\sec x+\tan x),\qquad -\frac{\pi}{2}<x<\pi/2$$

The general solution

$$y(x)=c_1\cos x+c_2\sin x-\cos x\ \log(\sec x+\tan x),\ -\pi/2<x<\pi/2$$

Note. The formula for a particular a solution ψ_p of L(y) = b(x) makes sense for some discontinuous functions b (x). Then ψ_p will be a solution of L(y) = b(x) at the continuity points of b **.**

Q 4. Find a particular solution of the equation.

$$y''+y=b(x),$$

Where,
$$b(x)=-1\qquad(-\Pi\le x<0),$$
$$=1\qquad(0\le x\le\Pi),$$
$$=0\qquad(1x1>\Pi).$$

Ans : Let us find out the particular solution of $y''+y=\alpha$ where α is a constant.

The characteristic polynomial is $p(r)=r^2+1$ and has roots $+i$, $-i$. Therefore the basic solutions (linearly independent solutions) are

$$\phi_1(x)=\cos x,\qquad \phi_2(x)=\sin x$$

Let $\psi_p=u_1(x)\phi_1(x)+u_2\phi_2(x)$ be a particular solution of the equation $y''+y=\alpha$. By method of separation of parameters we get,

$$u_1(x)=\int\frac{W_1(x)b(x)dx}{W(\phi_1,\phi_2)(x)}\quad,\quad u_2(x)=\int\frac{W_2(x)b(x)dx}{W(\phi_1,\phi_2)(x)}$$

where, $$W(\phi_1,\phi_2)(x)=\begin{vmatrix}\cos x & \sin x\\ -\sin x & \cos x\end{vmatrix}=1$$

$$W_1(x)=\begin{vmatrix}0 & \sin x\\ 1 & \cos x\end{vmatrix}=-\sin x,\ W_2(x)=\begin{vmatrix}\cos x & 0\\ -\sin x & 1\end{vmatrix}=\cos x$$

Then, $$u_1(x)=\int\frac{W_1(x)b(x)dx}{W(\phi_1,\phi_2)(x)}=\int\frac{-\alpha\sin x}{1}dx=\alpha\cos x$$

$$u_2(x) = \int \frac{W_2(x)\, b(x)}{W(\phi_1, \phi_2)(x)}\, dx = \int \frac{\alpha \cos x}{1}\, dx = \alpha \sin x$$

The particular solution

$$\psi_p = u_1 \phi_1 + u_2 \phi_2 = \alpha \cos^2 x + \alpha \sin^2 x = \alpha$$

Thus the general solution of $y'' + y = \alpha$ is

$$\psi = c_1 \cos x + c_2 \sin x + \alpha$$

The general solution on the real line becomes

$$\begin{aligned} \psi(x) &= c_1 \cos x + c_2 \sin x && ; \quad -\infty < x < -\pi \\ &= c_3 \cos x + c_4 \sin x - 1 && ; \quad -\pi \le x < 0 \\ &= c_5 \cos x + c_6 \sin x + 1 && ; \quad 0 \le x \le \pi \\ &= c_7 \cos x + c_8 \sin x && ; \quad \pi < x < \infty \end{aligned}$$

The continuity of ψ at $x = -\pi$, 0, π gives $-c_1 = -c_3 - 1$, $c_3 - 1 = c_5 + 1$, $-c_5 + 1 = -c_7$

Since we have three equations in 4 unknown, the particular solution ψ_p will not be unique

e.g. choose $c_3 = c_1^* + 1$ and $c_2 = c_4 = c_6 = c_8 = c$

Then $c_1^* \cos x + c \sin x$ is a complementary function or the solution of corresponding homogenous equation $y'' + y = 0$ and particular equation will be determined as follows.

If $c_3 = c_1^* + 1$ then $c_1 = c_3 + 1 = c_1^* + 2$

$$c_3 = c_1^* + 1$$

$$c_5 = c_3 - 2 = c_1^* + 1 - 2 = c_1^* - 1$$

$$c_7 = c_5 - 1 = c_1^* - 1 - 1 = c_1^* - 2$$

Thus, the particular solution becomes

$$\begin{aligned} \psi(x) &= 2\cos x && -\infty < x < \pi \\ &= \cos x - 1 && -\pi \le x < 0 \\ &= -\cos x + 1 && 0 \le x \le \pi \\ &= -2\cos x && \pi < x < \infty \end{aligned}$$

If we choose $c_3 = c_1^* + 2$ then $c_1 = c_3 + 1 = c_1^* + 3$,

$$c_3 = c_1^* + 2,$$

$$c_5 = c_3 - 2 = c_1^*,$$

$$c_7 = c_5 - 1 = c_1^* - 1,$$

and the particular solution becomes

$$\begin{aligned} \psi(x) &= 3\cos x && ; \quad -\infty < x < -\pi \\ &= 2\cos x - 1 && ; \quad -\pi \le x < 0 \\ &= 1 && ; \quad 0 \le x \le \pi \\ &= -\cos x && ; \quad \pi < x < \infty. \end{aligned}$$

Thus, we can generate infinitely many particular solutions that are piecewise continuous.

Method of undetermined coefficients :

The method described sofar is called the method of variation of parameters. Although this method yields a solution of the non-homogeneous equation it sometimes require more labor than necessary. We now explain a method which is often faster than a method of variation of parameters. This method is useful to solve the non-homogeneous equation $L(y) = b(x)$, when $b(x)$ is a solution of some homogeneous equation with constant coefficients. The procedure we are about to describe is called the method of undetermined coefficients.

For the given different equation $L(y) = b(x)$, suppose $b(x)$ is a solution of some homogeneous equation $M(y) = 0$ with constant coefficients. Then $M(b(x)) = 0$. If ψ is a solution of $L(y) = b(x)$ and $M(b) = 0$ then

$$M\left[L(\psi)\right] = M(b) = 0.$$

Therefore ψ is a solution of the homogeneous equation $M(L(y)) = 0$ with constant coefficients. If the order of differential operator $L$is n and that of M is m then $M(L(y)) = 0$ is a homogeneous differential equation of order $m + n$ and therefore there are $m + n$ linearly independent solutions of $M(L(x)) = 0$. Since $b(x)$ is a particular solution of $M(y) = 0$ every linear combination of these $n + m$ linearly independent solution will not be a solution of $L(y) = b(x)$. Thus, to fine the solution of $L(y) = b(x)$ we substitute the linear combination of solutions into $L(y) = b(x)$ and determine the set of coefficients other than the coefficients of the solutions corresponding to the homogeneous equation $L(y) = 0$.

We give an example to show the usefulness of this method. Suppose we consider

$$L(y) = y'' - 3y' + 2y = x^2$$

Since $(x^2)''' = 0,\ x^2$ is a solution of $M(y) = y''' = 0$.

Every solution ψ of $L(y) = x^2$ is a solution of

$$M(L(y)) = M(y''' - 3y' + 2y) = M(x^2) = 0.$$

But $$M(y''' - 3y' + 2y) = (y''' - 3y' + 2y)''' = 0$$

i.e. $$y^{(v)} - 3y^{(iv)} + 2y''' = 0.$$

The characteristic polynomial of this equation is $p(r) = r^5 - 3y^4 + 2r^3$ (just the product of characteristics polynomials of L and M). The roots of $p(r)$ are 0, 0, 0, 1, 2 and hence ψ must have the form $\psi = c_0 + c_1 x + c_2 x^2 + c_3 e^x + c_4 e^{2x}$ observe that $c_3 e^x + c_4 e^{2x}$ is a solution of $L(y) = 0$.

Since we are interested only in particular solution ψ_p of $L(y) = x^2,$ we can assume

$$\psi_p = c_0 + c_1 x + c_2 x^2$$

Since ψ_p is a solution, it should satisfy the differential equation $L(y) = x^2$.

$$L(\psi_p) = \psi_p'' - 3\psi_p' + 2\psi_p = 2\left[c_0 + c_1 x + c_2 x^2\right] - 3\left[c_1 + 2c_2 x\right] + \left[2c_2\right]$$

$L(\psi_p) = x^2$ gives $2c_2 x^2 + (2c_1 - 6c_2)x + (2c_0 - 3c_1 + 2c_2) = x^2$

Since the above equation should hold for all values of x, on equating the coefficients of equal powers of x we get ,

$$2c_2 = 1, \ 2c_1 - 6c_2 = 0, \ 2c_0 - 3c_1 + 2c_2 = 0.$$

By solving these equations simultaneously we get,

$$c_2 = \frac{1}{2}, \quad c_1 = \frac{3}{2}, \quad c_0 = \frac{7}{4}.$$

Therefore, $\psi_p = \frac{7}{4} + \frac{3}{2}x + \frac{1}{2}x^2$

$$= \frac{1}{4}(2x^2 + 6x + 7)$$

is a particular solution of $L(y) = x^2$

and $\psi = \psi_p + c_3 e^x + c_4 e^{2x}$

$\psi = \frac{1}{4}(2x^2 + 6x + 7) + c_3 e^x + c_4 e^{2x}$ is a general solution of $y'' - 3y' + 2y = x^2$.

This method is also called as annihilator method since to solve $L(y) = b(x)$, we find the operator M which annihilates $b(x)$. i.e. $M(b(x)) = 0$.

Once M has been found the problem becomes algebraic in nature.

EXAMPLES

Exp. 1. Using the annihilator method find a particular solution of each of the following equations.

(a) $y'' + 4y = \cos x$ (b) $y'' - 4y = 3e^{2x} + 4e^{-x}$

(c) $y'' - y' - 2y = x^2 + \cos x$

Ans. (a) : $\cos x$ is a solution of $y'' + y = 0$ therefore

$$M(y) = y'' + y$$

$L(y) = y'' + 4y$ therefore $M(L(y)) = [L(y)]'' + [L(y)]$

$$M[L(y)] = (y'' + 4y)'' + y'' + 4y$$

$$= y^{(iv)} + 5y'' + 4y$$

Thus, $M[L(y)] = 0$ implies $y^{(iv)} + 5y'' + 4y = 0.$

The characteristic polynomial of the above equation is

$$p(r) = r^4 + 5r^2 + 4 = (r^2 + 4)(r^2 + 1)$$

The root of $p(r)$ are $i, -i, +2i, -2i$ and hence the solutions ψ have the form

$$\psi = c_1 \cos x + c_2 \sin x + c_3 \cos 2x + c_4 \sin 2x$$

observe that $c_3 \cos 2x + c_4 \sin 2x$ is a solution of $L(y) = y'' + 4y = 0$. Since we are interested only in particular solution ψ_p of $L(y) = \cos x$, we can assume

$$\psi_p(x) = c_1 \cos x + c_2 \sin x$$

Since ψ_p is a solution it should satisfy the differential equation $y'' + 4y = \cos x$.

$$\psi_p'' + 4\psi_p = -c_1 \cos x - c_2 \sin x + 4(c_1 \cos x + c_2 \sin x) = \cos x$$

On equation the coefficients of $\cos x$ and $\sin x$ we get

$$3c_1 = 1,\ 3c_2 = 0 \quad \text{i.e.} \quad c_1 = \frac{1}{3} \text{ and } c_2 = 0$$

Thus, particular solution $\psi_p = \frac{1}{3} \cos x$

Ans. (b) : $3e^{2x} + 4e^{-x}$ is a solution of $(D-2)(D-1) = 0$ i.e. $y'' - y' - 2y = 0$.

Thus, $M(y) = y'' - y' - 2y$. since $L(y) = y'' - 4y$, $M(L(y)) = y^{(iv)} - 4y''' - 6y'' + 4y' + 8y$.

The differential equation

$$y^{(iv)} - 4y''' - 6y'' + 4y' + 8y = 0$$

has a characteristic polynomial $p(r) = r^4 - 4r^3 - 6r^2 + 4r + 8$. The roots of characteristic polynomial are 2, 2, –1, –2.

The solution ψ has the form

$$\psi(x) = c_1 e^{2x} + c_2 x e^{2x} + c_3 e^{-x} + c_4 e^{-2x}$$

Observe that $c_1 e^{2x} + c_4 e^{-2x}$ is a solution of the homogeneous equation $L(y) = 0$.

Since we are only interested in particular solution assume the solution

$$\psi_p = c_1 e^{2x} + c_2 x e^{2x} + c_3 e^{-x}$$

Since ψ_p is a particular solution is should satisfy the equation $y'' - 4y = 3e^{2x} + 4e^{-x}$

$$\psi_p' = [2c_1 + (1+2x)c_2]e^{2x} - c_3 e^{-x}$$

$$\psi_p'' = [4c_1 + (4+4x)c_2]e^{2x} + c_3 e^{-x}$$

$$\psi_p'' - 4\psi_p = [4c_1 + (4+4x)c_2 - 4c_1 - 4xc_2]e^{2x} - 3c_3 e^{-x}$$

But ψ_p satisfies $y'' - 4y = 3e^{2x} + 4e^{-x}$

Therefore $[4c_2]e^{2x} - 3c_3 e^{-x} = 3e^{2x} + 4e^{-x}$

By comparing coefficient of e^{2x} and e^{-x} we get $4c_2 = 3$ and $-3c_3 = 4$

Thus, $\psi = c_1 e^{2x} + \frac{3}{4} x e^{2x} - \frac{4}{3} e^{-x} + c_4 e^{-2x}$ and particular integral

$$\psi_p = \frac{3}{4} x e^{2x} - \frac{4}{3} e^{-x}.$$

Ans. (c) : $x^2 + \cos x$ is a solution of $D^3(D^2+1)y = 0$ i.e. $y^{(v)} + y''' = 0.$ Thus $M(y) = y^{(v)} + y'''$

$$L(y) = y'' - y' - 2y$$

Therefore $M[L(y)] = [y'' - y' - 2y]^{(v)} + [y'' - y' - 2y]'''$

$$= y^{(7)} - y^{(6)} - y^{(5)} - y^{(4)} - 2y''' = 0.$$

The differential equation $M[L(y)] = 0$ has a characteristic polynomial $p(r) = r^7 - r^6 - r^5 - r^4 - 2r^3$.

The roots of characteristic polynomial are $0, 0, 0, i, -i, \frac{1+\sqrt{5}}{2}$ and $\frac{1-\sqrt{5}}{2}$.

The solution ψ must have the form

$$\psi(x) = c_0 + c_1 x + c_2 x^2 + c_3 \cos x + c_4 \sin x + c_5 e^{\left(\frac{1+\sqrt{5}}{2}\right)x} + c_6 e^{\left(\frac{1-\sqrt{5}}{2}\right)x}$$

The expression $c_5 e^{\left(\frac{1+\sqrt{5}}{2}\right)x} + c_6 e^{\left(\frac{1-\sqrt{5}}{2}\right)x}$ is a solution of the homogeneous equation $L(y) = y'' - y' - 2y = 0.$

Since we are interested in particular solution assume the solution

$$\psi_p = c_0 + c_1 x + c_2 x^2 + c_3 \cos x + c_4 \sin x$$

The problem is to determine the constants c_0, c_1, c_2, c_3, c_4 so that $L(\psi_p) = x^2 + \cos x.$

$$\psi_p' = c_1 + 2c_2 x - c_3 \sin x + c_4 \cos x$$

$$\psi_p'' = 2c_2 - c_3 \cos x - c_4 \sin x$$

$$L(\psi_p) = \psi_p'' - \psi_p' - 2\psi_p$$

$$= (2c_2 - c_1 - 2c_0) - (2c_1 + 2c_2)x - 2c_2 x^2 - (3c_3 + c_4)\cos x$$
$$+ (c_3 - 3c_4)\sin x$$

$$= x^2 + \cos x$$

Thus, $2c_2 - c_1 - 2c_0 = 0, \quad 2c_1 + 2c_2 = 0, \; -2c_2 = 1, \; 3c_3 + c_4 = -1$ and $c_3 - 3c_4 = 0$

Simulataneous evaluation of above equation gives

$$c_2 = -\frac{1}{2}, \quad c_1 = \frac{1}{2}, \quad c_0 = -\frac{3}{4}, \quad c_3 = 3c_4, \quad c_4 = -\frac{1}{10}, \quad c_3 = -\frac{3}{10}$$

Therefore $\psi_p = -\frac{3}{4} + \frac{1}{2}x - \frac{1}{2}x^2 - \frac{3}{10}\cos x - \frac{1}{10}\sin x$

EXERCISE

Exp. 1. Use the method of variation of parameters and find the general solution of each of the following equation.

(a) $y'' - y = \sin^2 x$ (b) $y'' + y = 4x \sin x$

(c) $y'' + 3y' + 2y = 12e^x$ (d) $y'' + 2y' + y = x^2 e^{-x}$

(e) $y'' + 4y = \cos x$ (f) $y'' + 9y = \sin 3x$

(g) $y'' - 7y' + 6y = \sin x$ (h) $4y'' - y = e^x$

(i) $6y'' + 5y' - 6y = x$

Exp. 2. Find the particular solution of each of the following equation using the method of undetermined coefficients.

(a) $y'' + 4y' + 4y = 4x^2 + 6e^x$ (b) $y'' - 3y' + 2y = 2xe^{3x} + 3\sin x$

(c) $y'' + 4y' + 4y = 3xe^{-2x}$ (d) $y'' - 3y' + 2y = 6e^{-x}$

Ans.(1): (a) $y = c_1 e^x + c_2 e^{-x} - \frac{1}{5}\sin^2 x - \frac{2}{5}$ (b) $c_1 \cos x + c_2 \sin x - x^2 \cos x + x \sin x$

(c) $c_1 e^{-2x} + c_2 c^{-x} + 2e^x$ (d) $c_1 e^{-x} + c_2 x e^{-x} + \frac{x^4 e^{-x}}{12}$

(e) $c_1 \cos 2x + c_2 \sin 2x + \frac{1}{3}\cos x$ (f) $c_1 \cos 3x + c_2 \sin 3x - \frac{1}{6} x \cos 3x$

(g) $c_1 e^{6x} + c_2 e^x + \frac{1}{74}(7\cos x + 5\sin x)$ (b) $c_1 e^{\frac{x}{2}} + c_2 e^{-\frac{x}{2}} + \frac{1}{3}e^x$

(i) $c_1 e^{\frac{2x}{3}} + c_2 e^{-\frac{3x}{2}} - \frac{1}{6}x - \frac{5}{36}.$

Ans.(2): (a) $x^2 - 2x + \frac{3}{2} + \frac{2}{3}e^x$ (b) $xe^{3x} - \frac{3}{2}e^{3x} + \frac{3}{10}\sin x + \frac{9}{10}\cos x$

(c) $\frac{1}{2}x^3 e^{-2x}$ (e) e^{-x}

Chapter 2

Linear Equations with Variable Coefficients

Contents :

Introduction

Solutions to linear equations with variables coefficients are necessary to analysis most of the situations in science and technology. In the last chapter we have studied linear equations with constant coefficients. In this chapter we are going to study linear equations with variable coefficients. There is no standard procedure to find all possible solutions of a given equation. However it is possible to construct series solution if the coefficient functions and the control function are analytic on some open set.

Unit 1 : Homogeneous equations with variable coefficients.

A linear differential equation of order n with variable coefficients is an equation of the form $a_0(x)y^{(n)}+a_1(x)y^{(n-1)}+a_2(x)y^{(n-2)}+.....+a_n(x)y=b(x)$, where $a_0,\ a_1,\ a_2,\ ....,a_n,\ b$ are complex valued functions defined on some interval I $\subset$ R. Points where $a_0(x) = 0$ for x in I are called singular points. In this chapter we assume that $a_0(x) \neq 0$ on I. Since a_0 is non-zero we can divide the equation by a_0 and rename functions $a_i(x) / a_0(x)$ by new $a_i(x)$ and $b(x) / a_0(x)$ as new $b(x)$. Then above equation can be written as

$$y^{(n)}+a_1\,y^{(n-1)}+a_2y^{(n-2)}+.....+a_ny=b(x)$$

In this chapter we denote the left hand side of the above equation by an operator L. Thus, $L(y)=y^{(n)}+a_1(x)y^{(n-1)}+a_2(x)y^{(n-2)}+....+a_n(x)y$ and the equation becomes $L(y) = b(x)$.

If $b(x) = 0$ for all x in I we call equation $L(y) = 0$ a, homogeneous equation whereas if

$b(x) \neq 0$ for some x in I, the equation is called a non-homogeneous equation.

A function ϕ is a solution of $L(y) = 0$ on I if ϕ is n times differentiable and satisfies $L(\phi) = 0$ for all x in I.

Most of the results we developed in chapter I are valid in more general case we are now considering. The major difficulty with linear equations with variable coefficients, from a practical point of view, is that there are very few types of equations whose solutions can be expressed in terms of elementary functions and for which standard method of obtaining them, if they do exist, are available. However, in case $a_1, a_2, a_3, \ldots\ldots a_n$ have convergent power series expansions the solutions will have this property also and the series solutions can be obtained by a simple formal procedure. But there is no analogue of the theorem 1.3.1 of chapter I, which gives a procedure to find all possible solutions of given equation.

A. Initial value problems for the homogeneous equation

Although in many cases it is not possible to find the solution, we can prove that if the functions $a_i(x),\ \ i = 1, 2, 3, \ldots\ldots n$ are continous functions then there is a solution to $L(y) = 0$. Moreover if we know the initial values of the solution and its derivatives then the solution is unique.

Theorem 2.1.1 :

Let $b_1, b_2, b_3, \ldots., b_n$ be non-negative constants such that for all x in I

$$|a_i(x)| \leq b_i \qquad i = 1, 2, 3, \ldots., n \text{ and define } k \text{ by}$$

$$k = 1 + b_1 + b_2 + b_3 + \ldots. + b_n.$$

If x_0 is a point in I and ϕ is a solution of $L(y) = 0$ on I then

$$\|\phi(x_0)\| e^{-k|x-x_0|} \leq \|\phi(x)\| \leq \|\phi(x_0)\| e^{k|x-x_0|} \text{ for all } x \text{ in I.}$$

Proof : The proof of this theorem is similar to the proof of theorem 1.3.3.

Let $\qquad u(x) = \|\phi(x)\|^2 = |\phi|^2 + |\phi'(x)|^2 + |\phi''|^2 + \ldots. + |\phi^{(n-1)}|^2$

$$= \phi\bar{\phi} + \phi'\bar{\phi}' + \phi''\bar{\phi}'' + \ldots. + \phi^{(n)}\phi^{(n-1)} + \phi^{(n-1)}\bar{\phi}^{(n-1)}$$

Hence $\quad u'(x) = \phi'\bar{\phi} + \phi\bar{\phi}' + \phi''\bar{\phi}' + \phi'\bar{\phi}'' + \ldots. + \phi^{(n)}\bar{\phi}^{(n-1)} + \phi^{(n-1)}\bar{\phi}^{(n)}$

Therefore $\quad |u'(x)| \leq 2|\phi||\phi'| + 2|\phi'||\phi''| + 2|\phi''||\phi'''| + \ldots. + 2|\phi^{(n-1)}||\phi^n|$

(for any complex variable z, $|z| = |\bar{z}|$)

Since ϕ is solution of $L(y) = 0$,

$$\phi^{(n)} = -a_1(x)\phi^{(n-1)} - a_2(x)\phi^{(n-2)} - a_3\phi^{(n-3)} - \ldots.. - a_n\phi$$

$$|\phi^{(n)}(x)| \leq |a_1(x)||\phi^{(n-1)}(x)| + |a_2(x)||\phi^{(n-2)}| + a_3(x)|\phi^{(n-3)}| + \ldots. + |a_n||\phi|$$

For all x in I, $|a_i(x)| \leq b_i,\ \ i = 1, 2, 3, \ldots.., n$ and therefore

$$|\phi^{(n)}(x)| \leq b_1|\phi^{(n-1)}| + b_2|\phi^{(n-2)}| + b_3|\phi^{(n-3)}| + \ldots. + b_n|\phi|$$

and $\quad |u'| \leq 2|\phi||\phi'| + 2|\phi'||\phi''| + 2|\phi''||\phi'''| + \ldots..$

$$+2|\phi^{(n-2)}||\phi^{(n-1)}|+2|\phi^{(n-1)}|\left[b_1|\phi^{(n-1)}|+b_2\,|\phi^{(n-2)}|+....+b_n|\phi|\right]$$

The rest of the proof is on the same lines as that of theorem 1.3.3.

$$\left[(|a|-|b|)^2 \geq 0 \;\Rightarrow\; 2|a||b| \leq |a|^2+|b|^2\right]$$

$$|u'(x)| \leq (|\phi|^2+|\phi'|^2)+(|\phi'|^2+|\phi''|^2)+.....+(|\phi^{(n-2)}|^2+|\phi^{(n-1)}|^2)$$

$$+b_1(|\phi^{(n-1)}|^2+|\phi^{(n-1)}|^2)+b_2(|\phi^{(n-1)}|^2+|\phi^{(n-2)}|^2)+....$$

$$+b_n(|\phi^{(n-1)}|^2+|\phi|^2)$$

$$\leq (1+b_n)|\phi|^2+(2+b_{n-1})|\phi'|^2+....+(2+b_2)|\phi^{(n-2)}|^2$$

$$+(1+2b_1+b_2+b_3+....+b_n)|\phi^{(n-1)}|^2$$

Since each coefficient on the right hand side is less than 2 k we have

$$|u'(x)| \leq 2k\; u(x)$$

Consider the right inequality which can be written as

$$u'(x)-2k\,u(x) \leq 0.$$

Integrate above inequality from x_0 to x with $x > x_0$.

$$e^{-2kx}u(x)-e^{-2kx_0}\,u(x_0) \leq 0$$

or $$u(x) \;\leq\; e^{2k(x-x_0)}u(x_0)$$

$$\|\phi(x)\|^2 \;\leq\; e^{2k(x-x_0)}\,\|\phi(x_0)\|^2$$

i.e. $$\|\phi(x)\| \;\leq\; e^{k(x-x_0)}\,\|\phi(x_0)\|$$

Similarly $-2k\,u(x) \leq u'(x)$ gives

$$|\phi(x_0)||\,e^{-k(x-x_0)} \leq \|\phi(x)\|\;,\;\;(x > x_0)$$

and therefore

$$\|\phi(x_0)\|\,e^{-k(x-x_0)} \;\leq\; \|\phi(x)\| \;\leq e^{k(x-x_0)}\|\phi(x_0)\|,\;(x > x_0)$$

If $x < x_0$ repeat the same procedure and integrate the inequality from x to x_0. We get

$$\|\phi(x_0)\|\,e^{-k(x-x_0)} \;\leq\; \|\phi(x)\| \;\leq \|\phi(x_0)\|\,e^{-k(x-x_0)}\;(x < x_0)$$

which is the required inequality for $x < x_0$.

Observe that if interval I is closed and bounded interval and if $a_i(x)$ are continuous functions on I then these functions are bounded. [continuous function on closed and bounded intervals is bounded and the function attains it bounds]. Since $a_j(x)$ are bounded functions on I, there always exist finite constants b_j such that $|a_j(x)| \leq b_j$ for $j = 1, 2, 3, ..., n$.

Theorem 2.2.1 : (Uniqueness theorem)

Let x_0 be in I and let $\alpha_1,\alpha_2,\alpha_3,....,\alpha_n$ be any n constants. There is at most one solution ϕ of $L(y) = 0$ on I satisfying

$$\phi(x_0)=\alpha_1,\;\phi'(x_0)=\alpha_2,\;\phi''(x_0)=\alpha_3,.....,\phi^{(n-1)}(x_0)=\alpha_n.$$

Proof : Let x be any point in I other than x_0. Let J be closed and bounded interval in I containing x_0 and x. On the interval J continuous functions $a_j(x)$ are bounded, that is,

$$|a_j(x)| \le b_j \qquad (j = 1, 2, 3, ..., n),$$

for some constants b_j (These constants b_j may depend on the choice of $J \subset$ I).

Suppose ϕ and ψ are two solutions of $L(y) = 0$ on J satisfying the given initial conditions i.e. $\phi(x_0) = \psi(x_0) = \alpha_1,\ \phi'(x_0) = \psi'(x_0) = \alpha_2, ..., \phi^{(n-1)}(x_0) = \psi^{(n-1)}(x_0) = \alpha_n$. Define $\theta = \phi - \psi$ in J. Since ϕ and ψ satisfy $L(y) = 0$. $\theta(x_0) = \phi(x_0) - \psi(x_0) = 0$, and $L(\phi) = L(\psi) = 0$ therefore by linearity $L(\theta) = 0$. $\theta(x_0) = \phi(x_0) - \psi(x_0) = 0$similarly $\theta'(x_0) = \theta''(x_0) = ...$ $= \theta^{(n-1)}(x_0) = 0$. but $\|\theta(x_0)\|^2 = |\theta(x_0)|^2 + |\theta'(x_0)|^2 + |\theta''(x_0)|^2 + \quad ....+ |\theta^{(n-1)}(x_0)|^2 = 0$. Applying theorem 2.1.1 we obtain $\|\theta(x)\| = 0$ for all x in J. In particular $\theta(x) = 0$ for all x in J $\subset$ I. But x is any point in I and therefore $\theta(x) = 0$ for every x in I. This proves that $\phi(x) = \theta(x)$ for every x in I.

Here we state existence theorem without proof.

Theorem 2.1.3 : (Existence Theorem)

Let $a_1(x),\ a_2(x),\ a_3(x), ..., a_n(x)$ be continuous functions on an interval I containing the point x_0. If $\alpha_1, \alpha_2, ..., \alpha_n$ are any n constants, there exists a solution ϕ of

$$L(y) = y^{(n)} + a_1(x)\, y^{(n-1)} + a_2(x)\, y^{(n-2)} + ... + a_n(x) y = 0 \text{ on I satisfying}$$

$$\phi(x_0) = \alpha_1,\ \phi'(x_0) = \alpha_2,\ \phi''(x_0) = \alpha_3, ..., \phi^{(n-1)}(x_0) = \alpha_n.$$

(B) Solutions of homogeneous equation

Superposition principle :

If $\phi_1, \phi_2, \phi_3, ..., \phi_m$ are any m solutions of the $L(y) = 0$ on an interval I and $c_1, c_2, c_3, ..., c_m$ are any m constants then $c_1\phi_1 + c_2\phi_2 + c_3\phi_3 + ... + c_m\phi_m$ is also a solution of $L(y) = 0$.

The trivial solution is a function which is identically zero on I.

Theorem 2.1.4

There exist n linearly independent solutions (definition 1.3) of $L(y) = 0$ on I.

Proof : Let x_0 be a point in I. According to theorem 2.1.3 and theorem 2.1.2, there is a unique solution of $L(y) = 0$ satisfying given initial conditions at x_0.

Let ϕ_1 be a solution of $L(y) = 0$ satisfying

$$\phi_1(x_0) = 0,\ \phi_1'(x_0) = 0,\ \phi_1''(x_0) = 0, ..., \phi_1^{(n-1)}(x_0) = 0$$

Let ϕ_2 be a solution of $L(y) = 0$ satisfying

$$\phi_2(x_0) = 0,\ \phi_2'(x_0) = 1,\ \phi_2''(x_0) = 0, ..., \phi_2^{(n-1)}(x_0) = 0$$

In general Let ϕ_i be a solution of $L(y) = 0$ with

$\phi_i^{(i-1)}(x_0)=1, \quad \text{and} \quad \phi_i(x_0)=\phi_i'(x_0)=......=\phi_i^{(i-2)}(x_0)=\phi_i^{(i)}(x_0)=......\phi_i^{(n-1)}(x_0)=0$

i.e. $\phi_i^{(i-1)}(x_0)=1, \quad \text{and} \quad \phi_i^{(k)}(x_0)=0, \quad k=1,2,3,.....,n-1,\ k\neq i-1.$

We will prove that these solutions $\phi_1,\phi_2,\phi_3,....,\phi_n$ are linearly independent on I. Suppose there are constants c_1, c_2, c_3,..... c_n such that

$$c_1\phi_1(x)+c_2\phi_2(x)+c_3\phi_3(x)+.....+c_n\phi_n(x)=0 \text{ for all } x \text{ in I.}$$

Differentiating above equation (n – 1) times we get,

$$c_1\phi_1'(x)+c_2\phi_2'(x)+c_3\phi_3'(x)+.....+c_n\phi_n'(x)=0$$

$$c_1\phi_1''(x)+c_2\phi_2''(x)+c_3\phi_3''(x)+.....+c_n\phi_n''(x)=0$$

$$\vdots \qquad \vdots \qquad \vdots \qquad \vdots$$

$$c_1\phi_1^{(n-1)}(x)+c_2\phi_2^{(n-1)}(x)+c_3\phi_3^{(n-1)}(x)+.....+c_n\phi_n^{(n-1)}(x)=0$$

Above equations hold for all values of x in I.

In particular these equations are true for $x = x_0$.

Since $\phi_i^{(j-1)}(x_0)=0$ for $j=1,2,3,.....,n,\ j\neq i$ and $\phi_i^{(i-1)}(x_0)=1$ for $j=i$ we get

$$c_1.1+c_2.0+c_3.0+....+c_n.0=0$$

$$c_1.0+c_2.1+c_3.0+....+c_n.0=0$$

In general

$$c_1.0+c_2.0+....+c_{i-1}.0+c_i.1+c_{i+1}.0+.....+c_n.0=0$$

Thus, $c_i =$ 0 for i = 0, 1, 2, 3,...., n and therefore solutions $\phi_1,\phi_2,\phi_3,....,\phi_n$ are linearly independent.

(C) Reduction of order of a homogeneous equation

Suppose we have found one solution of the equation $L(y)=y''+a_1(x)y'+a_2(x)y=0$ then by using the variation of constants method we can reduce $L(y) = 0$ into a linear differential equation of order one and obtain the second solution of the differential equation.

Theorem 2.1.5

If $\phi_1(x)$ is a solution of $L(y)=y''+a_1(x)y'+a_2(x)y=0$ on an interval I and $\phi_1(x)\neq 0$ on I, the second solution $\phi_2(x)$ is given by

$$\phi_2(x)=\phi_1(x)\int_{x_0}^{x}\frac{1}{[\phi_1(s)]^2}\ \exp\left[-\int_{x_0}^{s}a_1(t)\,dt\right]ds.$$

The functions ϕ_1 and ϕ_2 are linearly independent.

Proof : Since ϕ_1 is a solution of $L(y) = 0$, $L(\phi_1) = 0$.

Let $\phi_2(x)=u(x)\phi_1(x)$ be second solution of $L(y) = 0$.

$$L(\phi_2)=L(u\phi_1)=(u\phi_1)''+a_1(x)(u\phi_1)'+a_2(x)(u\phi_1)=0.$$

i.e. $u''(x)\phi_1(x)+2u'(x)\phi_1'(x)+u(x)\phi_1''(x)$

$$+a_1(x)\left[u'(x)\phi_1(x)+u(x)\phi_1'(x)\right]+a_2(x)(u(x)\phi_1(x))=0.$$

Since $\quad L(\phi_1)=\phi_1''+a_1(x)\phi_1'+a_2(x)\phi_1=0,$

$$u''(x)\phi_1(x)+2u'(x)\phi_1'(x)+a_1(x)[u'(x)\phi_1(x)]=0$$

Thus, $\quad \phi_1(x)u''(x)+\left[2\phi_1'(x)+a_1(x)\phi_1(x)\right]u'(x)=0.$

If $v = u'$ then above equation is linear equation of order one and can always be solved explicitly provided $\phi_1(x)\neq 0$ on I.

$$\phi_1(x)v'(x)+\left[2\phi_1'(x)+a_1(x)\phi_1(x)\right]v(x)=0$$

$$\frac{v'(x)}{v(x)}+\left[\frac{2\phi_1'(x)}{\phi_1(x)}+a_1(x)\right]=0$$

On integrating above equation between the limits x_0 to x, we get

$$\log v(x)-\log v(x_0)+\int_{x_0}^{x}\left[\frac{2\phi_1'(t)}{\phi_1(t)}+a_1(t)\right]\mathrm{dt}=0$$

$$\log v(x)-\log v(x_0)+2\left[\log\phi_1(x)-\log\phi_1(x_0)\right]+\int_{x_0}^{x}a_1(t)\,dt=0.$$

$$\log\frac{v(x)\phi_1^2(x)}{v(x_0)\phi_1^2(x_0)}=-\int_{x_0}^{x}a_1(t)\,dt$$

$$\frac{v(x)\phi_1^2(x)}{v(x_0)\phi_1^2(x_0)}=e^{-\int_{x_0}^{x}a_1(t)\,dt}=\exp\left[-\int_{x_0}^{x}a_1(t)\,dt\right]$$

i.e. $$v(x)=\frac{v(x_0)\phi_1^2(x_0)}{\phi_1^2(x)}\quad\exp\left[-\int_{x_0}^{x}a_1(t)\,dt\right]$$

But $v(x_0)\phi_1^2(x_0)$ are the values of $v(x)\phi_1^2(x)$ evaluated at point x_0 and therefore is constant

Let $\quad c=v(x_0)\phi_1^2(x_0),$ then

$$v(x)=\frac{c}{\phi_1^2(x)}\quad\exp\left[-\int_{x_0}^{x}a_1(t)\,dt\right]$$

But $v(x)=u'(x)$ and therefore

$$u(x)=\int^{x}\frac{c}{\phi_1^2(s)}\quad\exp\left[-\int_{x_0}^{s}a_1(t)\,dt\right]ds.$$

Since, $\quad \phi_2(x)=u(x)\phi_1(x)$ we get the required result.

We can generalize above theorem for linear differential equation

$$L(y) = y^{(n)} + a_1(x)\, y^{(n-1)} + a_2(x) y^{(n-2)} + ... + a_n(x) y = 0.$$

Theorem 2.1.6 :

Let ϕ_1 be a solution of $L(y) = 0$ on an interval I and suppose $\phi_1(x) \neq 0$ on I. Then we can reduce the order of equation $L(y) = 0$ by one. If $v_2, v_3, ..., v_n$ are linearly independent solutions of the reduced differential equation of order $n - 1$ and if $v_k = u_k'$, $k = 1, 2, 3, ..., n$, then $\phi_1, u_1\phi_1, u_2\phi_2, ..., u_n\phi_n$ are linearly independent solutions of $L(y) = 0$ on I.

Proof : Let ϕ_1 be solution of $L(y) = 0$ on I. we try to find a solution ϕ of $L(y) = 0$ of the form $\phi = u(x)\phi_1(x)$, where $u(x)$ is n times differentiable function defined on an interval I. If $\phi(x) = u(x)\phi_1(x)$ is a solution of $L(y) = 0$ then $L(u(x)\phi_1(x)) = 0$.

$$\begin{aligned} L(u\phi_1) &= (u\phi_1)^{(n)} + a_1(x)(u\phi_1)^{(n-1)} + a_2(x)(u\phi_1)^{(n-2)} + ... + a_n(u\phi_1) = 0 \\ &= u^{(n)}\phi_1 + n u^{(n-1)}\phi_1' + ... + u\phi_1^{(n)} \\ &\quad + a_1(x)\left[u^{(n-1)}\phi_1 + (n-1)u^{(n-2)}\phi_1' + ... + u\phi_1^{(n-1)}\right] \\ &\quad + a_2(x)\left[u^{(n-2)}\phi_1 + (n-2)u^{(n-3)}\phi_1' + ... + u\phi_1^{(n-2)}\right] \\ &\quad + ... \\ &\quad + a_{n-1}\left[u'\phi_1 + u_1\phi_1'\right] + a_n \cdot u\phi_1 = 0 \end{aligned}$$

The coefficient of u in the above expression is $\phi_1^{(n)} + a_1(x)\phi_1^{(n-1)} + a_2(x)\phi_2^{(n-1)} + ... + \phi_1 = L(\phi_1) = 0.$ Therefore the right hand side of above equation consists of $u', u'', u''', ..., u^{(n)}$ Therefore if we substitute $v = u'$ then the above equation becomes a linear homogeneous equation of order $n - 1$ in v.

$$\phi_1 u^{(n)} + \left[n\phi_1' + a_1(x)\phi_1\right]u^{(n-1)} + ... + \left[n\phi_1^{(n-1)} + (n-1)a_1(x)\phi_1^{(n-2)} + ... + a_{n-1}\phi_1\right]u' = 0$$

Since $v(x) = u'(x)$ we get

$$\phi_1 v^{(n-1)} + \left[n\phi_1' + a_1(x)\phi_1\right]v^{(n-2)} + ... + \left[n\phi_1^{(n-1)} + (n-1)a_1(x)\phi_1^{(n-2)} + ... + a_{n-1}\phi_1\right]v = 0$$

Since, $\phi_1(x) \neq 0$ on I we can divide above equation by ϕ_1. Thus, we can reduce the order of differential equation by one. Suppose $v_2, v_3, v_4, ..., v_n$ are linearly independent solutions of the differential equation in v of order $n - 1$. Then

$$\phi_1 v_k^{(n-1)} + \left[n\phi_1' + a_1\phi_1\right]v_k^{(n-2)} + ... + \left[n\phi_1^{(n-1)} + (n-1)a_1\phi_1^{(n-2)} + ... + a_{n-1}\phi_1\right]v_k = 0$$

But then $v_k(x) = u_k'(x)$ for $k = 2, 3, 4, ..., n$

and $u_k(x) = \int_{x_0}^{x} v_k(t)\, dt \qquad k = 2, 3, 4, ..., n$

But then by assumption $u_k(x)\phi_1(x)$ is a solution of $L(y) = 0$. Thus the functions $\phi_1, u_2\phi_1, u_3\phi_1, \ldots\ldots, u_n\phi_1$ are solutions of $L(y) = 0$. These functions are linearly independent.

Suppose we have constants $c_1, c_2, c_3, \ldots., c_n$ such that

$$c_1\phi_1 + c_2 u_2 \phi_1 + c_3 u_3 \phi_1 + \ldots.. + c_n u_n\phi_1 = 0$$

Since, $\phi_1(x) \neq 0$ on I this implies

$$c_1 + c_2u_2 + c_3u_3 + \ldots. + c_nu_n = 0$$

Differentation above equation and substituting

$$u_k' = v_k \text{ for } k = 2, 3, \ldots., n \text{ we get}$$

$$c_2v_2 + c_3v_3 + c_4v_4 + \ldots. + c_nv_n = 0.$$

Since $v_2, v_3, v_4, \ldots.., v_n$ are linearly independent by definition 1.3 we get $c_2 = c_3 = c_4 = \ldots.. = c_n = 0$ and therefore $c_1 = 0$. Thus $\phi_1, u_2\phi_1, u_3\phi_1, \ldots.., u_n\phi_1$ are linearly independent solutions.

EXAMPLES

Q. 1. Consider the equation

$$y'' + \frac{1}{x}y' - \frac{1}{x^2}y = 0 \quad \text{for} \quad x > 0.$$

(a) Show that there is a solution of the form x^r, where r is a constant.

(b) Find two linearly independent solutions for $x > 0$ and prove that they are linearly independent.

(c) Find the two solutions ϕ_1, ϕ_2 satisfying

$$\phi_1(1) = 1 \quad ; \quad \phi_2(1) = 0$$

$$\phi_1'(1) = 0 \quad ; \quad \phi_2'(1) = 1$$

Ans (a) :

Let $\phi(x) = x^r$ be a solution to $L(y) = y'' + \frac{1}{x}y' - \frac{1}{x^2}y = 0$ Since ϕ is a solution $L(y) = 0$.

Therefore $r(r-1)x^{r-2} + r\,x^{r-2} - x^{r-2} = 0$

that is $(r^2 - 1)x^{r-2} = 0 \quad \text{for} \quad x > 0$

Thus, $r^2 - 1 = 0 \quad \text{or} \quad r = +1, -1.$

Therefore $\phi_1(x) = x$ and $\phi_2(x) = \frac{1}{x}$ are two solutions of $L(y) = 0$.

Ans (b) :

Let $c_1\phi_1 + c_2\phi_2 = 0$ then $c_1x + \frac{c_2}{x} = 0$. Differentiate this equation twice with respect to

x we get $\frac{2c_2}{x^3} = 0$ implies $c_2 = 0$ and therefore $c_1 = 0$. Thus, ϕ_1 and ϕ_2 are linearly independent.

Ans (c) :

$$\phi_1(x) = c_1 x + c_2 \frac{1}{x}$$

$$\phi_1(1) = 1 \quad \text{and} \quad \phi_1'(1) = 0 \text{ gives}$$

$$c_1 x + c_2 \frac{1}{x} = 1 \quad \text{at} \quad x = 1 \quad \text{i.e.} \quad c_1 + c_2 = 1$$

$$c_1 x - \frac{c_2}{x^2} = 0 \quad \text{at} \quad x = 1 \quad \text{i.e.} \quad c_1 - c_2 = 0$$

Thus, $c_1 = c_2 = \frac{1}{2}$ and $\phi_1(x) = \frac{1}{2}\left(x + \frac{1}{x}\right)$

Let, $\phi_2(x) = d_1 x + d_2 \frac{1}{x}$

$$\phi_2(1) = 0 \quad \text{and} \quad \phi_2'(1) = 1 \quad \text{gives}$$

$$d_1 + d_2 = 0 \quad \text{and} \quad d_1 - d_2 = 1. \quad \text{Then} \quad d_1 = \frac{1}{2} \quad \text{and} \quad d_2 = -\frac{1}{2}$$

and $\phi_2(x) = \frac{1}{2}\left(x - \frac{1}{x}\right)$.

Q. 2. Find two linearly independent solutions of the equation

$$(3x-1)^2 y'' + (9x-3)y' - 9y = 0 \quad \text{for} \quad x > \frac{1}{2}$$

Ans. : Put $t = 3x - 1$ then $\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx} = \dot{y} \cdot 3$

where $\cdot$ represents derivative with respect to t.

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\frac{dy}{dx} = \frac{d}{dx}\dot{y} \cdot 3 = \frac{d}{dt}(3\dot{y})\frac{dt}{dx} = 3\ddot{y} \cdot 3 = 9\ddot{y}$$

Therefore $9t^2\,\ddot{y} + 9t\,\dot{y} - 9y = 0$

or $t^2\ddot{y} + t\dot{y} - y = 0$

Let $y = t^r$ be a solution then

$$r(r-1)t^r + rt^{+r} - t^r = 0 \quad \text{implies} \quad (r^2 - 1)t^r = 0.$$

But $t > 0$ therefore $r = +1, -1$

and $\phi_1(t) = t$ and $\phi_2(t) = \frac{1}{t}$ are solutions

But $t = 3x - 1$ and therefore

$\phi_1(x) = 3x - 1$ and $\phi_2(x) = \frac{1}{3x-1}$ are two solutions of given equation.

$$c_1\phi_1(x)+c_2\phi_2(x)=0 \text{ implies } c_1(3x-1)+\frac{c_2}{3x-1}=0$$

On differentiating this equation two times with respect to x we get $-\frac{18c_2}{(3x-1)^2}=0$ and therefore $c_2 = 0$. Since $3x - 1 \neq 0$, $c_1 = 0$.

Thus ϕ_1 and ϕ_2 are linearly independent.

Q. 3. A differential equation and a function ϕ_1 are given in each of the following. Verify that the function ϕ_1 satisfies the equation and find a second independent solution.

(a) $x^2y''-7xy'+15y=0, \quad \phi_1(x)=x^3,\ (x>0)$

(b) $xy''-(x+1)y'+y=0, \quad \phi_1(x)=e^x,\ (x>0)$

(c) $(1-x^2)y''-2xy'+2y=0, \quad \phi_1(x)=x,\ (0<x<1)$

Ans (a) : $\phi_1(x)=x^3, \quad \phi_1'(x)=3x^2, \quad \phi_1''(x)=6x$

$$L(\phi_1)=x^2\phi_1''-7x\phi_1'+15\phi_1=x^2(6x)-7x(3x^2)+15x^3=0.$$

Since $L(\phi_1)=0$, ϕ_1 is a solution of $L(y) = 0$

To determine the second solution, since $x > 0$, we can divide the given equation by x^2. Consider $y''-\frac{7}{x}y'+\frac{15}{x^2}y=0.$

Let $\phi(x)=u(x)\phi_1(x)=x^3u(x)$ be a solution. Then $(x^3u(x))''-\frac{7}{x}(x^3u)'+\frac{15}{x^2}(x^3u)=0$ gives $(u''x^3+6x^2u'+6xu)-\frac{7}{x}(x^3u'+3x^2u)+\frac{15}{x^2}(x^3u)=0$ or $u''x^3+6x^2u'-7x^2u'=0$

i.e. $u''x-u'=0 \quad \Rightarrow \quad \frac{u''}{u'}=\frac{1}{x}$ (Integrate with respect to x)

$\log u'=\log x+\log k \Rightarrow \quad u'=kx$

But then $u=k\frac{x^2}{2}$ Let $k=2$ Then $\phi(x)=u(x)\phi_1(x)=x^2(x^3)=x^5$ is the second solution independent of ϕ_1 as $c_1x^3+c_2x^5=0$ implies $c_1=c_2=0$.

Ans (b) : $\phi_1(x)=e^x=\phi_1'(x)=\phi_1''(x)$ Let $L(y)=y''-(1+\frac{1}{x})y'+\frac{1}{x}y=0$. (We can divide the given equation by x as $x > 0$.)

$$L(\phi_1)=e^x-(1+\frac{1}{x})e^x+\frac{1}{x}e^x=0 \qquad \therefore \phi_1 \text{ is a solution.}$$

To determine second solution, let $\phi(x)=u(x)\phi_1(x)$ be a solution then by theorem 2.1.5

$$u(x)=\int\frac{1}{\phi_1^2(x)}\exp\left[-\int^x a_1(t)\,dt\right]dx$$

$$= \int e^{-2x} \exp\left[+\int\left(1+\frac{1}{x}\right)dx\right]dx$$

$$= \int e^{-2x} \exp[x+\log x]\,dx$$

$$= \int e^{-2x}\, xe^{x}\, dx = \int xe^{-x}\, dx \; = -(1 + x)\, e^{-x}$$

Thus, $\phi_2(x) = -(1 + x)$ is a second solution of the equation $L(y) = 0$

$$c_1\phi_1(x) + c_2\phi_2(x) = c_1 e^{x} + c_2(-1)(1+x) = 0 \quad \Rightarrow \quad c_1 + c_2(-1)(1+x)e^{-x} = 0$$

But then $c_2 = c_1 = 0$ therefore ϕ_1 and ϕ_2 are linearly independent solutions.

Ans (c) : $\phi_1(x) = x,\;\; \phi_1'(x) = 1,\;\; \phi_1''(x) = 0.$ Let $L(y) = y'' - \frac{2x}{1-x^2}y' + \frac{2y}{1-x^2} = 0.$

$L(\phi_1) = 0 - \frac{2x}{1-x^2}1 + \frac{2x}{1-x^2} = 0.$ Therefore ϕ_1 is a solition $L(y) = 0$. To determine second solution, let $\phi_2(x) = u(x)\phi_1(x)$ be a solution of $L(y) = 0$.

By Theorem 2.1.5.

$$u(x) = \int \frac{1}{\phi_1^2} \exp\left[-\int^{x} a_1(t)\,dt\right]dx$$

$$= \int \frac{1}{x^2} \exp\left[-\int^{x} \frac{-2t\,dt}{1-t^2}\right]dx = \int \frac{1}{x^2}\,\frac{1}{1-x^2}\,dx$$

$$= \int \frac{dx}{x^2(1-x^2)} = \int \frac{dx}{x^2} + \frac{1}{2}\int \frac{dx}{1-x} + \frac{1}{2}\int \frac{dx}{1+x}$$

$$= -x^{-1} + \frac{1}{2}\log\left(\frac{1+x}{1-x}\right).$$

Then $\phi_2(x) = \phi_1(x)u(x) = -1 + \frac{x}{2}\log\left(\frac{1+x}{1-x}\right)$ is a second solution.

Q. 4. One solution of $x^3y''' - 3x^2y'' - 6xy' - 6y = 0$ for x > 0 is $\phi_1(x) = x$ find the remaining two independent solutions for $x > 0$.

Ans : Let $\phi = xu$ be a solution of $L(y) = 0$. Then $\phi' = xu' + u,\; \phi'' = xu'' + 2u',\; \phi''' = xu''' + 3u''$.

$L(\phi) = x^3(xu''' + 3u'') - 3x^2(xu'' + 2u') + 6x(xu' + u) - 6xu = 0$ implies $x^4u''' = 0.$ Since $x \neq 0,\; u''' = 0$ gives $u = c_1x + c_2x^2.$ Thus $u = x$ and $u = x^2$ are two linear independent solutions of $u''' = 0$. But $\phi = xu$ is a solution. Therefore $\phi_2(x) = x^2$ and $\phi_3(x) = x^3$ are remaining two linearly independent solutions.

Q. 5. Consider the equation $L(y) = y''' + a_1(x)y'' + a_2(x)y' + a_3(x)y = 0.$ **Suppose** ϕ_1 **and** ϕ_2 **are given linearly independent solutions of** $L(y) = 0$**. Let** $\phi = u\phi_1$ **and compute the solution of order two satisfied by** u' **in order that** $L(\phi) = 0$**. Show that** $\left(\frac{\phi_2}{\phi_1}\right)'$ **is a solution of this equation of order two.**

Ans (c) : Let $\phi = u\phi_1$ be a solution of $L(y) = 0$.

$$\phi' = u'\phi_1 + u\phi_1', \ \phi'' = u''\phi_1 + 2u'\phi_1' + u\phi_1'', \ \phi''' = u'''\phi_1 + 3u''\phi_1' + +3u'\phi_1'' + u\phi_1'''$$

$$L(y) = \left[u'''\phi_1 + 3u''\phi_1' + 3u'\phi_1'' + u\phi_1'''\right] + a_1(x)\left[u''\phi_1 + 2u'\phi_1' + u\phi_1''\right]$$

$$+a_2(x)\left[u'\phi_1 + u\phi_1'\right] + a_3(x)u\phi_1 = 0.$$

Since ϕ_1 is a solution $\phi_1''' + a_1\phi_1'' + a_2\phi_1' + a_3\phi_1 = 0.$

$$L(y) = u'''\phi_1 + 3u''\phi_1' + 3u'\phi_1'' + a_1(x)\left[u''\phi_1 + 2u'\phi_1'\right] + a_2(x)u'\phi_1 = 0.$$

$$= \phi_1 u''' + \left[3\phi_1' + a_1(x)\phi_1\right]u'' + \left[3\phi_1'' + 2a_1\phi_1' + a_2\phi_1\right]u' = 0$$

Thus, $L(y) = 0$ is an equation of order two in u'. $\left(\frac{\phi_2}{\phi_1}\right)'$ is a solution of $L(v) = \phi_1 v'' + \left[3\phi_1' + a_1\phi_1\right]v' + \left[3\phi_1'' + 2a_1\phi_1' + a_2\phi_1\right]v = 0$ if it satisfies the equation, $L(v) = 0$.

$$v = \left(\frac{\phi_2}{\phi_1}\right)' = \frac{\phi_2'}{\phi_1} - \frac{\phi_2\phi_1'}{\phi_1^2}, \ v' = \frac{\phi_2''}{\phi_1} - \frac{\phi_1'\phi_2'}{\phi_1^2} - \frac{\phi_1'\phi_2'}{\phi_1^2} - \frac{\phi_2\phi_1''}{\phi_1^2} + \frac{2\phi_2\phi_1'^2}{\phi_1^3}$$

$$v' = \frac{\phi_2''}{\phi_1} - \frac{2\phi_1'\phi_2'}{\phi_1^2} - \frac{\phi_2\phi_1''}{\phi_1^2} + \frac{2\phi_2\phi_1'^2}{\phi_1^3}$$

$$v'' = \frac{\phi_2'''}{\phi_1} - \frac{3\phi_2''\phi_1'}{\phi_1^2} - \frac{3\phi_1''\phi_2'}{\phi_1^2} + \frac{6\phi_1'^2\phi_2'}{\phi_1^3} + \frac{6\phi_2\phi_1'\phi_1''}{\phi_1^3} - \frac{6\phi_2\phi_1'^3}{\phi_1^4} - \frac{\phi_2\phi_1'''}{\phi_1^2}$$

$$L(v) = \phi_1 v'' + \left[3\phi_1' + a_1\phi_1\right]v' + \left[3\phi_1'' + 2a_1\phi_1' + a_2\phi_1\right]v$$

$$= \left(\phi_2''' + a_1\phi_2'' + a_2\phi_2'\right) - \frac{\phi_2}{\phi_1}\left(\phi_1''' + a_1\phi_1'' + a_2\phi_1'\right)$$

Since ϕ_2 and ϕ_1 are solutions, $\phi_2''' + a_1\phi_2'' + a_2\phi_2' + a_3\phi_2 = 0$ and $\phi_1''' + a_1\phi_1'' + a_2\phi_1' + a_3\phi_1 = 0$ and therefore

$$L(v) = -a_3\phi_2 - \frac{\phi_2}{\phi_1}\left(-a_3\phi_1\right) = 0.$$

Thus, $v = \left(\frac{\phi_2}{\phi_1}\right)'$ is a solution of reduced equation.

EXERCISE

Use the reduction of order method and find the general solution of each of the following equations. Verify that ϕ_1 satisfies the equation.

(a) $x^2 y'' - xy' + y = 0, \quad \phi_1 = x$ (Ans. $y = c_1 x + c_2 x \log x$)

(b) $y'' - \frac{2}{x} y' + \frac{2}{x^2} y = 0, \phi_1 = x$ (Ans. $y = c_1 x + c_2 x^2$)

(c) $(2x^2 + 1)y'' - 4xy' + 4y = 0, \phi_1 = x$ (Ans. $y = c_1 x + c_2 (2x^2 - 1)$)

(d) $y'' + (x^2 - x)y' - (x-1)y = 0, \quad \phi_1 = x$ (Ans. $y = c_1 x + c_2 x \int \frac{e^{-\frac{x^3}{3} + \frac{x^2}{2}}}{x^2} dx$)

(e) $y'' + \left(\frac{x}{2} - \frac{1}{x}\right) y' - y = 0, \quad \phi_1 = x^2$ (Ans. $y = c_1 x^2 + c_2 x^2 \int \frac{e^{\frac{-x^2}{4}}}{x^3} dx$)

(f) $2x^2 y'' + 3xy' - y = 0, \quad \phi_1 = x^{1/2}$ (Ans. $y = c_1 x^{\frac{1}{2}} + c_2 x^{-1}$)

Unit 2 : Basis

In the course on linear algebra we learn about a vector space also called as linear space and the basis of a linear space. Suppose S is a set of functions with the following property.

If $\phi_1, \phi_2 \in$ S, $c_1 \phi_1 + c_2 \phi_2 \in$ S for any two constants c_1, c_2. Then the set S is called a linear space of functions. If a linear space of functions S contains n functions $\phi_1, \phi_2, \phi_3, ..., \phi_n$ which are linearly independent and every function from S can be represented as a linear combination of these functions $\phi_1, \phi_2, \phi_3, ..., \phi_n$ then the set $\{\phi_1, \phi_2, \phi_3, ..., \phi_n\}$ is called a basis for the linear space S. The number n is called dimension of S.

For a given linear differential equation $L(y) = y^{(n)} + a_1(x) y^{(n-1)} + a_2(x) y^{(n-1)} + \cdots + a_n(x) y = 0,$ the collection of all solutions denoted by S of $L(y) = 0$ is a linear space. Every basis of S contains n linearly independent functions and therefore dimension of solution space S is n.

To check the linear independence of functions $\phi_1, \phi_2, \phi_3, ..., \phi_n$, we consider the wronskian W$(\phi_1, \phi_2, \phi_3, ..., \phi_n)$. There is a relation between the linear indepence of functions and the Wronskian W$(\phi_1, \phi_2, \phi_3, ..., \phi_n)$. In chapter I we have proved this result for the linear differential equation with constant coefficients.

A. Linear Independence and Wronskian

In section 1(B) we have seen that for the differential equation $L(y) = y^{(n)} + a_1(x)y^{(n-1)} + a_2(x)y^{(n-2)} + \cdots + a_n y = 0$ there are n linearly independent solutions $\phi_1, \phi_2, \phi_3, ..., \phi_n$ satisfying the initial conditions $\phi_i^{(i-1)}(x_0) = 1,\ \phi_i^{(j-1)}(x_0) = 0,\ j \neq i.$ These linearly independent solutions is a basis of solution space of $L(y) = 0$. Every solution of $L(y) = 0$ can be represented as a linear combination of these functions $\phi_1, \phi_2, \phi_3, ..., \phi_n$.

Theorem 2.2.1

Let $\phi_1, \phi_2, \phi_3, ..., \phi_n$ be n solutions of $L(y) = 0$ on I satisfying the initial conditions.

$$\phi_i^{(i-1)}(x_0) = 1,\ \phi_i^{(j-1)}(x_0) = 0,\ j \neq i,\quad x_0 \in I$$

If ϕ is any solution of $L(y) = 0$ on I, there are n constants c_1, c_2, c_3,....,c_n such that $\phi = c_1\phi_1 + c_2\phi_2 + \cdots + c_n\phi_n$

Proof :

Let ϕ is any solution of $L(y) = 0$ on I. Let $\phi(x_0) = \alpha_1, \phi'(x_0) = \alpha_2, \phi''(x_0) = \alpha_3, \cdots \phi^{(n-1)}(x_0) = \alpha_n$ for some constants $\alpha_1, \alpha_2, \alpha_3, \cdots \alpha_n$.

Consider a function $\psi = \alpha_1\phi_1 + \alpha_2\phi_2 + \alpha_3\phi_3 + \cdots + \alpha_n\phi_n$

Since $\phi_1, \phi_2, ..., \phi_n$ are solutions of $L(y) = 0$, by superposition principle (chapter 2 unit 1(B)) ψ is also a solution of $L(y) = 0$ and clearly

$$\psi(x_0) = \alpha_1\phi_1(x_0) + \alpha_2\phi_2(x_0) + \alpha_3\phi_3(x_0) + \cdots + \alpha_n\phi_n(x_0) = \alpha_1$$

as $\phi_1(x_0) = 1$ and $\phi_i(x_0) = 0$ for $i = 2, 3, 4, ..., n.$

$$\psi'(x_0) = \alpha_1\phi_1'(x_0) + \alpha_2\phi_2'(x_0) + \alpha_3\phi_3'(x_0) + \cdots + \alpha_n\phi_n'(x_0) = \alpha_2$$

Since, $\phi_1'(x_0) = 0,\ \phi_2'(x_0) = 1,\ \phi_3'(x_0) = 0 \cdots \phi_n'(x_0) = 0.$

$$\psi''(x_0) = \alpha_1\phi_1''(x_0) + \alpha_2\phi_2''(x_0) + \alpha_3\phi_3''(x_0) + \cdots + \alpha_n\phi_n''(x_0) = \alpha_3$$

Since, $\phi_1''(x_0) = 0,\ \phi_2''(x_0) = 1,\ \phi_3''(x_0) = 0 \cdots \phi_n''(x_0) = 0.$

In general $\psi^i(x_0) = \alpha_i \quad i = 3, 4, 5, ..., n-1$

Thus, we see that

$$\psi(x_0) = \alpha_1,\ \psi'(x_0) = \alpha_2,\ \psi''(x_0) = \alpha_3, ..., \psi^{(n-1)}(x_0) = \alpha_n.$$

Thus, ψ is a solution of $L(y) = 0$ having the same initial conditions at x_0 as that of ϕ. By uniqueness theorem (chapter II Unit I theorem 2.1.2) we must have $\psi = \phi$.

i.e. $\phi = \alpha_1\phi_1 + \alpha_2\phi_2 + \alpha_3\phi_3 + ... + \alpha_n\phi_n$

Thus, every solution of $L(y) = 0$, can be uniquely represented as a linear combination of $\phi_1, \phi_2, ..., \phi_n$. Since $\phi_1, \phi_2, ..., \phi_n$ are linearly independent the set $\{\phi_1, \phi_2, \phi_3, ..., \phi_n\}$ is a basis for the solutions $L(y) = 0$.

Recall that the Wronkian of n functions $\phi_1, \phi_2, \phi_3, ..., \phi_n$ is defined as the determinant

$$W(\phi_1, \phi_2, \phi_3, ..., \phi_n) = \begin{vmatrix} \phi_1 & \phi_2 & \phi_3 & \cdots & \phi_n \\ \phi_1' & \phi_2' & \phi_3' & \cdots & \phi_n' \\ \phi_1'' & \phi_2'' & \phi_3'' & \cdots & \phi_n'' \\ \vdots & \vdots & \vdots & & \vdots \\ \phi_1^{(n-1)} & \phi_2^{(n-1)} & \phi_3^{(n-1)} & & \phi_n^{(n-1)} \end{vmatrix}$$

Theorem 2.2.2 :

If $\phi_1, \phi_2, \phi_3, ..., \phi_n$ are n solutions of $L(y) = 0$ where $L(y) = y^{(n)} + a_1(x)y^{(n-1)} + a_2(x)y^{(n-2)} + \cdots + a_n(x)y,$ on an interval I, then they are linearly independent on I if and only if $W(\phi_1, \phi_2, \phi_3, ..., \phi_n)(x) \neq 0$ for all x in I.

Proof :

Suppose $W(\phi_1, \phi_2, \phi_3, ..., \phi_n)(x) \neq 0$ for all x in I. We show that $\phi_1, \phi_2, \phi_3, ..., \phi_n$ are linearly independent on I. i.e. $\sum_{i=1}^{n} c_i \phi_i = 0 \Rightarrow c_1 = c_2 = c_3 = ... = c_n = 0$

If there are constants $c_1, c_2, c_3, ..., c_n$ such that

$\sum c_i \phi_i(x) = c_1 \phi_1(x) + c_2 \phi_2(x) + c_3 \phi_3(x) + ... + c_n \phi_n(x) = 0$ for all x in I then clearly,

$$c_1 \phi_1'(x) + c_2 \phi_2'(x) + c_3 \phi_3'(x) + ... + c_n \phi_n'(x) = 0$$

$$c_1 \phi_1''(x) + c_2 \phi_2''(x) + c_3 \phi_3''(x) + ... + c_n \phi_n''(x) = 0$$

$$\vdots$$

$$c_1 \phi_1^{(n-1)}(x) + c_2 \phi_2^{(n-1)}(x) + c_3 \phi_3^{(n-1)}(x) + ... + c_n \phi_n^{(n-1)}(x) = 0$$

Thus we get a system of linear equations

$$\begin{bmatrix} \phi_1(x) & \phi_2(x) & \phi_3(x) & \cdots & \phi_n(x) \\ \phi_1'(x) & \phi_2'(x) & \phi_3'(x) & \cdots & \phi_n'(x) \\ \phi_1''(x) & \phi_2''(x) & \phi_3''(x) & \cdots & \phi_n''(x) \\ \vdots & \vdots & \vdots & & \vdots \\ \phi_1^{(n-1)}(x) & \phi_2^{(n-1)}(x) & \phi_3^{(n-1)}(x) & & \phi_n^{(n-1)}(x) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

The above system can be represented by Ax = 0. If A is invertible then we can premultiply by A^{-1} and we get x = 0. The square matrix is invertible if it is non-singular i.e. determinant of A is non-zero.

The determinant of the matrix A is $W(\phi_1,\phi_2,\phi_3,....,\phi_n)(x)$. Therefore if $W(\phi_1,\phi_2,\phi_3,....,\phi_n)(x) \neq 0$ then $c_1 = c_2 = c_3 = ... = c_n = 0$. Since $\sum_{i=1}^{n} c_i \phi_i = 0 \Rightarrow c_i = 0$, $i = 1, 2, 3, ..., n$, $\phi_1,\phi_2,\phi_3,...,\phi_n$ are linearly independent.

Conversely, suppose $\phi_1,\phi_2,\phi_3,....,\phi_n$ are linearly independent solutions of $L(y) = 0$ defined on I. Suppose there is an x_0 in I such that $W(\phi_1,\phi_2,\phi_3,....,\phi_n)(x_0) = 0$.

Then system of n linear equations

$$\begin{bmatrix} \phi_1(x_0) & \phi_2(x_0) & \phi_3(x_0) & \cdots & \phi_n(x_0) \\ \phi_1'(x_0) & \phi_2'(x_0) & \phi_3'(x_0) & \cdots & \phi_n'(x_0) \\ \phi_1''(x_0) & \phi_2''(x_0) & \phi_3''(x_0) & \cdots & \phi_n''(x_0) \\ \vdots & \vdots & \vdots & & \vdots \\ \phi_1^{(n-1)}(x_0) & \phi_2^{(n-1)}(x_0) & \phi_3^{(n-1)}(x_0) & & \phi_n^{(n-1)}(x_0) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

has a solution $c_1, c_2, c_3,....,c_n$, where not all the constants $c_1, c_2, c_3,...,c_n$ are zero. Let $c_1, c_2, c_3,....,c_n$ be a non-zero solution of above system of equations and consider

$$\psi(x) = c_1\phi_1(x) + c_2\phi_2(x) + c_3\phi_3(x) + \cdots + c_n\phi_n(x).$$

Since, ϕ_i, $i = 1, 2, 3....n$ are solution of $L(y) = 0$, ψ is also a solution of equation $L(y) = 0$. Now $L(y) = 0$ and from above system of equations we get

$$\psi(x_0) = c_1\phi_1(x_0) + c_2\phi_2(x_0) + c_3\phi_3(x_0) + \cdots + c_n\phi_n(x_0) = 0.$$

$$\psi'(x_0) = c_1\phi_1'(x_0) + c_2\phi_2'(x_0) + c_3\phi_3'(x_0) + \cdots + c_n\phi_n'(x_0) = 0$$

In general

$$\psi^{(i)}(x_0) = c_1\phi_1^{(i)}(x_0) + c_2\phi_2^{(i)}(x_0) + c_3\phi_3^{(i)}(x_0) + \cdots + c_n\phi_n^{(i)}(x_0) = 0$$

for $i = 1, 2, 3, 4,...., n-1$.

Thus, $\psi(x_0) = \psi'(x_0) = \psi''(x_0) = ... = \psi^{(n-1)}(x_0) = 0.$

From theorem 2.1.1 it follows that $\psi(x) \equiv 0$ on I.

Therefore, $c_1\phi_1(x) + c_2\phi_2(x) + c_3\phi_3(x) + \cdots + c_n\phi_n(x) = 0$ for all x in I. Thus, we have $c_1, c_2, c_3,..... c_n$ not all zero such that $c_1\phi_1(x) + c_2\phi_2(x) + c_3\phi_3(x) + \cdots + c_n\phi_n(x) = 0$ for all x in I. Therefore the set $\phi_1,\phi_2,\phi_3,....,\phi_n$ is not linearly independent on I. But this contradicts the fact that $\phi_1,\phi_2,\phi_3,....,\phi_n$ are linearly independent on I. Therefore the assumption that there was a point x_0 in I such that $W(\phi_1,\phi_2,\phi_3,....,\phi_n)(x_0) = 0$. must be false i.e. $W(\phi_1,\phi_2,\phi_3,....,\phi_n)(x) \neq 0$ for all x in I.

Theorem 2.2.3

Let $\phi_1, \phi_2, \phi_3, ..., \phi_n$ be n linearly independent solutions of $L(y) = y^{(n)} + a_1(x)y^{(n-1)} + a_2(x)y^{(n-2)} + \cdots + a_n(x)y = 0$ on an interval I. If ϕ is any solution of $L(y) = 0$ on I, it can be represented in the form $\phi = c_1\phi_1 + c_2\phi_2 + c_3\phi_3 + ... + c_n\phi_n$, where $c_1, c_2, c_3, ... c_n$ are constants. Thus any set of n linearly independent solutions of $L(y) = 0$ on I is a basis for the solution space of $L(y) = 0$ on I.

Proof :

Let x_0 be a point in I. Suppose ϕ is any solution of $L(y) = 0$. Let $\phi(x_0) = \alpha_{1,}\ \phi'(x_0) = \alpha_{2,}\ \phi''(x_0) = \alpha_{3,} \cdots \phi^{(n-1)}(x_0) = \alpha_n$ We show that there exist unique constants $c_1, c_2, c_3, ... c_n$ such that $\psi = c_1\phi_1(x) + c_2\phi_2(x) + c_3\phi_3(x) + \cdots + c_n\phi_n(x)$ is a solution of $L(y) = 0$ satisfying $\psi(x_0) = \alpha_1,\ \psi'(x_0) = \alpha_2,\ \psi''(x_0) = \alpha_3, ..., \psi^{(n-1)}(x_0) = \alpha_n$. These initial conditions are equivalent to the following equations for $c_1, c_2, c_3, ... c_n$ (e.g. $\psi(x_0) = c_1\phi_1(x_0) + c_2\phi_2(x_0) + ... + c_n\phi_n(x_0) = \alpha_1$)

$$\begin{bmatrix} \phi_1(x_0) & \phi_2(x_0) & \phi_3(x_0) & \cdots & \phi_n(x_0) \\ \phi_1'(x_0) & \phi_2'(x_0) & \phi_3'(x_0) & \cdots & \phi_n'(x_0) \\ \phi_1''(x_0) & \phi_2''(x_0) & \phi_3''(x_0) & \cdots & \phi_n''(x_0) \\ \vdots & \vdots & \vdots & & \vdots \\ \phi_1^{(n-1)}(x_0) & \phi_2^{(n-1)}(x_0) & \phi_3^{(n-1)}(x_0) & & \phi_n^{(n-1)}(x_0) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \vdots \\ \alpha_n \end{bmatrix}$$

Since $\phi_1, \phi_2, \phi_3, ..., \phi_n$ are linearly independent by theorem 2.2.2, $W(\phi_1, \phi_2, \phi_3, ..., \phi_n)(x_0) \neq 0$. Therefore the coefficient matrix is inversible and there is a unique solution $c_1, c_2, c_3, ... c_n$ of the above system of equations.

Thus we have a unique solution

$$\psi = c_1\phi_1(x) + c_2\phi_2(x) + c_3\phi_3(x) + \cdots + c_n\phi_n(x)$$

Satisfying $\psi(x_0) = \alpha_1,\ \psi'(x_0) = \alpha_2,\ \psi''(x_0) = \alpha_3, ..., \psi^{(n-1)}(x_0) = \alpha_n$. But ϕ is a solution with identical initial conditions. Therefore by uniqueness theorem we have $\phi(x) = \psi(x)$ on I. Thus $\phi(x) = c_1\phi_1(x) + c_2\phi_2(x) + c_3\phi_3(x) + \cdots + c_n\phi_n(x)$ on I and any solution of $L(y) = 0$ can be represented as a linear combination of n linearly independent solutions $\phi_1, \phi_2, \phi_3, ..., \phi_n$.

In theorem 2.2.2 we have seen that the function $\phi_1, \phi_2, \phi_3, ..., \phi_n$ are linearly independent solutions of $L(y) = 0$ if and only if the Wronskian $W(\phi_1, \phi_2, \phi_3, ..., \phi_n)(x) \neq 0$ for all x in I. In the next theorem we show that it is sufficient to calculate the Wronskian $W(\phi_1, \phi_2, \phi_3, ..., \phi_n)$ at some point x_0 in I.

Theorem 2.2.4

Let $\phi_1, \phi_2, \phi_3, ..., \phi_n$ be n solutions of $L(y) = 0$ on an interval I and let x_0 be any point in I. Then

$$W(\phi_1, \phi_2, \phi_3, ..., \phi_n)(x) = \exp\left[-\int_{x_0}^{x} a_1(t)\,dt\right] W(\phi_1, \phi_2, \phi_3, ..., \phi_n)(x_0)$$

Note that since exponential function is non-zero function,

$W(\phi_1, \phi_2, \phi_3, ..., \phi_n)(x_0) \neq 0$ implies $W(\phi_1, \phi_2, \phi_3, ..., \phi_n)(x) \neq 0$ for all x in I.

Proof :

$$\text{Let } W = W(\phi_1, \phi_2, \phi_3, ..., \phi_n) = \begin{vmatrix} \phi_1 & \phi_2 & \phi_3 & \cdots & \phi_n \\ \phi_1' & \phi_2' & \phi_3' & \cdots & \phi_n' \\ \phi_1'' & \phi_2'' & \phi_3'' & \cdots & \phi_n'' \\ \vdots & \vdots & \vdots & & \vdots \\ \phi_1^{(n-1)} & \phi_2^{(n-1)} & \phi_3^{(n-1)} & & \phi_n^{(n-1)} \end{vmatrix}$$

On differentiating W row wise we get

$$W' = \begin{bmatrix} \phi_1' & \phi_2' & \phi_3' & \cdots & \phi_n' \\ \phi_1' & \phi_2' & \phi_3' & \cdots & \phi_n' \\ \phi_1'' & \phi_2'' & \phi_3'' & \cdots & \phi_n'' \\ \vdots & \vdots & \vdots & & \vdots \\ \phi_1^{(n-1)} & \phi_2^{(n-1)} & \phi_3^{(n-1)} & & \phi_n^{(n-1)} \end{bmatrix} + \begin{bmatrix} \phi_1 & \phi_2 & \phi_3 & \cdots & \phi_n \\ \phi_1'' & \phi_2'' & \phi_3'' & \cdots & \phi_n'' \\ \phi_1'' & \phi_2'' & \phi_3'' & \cdots & \phi_n'' \\ \vdots & \vdots & \vdots & & \vdots \\ \phi_1^{(n-1)} & \phi_2^{(n-1)} & \phi_3^{(n-1)} & & \phi_n^{(n-1)} \end{bmatrix} + ...$$

$$+ \begin{bmatrix} \phi_1 & \phi_2 & \phi_3 & \cdots & \phi_n \\ \phi_1' & \phi_2' & \phi_3' & \cdots & \phi_n' \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ \phi_1^{(n-2)} & \phi_2^{(n-2)} & \phi_3^{(n-2)} & & \phi_n^{(n-2)} \\ \phi_1^{(n)} & \phi_2^{(n)} & \phi_3^{(n)} & & \phi_n^{(n)} \end{bmatrix}$$

$$= V_1 + V_2 + V_3 + ... + V_n \text{ (say)}$$

Where V_k differs from W only in its k^{th} row and the k^{th} row of V_k is obtained by differentiating the k^{th} row of W. The first $n - 1$ determinants are all zero, since they each have two identical rows. Observe that V_k has kth and $(k + 1)^{th}$ row identical.

Since $\phi_1, \phi_2, ..., \phi_n$ are solution of $L(y) = 0$, we have

$$\phi_i^{(n)} = -a_1\,\phi_i^{(n-1)} - a_2\,\phi_i^{(n-2)} - a_3\,\phi_i^{(n-3)} .... - a_n\,\phi_i \;\; (i = 1, 2, 3, ..., n)$$

$$= -\sum_{j=0}^{n-1} a_{n-j}\ \phi_i^{(j)}$$

Therefore

$$W' = V_n = \begin{bmatrix} \phi_1 & \phi_2 & \phi_3 & \cdots & \phi_n \\ \phi_1{}' & \phi_2 & \phi_3 & \cdots & \phi_n{}' \\ \phi_1{}'' & \phi_2{}'' & \phi_3{}'' & \cdots & \phi_n{}'' \\ \vdots & \vdots & \vdots & & \vdots \\ \phi_1{}^{(n-2)} & \phi_2{}^{(n-2)} & \phi_3{}^{(n-2)} & & \phi_n{}^{(n-2)} \\ -\sum_{j=0}^{n-1} a_{n-j}\ \phi_1^{(j)} & -\sum_{j=0}^{n-1} a_{n-j}\ \phi_2^{(j)} & -\sum_{j=0}^{n-1} a_{n-j}\ \phi_3^{(j)} & \cdots & -\sum_{j=0}^{n-1} a_{n-j}\ \phi_n^{(j)} \end{bmatrix}$$

Elimentary row transformations do not change the value of the determinant. Perform the transformation

$R_n + a_n R_1 + a_{n-1} R_3 + a_{n-3} R_3 + \ldots + a_2 R_{n-1}.$ we get

$$W' = \begin{bmatrix} \phi_1 & \phi_2 & \phi_3 & \cdots & \phi_n \\ \phi_1{}' & \phi_2{}' & \phi_3{}' & \cdots & \phi_n{}' \\ \vdots & \vdots & \vdots & & \vdots \\ \phi_1{}^{(n-2)} & \phi_2{}^{(n-2)} & \phi_3{}^{(n-2)} & \cdots & \phi_1{}^{(n-2)} \\ -a_1\phi_1{}^{(n-1)} & -a_1\phi_2{}^{(n-1)} & -a_1\phi_3{}^{(n-1)} & \cdots & -a_1\phi_n{}^{(n-1)} \end{bmatrix}$$

$$= -a_1 \begin{bmatrix} \phi_1 & \phi_2 & \phi_3 & \cdots & \phi_n \\ \phi_1{}' & \phi_2{}' & \phi_3{}' & \cdots & \phi_n{}' \\ \phi_1{}'' & \phi_2{}'' & \phi_3{}'' & \cdots & \phi_n{}'' \\ \vdots & \vdots & \vdots & & \vdots \\ \phi_1{}^{(n-1)} & \phi_2{}^{(n-1)} & \phi_3{}^{(n-1)} & & \phi_n{}^{(n-1)} \end{bmatrix} = -a_1 W(\phi_1, \phi_2, \phi_3, \ldots., \phi_n)$$

Therefore $W' + a_1 W = 0$ and we get,

$$W(x) = e^{-\int_{x_0}^{x} a_1(t)\,dt}\ W(x_0)$$

i.e. $W(\phi_1, \phi_2, \phi_3, \ldots., \phi_n)(x) = \exp\left[-\int_{x_0}^{x} a_1(t)\,dt\right] W(\phi_1, \phi_2, \phi_3, \ldots., \phi_n)(x_0).$

Corollatory : If the coefficient a_1 is constant then

$$W(\phi_1, \phi_2, \phi_3, \ldots., \phi_n)(x) = e^{-a_1(x-x_0)}\ W(\phi_1, \phi_2, \phi_3, \ldots., \phi_n)(x_0).$$

From theorem 2.2.2 and theorem 2.2.4 it follows that n solutions $\phi_1, \phi_2, \phi_3, \ldots., \phi_n$ of $L(y) = 0$ on I are linearly independent if and only if $W(\phi_1, \phi_2, \phi_3, \ldots, \phi_n)(x_0) \neq 0$ for some point x_0 in I.

B. Solutions of non-homogeneous equation

The equation $L(y) = y^{(n)} + a_1(x)y^{(n-1)} + a_2(x)y^{(n-2)} + ... + a_n y = b(x)$ where $a_1, a_2, a_3,....,$ a_n, b are continuous functions on an interval I is a non-homogeneous linear equation of order n with variable coefficients. The solutions of this equation can be determined by the variations of constant method.

Theorem 2.2.5

Let $b(x)$ be a continuous function on an interval I and let $\phi_1, \phi_2, \phi_3, ..., \phi_n$ be a basis for the solutions of $L(y) = 0$ on I. Every solution ψ of $L(y) = y^{(n)} + a_1(x)y^{(n-1)} + a_2(x)y^{(n-2)} + ... + a_n(x)y = b(x)$ can be written as $\psi = \psi_p + c_1\phi_1 + c_2\phi_2 + c_3\phi_3 + ... + c_n\phi_n$ where is a ψ_p is a particular solution of $L(y) = b(x)$ and $c_1, c_2, c_3, ... c_n$ are constants. Every such ψ is a solution of $L(y) = b(x)$. A particular solution ψ_p is given by

$$\psi_p = \sum_{k=1}^{n} \phi_k(x) \int_{x_0}^{x} \frac{W_k(t)\, b(t)}{W(\phi_1, \phi_2, \phi_3, ..., \phi_n)(t)}\, dt$$

where $W(\phi_1, \phi_2, \phi_3, ..., \phi_n)$ is a wronkian of $\phi_1, \phi_2, \phi_3, ..., \phi_n$ and W_k is the determinant obtained from $W(\phi_1, \phi_2, \phi_3, ..., \phi_n)$ by replacing the k^{th} column $(\phi_k\ \phi_k'\ \phi_k'' .... \phi_k^{(n-1)})^T$ by $(0, 0, 0..., 0, 1)^T$.

Proof :

If ψ_p is a particular solution of $L(y) = b(x)$ and ψ is any other solution of $L(y) = b(x)$, then

$$L(\psi - \psi_p) = L(\psi) - L(\psi_p) = b(x) - b(x) = 0.$$

Therefore $\psi - \psi_p$ is a solution of corresponding homogeneous equation $L(y) = 0$. Since $\phi_1, \phi_2, \phi_3, ..., \phi_n$ is a basis for the solution of $L(y) = 0$ on I, every solution of $L(y) = 0$ can be expressed as a linear combination of $\phi_1, \phi_2, \phi_3, ..., \phi_n$.

$$\psi - \psi_p = c_1\phi_1 + c_2\phi_2 + c_3\phi_3 + ... + c_n\phi_n$$

$$\psi = \psi_p + c_1\phi_1 + c_2\phi_2 + c_3\phi_3 + ... + c_n\phi_n.$$

A particular solution ψ_p can be found by variation of constants method. Let ψ_p be of the form

$$\psi_p = u_1(x)\phi_1(x) + u_2(x)\phi_2(x) + u_3(x)\phi_3(x) + ... + u_n(x)\phi_n(x)$$

Since ψ_p is a solution, $L(\psi_p) = b(x)$.

$$\psi_p' = u_1'\phi_1 + u_2'\phi_2 + u_3'\phi_3 + ... + u_n'\phi_n + u_1\phi_1' + u_2\phi_2' + u_3\phi_3' + ... + u_n\phi_n'$$

Choose $u_1, u_2, u_3, ... u_n$ such that $u_1'\phi_1 + u_2'\phi_2 + u_3'\phi_3 + ... + u_n'\phi_n = 0$

Then $\quad \psi_p' = u_1\phi_1' + u_2\phi_2' + u_3\phi_3' + ... + u_n\phi_n'$

$$\psi_p'' = u_1'\phi_1' + u_2'\phi_2' + u_3'\phi_3' + ... + u_n'\phi_n' + u_1\phi_1'' + u_2\phi_2'' + u_3\phi_3'' + ... + u_n\phi_n''$$

Let $\quad u_1'\phi_1' + u_2'\phi_2' + u_3'\phi_3' + ... + u_n'\phi_n' = 0$ then

$$\psi_p'' = u_1\phi_1'' + u_2\phi_2'' + u_3\phi_3'' + \ldots + u_n\phi_n''$$

In general choose $u_1'\phi_1^{(k)} + u_2'\phi_2^{(k)} + u_3'\phi_3^{(k)} + \ldots + u_n'\phi_n^{(k)} = 0$

Then $\psi_p^{(k+1)} = u_1\phi_1^{(k+1)} + u_2\phi_2^{(k+1)} + u_3\phi_3^{(k+1)} + \ldots + u_n\phi_n^{(k+1)}$

and $\psi_p^{(n)} = u_1'\phi_1^{(n-1)} + u_2'\phi_2^{(n-1)} + u_3'\phi_3^{(n-1)} + \ldots + u_n'\phi_n^{(n-1)} + u_1\phi_1^{(n)} + u_2\phi_2^{(n)} + \ldots + u_n\phi_n^{(n)}$

If we choose $u_1'\phi_1^{(n-1)} + u_2'\phi_2^{(n-1)} + u_3'\phi_3^{(n-1)} + \ldots + u_n'\phi_n^{(n-1)} = b(x)$. Then

$$\psi_p^{(n)} = u_1\phi_1^{(n)} + u_2\phi_2^{(n)} + u_3\phi_3^{(n)} + \ldots + u_n\phi_n^{(n)} + b(x)$$

Thus we have the following equations

$$\psi_p = u_1\phi_1 + u_2\phi_2 + u_3\phi_3 + \ldots + u_n\phi_n$$

$$\psi_p' = u_1\phi_1' + u_2\phi_2' + u_3\phi_3' + \ldots + u_n\phi_n' \quad ; \quad u_1'\phi_1 + u_2'\phi_2 + u_3'\phi_3 + \ldots + u_n'\phi_n = 0$$

$$\psi_p'' = u_1\phi_1'' + u_2\phi_2'' + u_3\phi_3'' + \ldots + u_n\phi_n'' \quad ; \quad u_1'\phi_1' + u_2'\phi_2' + u_3'\phi_3' + \ldots + u_n'\phi_n' = 0$$

$$\psi_p''' = u_1\phi_1''' + u_2\phi_2''' + u_3\phi_3''' + \ldots + u_n\phi_n''' \quad ; \quad u_1'\phi_1'' + u_2'\phi_2'' + u_3'\phi_3'' + \ldots + u_n'\phi_n'' = 0$$

$$\vdots$$

$$\psi_p^{(n-1)} = u_1\phi_1^{(n-1)} + u_2\phi_2^{(n-1)} + \ldots + u_n\phi_n^{(n-1)} \quad ; \quad u_1'\phi_1^{(n-2)} + u_2'\phi_2^{(n-2)}$$

$$+ u_3'\phi_3^{(n-2)} + \ldots + u_n'\phi_n^{(n-2)} = 0$$

$$\psi_p^{(n)} = u_1\phi_1^{(n)} + u_2\phi_2^{(n)} + u_3\phi_3^{(n)} + \ldots + u_n\phi_n^{(n)} + b(x) \quad ; \quad u_1'\phi_1^{(n-1)} + u_2'\phi_2^{(n-1)}$$

$$+ u_3'\phi_3^{(n-1)} + \ldots + u_n'\phi_n^{(n-1)} = b(x)$$

Adding the terms columwise on left we get

$$L(\psi_p) = u_1 L(\phi_1) + u_2 L(\phi_2) + u_3 L(\phi_3) + \ldots + u_n L(\phi_n) + b(x)$$

Since $\phi_1, \phi_2, \phi_3, \ldots., \phi_n$ are n solutions of homogeneous equation $L(y) = 0$, $L(\phi_1) = L(\phi_2) = L(\phi_3) = \ldots = L(\phi_n) = 0$ and $L(\psi_p) = b(x)$.

The right hand side equations are the following system of linear equations.

$$\begin{bmatrix} \phi_1 & \phi_2 & \phi_3 & \cdots & \phi_n \\ \phi_1' & \phi_2' & \phi_3' & \cdots & \phi_n' \\ \phi_1'' & \phi_2'' & \phi_3'' & \cdots & \phi_n'' \\ \vdots & \vdots & \vdots & & \vdots \\ \phi_1^{(n-1)} & \phi_2^{(n-1)} & \phi_3^{(n-1)} & & \phi_n^{(n-1)} \end{bmatrix} \begin{bmatrix} u_1' \\ u_2' \\ u_3' \\ \vdots \\ u_n' \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ b \end{bmatrix}$$

We solve the above system of equations by Cramer's rule.

Thus, $u_k' = \dfrac{\overline{W}_k}{\mathrm{W}(\phi_1, \phi_2, \phi_3, \ldots, \phi_n)}$

Where $\mathrm{W}(\phi_1, \phi_2, \phi_3, \ldots, \phi_n)$ is a Wronksian of $\phi_1, \phi_2, \phi_3, \ldots, \phi_n$ and $\overline{W}_k$ is the determinant obtained from $W(\phi_1,\phi_2,\phi_3,...,\phi_n)$ by replacing k^{th} column by $(0,0,0,.....,0,b)^T$. Thus

$$\bar{W}_k = \begin{vmatrix} \phi_1 & \phi_2 & \cdots & \phi_{k-1} & 0 & \phi_{k+1} & \cdots & \phi_n \\ \phi_1' & \phi_2' & & \phi_{k-1}' & 0 & \phi_{k+1}' & \cdots & \phi_n' \\ \phi_1'' & \phi_2'' & & \phi_{k-1}'' & 0 & \phi_{k+1}'' & \cdots & \phi_n'' \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ \phi_1^{(n-1)} & \phi_2^{(n-1)} & & \phi_{k-1}^{(n-1)} & b(x) & \phi_{k+1}^{(n-1)} & & \phi_n^{(n-1)} \end{vmatrix}$$

$$= b(x) \begin{vmatrix} \phi_1 & \phi_2 & \cdots & \phi_{k-1} & 0 & \phi_{k+1} & \cdots & \phi_n \\ \phi_1' & \phi_2' & & \phi_{k-1}' & 0 & \phi_{k+1}' & \cdots & \phi_n' \\ \phi_1'' & \phi_2'' & & \phi_{k-1}'' & 0 & \phi_{k+1}'' & \cdots & \phi_n'' \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ \phi_1^{(n-1)} & \phi_2^{(n-1)} & & \phi_{k-1}^{(n-1)} & 1 & \phi_{k+1}^{(n-1)} & & \phi_n^{(n-1)} \end{vmatrix} = b(x) W_k$$

Thus, $$u_k' = \frac{b(x)W_k}{W(\phi_1,\phi_2,\phi_3,...,\phi_n)} \qquad \text{i.e. } u_k = \int_{x_0}^{x} \frac{b(t)W_k(t)}{W(\phi_1,\phi_2,\phi_3,...,\phi_n)(t)}\,dt.$$

and $$\psi_p = u_1\phi_1 + u_2\phi_2 + u_3\phi_3 + ... + u_n\phi_n = \sum_{k=1}^{n} \phi_k u_k$$

$$= \sum_{k=1}^{n} \phi_k(x) \int_{x_0}^{x} \frac{b(t)W_k(t)}{W(\phi_1,\phi_2,\phi_3,...,\phi_n)(t)}\,dt.$$

EXAMPLES

Q. 1. Consider the equation $L(y) = y'' + a_1(x)y' + a_2(x)y = 0$, where a_1, a_2 are continuous on some interval I. Let ϕ_1, ϕ_2 and ψ_1, ψ_2 be two bases for the solution $L(y) = 0$. Show that there is a non-zero constant k. Such that $W(\psi_1,\psi_2)(x) = k\,W(\phi_1,\phi_2)(x)$

Ans. : Since ϕ_1, ϕ_2 is bases for the solutions of $L(y) = 0$ and ψ_1, ψ_2 are solutions of $L(y) = 0$.

$$\psi_1 = c_1\phi_1 + c_2\phi_2 \quad \text{and} \quad \psi_2 = d_1\phi_1 + d_2\phi_2$$

for some constants c_1, c_2, d_1, d_2.

$$W(\psi_1,\psi_2)(x) = \begin{vmatrix} c_1\phi_1 + c_2\phi_2 & d_1\phi_1 + d_2\phi_2 \\ c_1\phi_1' + c_2\phi_2' & d_1\phi_1' + d_2\phi_2' \end{vmatrix}$$

$$= \begin{vmatrix} c_1\phi_1 & d_1\phi_1 + d_2\phi_2 \\ c_1\phi_1' & d_1\phi_1' + d_2\phi_2' \end{vmatrix} + \begin{vmatrix} c_2\phi_2 & d_1\phi_1 + d_2\phi_2 \\ c_2\phi_2' & d_1\phi_1' + d_2\phi_2' \end{vmatrix}$$

$$= c_1\left[\begin{vmatrix} \phi_1 & d_1\phi_1 \\ \phi_1' & d_1\phi_1' \end{vmatrix} + \begin{vmatrix} \phi_1 & d_2\phi_2 \\ \phi_1' & d_2\phi_2' \end{vmatrix}\right] + c_2\left[\begin{vmatrix} \phi_2 & d_1\phi_1 \\ \phi_2' & d_1\phi_1' \end{vmatrix} + \begin{vmatrix} \phi_2 & d_2\phi_2 \\ \phi_2' & d_2\phi_2' \end{vmatrix}\right]$$

$$= c_1 d_2 \left[\begin{vmatrix} \phi_1 & \phi_2 \\ \phi_1' & \phi_2' \end{vmatrix} + c_2 d_1 \begin{vmatrix} \phi_2 & \phi_1 \\ \phi_2' & \phi_1' \end{vmatrix} \right]$$

$$= (c_1 d_2 - c_2 d_1) \begin{vmatrix} \phi_1 & \phi_2 \\ \phi_1' & \phi_2' \end{vmatrix} = (c_1 d_2 - c_2 d_1)\ \mathrm{W}(\phi_1, \phi_2)(x)$$

Thus, $\mathrm{W}(\psi_1, \psi_2)(x) = (c_1 d_2 - c_2 d_1)\ \mathrm{W}(\phi_1, \phi_2)(x)$

Since ψ_1, ψ_2 are independent $c_1 d_2 - c_2 d_1 \neq 0$.

Therefore there is a non-zero constant $k = c_1 d_2 - c_2 d_1$ such that $\mathrm{W}(\psi_1, \psi_2)(x) = k\ \mathrm{W}(\phi_1, \phi_2)(x)$.

Q. 2. Consider $L(y) = y'' + a_1(x) y' + a_2(x) y = 0$. Show that a_1 and a_2 are uniquely determined by any basis ϕ_1, ϕ_2 for the solutions of $L(y) = 0$. Show that

$$a_1 = -\frac{\begin{vmatrix} \phi_1 & \phi_2 \\ \phi_1'' & \phi_2'' \end{vmatrix}}{\mathrm{W}(\phi_1, \phi_2)}, \quad a_2 = -\frac{\begin{vmatrix} \phi_1' & \phi_2' \\ \phi_1'' & \phi_2'' \end{vmatrix}}{\mathrm{W}(\phi_1, \phi_2)}$$

Ans. : Since ϕ_1, ϕ_2 is basis for solutions of $L(y) = 0$, ϕ_1, ϕ_2 are solutions of $L(y) = 0$.

$$L(\phi_1) = \phi_1'' + a_1 \phi_1' + a_2 \phi_1 = 0$$

$$L(\phi_2) = \phi_2'' + a_1 \phi_2' + a_2 \phi_2 = 0$$

Solving above two equations for a_1 and a_2 by Cramers rule, we get.

$$a_1 = -\frac{\begin{vmatrix} -\phi_1'' & \phi_1 \\ -\phi_2'' & \phi_2 \end{vmatrix}}{\begin{vmatrix} \phi_1' & \phi_1 \\ \phi_2' & \phi_2 \end{vmatrix}} = \frac{\begin{vmatrix} \phi_1 & \phi_2 \\ \phi_1'' & \phi_2'' \end{vmatrix}}{\mathrm{W}(\phi_1, \phi_2)}$$

$$a_2 = -\frac{\begin{vmatrix} +\phi_1' & -\phi_1'' \\ +\phi_2' & -\phi_2'' \end{vmatrix}}{\begin{vmatrix} \phi_1' & \phi_1 \\ \phi_2' & \phi_2 \end{vmatrix}} = \frac{\begin{vmatrix} \phi_1' & \phi_2' \\ \phi_1'' & \phi_2'' \end{vmatrix}}{\mathrm{W}(\phi_1, \phi_2)}$$

[We use the elementory properties of determinants det A = det A^T and if we interchange row / column, the value of det change its sign.]

Q. 3. Consider the equation $y'' + \alpha(x) y = 0$ where α is a continuous function on $-\infty < x < \infty$. Let, ϕ_1, ϕ_2 be the basis for the solutions satisfying

$$\phi_1(0) = 1,\ \phi_2(0) = 0,\ \phi_1'(0) = 1,\ \phi_2'(0) = 1,$$

Show that $\mathrm{W}(\phi_1, \phi_2)(x) = 1$ for all x

Ans. : For the differential equation

$y'' + a_1(x)y' + a_2(x)y = 0$, if ϕ_1 and ϕ_2 are two solutions then

$$W(\phi_1,\phi_2)(x) = e^{-\int_{x_0}^{x} a_1(t)\,dt} \quad W(\phi_1,\phi_2)(x_0)$$

For $y'' + \alpha(x)y = 0,\ a_1 = 0$. Therefore $W(\phi_1,\phi_2)(x) = W(\phi_1,\phi_2)(0)$

$$= \begin{vmatrix} \phi_1(0) & \phi_2(0) \\ \phi_1'(0) & \phi_2'(0) \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1$$

Thus, $W(\phi_1,\phi_2)(x) = 1$ for all x.

Q. 4. Find a general solution of $y'' - \dfrac{2}{x^2}y = x\ (0 < x < a)$

Ans. : Assume that the solution of homogeneous equation $L(y) = y'' - \dfrac{2}{x^2}y = 0$ is of the form x^r. Then $y = x^r$ implies $L(x^r) = r(r-1)x^{r-2} - 2x^{r-2} = 0$ gives $r(r-1) - 2 = 0$. Then $r^2 - r - 2 = 0$ implies $r = 2$ and $r = -1$. Thus, $\phi_1(x) = x^2$ and $\phi_2(x) = \dfrac{1}{x}$ are solutions of homogeneous equation $L(y) = y'' - \dfrac{2}{x^2}y = 0.$

A solution ψ_p of the non-homogeneous equation has the form

$$\psi_p = u_1(x)\phi_1(x) + u_2(x)\phi_2(x)$$

$$= u_1(x)x^2 + u_2(x)\frac{1}{x}$$

Where, $u_1' = \dfrac{b(x)W_1}{W(\phi_1,\phi_2)}$ and $u_2' = \dfrac{b(x)W_2}{W(\phi_1,\phi_2)}$

Here $b(x) = x$

$$W_1 = \begin{vmatrix} 0 & \frac{1}{x} \\ 1 & -\frac{1}{x^2} \end{vmatrix} = -\frac{1}{x} \quad , \quad W_2 = \begin{vmatrix} x^2 & 0 \\ 2x & 1 \end{vmatrix} = x^2$$

$$W(\phi_1,\phi_2) = \begin{vmatrix} \phi_1 & \phi_2 \\ \phi_1' & \phi_2' \end{vmatrix} = \begin{vmatrix} x^2 & \frac{1}{x} \\ 2x & -\frac{1}{x^2} \end{vmatrix} = -1 - 2 = -3 \text{ and we find that}$$

$$u_1' = \frac{x(-1/x)}{-3} = \frac{1}{3} \quad \text{and} \quad u_2' = \frac{b(x)W_2}{W} = \frac{x(x^2)}{-3} = -\frac{x^3}{3}$$

We may take $u_1 = \dfrac{x}{3}$ and $u_2 = -\dfrac{x^4}{12}$. We skip the constants of integration as they correspond to the solution of corresponding homogeneous equation.

Thus, the solution of non-homogeneous equation becomes

$$\psi_p = \frac{x}{3} \cdot x^2 - \frac{x^4}{12} \cdot \frac{1}{x} = \frac{x^3}{4}.$$

Every solution ϕ of $L(y) = x$ has the form

$$\phi(x) = \psi_p + c_1 \phi_1(x) + c_2 \phi_2(x) = \frac{x^3}{4} + c_1 x^2 + \frac{c_2}{x}$$

where, c_1 and c_2 are constants.

Q. 5. One solution of $x^2 y'' - 2y = 0$ on $0 < x < \infty$ is $\phi_1(x) = x^2$. Find all solutions of $x^2 y'' - 2y = 2x - 1$ on $0 < x < \infty$.

Ans. : $\phi_1(x) = x^2$ Let $\phi_2(x) = u(x)\phi_1(x)$ be a solution of $x^2 y'' - 2y = 0$. Then

$$L(\phi_2) = x^2\left[u''(x)\phi_1(x) + 2u'(x)\phi_1'(x) + u(x)\phi_1''(x)\right] - 2u(x)\phi_1(x) = 0$$

$$= x^2\left[u'' x^2 + 2u'(x)\cdot 2x + u(x)\cdot 2\right] - 2u(x)x^2 = 0$$

$$L(\phi_2) = 0 \text{ gives } u'' x^4 + 4u' x^3 = 0 \text{ i.e. } \frac{u''}{u'} = -\frac{4}{x}$$

$$\log u' = -4\log x \text{ and } u' = x^{-4} \text{ or } u = -\frac{1}{3x^3}$$

Therefore $\phi_2(x) = u(x)\phi_1(x) = -\frac{1}{3x^3} \cdot x^2 = -\frac{1}{3x}$

Since $L(y)$ is a linear differential operator $\phi_2(x) = \frac{1}{x}$ is a second solution.

Thus, $\phi_1(x) = x^2$ and $\phi_2(x) = \frac{1}{x}$ are solutions of the homogenous equation $L(y) = x^2 y'' - 2y = 0$ or $y'' - \frac{2}{x} y = 0$.

Equation $x^2 y'' - 2y = 2x - 1$ is the given differential equation. To reduce the equation in standard form we have to divide the given equation by x^2 we can do so since x is positive. Therefore consider the eqution $y'' - \frac{2}{x^2} y = \frac{2}{x} - \frac{1}{x^2}$. Solution of this equation will be a solution of given equation.

A solution y_p of a non-homogeneous equation $y'' - \frac{2}{x^2} y = \frac{2}{x} - \frac{1}{x^2}$ has the form

$$\psi_p = u_1(x)\phi_1(x) + u_2(x)\phi_2(x) = u_1(x)x^2 + u_2(x)\frac{1}{x}$$

Where, $u_1'(x) = \frac{b(x)W_1}{W(\phi_1, \phi_2)}$ and $u_2'(x) = \frac{b(x)W_2}{W(\phi_1, \phi_2)}$

$$b(x) = \frac{2}{x} - \frac{1}{x^2}, \quad W(\phi_1, \phi_2) = \begin{vmatrix} x^2 & 1/x \\ 2x & -1/x^2 \end{vmatrix} = -3$$

$$W_1 = \begin{vmatrix} 0 & 1/x \\ 1 & -1/x^2 \end{vmatrix} = -\frac{1}{x}, \quad W_2 = \begin{vmatrix} x^2 & 0 \\ 2x & 1 \end{vmatrix} = x^2$$

$$u_1'(x) = \frac{\left(\frac{2}{x} - \frac{1}{x^2}\right)\left(-\frac{1}{x}\right)}{-3} = \frac{2}{3x^2} - \frac{1}{3x^3} \quad \text{and} \quad u_1(x) = \frac{2}{3}\left(-\frac{1}{x}\right) - \frac{1}{3}\left(-\frac{1}{2x^2}\right)$$

$$u_2'(x) = \frac{\left(\frac{2}{x} - \frac{1}{x^2}\right)(x^2)}{-3} = -\frac{2}{3}x + \frac{1}{3} \quad \text{and} \quad u_2(x) = -\frac{1}{3}x^2 + \frac{1}{3}x$$

Thus $\quad \psi_p = u_1(x)\phi_1(x) + u_2(x)\phi_2(x)$

$$= \left[-\frac{2}{3x} + \frac{1}{6x^2}\right]x^2 + \left[-\frac{x^2}{3} + \frac{x}{3}\right]\frac{1}{x} = -x + \frac{1}{2}.$$

The general solution of given non-homogeneous equation is $\psi = -x + \frac{1}{2} + c_1 x^2 + \frac{c_2}{x}$, where c_1 and c_2 are constants.

Q. 6. One solution of $x^2 y'' - xy' + y = 0 \ (x > 0)$ is $\phi_1(x) = x$. Find the solutions ψ of $x^2 y'' - xy' + y = x^2$ satisfying $\psi(1) = 1, \ \psi'(1) = 0$.

Ans. : The given non-homogeneous equation is $y'' - \frac{1}{x}y' + \frac{1}{x^2}y = 1$. (We can divide the equation by x^2 as x^2 is positive)

Let $\phi_2(x) = u(x)\phi_1(x) = u(x)x$ be an other solution.

$L(\phi_2) = [u''x + 2u'] - \frac{1}{x}[u'x + u] + \frac{1}{x^2}u(x)x = 0$ gives $u''x + u' = 0$. Therefore $u' = \frac{c_1}{x}$ and $u(x) = c_1 \log x.$

$\phi_2(x) = u(x)\phi_1(x) = c_1 x \log x$ is second solution. Without loss of generality we choose $c_1 = 1$.

Thus, $\phi_1(x) = x$ and $\phi_2(x) = x \log x$ are two solutions of homogeneous equation

$$y'' - \frac{1}{x}y' + \frac{1}{x^2}y = 0.$$

A solution ψ_p of a non-homogeneous equation

$y'' - \frac{1}{x}y' + \frac{1}{x^2}y = 1$ has the form

$$\psi_p = u_1(x)\phi_1(x) + u_2(x)\phi_2(x)$$

Then $u_1'(x) = \frac{b(x)W_1}{W(\phi_1, \phi_2)}$, $u_2'(x) = \frac{b(x)W_2}{W(\phi_1, \phi_2)}$

where $\quad b(x)=1, \quad W(\phi_1,\phi_2)=\begin{vmatrix} x & x\log x \\ 1 & 1+\log x \end{vmatrix}=x$

$$W_1=\begin{vmatrix} 0 & x\log x \\ 1 & 1+\log x \end{vmatrix}=-x\log x, \quad W_2=\begin{vmatrix} x & 0 \\ 1 & 1 \end{vmatrix}=x$$

$$u_1'(x)=\frac{-x\log x}{x}=-\log x \text{ and } u_1(x)=-(x\log x-x)$$

$$u_2'(x)=\frac{x}{x}=1 \text{ and } u_2(x)=x$$

Therefore $\psi_p=(x-x\log x)x+x\cdot x\log x=x^2$

The general solution of given non-homogeneous equation is

$$\psi=x^2+c_1x+c_2x\log x$$

Since $\psi(1)=1$ and $\psi'(1)=0, 1+c_1=1$ and $c_1=0$

$$\psi'(x)=2x+C_2(\log x+1), \psi'(1)=2+c_2=0 \text{ and } c_2=-2$$

Therefore the solution satisfying given initial condition is

$$\psi(x)=x^2-2x\log x.$$

Q. 7.

(a) Show that there is a basis ϕ_1, ϕ_2 for the solutions of $x^2y''+4xy'+(2+x^2)y=0 \ (x>0)$ of the form

$$\phi_1(x)=\frac{\psi_1(x)}{x^2}, \quad \phi_2(x)=\frac{\psi_2(x)}{x^2}$$

(b) Find all solutions of

$$x^2y''+4xy'+(2+x^2)y=x^2 \text{ for } x>0\cdot$$

Ans. :

(a) Let $\phi=\frac{v}{x^2}$ be a solution of the given homogeneous equation.

$$L(y)=y''+\frac{4}{x}y'+\left(\frac{2}{x^2}+1\right)y=0$$

Then, $\quad \phi'=\frac{v'}{x^2}-\frac{2v}{x^3}, \quad \phi''=\frac{v''}{x^2}-\frac{4v'}{x^3}+\frac{6v}{x^4}$ and

$$L(y)=\left(\frac{v''}{x^2}-\frac{4v'}{x^3}+\frac{6v}{x^4}\right)+\frac{4}{x}\left(\frac{v'}{x^2}-\frac{2v}{x^3}\right)+\left(\frac{2}{x^2}+1\right)\frac{v}{x^2}=0.$$

Therefore $L(y)=0$ implies $v''+v=0.$

$\psi_1(x)=\cos x$ and $\psi_2(x)=\sin x$ are two linearly independent solutions of $v''+v=0.$

Thus, $\phi_1(x)=\dfrac{\cos x}{x^2}$ and $\phi_2(x)=\dfrac{\sin x}{x^2}$ are two linearly independent solutions of given equation.

(b) $\phi_1(x)=\dfrac{\cos x}{x^2}$, $\phi_2(x)=\dfrac{\sin x}{x^2}$

$$L(y)=y''+\frac{4}{x}y'+\left(\frac{2}{x^2}+1\right)y=1$$

A solution ψ_p of $L(y)$ = 1 has the form

$$\psi_p=u_1(x)\phi_1(x)+u_2(x)\phi_2(x).$$

Then $u_1'(x)=\dfrac{b(x)W_1}{W(\phi_1,\phi_2)}$, $u_2'(x)=\dfrac{b(x)W_2}{W(\phi_1,\phi_2)}$

where $b(x)=1$

$$W(\phi_1,\phi_2)=\begin{vmatrix}\dfrac{\cos x}{x^2} & \dfrac{\sin x}{x^2}\\ -\dfrac{\sin x}{x^2}-\dfrac{2\cos x}{x^3} & \dfrac{\cos x}{x^2}-\dfrac{2\sin x}{x^3}\end{vmatrix}=\frac{1}{x^4}$$

$$W_1=\begin{vmatrix}0 & \dfrac{\sin x}{x^2}\\ 1 & \dfrac{\cos x}{x^2}-\dfrac{2\sin x}{x^3}\end{vmatrix}=-\frac{\sin x}{x^2},\quad W_2=\begin{vmatrix}\dfrac{\cos x}{x^2} & 0\\ -\dfrac{\sin x}{x^2}-\dfrac{2\cos x}{x^3} & 1\end{vmatrix}=\frac{\cos x}{x^2}$$

$$u_1'(x)=\frac{-\sin x/x^2}{1/x^4}=-x^2\sin x,\ u_1(x)=x^2\cos x-2x\sin x-2\cos x$$

$$u_2'(x)=\frac{\cos x/x^2}{1/x^4}=+x^2\cos x,\ u_2(x)=x^2\sin x+2x\cos x-2\sin x$$

$$\psi_p=u_1(x)\phi_1(x)+u_2(x)\phi_2(x)$$

$$=(x^2\cos x-2x\sin x-2\cos x)\frac{\cos x}{x^2}+(x^2\sin x+2x\cos x-2\sin x)\frac{\sin x}{x^2}$$

$$=1-\frac{2}{x^2}$$

Therefore the general solution of non-homogeneous equation is

$$\psi=\psi_p+c_1\phi_1+c_2\phi_2=1-\frac{2}{x^2}+c_1\frac{\cos x}{x^2}+c_2\frac{\sin x}{x^2}.$$

Q. 8. Consider the equation $y''+y=b(x)$ where b is a continuous function on $1\le x<\infty$ satisfying $\int_1^\infty |b(t)|\,dt<\infty.$ show that particular solution ψ_p is given by

$$\psi_p(x) = \int_1^x \sin(x-t)b(t)\,dt$$

Ans. : The homogeneous equation $y'' + y = 0$ has two solutions $\phi_1(x) = \cos x$, $\phi_2(x) = \sin x$.

The particular solution ψ_p has the form

$$\psi_p(x) = u_1(x)\phi_1(x) + u_2(x)\phi_2(x) \quad \text{where}$$

$$u_1'(x) = \frac{b(x)W_1}{W_1(\phi_1, \phi_2)}, \quad u_2'(x) = \frac{b(x)W_2}{W_1(\phi_1, \phi_2)}$$

$$W(\phi_1, \phi_2) = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = 1$$

$$W_1 = \begin{vmatrix} 0 & \sin x \\ 1 & \cos x \end{vmatrix} = -\sin x, \quad W_2 = \begin{vmatrix} \cos x & 0 \\ -\sin x & 1 \end{vmatrix} = \cos x$$

$$u_1(x) = \int_1^x \frac{b(t)W_1(t)}{W(\phi_1, \phi_2)(t)}\,dt = \int_1^x \frac{-b(t)\sin t}{1}\,dt$$

$$u_2(x) = \int_1^x \frac{b(t)W_2(t)}{W(\phi_1, \phi_2)(t)}\,dt = \int_1^x \frac{b(t)\cos t}{1}\,dt$$

$$\psi_p = -\cos x \int_1^x b(t)\sin t\,dt + \sin x \int_1^x b(t)\cos t\,dt$$

$$= \int_1^x b(t)\left[\sin x \cos t - \cos x \sin t\right]dt$$

$$= \int_1^x b(t)\sin(x-t)\,dt$$

EXERCISE

1. Consider the equation $y'' + a_1(x)y' + a_2(x)y = 0$ where $a_1(x)$ and $a_2(x)$ are continuous functions on $-\infty < x < \infty$ and are periodic with period $\theta > 0$ i.e. $a_1(x+\theta) = a_1(x)$, $a_2(x+\theta) = a_2(x)$ for all x. Let ϕ be a non-trival solution and let $\psi(x) = \phi(x+\theta)$. Shown that ψ is also a solution.

2. Consider the equation $y'' + \alpha(x)y = 0$ where α is a continuous functions on $-\infty < x < \infty$ which is of period $\theta > 0$. Let ϕ_1, ϕ_2 be the basis for solution satisfying

$$\phi_1(0) = 1, \quad \phi_2(0) = 0$$

$$\phi_1'(0) = 0, \quad \phi_2'(0) = 1$$

Show that there is at least one non-trival solution ϕ of period θ if and only if $\phi_1(\theta) + \phi_2'(\theta) = 2$.

3. One solution of $L(y) = y'' + \frac{1}{4x^2} y = 0$ for $x > 0$ is $\phi_1(x) = x^{1/2}$ show that there is another solution ψ of the form $\psi = u\phi$ where u is some function.

4. Use the method of variation of parameter and find the particular solution of the following equations where the solutions for the related homogeneous equation are given.

(a) $y'' - \frac{2}{x} y' + \frac{2}{x^2} y = x \log x,\ \phi_1(x) = x,\ \phi_2(x) = x^2$ $\left[\text{Ans. : } \psi_p = \frac{1}{2} x^3 \log x - \frac{3}{4} x^3\right]$

(b) $x^2 y'' + xy' - 4y = x^3,\ \phi_1 = x^2,\ \phi_2 = \frac{1}{x^2}$ $\left[\text{Ans. : } \psi_p = x^3/5\right]$

(c) $x^2 y'' + xy' - y = x^2 e^{-x},\ \phi_1 = x,\ \phi_2 = 1/x$ $\left[\text{Ans. : } \psi_p = e^{-x}(1 + x^{-1})\right]$

(d) $2x^2 y'' + 3xy' - y = x^{-1};\ \phi_1 = x^{1/2},\ \phi_2 = x^{-1}$ $\left[\text{Ans. : } \psi_p = -\frac{1}{3} x^{-1} \log x\right]$

Unit 3 : Homogeneous equations with analytic coefficients

So far we have shown how to construct solutions of various special types of differential equations using the exponential function, polynomials and the fundamental theorem of calculus - that is how to reduce the integration of these differential equations to one or more quadratures. The major difficulty with linear equations with variable coefficients, from a practical point of view, is that it is rare that we can solve equations in terms of elementary functions, such as exponential and trignometric functions. . However in case the coefficients $a_1, a_2, a_3, ..., a_n$, and b have convergent power series expansions the solutions will have this property also and these series solutions can be obtained by a simple formal process.

An infinite series of the form $\sum_{n=0}^{\infty} a_n (z - z_0)^n$ is called a power series in $z - z_0$. Here a_n, z, z_0 are complex numbers. With every power series there is associated a disk, called the disk of convergence such that a series converges absoulately for every z interior to this disk. The center of the disk is at z_0 and its radius is called the radius of convergence of the power series.

Given a power series $\sum_{n=0}^{\infty} a_n (z - z_0)^n$, let $\lambda = \operatorname{Lim\,sup}_{n \to \infty} \sqrt[n]{|a_n|}$, $r = \frac{1}{\lambda}$ (where $r = 0$ if $\lambda = +\infty$ and $r = \infty$ if $\lambda = 0$). The series converges absoulately if $|z - z_0| < r$ and diverges if $|z - z_0| > r$.

If x_0, x and a_n are real numbers the series $\sum_{n=0}^{\infty} a_n (x - x_0)^n$ is called a real power series. Its disk of convergence intersects the real axis in an interval $(x_0 - r, x_0 + r)$ called the interval of convergence.

If g is a function defined on an interval I containing point x_0 we say that g is analytic at x_0 if g can be expanded in a power series about x_0 which has a positive radius of convergence. Thus g is analytic at x_0 if it can be representd in the form

$$g(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n$$

Where a_n are constants and the series converges for $|x - x_0| < r,\ r > 0$. If g has a power series expansion then all the derivatives of g exist on $|x - x_0| < r$ and they may be computed by differentating the series term by term that is

$$g'(x) = \sum_{n=0}^{\infty} n\,a_n (x - x_0)^{n-1}\ ,\quad g''(x) = \sum_{n=0}^{\infty} n(n-1)\,a_n (x - x_0)^{n-2} \text{ etc.}$$

The differentiated series converges on $|x - x_0| < r$.

In calculus there are certain tests by which one could determine an interval of converge of a real power series. A simple one and one which is frequently used is known as ratio test.

The series $\sum_{n=0}^{\infty} u_i$ converges absolutely if $\underset{n\to\infty}{\text{Lt}} \left|\frac{u_{n+1}}{u_n}\right| = k < 1.$

Example 1 : For the power series $\sum_{n=0}^{\infty} \frac{x^n}{n}$,

$$\underset{n\to\infty}{\text{Lt}} \left|\frac{u_{n+1}}{u_n}\right| = \underset{n\to\infty}{\text{Lt}} \left|\frac{x^{n+1}/n+1}{x^n/n}\right| = \underset{n\to\infty}{\text{Lim}} \left|\frac{n}{n+1}\cdot x\right| = |x|$$

Hence the series converges absoulately if | x | < |.

Example 2 : For the power series

$$1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots. + \frac{(-1)^{n-1}}{(2n-2)!} x^{2n-2} + \ldots.,$$

$$u_n = \frac{(-1)^{n-1}}{(2n-2)!} x^{2n-2} \quad \text{and} \quad u_{n+1} = \frac{(-1)^n}{(2n)!} x^{2n}$$

Therefore

$$\underset{n\to\infty}{\text{Lim}} \left|\frac{u_{n+1}}{u_n}\right| = \underset{n\to\infty}{\text{Lim}} \left|\frac{(-1)^n x^{2n}}{(2n)!} \times \frac{(2n-2)!}{(-1)^{n-1} x^{2n-2}}\right| = \underset{n\to\infty}{\text{Lim}} \frac{x^2}{2n(2n-1)} = 0 \text{ for each } x$$

Hence the series converges absoulately for all x. Its interval of convergence is the entire real axis.

Theorem 2.3.1 : (Existence theorem)

Let x_0 be a real number and suppose that the coefficients $a_1, a_2, a_3, \ldots.. a_n$ in

$$L(y) = y^{(n)} + a_1(x) y^{(n-1)} + a_2(x) y^{(n-2)} + \ldots. + a_n(x) y$$

have convergent power series expansions in powers of $(x - x_0)$ on an interval $|x - x_0| < r,\ r > 0$.

If $\alpha_1, \alpha_2, \alpha_3, \ldots.. \alpha_n$ are any n constants, there exists a solution ϕ of the problem

$$L(y) = 0,\ \ y(x_0) = \alpha_1,\ \ y'(x_0) = \alpha_2, \ldots., y^{(n-1)}(x_0) = \alpha_n$$

with a power series expansion

$$\phi(x) = \sum_{k=0}^{\infty} c_k (x - x_0)^k$$

is convergent for $|x - x_0| < r$. We have

$k!\, c_k = \alpha_{k+1}$ $(k = 0, 1, 2, 3, ..., n-1)$, and c_k for $k \geq n$ may be computed in terms of c_0, c_1, c_2, c_3,..... c_{n-1} by substituting the series into $L(y) = 0$.

If the coefficients a_1, a_2, a_3,..... a_n are analytic at x_0 then the solutions are also analytic. The solutions can be computed by a formal algebraic process.

Illustration :

$$L(y) = y'' - xy = 0$$

Here $a_1(x) = 0$, $a_2(x) = -x$ are analytic for all real x.

Let the solution of the equation $L(y) = 0$ be ϕ defined by

$$\phi(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + ... = \sum_{k=0}^{\infty} a_n x^n$$

Then $$\phi'(x) = a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + ... = \sum_{n=1}^{\infty} n\, a_n x^{n-1}$$

$$\phi''(x) = 2a_2 + 6a_3 x + 12a_4 x^2 + ... = \sum_{n=0}^{\infty} n(n-1) a_n x^{n-2}$$

$$\phi''(x) - x\phi(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} - x \sum_{n=0}^{\infty} a_n x^n$$

$$= 2a_2 + 6a_3 x + 12a_4 x^2 + 20a_5 x^3 + ... - \{a_0 x + a_1 x^2 + a_2 x^3 + a_3 x^4 + ...\}$$

$$= 2a_2 + (6a_3 - a_0)x + (12a_4 - a_1)x^2 + (20a_5 - a_2)x^3 + ...$$

$$= 2a_2 + \sum_{n=1}^{\infty} \left[(n+2)(n+1)\, a_{n+2} - a_{n-1}\right] x^n$$

ϕ is a solutin of $L(y) = y'' - xy = 0$ if $\phi'' - x\phi = 0$ or

$$2a_2 + \sum_{n=1}^{\infty} \left[(n+2)(n+1)\, a_{n+2} - a_{n-1}\right] x^n = 0$$

Above equation is true only if all the coefficients of the power series of x are zero. Thus,

$$2a_2 = 0,\ (n+2)(n+1)\, a_{n+2} - a_{n-1} = 0,\ n = 1, 2, 3....$$

This gives an infinite set of equations, and can be solved for a_n. Thus, for $n = 1$ we have

$$(3)\cdot(2)\cdot a_3 = a_0 \quad \text{or} \quad a_3 = \frac{a_0}{(3)\cdot(2)}$$

For $n = 2$ we find

$$(4)\cdot(3)\, a_4 = a_1 \quad \text{or} \quad a_4 = \frac{a_1}{(4)\cdot(3)}$$

Continuing in this way we see that

$$a_0; \qquad a_1 \qquad a_2 = 0$$

$$a_3 = \frac{a_0}{(3)\cdot(2)}; \qquad a_4 = \frac{a_1}{(4)\cdot(3)}; \qquad a_5 = \frac{a_2}{(5)\cdot(4)} = 0$$

$$a_6 = \frac{a_3}{(6)\cdot(5)} = \frac{a_0}{(6)\cdot(5)\cdot(3)\cdot(2)}; \qquad a_7 = \frac{a_4}{(7)\cdot(6)} = \frac{a_1}{(7)\cdot(6)\cdot(4)\cdot(3)}; \qquad a_8 = 0$$

$$a_9 = \frac{a_6}{(9)\cdot(8)} = \frac{a_0}{(9)\cdot(8)\cdot(6)\cdot(5)\cdot(3)\cdot(2)}; \qquad a_{10} = \frac{a_7}{(10)\cdot(9)} = \frac{a_1}{(10)\cdot(9)\cdot(7)\cdot(6)\cdot(4)\cdot(3)}; \qquad a_{11} = 0.$$

In general

$$a_{3m} = \frac{a_0}{(2)\cdot(3)\cdot(5)\cdot(6)\cdot(8)\cdot(9)\cdots(3m-1)(3m)};$$

$$a_{3m+1} = \frac{a_1}{(3)\cdot(4)\cdot(6)\cdot(7)\cdot(9)\cdot(10)\cdots(3m)(3m+1)}; a_{3m+2} = 0$$

Thus all the constants are determined in terms of a_0 and a_1. Collecting together terms containing a_0 and a_1 as a factor we have

$$\phi(x) = a_0\left[1 + \frac{x^3}{(3)\cdot(2)} + \frac{x^6}{(2)\cdot(3)\cdot(5)\cdot(6)} + \ldots\right] + a_1\left[x + \frac{x^4}{(4)\cdot(3)} + \frac{x^7}{(3)\cdot(4)\cdot(6)\cdot(7)} + \ldots\right]$$

Let ϕ_1 and ϕ_2 represent the two series in the brackets.

Thus, $$\phi_1(x) = 1 + \sum_{m=1}^{\infty} \frac{x^{3m}}{2\cdot3\cdot5\cdot6\cdot8\cdot9....(3m-1)(3m)},$$

$$\phi_2(x) = x + \sum_{m=1}^{\infty} \frac{x^{3m+1}}{3\cdot4\cdot6\cdot7\cdot9\cdot10....(3m)(3m+1)}.$$

We have shown, in a formal way that ϕ satisfies $y'' - xy = 0$ for any two contants a_0 and a_1

In particular the choice $a_0 = 0$ and $a_1 = 1$ implies $\phi_2(x)$ satisfies the equation and $a_0 = 1$, $a_1 = 0$ implies $\phi_1(x)$ satisfies the equation.

The only question that remains is about the convergence of the series, defining $\phi_1(x)$ and $\phi_2(x)$.

$$\phi_1(x) = 1 + \sum d_m(x) = 1 + \sum_{m=1}^{\infty} \frac{x^{3m}}{2\cdot3\cdot5\cdot6\cdot8\cdot9....(3m-1)(3m)}$$

$$\frac{d_{m+1}}{dm} = \frac{x^{3m+3}}{2\cdot3\cdot5\cdot6\cdot8\cdot9\cdots(3m)(3m+2)(3m+3)} \times \frac{2\cdot3\cdot5\cdot6\cdot8\cdot9\cdots(3m-1)(3m)}{x^{3m}}$$

$$= \frac{x^3}{(3m+2)(3m+3)}$$

$$\operatorname*{Lt\ sup}_{m\to\infty} \frac{1}{(3m+2)(3m+3)} = \operatorname*{Lt\ inf}_{m\to\infty} \frac{1}{(3m+2)(3m+3)} = 0.$$

The series converges if $|x| < \infty$.

Similarly $\phi_2(x)$ is convergent series.

EXAMPLES

1. Find two linearly independent power series solutions (in powers of x) of the following equations.

(a) $y'' - xy' + y = 0$ (b) $y'' + 3x^2 y' - xy = 0$

(c) $y'' - x^2 y = 0$ (d) $y'' + 3x^3 y' + x^2 y = 0$

Ans. (a) : Let $\phi(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots + a_n x^n + \ldots = \sum_{n=0}^{\infty} a_n x^n$ be a solution of $L(y) = y'' - xy' + y = 0$. Since it is a solution it satisfies the equation $L(\phi) = 0$.

$$\phi(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots + a_n x^n + \ldots = \sum_{n=0}^{\infty} a_n x^n$$

Then $$\phi'(x) = a_1 + 2a_2 x + 3a_3 x^2 + \ldots + n\,a_n x^{n-1} + \ldots = \sum_{n=1}^{\infty} n\,a_n x^{n-1}$$

$$\phi''(x) = 2a_2 + 3\cdot 2a_3 x + \ldots + n(n-1)\,a_n x^{n-2} + \ldots = \sum_{n=2}^{\infty} n(n-1)\,a_n x^{n-2}$$

Thus, $$L(\phi) = \sum_{n=2}^{\infty} n(n-1)\,a_n x^{n-2} - x\sum_{n=1}^{\infty} n\,a_n x^{n-1} + \sum_{n=0}^{\infty} a_n x^n = 0.$$

$$= \sum_{n=0}^{\infty} (n+2)(n+1)\,a_{n+2} x^n - \sum_{n=1}^{\infty} n\,a_n x^n + \sum_{n=0}^{\infty} a_n x^n = 0$$

$$= (2a_2 + a_0) + \sum_{n=1}^{\infty} \{(n+2)(n+1)\,a_{n+2} - na_n + a_n\} x^n = 0$$

$$L(\phi) = (2a_2 + a_0) + \sum_{n=1}^{\infty} \{(n+2)(n+1)\,a_{n+2} - (n-1)a_n\} x^n = 0$$

We see that $L(\phi) = 0$ if and only if $2a_2 + a_0 = 0$ and $(n+2)(n+1)\,a_{n+2} - (n-1)\,a_n = 0$ for $n = 1, 2, 3, \ldots.$ $a_2 = -\frac{1}{2}a_0$; $a_{n+2} = \frac{(n-1)\,a_n}{(n+2)(n+1)}$ is called recurrence relation.

a_0 ; a_1

$a_2 = \frac{-a_0}{2\cdot 1}$; ; $a_3 = 0.a_1$

$a_4 = \frac{a_2}{4\cdot 3} = -\frac{a_0}{2\cdot 3\cdot 4}$; $a_5 = \frac{2}{5\cdot 4}.0 = 0$

$a_6 = \frac{3a_4}{6\cdot 5} = -\frac{3a_0}{2\cdot 3\cdot 4\cdot 5\cdot 6}$; $a_7 = 0$

$a_8 = -\frac{3\cdot 5\,a_0}{2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7\cdot 8}$; $a_9 = 0$

$a_{10} = -\frac{3\cdot 5\cdot 7\,a_0}{10!}$; $a_{11} = 0$

In general $a_{2n+1}=0 \quad n=1,2,3,....$

$$a_{2n}=-\frac{3\cdot 5\cdot 7\cdot 9\cdots(2n-3)\,a_0}{(2n)!}$$

$$=-\frac{2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7\cdot 8\cdot 9\cdots(2n-3)(2n-2)(2n-1)(2n)}{2\cdot 4\cdot 6\cdot 8\cdots 2n(2n-1)(2n)!}a_0$$

$$=-\frac{(2n)\,!}{2^n n!(2n-1)(2n)!}a_0$$

$$=-\frac{a_0}{2^n n!(2n-1)}$$

$$\phi(x)=\sum_{n=0}^{\infty}a_{2n}x^{2n}+\sum_{n=0}^{\infty}a_{2n+1}x^{2n+1}$$

$$=a_0-\sum_{n=1}^{\infty}\frac{a_0}{2^n n!(2n-1)}x^{2n}+a_1x$$

$$=a_0\left[1-\sum_{n=1}^{\infty}\frac{x^{2n}}{2^n n!(2n-1)}\right]+a_1x$$

$\phi_1(x)=1-\sum_{n=1}^{\infty}\frac{x^{2n}}{2^n n!(2n-1)}$ and $\phi_2(x)=x$ are two solutions of the equation

Let $\qquad \phi_1(x)=\sum_{m=0}^{\infty}d_m(x)$

$$\frac{d_{n+1}}{dn}=\frac{\dfrac{x^{2(n+1)}}{2^{n+1}(n+1)!(2n+1)}}{\dfrac{x^{2n}}{2^n n!(2n-1)}}=\frac{x^2(2n-1)}{2(n+1)(2n+1)}$$

$$\underset{n\to\infty}{\text{Lt}}\frac{d_{n+1}}{d_n}=\underset{n\to\infty}{\text{Lt}}\frac{(2n-1)}{2(n+1)(2n+1)}=0$$

Redius of convergence = ∞

The series converges if $|x|<\infty$ i.e. all values of x. Both the solutions are convergent for all values of x.

Ans. (b) : Let $\phi(x)=\sum_{n=0}^{\infty}a_nx^n$ be a solution.

$$\phi'(x)=\sum_{n=1}^{\infty}na_nx^{n-1}\;,\;\phi''(x)=\sum_{n=2}^{\infty}n(n-1)a_nx^{n-2}$$

$$L(\phi)=\phi''+3x^2\phi'-x\phi$$

$$= \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + 3x^2 \sum_{n=1}^{\infty} n\, a_n x^{n-1} - x \sum_{n=0}^{\infty} a_n x^n$$

$L(\phi) = 0$ implies

$$2\cdot 1a_2 + 3\cdot 2a_3 x + 4\cdot 3a_4 x^2 + \cdots + 3\left[a_1 x^2 + 2a_2 x^3 + 3a_3 x^4 + 4a_4 x^5 + ...\right]$$

$$-\left[a_0 x + a_1 x^2 + a_2 x^3 + a_3 x^4 + ...\right] = 0.$$

$$2\cdot 1a_2 + (3\cdot 2a_3 - a_0)x + \left[4\cdot 3a_4 + (3-1)a_1\right]x^2 + \left[5\cdot 4a_5 + (3(2)-1)a_2\right]x^3$$

$$-\left[6.5a_6 + (3(3)-1)a_3\right]x^4 + \cdots = 0$$

$$2\cdot 1a_2 + (3\cdot 2a_3 - a_0)x + \sum_{n=1}^{\infty} \left[(n+3)(n+2)\, a_{n+3} + (3n-1)\, a_n\right]x^{n+1} = 0$$

Then $a_2 = 0$; $a_3 = \dfrac{a_0}{2\cdot 3}$; $a_{n+3} = -\dfrac{(3n-1)\, a_n}{(n+3)(n+2)}$

a_0 ; a_1 ; $a_2 = 0$

$a_3 = -\dfrac{a_0}{2\cdot 3}$; $a_4 = -\dfrac{2\, a_1}{4\cdot 3}$; $a_5 = 0$

$a_6 = -\dfrac{8\, a_3}{6\cdot 5} = +\dfrac{8\, a_0}{2\cdot 3\cdot 5\cdot 6}$; $a_7 = -\dfrac{11 a_4}{7\cdot 6} = \dfrac{11\cdot 2\, a_1}{7\cdot 6\cdot 4\cdot 3}$; $a_8 = 0$

$a_9 = -\dfrac{(18-1)(9-1)a_0}{9\cdot 8\cdot 6\cdot 5\cdot 3\cdot 2}$; $a_{10} = -\dfrac{20\cdot 11\cdot 2\, a_1}{10\cdot 9\cdot 7\cdot 6\cdot 4\cdot 3}$; $a_{11} = 0.$

The solution

$$\phi_1(x) = 1 + \sum_{m=1}^{\infty} \frac{(-1)^m (-1)\, 8\cdot 17\cdots(9m-1)}{2\cdot 3\cdot 5\cdot 6\cdot 8\cdot 9\cdots\cdot(3m-1)(3m)}\, x^{3m}$$

$$\phi_2(x) = x + \sum_{m=1}^{\infty} \frac{(-1)^m\, 2\cdot 11\cdot 20\cdots(3(3m-2)-1)}{3\cdot 4\cdot 6\cdot 7\cdot 9\cdot 10\cdots\cdot(3m)(3m+1)}\, x^{3m+1}$$

Ans. (c) : Let $\phi(x) = \sum_{n=0}^{\infty} a_n x^n$ be a solution of $y'' - x^2 y = 0.$ Since it is a solution $\phi(x)$ satisfies

$L(\phi) = \phi'' - x^2\phi = 0.$

$$\phi(x) = \sum_{n=0}^{\infty} a_n x^n, \quad \phi'(x) = \sum_{n=1}^{\infty} n\, a_n x^{n-1}, \quad \phi''(x) = \sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}$$

$$L(\phi) = \sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2} - x^2 \sum_{n=0}^{\infty} a_n x^n = 0.$$

$$2\cdot 1a_2 + 3\cdot 2\, a_3\, x + \sum_{n=4}^{\infty} n(n-1)a_n x^{n-2} - \sum_{n=0}^{\infty} a_n x^{n+2} = 0$$

$$2\cdot 1 a_2 + 3\cdot 2 a_3 x + \sum_{n=0}^{\infty} (n+4)(n+3) a_{n+4} x^{n+2} - \sum_{n=0}^{\infty} a_n x^{n+2} = 0$$

Here we have replaced n by $n + 4$ and therefore the sum is from 0 to ∞.

$$2\cdot 1 a_2 + 3\cdot 2 a_3 x + \sum_{n=0}^{\infty} \left[(n+4)(n+3) a_{n+4} - a_n\right] x^{n+2} = 0.$$

Thus, $a_2 = 0,$ $a_3 = 0$ and $a_{n+4} = \dfrac{a_n}{(n+4)(n+3)}.$

a_0	a_1	$a_2 = 0$	$a_3 = 0$
$a_4 = \dfrac{a_0}{3\cdot 4}$;	$a_5 = \dfrac{a_1}{5\cdot 4}$; ;	$a_6 = 0$;	$a_7 = 0$
$a_8 = \dfrac{a_4}{8\cdot 7} = \dfrac{a_0}{3\cdot 4\cdot 7\cdot 8}$;	$a_9 = \dfrac{a_5}{9\cdot 8} = \dfrac{a_1}{4\cdot 5\cdot 8\cdot 9}$;	$a_{10} = 0$;	$a_{11} = 0$
$a_{12} = \dfrac{a_0}{3\cdot 4\cdot 7\cdot 8\cdot 11\cdot 12}$;	$a_{13} = \dfrac{a_1}{4\cdot 5\cdot 8\cdot 9\cdot 12\cdot 13}$;	$a_{14} = 0$;	$a_{15} = 0$
⋮	⋮	⋮	⋮

Thus all the coefficients a_n's are determined in terms of a_0 and a_1 since $a_2 = a_3 = 0$ implies a_{4m+2} and $a_{4m+3} = 0$ for $m = 0, 1, 2, 3,.....$ Therefore

$$\phi(x) = \sum_{m=0}^{\infty} a_{4m} x^m + \sum_{m=0}^{\infty} a_{4m+1} x^{4m+1}$$

$$= a_0\left[1 + \sum_{m=1}^{\infty} \frac{x^{4m}}{3\cdot 4\cdot 7\cdot 11\cdot 12\cdots(4m-1)(4m)}\right]$$

$$+ a_1\left[x + \sum_{m=1}^{\infty} \frac{x^{4m+1}}{4\cdot 5\cdot 8\cdot 9\cdot 12\cdot 13\cdots(4m)(4m+1)}\right]$$

Therefore two linearly independent solutions are

$$\phi_1(x) = 1 + \sum_{m=1}^{\infty} \frac{x^{4m}}{3\cdot 4\cdot 7\cdot 11\cdot 12\cdot 13\cdots(4m-1)(4m)} \text{ and}$$

$$\phi_2(x) = x + \sum_{m=1}^{\infty} \frac{x^{4m+1}}{4\cdot 5\cdot 8\cdot 9\cdot 12\cdot 13\cdots(4m)(4m+1)}.$$

Ans. (d) : Let $\phi_2(x) = \sum_{n=0}^{\infty} a_n x^n$ be a solution of $L(y) = y'' + x^3 y' + x^2 y = 0.$ Therefore

$$L(\phi) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} + x^3 \sum_{n=1}^{\infty} n a_n x^{n-1} + x^2 \sum_{n=0}^{\infty} a_n x^n = 0$$

that is $$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} + \sum_{n=1}^{\infty} n a_n x^{n+2} + \sum_{n=0}^{\infty} a_n x^{n+2} = 0.$$

The first term starts from x^0 where as last two series start from x^3 and x^2 respectively. To get the common base we write the expansion in the following form

$$2\cdot 1a_2+3\cdot 2a_3x+4\cdot 3a_4x^2+\sum_{n=5}^{\infty} n(n-1)a_nx^{n-2}+\sum_{n=1}^{\infty} n\,a_nx^{n+2}+a_0x^2+\sum_{n=1}^{\infty} a_nx^{n+2}=0.$$

Therefore

$$2\cdot 1a_2+3\cdot 2a_3x+(4\cdot 3a_4+a_0)x^2+\sum_{n=5}^{\infty} n(n-1)a_nx^{n-2}+\sum_{n=1}^{\infty}(n+1)a_nx^{n+2}=0$$

We replace n by $n+4$ in the first series.

$$2\cdot 1a_2+3\cdot 2a_3x+(4\cdot 3a_4+a_0)x^2+\sum_{n=1}^{\infty}\left[(n+4)(n+3)a_{n+4}+(n+1)a_n\right]x^{n+2}=0.$$

Above equation is true for all values of x and therefore

$$a_2=0;\qquad a_3=0;\qquad 4\cdot 3a_4+a_0=0;\qquad (n+4)(n+3)a_{n+4}+(n+1)a_n=0$$

a_0	a_1	$a_2=0$	$a_3=0$
$a_4=\dfrac{-a_0}{3\cdot 4}$; ;	$a_5=\dfrac{-2a_1}{5\cdot 4}$;	$a_6=0$;	$a_7=0$
$a_8=+\dfrac{5a_0}{3\cdot 4\cdot 7\cdot 8}$;	$a_9=\dfrac{6\cdot 2a_1}{4\cdot 5\cdot 8\cdot 9}$;	$a_{10}=0$;	$a_{11}=0$
$a_{12}=-\dfrac{9\cdot 5a_0}{3\cdot 4\cdot 7\cdot 8\cdot 11\cdot 12}$;	$a_{13}=-\dfrac{10\cdot 6\cdot 2a_1}{4\cdot 5\cdot 8\cdot 9\cdot 12\cdot 13}$;	$a_{14}=0$;	$a_{15}=0$
$\vdots$	$\vdots$	$\vdots$	$\vdots$

$$a_{4m}=\frac{(-1)^m 5\cdot 9\cdot 13\cdots(4m-3)}{3\cdot 4\cdot 7\cdot 8\cdot 11\cdot 12\cdots(4m-1)(4m)}\ ;$$

$$a_{4m+1}=\frac{(-1)^m 2\cdot 6\cdot 10\cdots(4m-2)}{4\cdot 5\cdot 8\cdot 9\cdots(4m)(4m+1)}\ ;\ a_{4m+2}=a_{4m+3}=0.$$

Therefore two linearly independent solutions are

$$\phi_1(x)=1+\sum_{m=1}^{\infty}\frac{(-1)^m 5\cdot 9\cdot 13\cdots(4m-3)}{3\cdot 4\cdot 7\cdot 8\cdot 11\cdot 12\cdots(4m-1)(4m)}x^{4m}$$

$$\phi_2(x)=x+\sum_{m=1}^{\infty}\frac{(-1)^m 2\cdot 6\cdot 10\cdots(4m-2)}{4\cdot 5\cdot 8\cdot 9\cdots(4m)(4m+1)}x^{4m+1}$$

2. Find the solution ϕ of $y''+(x-1)^2y'-(x-1)y=0$ in the form $\phi(x)=\sum_{k=0}^{\infty}a_k(x-1)^k$ which satsfies $\phi(1)=1,\ \phi'(1)=0.$

Ans. : Let $\phi(x)=\sum_{k=0}^{\infty}a_k(x-1)^k$ be a solution of $L(y)=y''+(x-1)^2y'-(x-1)y=0.$

$$L(\phi) = \sum_{k=2}^{\infty} k(k-1)\, a_k\, (x-1)^{k-2} + (x-1)^2 \sum_{k=1}^{\infty} k\, a_k\, (x-1)^{k-1} - \sum_{k=0}^{\infty} a_k\, (x-1)^{k+1} = 0.$$

that is $$\sum_{k=2}^{\infty} k(k-1)\, a_k\, (x-1)^{k-2} + \sum_{k=1}^{\infty} k\, a_k\, (x-1)^{k+1} - \sum_{k=0}^{\infty} a_k\, (x-1)^{k+1} = 0.$$

$$2\cdot 1 a_2 + 3\cdot 2 a_3 (x-1) + \sum_{k=4}^{\infty} k(k-1)\, a_k\, (x-1)^{k-2} + \sum_{k=1}^{\infty} k\, a_k\, (x-1)^{k+1}$$

$$-\, a_0 (x-1) - \sum_{k=1}^{\infty} a_k\, (x-1)^{k+1} = 0$$

In the third term replace k by $k + 3$ we get

$$2\cdot 1 a_2 + \left[3\cdot 2 a_3 - a_0\right](x-1) + \sum_{k=1}^{\infty} \left[(k+3)(k+2)\, a_{k+3} + (k-1)\, a_k\right] x^{k+1} = 0.$$

Thus,

$$a_0 \qquad ; a_1 \qquad ; a_2 = 0$$

$$a_3 = \frac{a_0}{3\cdot 2} \qquad ; a_4 = 0 \qquad ; a_5 = 0$$

$$a_6 = -\frac{2\, a_3}{6\cdot 5} = \frac{-2\, a_0}{2\cdot 3\cdot 5\cdot 6} \qquad ; a_7 = 0 \qquad ; a_8 = 0$$

$$a_9 = \frac{-5 a_6}{9\cdot 8} = \frac{5\cdot 2\, a_0}{2\cdot 3\cdot 5\cdot 6\cdot 8\cdot 9} \qquad ; a_{10} = 0 \qquad ; a_{11} = 0.$$

$$\vdots \qquad \vdots \qquad \vdots$$

$$a_{3m} = \frac{(-1)^m\, 2\cdot 5\cdot 8 \cdots (3m-1)\, a_0}{2\cdot 3\cdot 5\cdot 6\cdot 8\cdot 9 \cdots (3m-1)(3m)} \quad ; a_{3m+1} = 0 \text{ for } m = 1, 2 \ldots \quad ; a_{3m+2} = 0.$$

Corresponding to the coefficients a_0 and a_1 we get the following two linearly independent solutions.

$$\phi_1(x) = 1 + \sum_{m=1}^{\infty} \frac{(-1)^m\, 2\cdot 5\cdot 8 \cdots (3m-1)}{2\cdot 3\cdot 5\cdot 6\cdot 8\cdot 9 \cdots (3m-1)(3m)}\, (x-1)^{3m} \quad \text{and}$$

$$\phi_1(x) = (x-1)$$

The general solution ϕ is

$$\phi(x) = a_0 \left[1 + \sum_{m=1}^{\infty} \frac{(-1)^m\, 2\cdot 5\cdot 8 \cdots (3m-1)}{2\cdot 3\cdot 5\cdot 6 \cdots (3m-1)(3m)} (x-1)^{3m}\right] + a_1 (x-1)$$

$$\phi_1(1) = 1 \text{ give } a_0 = 1$$

$$\phi'(x) = a_0 \sum_{m=1}^{\infty} \frac{(-1)^m\, 2\cdot 5\cdot 8 \cdots (3m-1)(3m)}{2\cdot 3\cdot 5\cdot 6 \cdots (3m-1)(3m)}\, (x-1)^{3m-1} + a_1$$

$$\phi'(1) = 0 \text{ give } a_1 = 0.$$

Thus, the required solution is

$$\phi(x) = 1 + \sum_{m=1}^{\infty} \frac{(-1)^m 2\cdot 5\cdot 8\cdots(3m-1)}{2\cdot 3\cdot 5\cdot 6\cdots(3m-1)(3m)} (x-1)^{3m}.$$

3. Compute the solution ϕ of $y''' - xy = 0$ which satisfies $\phi(0) = 1,\ \phi'(0) = 0,\ \phi''(0) = 0.$

Ans. : Let $\phi(x) = \sum_{k=0}^{\infty} a_n x^n$ be a solution of $L(y) = y''' - xy = 0.$ Then

$$L(\phi) = \sum_{n=3}^{\infty} n(n-1)(n-2)a_n x^{n-3} - x\sum_{n=0}^{\infty} a_n x^n = 0$$

Then $\quad 3\cdot 2\cdot 1 a_3 + \sum_{n=4}^{\infty} n(n-1)(n-2)a_n x^{n-3} - \sum_{n=0}^{\infty} a_n x^{n+1} = 0$

In the first sum replace n by $n + 4$, then

$$3\cdot 2\cdot 1 a_3 + \sum_{n=0}^{\infty} \left[(n+4)(n+3)(n+2)a_{n+4} - a_n\right]x^{n+1} = 0$$

Thus, $a_3 = 0$ and $a_{n+4} = \dfrac{a_n}{(n+4)(n+3)(n+2)}.$

a_0 ; a_1 ; a_2 ; $a_3 = 0$

$a_4 = \dfrac{a_0}{4\cdot 3\cdot 2}$; $a_5 = \dfrac{a_1}{5\cdot 4\cdot 3}$; $a_6 = \dfrac{a_2}{6\cdot 5\cdot 4}$; $a_7 = 0$

$a_8 = \dfrac{a_0}{8\cdot 7\cdot 6\cdot 4\cdot 3\cdot 2}$; $a_9 = \dfrac{a_1}{9\cdot 8\cdot 7\cdot 5\cdot 4\cdot 3}$; $a_{10} = \dfrac{a_2}{10\cdot 9\cdot 8\cdot 6\cdot 5\cdot 4}$; $a_{11} = 0$

$a_{12} = \dfrac{a_0}{2\cdot 3\cdot 4\cdot 6\cdot 7\cdot 8\cdot 10\cdot 11\cdot 12}$; $a_{13} = \dfrac{a_1}{3\cdot 4\cdot 5\cdot 7\cdot 8\cdot 9\cdot 11\cdot 12\cdot 13}$; $a_{14} = \dfrac{a_2}{4\cdot 5\cdot 6\cdot 8\cdot 9\cdot 10\cdot 12\cdot 13\cdot 14}$; $a_{15} = 0$

$\vdots \qquad \vdots \qquad \vdots \qquad \vdots$

$a_{4m} = \dfrac{a_0}{2\cdot 3\cdot 4\cdot 6\cdot 7\cdot 8\cdots(4m-2)(4m-1)(4m)}$; $a_{4m+1} = \dfrac{a_1}{3\cdot 4\cdot 5\cdots(4m-1)(4m)(4m+1)}$

; $a_{4m+2} = \dfrac{a_2}{4\cdot 5\cdot 6\cdot 8\cdot 9\cdot 10\cdots(4m)(4m+1)(4m+2)}$; $a_{4m+3} = 0$

The general solution $\phi(x)$ of the given equation contains three parameters $a_0,\ a_1,\ a_2$. The solution $\phi(x)$ becomes

$$\phi(x) = a_0\left[1 + \sum_{m=1}^{\infty} \frac{x^{4m}}{2\cdot 3\cdot 4\cdot 6\cdot 7\cdot 8\cdots(4m-2)(4m-1)(4m)}\right]$$

$$+ a_1\left[x + \sum_{m=1}^{\infty} \frac{x^{4m+1}}{3\cdot 4\cdot 5\cdot 7\cdot 8\cdot 9\cdots(4m-1)(4m)(4m+1)}\right]$$

$$+a_2\left[x^2+\sum_{m=1}^{\infty}\frac{x^{4m+2}}{4\cdot 5\cdot 6\cdot 8\cdot 9\cdot 10\cdots(4m)(4m+1)(4m+2)}\right]$$

$\phi(0)=1$ gives $a_0=1$

$$\phi'(x)=a_0\sum_{m=1}^{\infty}\frac{4mx^{(4m-1)}}{2\cdot 3\cdot 4\cdot 6\cdot 7\cdot 8\cdots(4m-2)(4m-1)(4m)}$$

$$+a_1\left[1+\sum_{m=1}^{\infty}\frac{(4m+1)\,x^{4m}}{3\cdot 4\cdot 5\cdot 7\cdot 8\cdot 9\cdots(4m-1)(4m)(4m+1)}\right]$$

$$+a_2\left[2x+\sum_{m=1}^{\infty}\frac{(4m+2)\,x^{4m+1}}{3\cdot 4\cdot 5\cdot 7\cdot 8\cdot 9\cdots(4m-1)(4m)(4m+1)}\right]$$

$\phi'(0)=0$ gives $a_1=0.$

Similarly $\phi''(0)=0$ gives $a_2=0.$

Thus, the required solution is

$$\phi(x)=1+\sum_{m=1}^{\infty}\frac{x^{4m}}{2\cdot 3\cdot 4\cdot 6\cdot 7\cdot 8\cdots(4m-2)(4m-1)(4m)}.$$

4. Legendre equation is an important differential equation occur in physical problems. The equation

$$L(y)=(1-x^2)y''-2xy'+\alpha(\alpha+1)y=0$$

where α is constant is called Legendre equation.

If we write this equation as

$$y''-\frac{2x}{1-x^2}y'+\frac{\alpha(\alpha+1)}{1-x^2}y=0,$$

we see that a_1, a_2 are given by

$$a_1(x)=\frac{2x}{1-x^2}\ \text{ and }\ a_2(x)=\frac{\alpha(\alpha+1)}{1-x^2}.$$

Both these functions are analytic at $x = 0$. Indeed, $\frac{1}{1-x^2}=\sum_{k=0}^{\infty}x^{2k}$ and the series converges for $|x|<|$.

Thus, $a_1(x)$ and $a_2(x)$ have the series expansions. Both there series converge for $|x|<|$. Thus by existence theorem the solution $L(y) = 0$ on $|x|<|$ have convergent power series expansions.

Let ϕ be any solution of $L(y) = 0$ on $|x|<|$.

Suppose $\phi(x)=\sum_{k=0}^{\infty}a_nx^n$ then

$$L(\phi)=(1-x^2)\sum_{n=2}^{\infty}n(n-1)a_nx^{n-2}-2x\sum_{n=1}^{\infty}na_nx^{n-1}+\alpha(\alpha+1)\sum_{n=0}^{\infty}a_nx^n$$

$$= \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - \sum_{n=2}^{\infty} n(n-1)\, a_n x^n - 2\sum_{n=1}^{\infty} na_n x^n + \alpha(\alpha+1)\sum_{n=0}^{\infty} a_n x^n$$

$$= \sum_{n=0}^{\infty} \left[(n+2)(n+1)a_{n+2} - \{n(n-1)+2n-\alpha(\alpha+1)\}\, a_n \right] x^n$$

For ϕ to satisfy $L(\phi) = 0$ we must have all the coefficients of the powers of x equal to zero.

Hence, $\quad (n+2)(n+1)\, a_{n+2} - \left[n(n+1) - \alpha(\alpha+1)\right] a_n = 0,\ n = 0, 1, 2, 3, \ldots$

This is a recurrance relation which gives a_{n+2} in terms of a_n.

$$a_{n+2} = \frac{n(n+1) - \alpha(\alpha+1)}{(n+1)(n+2)}\, a_n$$

$$= \frac{-(\alpha+n+1)(\alpha-n)}{(n+1)(n+2)}\, a_n.$$

for n = 0 we get

$$a_2 = -\frac{\alpha(\alpha+1)}{2}\, a_0$$

$$a_3 = -\frac{(\alpha+2)(\alpha-1)}{2\cdot 3}\, a_1$$

Similarly,

$$a_4 = -\frac{(\alpha+3)(\alpha-2)}{3\cdot 4}\, a_2 \qquad ;\ a_5 = -\frac{(\alpha+4)(\alpha-3)}{4\cdot 5}\, a_3$$

$$= +\frac{\alpha(\alpha+1)(\alpha+3)(\alpha-2)}{2\cdot 3\cdot 4}\, a_0 \qquad = +\frac{(\alpha+4)(\alpha+2)(\alpha-1)(\alpha-3)}{2\cdot 3\cdot 4\cdot 5}\, a_1$$

In general

$$a_{2m} = (-1)^m \frac{(\alpha+2m-1)(\alpha+2m-3)\cdots(\alpha+1)\alpha(\alpha-2)\cdots(\alpha-2m+2)}{(2m)!}\, a_0$$

$$a_{2m+1} = (-1)^m \frac{(\alpha+2m)(\alpha+2m-2)\cdots(\alpha+2)(\alpha-1)(\alpha-3)\cdots(\alpha-2m+1)}{(2m+1)!}\, a_1$$

All the coefficients are determined in terms of a_0 and a_1 and we have

$$\phi(x) = a_0\, \phi_1(x) + a_1\, \phi_2(x)$$

where, $\quad \phi_1(x) = 1 + \sum_{m=1}^{\infty} \frac{(-1)^m (\alpha+2m-1)(\alpha+2m-3)\cdots\alpha(\alpha-2)\cdots(\alpha-2m+2)}{(2m)!}\, x^{2m}$

$$\phi_2(x) = x + \sum_{m=1}^{\infty} \frac{(-1)^m (\alpha+2m)(\alpha+2m-2)\cdots(\alpha+2)(\alpha-1)(\alpha-3)\cdots(\alpha-2m+1)}{(2m+1)!}\, x^{2m+1}$$

Both ϕ_1 and ϕ_2 are solutions of Legendre equation, corresponding to the choices

$$c_0 = 1,\ \ c_1 = 0 \qquad \text{and} \qquad c_0 = 0,\ c_1 = 1,$$

respectively. They form a basis for the solutions, since

$$\phi_1(0) = 1,\quad \phi_2(0) = 0 \qquad ; \qquad \phi_1'(0) = 0,\quad \phi_2'(0) = 1$$

$$\therefore \qquad W(\phi_1, \phi_2)(0) = \begin{vmatrix} \phi_1(0) & \phi_2(0) \\ \phi_1'(0) & \phi_2'(0) \end{vmatrix} = \phi_1(0)\phi_2'(0) - \phi_2(0)\phi_1'(0) = 1$$

Since Wronkian $W(\phi_1, \phi_2) \neq (0)$, ϕ_1, ϕ_2 are linearly independent and therefore forms a basis.

If α is a non-negativ even integer $\alpha = 2n$, then ϕ_1 has only a finite number of non-zero terms. In this case ϕ_1 is a polynomial of degree $2n$ containing only even powers of x. for example,

$$\alpha = 0, \qquad \phi_1(x) = 1 = p_0(x)$$

$$\alpha = 2, \qquad \phi_1(x) = 1 + \frac{(-1)(2+1)}{2}\alpha x^2 = 1 - 3x^2 = p_2(x)$$

or the recurrance relation

$$a_{n+2} = \frac{n(n+1) - \alpha(\alpha+1)}{(n+1)(n+2)} a_n \quad \text{implies}$$

$$a_2 = \frac{0(0) - 2(3)}{1 \cdot 2} a_0 = -3a_0$$

$$a_4 = \frac{2(3) - 2(3)}{3 \cdot 4} a_2 = 0$$

with $a_0 = 1$ we get $\phi_1(x) = 1 - 3x^2 = p_2(x)$

for $\alpha = 4$, $\qquad a_2 = \dfrac{0(0) - 4(5)}{1 \cdot 2} a_0 = -10a_0$

$$a_4 = \frac{2(3) - 4(5)}{3 \cdot 4} a_2$$

$$= \frac{6 - 20}{12}(-10a_0)$$

$$= +\frac{140}{12} a_0$$

$$= \frac{35}{3} a_0$$

$$a_6 = \frac{4(5) - 4(5)}{5 \cdot 6}$$

$$= 0$$

$$\phi_1(x) = a_0 \left[1 - 10x^2 + \frac{35}{3}x^4\right] \quad \text{with} \quad a_0 = 1$$

$$\phi_1(x) = 1 - 10x^2 + \frac{35}{3}x^4 = p_4(x)$$

The solution ϕ_2 is not a polynomial in this case since none of the coefficients in the series of ϕ_2 vanish.

A similar situation occurs when α is a positive odd integer n. Then ϕ_2 is a polynomial of degree n having only odd powers of x and ϕ_1 is not a polynomial.

for example

$$(\alpha = 1) \qquad a_{n+2} = \frac{n(n+1)-\alpha(\alpha+1)}{(n+1)(n+2)} a_n$$

$$a_3 = \frac{1(2)-1(2)}{2\cdot 3} a_1 = 0$$

$$\phi_2(x) = x = p_1(x) \qquad \text{(say)}$$

$$(\alpha = 3) \qquad a_3 = \frac{1(2)-3(4)}{2\cdot 3} a_1$$

$$= -\frac{5}{3} a_1$$

$$a_5 = \frac{3(4)-3(4)}{4\cdot 5} a_3 = 0$$

$$\phi_2(x) = x - \frac{5}{3}x^3 = p_3(x) \qquad \text{(say)}$$

$$(\alpha = 5) \qquad a_3 = \frac{1(2)-5\cdot 6}{2\cdot 3} a_1$$

$$= -\frac{14}{3} a_1$$

$$a_5 = \frac{3(4)-5\cdot 6}{4\cdot 5} a_3$$

$$= -\frac{18}{20}\left(-\frac{14}{3}\right) a_1$$

$$= \frac{21}{5} a_1$$

$$a_7 = \frac{5\cdot 6 - 5\cdot 6}{6\cdot 7} a_5 = 0.$$

$$\phi_2(x) = x - \frac{14}{3}x^3 + \frac{21}{5}x^5 = p_5(x) \qquad \text{(say)}$$

Definition : 2.1.3

A polynomial solution p_n of degree n of $(1-x^2)y'' - 2xy' + n(n+1) = 0,$

Satisfying $P_n(1) = 1$ is called the n^{th} Legendre polynomial and the differential equation is called Legendre equation.

Let ϕ be a polynomial of degree n defined by

$$\phi(x) = \frac{d^n}{dx^n}(x^2-1)^n$$

$u(x) = (x^2-1)^n$ implies $u'(x) = n(x^2-1)^{n-1}2x$ gives

$$(x^2-1)u'(x) - 2nx\ u(x) = 0$$

Differentiate this equation (n + 1) times.

First differentiation gives

$$(x^2-1)u''(x)+2x(1-n)u'(x)-2n\,u(x)=0$$

Second differentiation gives

$$(x^2-1)u'''(x)+2x\left[(1+1)-n\right]u''+2\,\left[(1-n)+(0-n)\right]u'(x)=0$$

Third differentiation gives

$$(x^2-1)u^{(iv)}+2x\left[(1+1+1)-n\right]u'''+2\left[(2-n)+(1-n)+(0-n)\right]u''=0.$$

i.e. $$(x^2-1)u^{(iv)}+2x(3-n)u'''-2\left[(n-2)+(n-1)+n\right]u''=0.$$

In general (n + 1)$^{\text{th}}$ differentiation gives

$$(x^2-1)u^{(n+2)}+2x((n+1)-n)u^{(n+1)}-2\left[1+2+3+4+...+(n-1)+n\right]u^{(n)}=0.$$

i.e. $$(x^2-1)u^{(n+2)}+2x\,u^{(n+1)}-n(n+1)u^{(n)}=0. \quad \text{or}$$

$$(1-x^2)u^{(n+2)}-2x\,u^{(n+1)}+n(n+1)u^{(n)}=0.$$

Since $$\phi(x)=\frac{d^n}{dx^n}(x^2-1)^n=\frac{d^n}{dx^n}u(x)=u^{(n)}(x),$$

$$(1-x^2)\phi''-2x\,\phi'(x)+n(n+1)\phi(x)=0$$

Thus the function $\phi(x)=\dfrac{d^n}{dx^n}(x^2-1)^n$ is a solution of Legendre equation.

$$\frac{d^n}{dx^n}(x^2-1)^n=\frac{d^n}{dx^n}\left[(x+1)^n(x-1)^n\right]$$

$$=\left\{\frac{d^n}{dx^n}\left[(x-1)^n\right]\right\}(x+1)^n+\left\{\frac{d^{n-1}}{dx^{n-1}}(x-1)^n\right\}\frac{d}{dx}(x+1)^n+...$$

$$=n(n-1)(n-2)....2\cdot 1(x+1)^n+\text{ terms containing }(x-1)\text{ as factor.}$$

$$= n\,!\,(x+1)^n+\text{ terms containing }(x-1)\text{ as factor.}$$

Thus, at $x = 1$,

$$\frac{d^n}{dx^n}(x^2-1)^n=2^n\cdot n!$$

Define $P_n(x)=\dfrac{1}{2^n\,n!}\phi(x)=\dfrac{1}{2^n\,n!}\dfrac{d^n}{dx^n}(x^2-1)^n$ then P_n (x) is a solution of Legendre equation with $\alpha=n$ $P_n(1)=\dfrac{1}{2^n\,n!}\left[2^n\cdot n!\right]=1$. Thus, $P_n(x)$ is a Legendre polynomial of degree n.

Suppose ψ is a polynomial solution of Legendre equation with $\alpha = n$. Since ϕ_1 and ϕ_2 are basic solutions of Legendre equation $\psi=c_1\phi_1+c_2\phi_2$ on $|x|<|$ for some constants c_1 and c_2 is a solution. If n is even ϕ_1 is polynomial solution and ϕ_2 is not a polynomial $\psi-c_1\phi_1$ is polynomial where as $c_2\ \phi_2$ is not a polynomial and therefore $c_2 = 0$. In particular the function P_n satisfies $P_n(x) = c_1\phi_1(x)$ for some constant c_1 if n is even. Since $P_n(1) = 1 = c_1\phi_1(1)$ therefore $\phi_1(1) \neq 0$. Thus no nontrivial polynomial solution of Legendre equation can be zero at $x = 1$. A similar result is valid for n odd.

The formula

$$P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n$$

is known as Rodriges formula. This expression can be used to prove properties of Legendre polynomials.

EXERCISES

1. The equation $(1-x^2)y'' - xy' + \alpha^2 y = 0$ where α is a constant is called the Chebyshev equation.

(a) Compute two linearly independent series solutions for $|x| < |$.

(b) Show that for every non negative integer $\alpha = n$ there is a polynomial solution of degree n.

2. The equation $y'' - 2xy' + 2\alpha y = 0,$ where α is a constant, is called the Hermite equation.

(a) Find two linearly independent solutions on $-\infty < x < \infty$.

(b) Show that there is a polynomial solution of degree n, in case $\alpha = n$ is a non-negative integer.

3. Find the general solution valid near the origin

(i) $y'' + 3xy' + 3y = 0$

(ii) $(1+4x^2)y'' - 8y = 0$

(iii) $(1+x^2)y'' - 4xy' + 6y = 0$

(iv) $2y'' + xy' - 4y = 0$

(v) $y'' + x^2 y = 0$

Answers :

1. (a) $$\phi_1(x) = 1 + \sum_{m=1}^{\infty}\frac{(-\alpha^2)(2^2-\alpha^2)\cdots\left[(2m-2)^2-\alpha^2\right]}{(2m)!}x^{2m}$$

$$\phi_2(x) = x + \sum_{m=1}^{\infty}\frac{(1^2-\alpha^2)(3^2-\alpha^2)\cdots\left[(2m-1)^2-\alpha^2\right]}{(2m+1)!}x^{2m+1}$$

(b) ϕ_1 is a polynomial if α is an even integer,

ϕ_2 is a polynomial if α is an odd integer.

2. (a) $$\phi_1(x) = 1 + \sum_{m=1}^{\infty} \frac{2^m(-\alpha)(2-\alpha)\cdots(2m-2-\alpha)}{(2m)!} x^{2m}$$

$$\phi_2(x) = x + \sum_{m=1}^{\infty} \frac{2^m(1-\alpha)(3-\alpha)\cdots(2m-1-\alpha)}{(2m+1)!} x^{2m+1}$$

(b) ϕ_1 is a polynomial if a is an even integer, ϕ_2 is a polynomial if α is an odd integer.

3. (i) $$y(x) = a_0\left[1 + \sum_{k=1}^{\infty} \frac{(-3)^k x^{2k}}{2^k k!}\right]$$

$$+a_1\left[x + \sum_{m=1}^{\infty} \frac{(-3)^k x^{2k+1}}{3\cdot 5\cdot 7,\cdots(2k+1)}\right]$$

(ii) $$y = a_0(1+4x^2) + a_1 \sum_{k=0}^{\infty} \frac{(-1)^{k+1} 2^{2k} x^{2k+1}}{4k^2-1}$$

(iii) $$y(x) = a_0(1-3x^2) + a_1\left(x - \frac{x^3}{3}\right)$$

(iv) $$y(x) = a_0\left(1 + x^2 + \frac{1}{12}x^4\right) + a_1 \sum_{k=0}^{\infty} \frac{3(-1)^k x^{2k+1}}{2^{2k} k!(2k-3)(2k-1)(2k+1)}$$

(v) $$y(x) = a_0\left[1 + \sum_{k=1}^{\infty} \frac{(-1)^k x^{4k}}{2^{2k} k!\, 3\cdot 7\cdot 11\cdots(4k-1)}\right]$$

$$+a_1\left[x + \sum_{k=1}^{\infty} \frac{(-1)^k x^{4k+1}}{2^{2k} k!\, 5\cdot 9\cdot 13\cdots(4k+1)}\right]$$

Chapter 3

Linear Equations with Regular Singular Points

Contents :

Introduction

For a linear differential equation $a_0(x)\,y^{(n)}+a_1(x)y^{(n-1)}+a_2(x)y^{(n-2)}+....+a_n(x)y=0$, where the coefficient functions a_0, a_1, a_2, a_3,..... a_n are analytic at some point x_0, the point x_0 is called an ordinary point of the equation if $a_0\,(x_0) \neq 0$. In the last chapter we have obtain power series solutions valid near an ordinary point of a linear equation.

A singular point of the above linear equation is any point $x = x_1$ for which $a_0\,(x_1) = 0$. In this chapter we shall get power series solutions valid near a certain kind of singular points of the equation. It is usually difficult to determine the nature of the solutions in the vicinity of singular points. However there is a large class of equations for which the singularity is rather weak in the sense that slight modification of the methods used for solving equations with analytic coefficients discussed in chapter II unit 3, serve to yield solutions near the singularities.

Definition 3.1.1 (a)

A point $x = x_0$ is a regular singular point of $L(y)=a_0(x)y^{(n)}+a_1(x)y^{(n-1)}+a_2(x)y^{(n-2)}+\cdots+a_n(x)y=0$ if the equation can be written in the form $L(y)=(x-x_0)^n\,y^{(n)}+b_1(x)\,(x-x_0)^{(n-1)}\,y^{(n-1)}+\cdots+b_n(x)y=0$ where b_1, b_2, b_3,..., b_n are analytic at x_0.

If the functions b_1, b_2, b_3,..., b_n can be written in the form

$$b_k(x)=(x-x_0)^k\,\beta_k(x) \qquad k = 1, 2, 3,.......,n$$

Where β_1, β_2, β_3,..., β_n are analytic at x_0 then $L(y) = 0$ becomes

$$y^{(n)}+\beta_1(x)y^{(n-1)}+\beta_2(x)y^{(n-2)}+.....+\beta_n(x)y=0$$

Definition 3.1.1 (b)

A equation of the form $c_0(x)(x-x_0)^n y^{(n)} + c_1(x)(x-x_0)^{n-1} y^{(y-1)} + c_2(x)(x-x_0)^{(n-2)}$ $y^{(n-2)} + ... + c_n(x)y = 0$ has a regular singular point at x_0 if c_0, c_1, c_2, c_3,..... c_n are analytic at $x = x_0$ and $c_0(x_0) \neq 0$.

Definition 3.1.2

If $x = x_0$ is a singular point but is not a regular singular point, then it is called irregular singular point. For example, consider the equation

$$x^2 y'' - y' - \frac{3}{4} y = 0.$$

The origin $x = 0$is a singular point but not regular therefore $x = 0$ is irregular singular point. The coefficient of y' is not of the form $xb_1(x)$ where $b_1(x)$ analytic.

In the first unit we study the differential equation that has a regular singular point at origin and all the analytic functions b_1, b_2, b_3,..., b_n are constants.

Unit 1 : The Euler Equation

The simplest example of a second order equation that follows defination 3.1.1(a) is the Euler equation

$$L(y) = x^2 y'' + a\,x\,y' + b\,y = 0$$

where a, b are constants.

Theorem 3.1.1

Consider the second order Euler equation

$$L(y) = x^2 y'' + a\,x\,y' + b\,y = 0 \qquad (a, b \text{ constants}),$$

and the polynomial q given by

$$q(r) = r(r-1) + ar + b$$

A basis for the solutions of the Euler equation on any interval not containing $x = 0$ is given by

$$\phi_1(x) = |x|^{r_1} \quad , \quad \phi_2(x) = |x|^{r_2},$$

in case r_1, r_2 are distinct roots of q and by

$$\phi_1(x) = |x|^{r_1} \quad , \quad \phi_2(x) = |x|^{r_1} \log|x|,$$

if r_1 is a root of equation q of multiplicity two.

Proof :

Case 1 : $r_1 \neq r_2$

(a) We first consider the equation for $x > 0$. Let x^r be a solution of Euler equation

$$L(y) = x^2 y'' + a\,x\,y' + b\,y = 0$$

$$L(x^r) = x^2\left[r(r-1)x^{r-2}\right] + a\,x\left[r\,x^{r-1}\right] + bx^r = 0$$

implies $\left[r(r-1)+ar+b\right]\left[x^r\right]=0$

q is a polynomial defined by $q(r)=r(r-1)+ar+b$

Thus, we have

$$L(x^r)=q(r)x^r$$

If r_1 is a root of $q(x)$ then $q(r_1) = 0$ and therefore $L(x^{r_1})=0$. i.e. $\phi_1(x)=x^{r_1}$ is a solution of $L(y) = 0$. If r_2 is another root of q and $r_2 \neq r_1$ then $\phi_2(x)=x^{r_2}$ is another solution of $L(y) = 0$. Thus, $\phi_1(x)=x^{r_1}$ and $\phi_2(x)=x^{r_2}$ is a basis for the solution of the Euler equation as ϕ_1 and ϕ_2 are linearly independent.

(b) If $x < 0$, Let $(-x)^r$ be a solution (if $x < 0$, $-x > 0$).

$$\left[(-x)^r\right]'=-r(-x)^{r-1},\left[(-x)^r\right]''=r(r-1)(-x)^{r-2}$$

$$x\left[(-x)^r\right]'=r(-x)(-x)^{r-1}=r(-x)^r \text{ and}$$

$$L(y)=r(r-1)(-x)^r+ar(-x)^r+b(-x)^r=q(r)(-x)^r$$

if $r_1 \neq r_2$ then

$\phi_1(x)=(-x)^{r_1}$, $\phi_2(x)=(-x)^{r_2}$ are solutions of $L(y) = 0$. If r_1 and r_2 are complex roots of $q(r) = 0$, we define x^r for r complex by

$$x^r=e^{r\log x}\ (x>0)$$

then $(x^r)'=(r\cdot\log x)'\cdot e^{r\log x}=\dfrac{r}{x}\cdot x^r=r\,x^{r-1}$ and the result follows on the same lines for complex roots also.

Thus, we have proved that if $(x > 0)$ $r_1 \neq r_2$ $\phi_1(x)=x^{r_1}$ and $\phi_2(x)=x^{r_2}$ are solutions of $L(y) = 0$ and for $x < 0$, $r_1 \neq r_2$ we have $\phi_1(x)=(-x)^{r_1}$ and $\phi_2(x)=(-x)^{r_2}$ are solution of $L(y) = 0$. Since $|x| = x$ for $x > 0$ and $|x| = -x$ for $x < 0$ $\phi_1(x)=|x|^{r_1}$ and $\phi_2(x)=|x|^{r_2}$ are solutions of $L(y) = 0$ if r_1, r_2 are distinct roots of $q(r) = 0$.

We prove that ϕ_1 and ϕ_2 are linearly independent.

Let $c_1\phi_1+c_2\phi_2=0$ i.e. $c_1|x|^{r_1}+c_2|x|^{r_2}=0$ then $c_1+c_2|x|^{r_2-r_1}=0$ for every $x\in\mathbb{R}$. Differentiating above equation w.r.t. x for $x > 0$ or $x < 0$ we get,

$$c_2(r_2-r_1)|x|^{r_2-r_1}=0$$

But $r_1 \neq r_2$ and $x \neq 0$ therefore $c_2 = 0$ and $c_1\phi_1+c_2\phi_2=0$ for all x implies $c_1 = 0$ since $c_2 = 0$ and $\phi_1(x)\neq 0$.

Thus, ϕ_1 and ϕ_2 are linearly independent solutions.

Therefore if r_1 and r_2 are distinct roots of $q(x) = 0$, then $\phi_1(x)=|x|^{r_1}$ and $\phi_2(x)=|x|^{r_2}$ forms a basis for the solutions of $L(y) = 0$.

Case 2 : $r_1 = r_2$

(a) $x > 0$: If $r_1 = r_2$ then $q(r_1) = 0$ and $q'(r_1) = 0$. We have proved that if r_1 is a root of $q(x) = 0$ then $\phi_1(x) = x^r$ is a solution. To construct second solution consider

$$\frac{\partial}{\partial r} L(x^r) = \frac{\partial}{\partial r}\left[q(r)x^r\right]$$

$$= \left[q'(r) + q(r)\log x\right]x^r$$

Since, $\frac{\partial}{\partial r} x^r = x^r \log x$

But if $r_1 = r_2 = r$ then $q(r) = 0$ and $q'(r) = 0$ and we have

$$\frac{\partial}{\partial r}\left[L(x^r)\right] = 0 .$$

$$\frac{\partial}{\partial r}\left[L(x^r)\right] = L\left(\frac{\partial}{\partial r} x^r\right) = L(x^r \log x)$$

Thus, $L(x^r \log x) = 0$ implies $x^r \log x$ is a solution of $L(y) = 0$.

If r_1 is a root of $q(r) = 0$ of multiplicity two then $\phi_1(x) = x^{r_1}$ and $\phi_2(x) = x^{r_1} \log x$ are two solutions of $L(y) = 0$.

(b) $x < 0$: If $x < 0$, then $-x > 0$ and $\phi_1(x) = (-x)^{r_1}$ and $\phi_2(x) = (-x)^{r_1} \log(-x)$ are solution of $L(y) = 0$.

Thus $\phi_1(x) = |x|^{r_1}$ and $\phi_2(x) = |x|^{r_1} \log|-x|$ are two solution of $L(y) = 0$.

$c_1\phi_1 + c_2\phi_2 = 0$ implies $c_1 + c_2 \log|x| = 0$ for all x and therefore $c_1 = c_2 = 0$ and ϕ_1, ϕ_2 are linearly independent.

Thus if r_1 is a repeated root of $q(r) = 0$ then $\phi_1(x) = |x|^{r_1}$ and $\phi_2(x) = |x|^{r_1} \log|x|$ is a basis for solutions of the Euler equation $L(y) = x^2 y'' + a\,xy' + by = 0$.

Illustration :

$x^2 y'' + xy' + y = 0$ for $x \neq 0$ is Euler equation with $a = b = 1$.

The polynomial $q(r) = r(r-1) + r + 1 = r^2 + 1$ and $r = +i, -i$ are roots of $q(r)$. A basis for the solutions by theorem 3.1.1 are

$$\phi_1(x) = |x|^i \quad \text{and} \quad \phi_2(x) = |x|^{-i} \qquad (x \neq 0)$$

$$|x|^i = e^{i\log|x|} = \cos(\log|x|) + i\sin(\log|x|)$$

Thus $\psi_1(x) = \cos(\log|x|)$ and $\psi_2(x) = \sin(\log|x|)$ is another basis for solution of $L(y) = x^2 y'' + xy' + y = 0$.

Theorem 3.1.2

Consider the Euler equation of order n.

$$L(y) = x^n y^{(n)} + a_1 x^{n-1} y^{(n-1)} + a_2 x^{n-2} y^{(n-2)} + \ldots + a_n y = 0,$$

where a_1, a_2, a_3,..., a_n are constants. Let r_1, r_2,..., r_s be distinct roots of the indicial polynomial $q(r) = r(r-1)(r-2)...(r-n+1) + a_1 r(r-1)...(r-n+2) + ... + a_n$ and suppose r_i has multiplicity m_i. Then the n functions

$$|x|^{r_1}, |x|^{r_1} \log|x|, ..., |x|^{r_1} (\log|x|)^{m_1-1}; |x|^{r_2}, |x|^{r_2} \log|x|, ..., |x|^{r_2} (\log|x|)^{m_2-1}; ...;$$
$$|x|^{r_s}, |x|^{r_s} \log|x|, ..., |x|^{r_s} (\log|x|)^{m_s-1}$$

form a basis for the solution of $L(y) = 0$ on any interval not containing zero.

Proof : Let $|x|^r$ be a solution of $L(y) = 0$.

$$(|x|^r)' = r|x|^{r-1}, \ (|x|^r)'' = r(r-1)|x|^{r-2}, ...$$
$$(|x|^r)^{(n)} = r(r-1)(r-2)...(r-n+1)|x|^{r-n}$$

Hence, $L(|x|^r) = r(r-1)(r-2)...(r-n+1)|x|^r + a_1 r(r-1)(r-2)$

$$...(r-n+2)|x|^r + ... + a_n|x|^r$$
$$= q(r)|x|^r$$

where $\quad q(r) = (r)(r-1)(r-2)...(r-n+1) + r(r-1)(r-2)...(r-n+2)a_1 + ... + a_n.$

The polynomial $q(r)$ is called indicial polynomial. Thus, $|x|^r$ is a solution of $L(y) = 0$ if $q(r) = 0$ i.e. if r is a root of indicial polynomial then $|x|^r$ is a solution of $L(y) = 0$.

Differentiating $L(|x|^r) = q(r)|x|^r$ with respect to 'r' we get

$$\frac{\partial}{\partial r} L(|x|^r) = L\left(\frac{\partial}{\partial r}|x|^r\right)$$
$$= \left(q'(r) + q(r)\log|x|\right)|x|^r$$

In general k times differentiation gives

$$\frac{\partial^k}{\partial r^k} L(|x|^r) = L\left(\frac{\partial^k}{\partial r^k}|x|^r\right)$$
$$= \left[q^{(k)}(r) + kq^{(r-1)}(r)\log|x| + k(k-1)q^{(k-2)}(x)(\log|x|)^2 + ... + q(r)(\log|x|)^k\right]|x|^r.$$

If r is a root of $q(r)$ with multiplicity $(k + 1)$ then $q(r) = 0, q'(r) = 0, q''(r) = 0...,$ $q^{(k)}(r) = 0$ and therefore $L\left(\frac{\partial^i}{\partial r^i}|x|^r\right) = 0$ for $i = 1, 2, 3, ..., k.$

Thus $\phi(x) = \frac{\partial^i}{\partial r^i}|x|^r, \ i = 1, 2, 3, ..., k.$ are solution of $L(y) = 0$.

If r_1 is a root of $q(r)$ of multiplicity m_1 then

$$|x|^{r_1}, \frac{\partial}{\partial r_1}|x|^{r_1} = |x|^{r_1}\log|x|, \frac{\partial^2}{\partial r_1^2}|x|^{r_1} = |x|^{r_1}(\log|x|)^2, ..., \frac{\partial^{m_1-1}}{\partial r_1^{m_1-1}}|x|^{r_1}$$

$= |x|^{r_1} (\log|x|)^{m_1-1}$ are solutions of $L(y) = 0$.

Repeating this process for each root of $q(r)$ we obtain all the solution and the result follows.

All these solutions are linearly independent and therefore form a basis for the solutions of $L(y) = 0$ on any interval not containing zero.

EXAMPLES

Q. 1. Find all solutions of the following equations for $x > 0$

(a) $x^2 y'' + 2xy' - 6y = 0$ **(b)** $2x^2 y'' + xy' - y = 0$

(c) $x^2 y'' + xy' - 4y = 0$ **(d)** $x^2 y'' - 5xy' + 9y = x^3$

(e) $x^3 y''' + 2x^2 y'' - xy' + y = 0.$

Ans.:

(a) The indicial equation

$q(r) = r(r-1) + 2r - 6 = r^2 + r - 6$ has root $r = 3, -2$.

Therefore $\phi_1(x) = x^3$ and $\phi_2(x) = x^{-2}$ are basic solutions and $\phi(x) = c_1 x^3 + c_2 x^{-2}$ is general solution for constants c_1, c_2.

(b) The indicial equation

$q(r) = 2r(r-1) + r - 1 = 2r^2 - r - 1$ has root $r = 1, -\frac{1}{2}$ and $\phi_1(x) = x, \phi_2(x) = x^{-\frac{1}{2}}$ are basic solution, $\phi(x) = c_1 x + c_2 x^{-\frac{1}{2}}$ is general solution for constants c_1, c_2.

(c) The indicial equation

$q(r) = r(r-1) + r - 4 = r^2 - 4$ has root 2, – 2 Then $\phi_1(x) = x^2$ and $\phi_2(x) = x^{-2}$ are basic solution, $\phi(x) = c_1 x^2 + c_2 x^{-2}$ is general solution.

(d) The indicial equation

$q(r) = r(r-1) - 5r + 9 = r^2 - 6r + 9$ has root 3, 3. Since the root 3 is repeated root of multiplicity two $\phi_1(x) = x^3$ and $\phi_2(x) = x^3 \log x$ are basic solution of corresponding homogeneous equation $x^2 y'' - 5xy' + 9y = 0$.

The particular solution will be determined by using variation of constant method.

Let $\psi = u_1(x)\phi_1(x) + u_2(x)\phi_2(x)$ be a solution of equation

$$y'' - \frac{5}{x} y' + \frac{9}{x^2} y = x \text{ then}$$

$$u_k(x) = \int \frac{W_k(t)\, b(t)\, dt}{W(\phi_1, \phi_2)} \quad \text{Here } b(t) = t,$$

$$W(\phi_1, \phi_2) = \begin{vmatrix} x^3 & x^3 \log x \\ 3x^2 & x^2 + 3x^2 \log x \end{vmatrix} = x^5,$$

$$W_1 = \begin{vmatrix} 0 & x^3 \log x \\ 1 & x^2 + 3x^2 \log x \end{vmatrix} = -x^3 \log x,$$

$$W_2 = \begin{vmatrix} x^3 & 0 \\ 3x^2 & 1 \end{vmatrix} = +x^3,$$

$$u_1(x) = \int \frac{-x^3 \log x \cdot x}{x^5} dx = -\int \frac{\log x}{x} dx = -\frac{1}{2}(\log x)^2$$

$$u_2(x) = \int \frac{x^3 \cdot x\, dx}{x^5} = \int \frac{dx}{x} = \log x$$

$$\psi(x) = u_1(x)\phi_1(x) + u_2(x)\phi_2(x)$$

$$= -\frac{1}{2}(\log x)^2 x^3 + (\log x)\, x^3 \log x = \frac{1}{2} x^3 (\log x)^2$$

The general solution

$$\phi = c_1\phi_1 + c_2\phi_2 + \psi = c_1 x^3 + c_2 x^3 \log x + \frac{1}{2} x^3 (\log x)^2.$$

(e) The indicial equation

$$q(r) = r(r-1)\,(r-2) + 2r(r-1) - r + 1$$

$$= (r-1)\left[r^2 - 2r + 2r - 1\right] = (r-1)(r^2 - 1) \text{ has root } 1, 1, -1.$$

Since one is a root of multiplicity two, $\phi_1(x) = x$, $\phi_2(x) = x \log x$ and corresponding to –1, $\phi_3(x) = x^{-1}$.

The general solution

$$\phi(x) = c_1 x + c_2 x \log x + c_3 x^{-1}.$$

Q. 2. Find all solutions of the following equations for $|x| > 0$.

(a) $x^2 y'' + xy' + 4y = 1$ **(b)** $x^2 y'' - 3xy' + 5y = 0$

(c) $x^2 y'' + xy' - 4\pi y = x$

Ans.:

(a) The indicial equation $q(r) = r(r-1) + r + 4$ has root $r = \pm 2i$.

Since both the roots are distinct, $\phi_1(x) = |x|^{2i}$ and $\phi_2(x) = |x|^{-2i}$. The general solution of homogeneous equation is

$$\phi(x) = c_1 |x|^{2i} + c_2 |x|^{-2i}$$

The particular solution will be calculated by variation of constant method.

Case 1 : $x > 0$,

If $x > 0$ then $|x| = x$

$$\phi_1(x) = x^{2i} \text{ and } \phi_2(x) = x^{-2i}$$

Let $\psi(x) = u_1(x)\phi_1(x) + u_2(x)\phi_2(x)$ be a solution of

$x^2y'' + xy' + 4y = 1$. then $u_1(x) = \int \frac{W_1(x)b(x)\,dx}{W(\phi_1, \phi_2)}$ and

$u_2(x) = \int \frac{W_2(x)b(x)}{W(\phi_1, \phi_2)}\,dx$ where $b(x) = \frac{1}{x^2}$,

$$W(\phi_1, \phi_2) = \begin{vmatrix} x^{2i} & x^{-2i} \\ 2i\,x^{2i-1} & -2ix^{-2i-1} \end{vmatrix} = -2i\,x^{-1} - 2i\,x^{-1} = -\frac{4i}{x}$$

$$W_1(x) = \begin{vmatrix} 0 & x^{-2i} \\ 1 & -2ix^{-2i-1} \end{vmatrix} = -x^{-2i}; \quad W_2 = \begin{vmatrix} x^{2i} & 0 \\ 2i\,x^{2i-1} & 1 \end{vmatrix} = x^{2i}$$

$$u_1(x) = \int \frac{-x^{-2i}\left(\frac{1}{x^2}\right)}{\left(-\frac{4i}{x}\right)}\,dx = \int \frac{x^{-2i-2+1}}{4i}\,dx = \frac{x^{-2i}}{4i(-2i)} = \frac{x^{-2i}}{8}$$

$$u_2(x) = \int \frac{x^{2i}\left(\frac{1}{x^2}\right)}{\left(-\frac{4i}{x}\right)}\,dx = \int \frac{x^{2i-2+1}}{-4i}\,dx = \frac{x^{2i}}{-4i(+2i)} = \frac{x^{2i}}{8}.$$

Thus, $\psi(x) = u_1(x)\phi_1(x) + u_2(x)\phi_2(x) = \frac{x^{-2i}}{8} \cdot x^{2i} + \frac{x^{2i}}{8} \cdot x^{-2i} = \frac{1}{4}$. For $x > 0$, $\phi(x) = c_1x^{2i} + c_2x^{-2i} + \frac{1}{4}$ is a solution of given equation.

Case 2 : $x < 0$

If $x < 0$ then $|x| = -x$ and $\phi_1(x) = (-x)^{2i}$ and $\phi_2(x) = (-x)^{-2i}$

Let $\psi(x) = u_1(x)\phi_1(x) + u_2(x)\phi_2(x)$ be a solution of the given differential equation.

$$W(\phi_1, \phi_2) = \begin{vmatrix} (-x)^{2i} & (-x)^{-2i} \\ -2i(-x)^{2i-1} & +2i(-x)^{-2i-1} \end{vmatrix} = \frac{2i}{(-x)} + \frac{2i}{(-x)} = \frac{4i}{-x}$$

$$W_1 = \begin{vmatrix} 0 & (-x)^{-2i} \\ 1 & 2i(-x)^{-2i-1} \end{vmatrix} = -(-x)^{-2i}, \quad W_2 = \begin{vmatrix} (-x)^{2i} & 0 \\ -2i(-x)^{2i-1} & 1 \end{vmatrix} = (-x)^{+2i},$$

$$b(x) = \frac{1}{x^2}$$

$$u_1(x) = \int \frac{W_1(x)b(x)}{W(\phi_1, \phi_2)}\,dx = \int \frac{-(-x)^{-2i} \cdot \frac{1}{(+x)^2}}{\frac{4i}{(-x)}}\,dx = -\frac{1}{4i}\frac{(-x)^{-2i}}{2i} = \frac{(-x)^{-2i}}{8}.$$

$$u_2(x) = \int \frac{W_2(x)\,b(x)}{W(\phi_1,\phi_2)}\,dx = \int \frac{\frac{(-x)^{2i}}{(-x)^2}}{\frac{4i}{(-x)}}\,dx = \frac{1}{4i}\,\frac{(-x)^{+2i}}{-2i} = \frac{(-x)^{+2i}}{8}$$

$$\psi(x) = u_1(x)\,\phi_1(x) + u_2(x)\,\phi_2(x)$$

$$\psi(x) = \frac{(-x)^{-2i}}{8}\cdot(-x)^{2i} + \frac{(-x)^{2i}}{8}(-x)^{-2i} = \frac{1}{4}\cdot$$

for $x < 0$, $\phi(x) = c_1(-x)^{2i} + c_2(-x)^{-2i} + \frac{1}{4}$ is a solution of the given differential equation.

Thus $\phi(x) = c_1\,|\,x\,|^{2i} + c_2\,|\,x\,|^{-2i} + \frac{1}{4}$ is a solution of the given differential equation if $x \neq 0$.

(b) The indical equation $q(r) = r(r-1) - 3r + 5$ has roots $2 + i$, $2 - i$. Since both the roots are distinct $\phi_1(x) = |\,x\,|^{2+i}$ and $\phi_2(x) = |\,x\,|^{2-i}$ are two independent solutions. The general solution

$$\phi = c_1\phi_1(x) + c_2\phi_2(x) = c_1\,|\,x\,|^{2+i} + c_2\,|\,x\,|^{2-i} = x^2\left(c_1\,|\,x\,|^{i} + c_2\,|\,x\,|^{-i}\right)\cdot$$

(c) The indical equation $q(r) = r(r-1) + r - 4\pi$ has roots $2\sqrt{\pi}$ and $-2\sqrt{\pi}$. Since both the roots are distinct $\phi_1(x) = |\,x\,|^{2\sqrt{\pi}}$ and $\phi_2(x) = |\,x\,|^{-2\sqrt{\pi}}$ are two solutions. The general solution of corresponding homogeneous equation is $\phi(x) = c_1\,|\,x\,|^{2\sqrt{\pi}} + c_2\,|\,x\,|^{-2\sqrt{\pi}}$.

We solve the non-homogeneous equation using the variation of constants method.

Case 1 : $x > 0$

If $x > 0$ then $|\,x\,|^{2\sqrt{\pi}} = x^{2\sqrt{\pi}}$, $|\,x\,|^{-2\sqrt{\pi}} = x^{-2\sqrt{\pi}}$

Let $\psi(x) = u_1(x)\,\phi_1(x) + u_2(x)\,\phi_2(x)$ be a solution of given equation then

$$W(\phi_1,\phi_2) = \begin{vmatrix} x^{2\sqrt{\pi}} & x^{-2\sqrt{\pi}} \\ 2\sqrt{\pi}\;x^{2\sqrt{\pi}-1} & -2\sqrt{\pi}\;x^{-2\sqrt{\pi}-1} \end{vmatrix} = \frac{-2\sqrt{\pi}}{x} - \frac{2\sqrt{\pi}}{x} = \frac{-4\sqrt{\pi}}{x}$$

$$W_1(x) = \begin{vmatrix} 0 & x^{-2\sqrt{\pi}} \\ 1 & -2\sqrt{\pi}\;x^{-2\sqrt{\pi}-1} \end{vmatrix} = -x^{-2\sqrt{\pi}},\quad W_2(x) = \begin{vmatrix} x^{2\sqrt{\pi}} & 0 \\ +2\sqrt{\pi}\;x^{+2\sqrt{\pi}-1} & 1 \end{vmatrix} = x^{2\sqrt{\pi}}$$

$$b(x) = \frac{1}{x},\ \text{since the given equation is } y'' + \frac{1}{x}y' - \frac{4\pi}{x}y = \frac{1}{x}.$$

$$u_1(x) = \int \frac{W_1(x)\,b(x)}{W(\phi_1,\phi_2)}\,dx = \int \frac{-x^{-2\sqrt{\pi}}\left(\frac{1}{x}\right)}{\frac{-4\sqrt{\pi}}{x}}\,dx = +\frac{1}{4\sqrt{\pi}}\int x^{-2\sqrt{\pi}-1+1}dx = +\frac{1}{4\sqrt{\pi}}\,\frac{x^{-2\sqrt{\pi}+1}}{(-2\sqrt{\pi}+1)}$$

Thus, $u_1(x) = +\dfrac{x^{-2\sqrt{\pi}+1}}{4\sqrt{\pi}\,(1-2\sqrt{\pi})}$

$$u_2(x) = \int \frac{W_2(x)b(x)}{W(\phi_1,\phi_2)}dx = \int \frac{x^{2\sqrt{\pi}}\cdot\frac{1}{x}}{-4\sqrt{\pi}\cdot\frac{1}{x}}dx = -\frac{1}{4\sqrt{\pi}}\frac{x^{2\sqrt{\pi}+1}}{2\sqrt{\pi}+1}$$

$$\psi(x) = +\frac{x^{-2\sqrt{\pi}+1}}{4\sqrt{\pi}(1-2\sqrt{\pi})}\cdot x^{2\sqrt{\pi}} - \frac{1}{4\sqrt{\pi}}\frac{x^{2\sqrt{\pi}+1}}{2\sqrt{\pi}+1}$$

$$= +\frac{x}{4\sqrt{\pi}}\left[\frac{1}{1-2\sqrt{\pi}} - \frac{1}{2\sqrt{\pi}+1}\right] = \frac{x}{4\sqrt{\pi}}\cdot\frac{4\sqrt{\pi}}{1-4\pi} = \frac{x}{1-4\pi}$$

For $x > 0$, $\phi(x) = c_1x^{2\sqrt{\pi}} + c_2x^{-2\sqrt{\pi}} + \frac{x}{1-4\pi}$ is a solution of given equation. For $x < 0$ also we get $\psi(x) = \frac{x}{1-4\pi}$.

Thus the general solution of the given equation is $\phi(x) = c_1|x|^{2\sqrt{\pi}} + c_2|x|^{-2\sqrt{\pi}} + \frac{x}{1-4\pi}$.

Till now we have considered Euler equation having a regular singular point at origin. At the beginning of this chapter we defined singular points, regular singular points and irregular singular points. We present some definitions of singularities which can be used to classify the singularities of the given differential equation.

Definition 3.1.3 (a) :

A second order differential equation

$$y'' + p(z)y' + q(z)y = 0,$$

analytic for $0 < |z - z_0| < r$, has a regular singular point at z_0 when p(z) has at worst a simple pole at $z = z_0$ and $q(z)$ has at worst a double pole at $z = z_0$.

Definition 3.1.3 (b) :

For a second order differential equation

$$y'' + p(x)y' + q(x)y = 0,$$

if $x = x_0$ is a singular point and if the denominator of $p(x)$ does not contain the factor $(x - x_0)$ to a power higher than one and if the denominator of $q(x)$ does not contain the factor $(x - x_0)$ to a power higher than two, then $x - x_0$ is called a regular singular point.

EXAMPLES

Q. 1. Classify the singular points, in the finite plane, of the equation

$$x(x-1)^2(x+2)y'' + x^2y' - (x^3+2x-1)y = 0$$

Ans.: $a_0(x) = x(x-1)^2(x+2) = 0$ gives $x = 0, 1, -2$

Thus the singular points in a finite plane are at $x = 0, 1, -2$.

Given equation can be written as

$$y'' + \frac{x^2}{x(x-1)^2(x+2)}\ y' - \frac{(x^3+2x-1)}{x(x-1)^2(x+2)}y = 0$$

therefore $p(x) = \frac{x}{(x-1)^2(x+2)}$ and $q(x) = \frac{-(x^3+2x-1)}{x(x-1)^2(x+2)}$

Since the denominator of $p(x)$ does not contain the factor $(x-0)$ and the denominator of $q(x)$ does not contain a factor $(x-0)^p$ for $p > 2$. Hence, $x = 0$ is a regular singular point. Now consider $x = 1$. Since the denominator of $p(x)$ contains the factor $(x-1)^p$ where $p = 2 > 1$ therefore $x = 1$ is not a regular singular point i.e. $x = 1$ is irregular singular point.

At $x = -2$, the factor $(x + 2)$ appears to the first power in the denominator of $p(x)$ which is not higher than 1 and the factor $(x + 2)$ appears to the first power in the denominator of $q(x)$ which is not higher than 2. so $x = -2$ is a regular singular point.

Q. 2. Classify the singular points in the finite plane for the equation

$$x^4(x^2+1)(x-1)^2 y'' + 4x^3(x-1)y' + (x+1)y = 0$$

Ans.: $a_0(x) = x^4(x^2+1)(x-1)^2 = 0$ gives $x = 0$, $x = \pm i$, $x = 1$ are roots of $a_0(x) = 0$.

Thus, the singular points in a finite plane are at $x = 0, +i, -i, 1$.

Given equation is of the form

$$y'' + \frac{4}{x(x^2+1)(x-1)} y' + \frac{(x+1)}{x^4(x^2+1)(x-1)^2} y = 0.$$

Here $p(x) = \frac{4}{x(x^2+1)(x-1)}$ and $q(x) = \frac{x+1}{x^4(x^2+1)(x-1)^2}$

(i) $x = 0$

The denominator of $p(x)$ contains a factor $(x-0)^r$ where $r = 1 \not> 1$ and the denominator of $q(x)$ contains a factor $(x-0)^r$ where $r = 4 > 2$. Therefore $x = 0$ is an irregular singular point.

(ii) $x = i$

The denominator of $p(x)$ contains a factor $(x-i)^r$ where $r = 1 \not> 1$ and the denominator of $q(x)$ contains a factor $(x-i)^r$ where $r = 1 \not> 2$. Therefore $x = i$ is a regular singular point.

(iii) $x = -i$

The denominator of $p(x)$ contains a factor $(x+i)^r$ where $r = 1 \not> 1$ and the denominator of $q(x)$ contains a factor $(x+i)^r$ where $r = 1 \not> 2$. Therefore $x = -i$ is a regular singular point.

(iv) $x = 1$

The denominator of $p(x)$ contains a factor $(x-1)^r$ where $r = 1 \not> 1$ and the denominator of $q(x)$ contains a factor $(x-1)^r$ where $r = 2 \not> 2$. Therefore $x = 1$ is a regular singular point.

Thus $x = i, -i, 1$ are regular singular points and $x = 0$ is an irregular singular point.

Q. 3. For each equation, locate and classify all its singular points in the finite plane.

(a) $x^3(x-1)y'' + (x-1)y' + 4xy = 0.$ **(b)** $x^2(x^2-4)y'' + 2x^3 y' + 3y = 0$

(c) $y'' + xy = 0$ **(d)** $x^2(x-4)^2 y'' + 3x y' - (x-4)y = 0$

Ans.:

(a) $a_0(x) = x^3(x-1),\ a_0(x) = 0$ give $x = 0, x = 1$. Therefore $x = 0$ and $x = 1$ are singularities. Given equation can be put in the form

$$y'' + \frac{1}{x^3}y' + \frac{4}{x^2(x-1)}y = 0$$

for $x = 0$, denominator of $p(x)$ contains a factor x^r where $r = 3 > 1$ and therefore $x = 0$ is irregular singular point. For $x = 1$, denominator of $p(x)$ contains a factor $(x-1)^r$ where $r = 0 \ngtr 1$ and the denominator of $q(x)$ contains a factor $(x-1)^r$ where $r = 1 \ngtr 2$.

Therefore $x = 1$ is a regular singular point.

(b) $a_0(x) = x^2(x^2-4) = x^2(x+2)(x-2)$. $a_0(x) = 0$ gives $x = 0, 2, -2$. Therefore 0, 2, –2 are singular points. Given equation is

$$y'' + \frac{2x}{(x+2)(x-2)}y' + \frac{3}{x^2(x+2)(x-2)}y = 0.$$

For $x = 0$, the denominator of $p(x)$ contains a factor x^r where $r = 0 \ngtr 1$ and denominator of $q(x)$ contains a factor x^r for $r = 2 \ngtr 2$. Therefor $x = 0$ is a regular singular point.

For $x = 2$, the denominator of $p(x)$ contains a factor $(x-2)^r$ for $r = 1 \ngtr 1$ and the denominator of $q(x)$ contains a factor $(x-2)^r$ for $r = 1 \ngtr 2$. Therefor $x = 1$ is a regular singular point.

For $x = -2$, the denominator of $p(x)$ contains a factor $(x+2)^r$ for $r = 1 \ngtr 1$ and the denominator of $q(x)$ contains a factor $(x+2)^r$ for $r = 1 \ngtr 2$. Therefor $x = -2$ is a regular singular point.

Thus, all the singular points are regular.

(c) $a_0(x) = 1 \neq 0$ for any x therefore equation do not have any finite singular point.

(d) $a_0(x) = x^2(x-4)^2$. $a_0(x) = 0$ gives $x = 0, 4$. $x = 0, 4$ are singular point of the given equation. Given equation is

$$y'' + \frac{3}{x(x-4)^2}y' - \frac{1}{x^2(x-4)}y = 0$$

For $x = 0$, the denominator of $p(x)$ contains a factor x^r for $r = 1 \ngtr 1$ and the denominator of $q(x)$ contains a factor x^r for $r = 2 \ngtr 2$. Therefor $x = 0$ is a regular singular point.

For $x = 4$, the denominator of $p(x)$ contains a factor $(x-4)^r$ for $r = 2 > 1$ therefor $x = 4$ is not a regular singular point.

Thus $x = 0$ is regular and $x = 4$ is irregular singular point.

EXERCISE

1. **For each equation, locate and classify all its singular points in the finite plane**

   (a) $x^2 y'' + y = 0$ (Ans.: $x = 0$ is regular, no irregular)

   (b) $(x^2+1)(x-4)^2 y'' + (x-4)^2 y' + y = 0$ (Ans.: $x = i, -i$ regular, $x = 4$ irregular)

   (c) $x^2(x-2)y'' + 3(x-2)y' + y = 0$ (Ans.: $x = 2$ is regular, $x = 0$ irregular)

   (d) $(1+4x^2)^2 y'' + 6x(1+4x^2)y' - 9y = 0$ (Ans.: $x = \pm\frac{i}{2}$ are regular)

2. **Find all solutions of the following equations.**

   (a) $x^2 y'' + 2xy' - 12y = 0$ (Ans.: $y = c_1 x^3 + c_2 x^{-4}$)

   (b) $x^2 y'' + xy' - 9y = 0$ (Ans.: $y = c_1 x^3 + c_2 x^{-3}$)

   (c) $x^2 y'' + xy' - 4y = x$ (Ans.: $x = c_1 x^2 + c_2 x^{-2} - \frac{x}{3}$)

   (d) $x^2 y'' - 3xy' + 4y = 0$ (Ans.: $y = x^2(c_1 + c_2 \ln x)$)

   (e) $x^2 y'' + 5xy' + 5y = 0$ (Ans.: $y = x^{-2}\left[c_1 \cos(\ln x) + c_2 \sin(\ln x)\right]$)

3. **Find all solutions of the following equations.**

   (a) $x^2 y'' - 5xy' + 9y = 0$ (Ans.: $y = |x|^3 (c_1 + c_2 \ln|x|)$)

   (b) $9x^2 y'' + 2y = 0$ (Ans.: $y = c_1 |x|^{\frac{1}{3}} + c_2 |x|^{\frac{2}{3}}$)

   (c) $2x^2 y'' - 3xy' + 2y = 0$ (Ans.: $y = c_1 |x|^2 + c_2 |x|^{\frac{1}{2}}$)

Unit 2 : Second order equation with Regular Singular Points

A second order equation with a regular singular point at x_0 has the form

$$L(y) = (x-x_0)^2 y'' + (x-x_0)\,a(x)y' + b(x)y = 0,$$

where $a(x)$, $b(x)$ are analytic functions at x_0 i.e. they have power series expansions

$$a(x) = \sum_{k=0}^{\infty} \alpha_k (x-x_0)^k \text{ and } b(x) = \sum_{k=0}^{\infty} \beta_k (x-x_0)^k$$

which are convergent on some interval $|x - x_0| < r_0$ for some $r_0 > 0$.

Without loss of generality we assume $x_0 = 0$. Then

$$L(y) = x^2 y'' + x\,a(x)y' + b(x)y = 0 \text{ and}$$

$a(x) = \sum_{k=0}^{\infty} \alpha_k x^k$, $b(x) = \sum_{k=0}^{\infty} \beta_k x^k$ which are convergent on an interval $|x| < r_0$, $r_0 > 0$. The Euler equation is a particular case of $L(y) = 0$ with a, b constants.

A second order equation with regular singular point has a power series solution. If functions $a(x)$, $b(x)$ have power series expansion on some interval $|x| < r_0$ then the power series solution converges on the interval $|x| < r_0$.

Theorem 3.2.1

Consider the equation

$$x^2 y'' + a(x)xy' + b(x)y = 0,$$

where a and b have convergent power series expansions for $|x| < r_0$, $r_0 > 0$. Let r_1, r_2 ($\text{Re } r_1 \geq \text{Re } r_2$) be the roots of the indicial polynomial

$$q(r) = r(r-1) + a(0)r + b(0)$$

for $0 < |x| < r_0$ there is a solution ϕ_1 of the form

$$\phi_1(x) = |x|^{r_1} \sum_{k=0}^{\infty} c_k x^k \qquad (c_0 = 1),$$

where the series converges for $|x| < r_0$. If $r_1 - r_2$ is not zero or a positiove integer, there is a second solution ϕ_2 for $0 < |x| < r_0$ of the form

$$\phi_2(x) = |x|^{r_2} \sum_{k=0}^{\infty} \tilde{c}_k x^k \qquad (\tilde{c}_0 = 1),$$

where the series converge for $|x| < r_0$.

The coefficients c_k, $\tilde{c}_k$ can be obtained by substitution of the solution into the differential equation.

Proof :

Suppose we have a solution ϕ of the form

$$\phi(x) = x^r \sum_{k=0}^{\infty} c_k x^k \qquad (c_0 \neq 0, \ x > 0)$$

for the equation $L(y) = x^2 y'' + a(x)xy' + b(x)y = 0$.

Where $a(x) = \sum_{k=0}^{\infty} \alpha_k x^k$ and $b(x) = \sum_{k=0}^{\infty} \beta_k x^k$ for $|x| < r_0$. Then

$$\phi(x) = x^r \sum_{k=0}^{\infty} c_k x^k = \sum_{k=0}^{\infty} c_k x^{k+r},$$

$$\phi'(x) = \sum_{k=0}^{\infty} (k+r)c_k x^{k+r-1} = x^{r-1} \sum_{k=0}^{\infty} (k+r)c_k x^k,$$

$$\phi''(x) = \sum_{k=0}^{\infty} (k+r)(k+r-1)c_k x^{k+r-2} = x^{r-2} \sum_{k=0}^{\infty} (k+r)(k+r-1)c_k x^k$$

$$b(x)\phi(x) = \left(\sum_{k=0}^{\infty} \beta_k x^k \right) \left(x^r \sum_{k=0}^{\infty} c_k x^k \right)$$

$$= x^r \sum_{k=0}^{\infty} \tilde{\beta}_k x^k \quad \text{where} \quad \tilde{\beta}_k = \sum_{j=0}^{k} c_j \beta_{k-j}$$

$$xa(x)\phi'(x) = x\left(\sum_{k=0}^{\infty} \alpha_k x^k\right)\left(x^{r-1} \sum_{k=0}^{\infty} (k+r) c_k x^k\right)$$

$$= x^r \left(\sum_{k=0}^{\infty} \alpha_k x^k\right)\left(\sum_{k=0}^{\infty} (k+r) c_k x^k\right)$$

$$= x^r \sum_{k=0}^{\infty} \tilde{\alpha}_k x^k \quad \text{where} \quad \tilde{\alpha}_k = \sum_{j=0}^{k} (j+r) c_j \alpha_{k-j}$$

$$x^2 \phi''(x) = x^r \sum_{k=0}^{\infty} (k+r)\ (k+r-1) c_k x^k.$$

Thus, $$L(\phi)(x) = x^r \sum_{k=0}^{\infty} (k+r)\ (k+r-1) c_k x^k + x^r \sum_{k=0}^{\infty} \tilde{\alpha}_k x^k + x^r \sum_{k=0}^{\infty} \tilde{\beta}_k x^k$$

$$= x^r \sum_{k=0}^{\infty} \left[(k+r)(k+r-1) c_k + \tilde{\alpha}_k + \tilde{\beta}_k \right] x^k$$

$L(\phi) = 0$ implies $[\quad]_k = \left[(k+r)(k+r-1) c_k + \tilde{\alpha}_k + \tilde{\beta}_k) \right] = 0$

$k = 0, 1, 2, 3, \ldots$

Using the definitions of $\tilde{\alpha}_k$, $\tilde{\beta}_k$ we can write $[\quad]_k$ as

$$[\quad]_k = (k+r)(k+r-1) c_k + \sum_{j=0}^{k} (j+r) c_j \alpha_{k-j} + \sum_{j=0}^{k} c_j \beta_{k-j}$$

$$= \left[(k+r)(k+r-1) + (k+r)\alpha_0 + \beta_0 \right] c_k + \sum_{j=0}^{k-1} \left[(j+r)\alpha_{k-j} + \beta_{k-j} \right] c_j$$

for $k = 0$ we must have

$$r(r-1) + r\alpha_0 + \beta_0 = 0.$$

Since $c_0 \neq 0$ the second degree polynomial q given by

$$q(r) = r(r-1) + r\alpha_0 + \beta_0$$

is called the indicial polynomial and the only admissible values of r are the roots of q.

$$[\quad]_k = q(r+k) c_k + d_k = 0 \qquad (k = 1, 2, 3, \ldots) \quad \ldots\ldots(3.2.1)$$

where $$d_k = \sum_{j=0}^{k-1} \left[(j+r)\alpha_{k-j} + \beta_{k-j} \right] c_j \qquad (k = 1, 2, 3, \ldots) \quad \ldots\ldots(3.2.2)$$

Note that d_k is a linear combination of c_0, c_1, c_2,..... c_{k-1} with coefficients involving the known functions a, b and r. Leaving r and c_0 indeterminant for the moment we solve equations (3.2.1) and (3.2.2) successively in terms of c_0 and r. The solutions we denote by $C_k(r)$ and the corresponding d_k by D_k(r). Thus,

$$D_1(r) = (r\alpha_1 + \beta_1) c_0, \qquad C_1(r) = -\frac{D_1(r)}{q(r+1)},$$

and in general

$$D_k(r) = \sum_{j=0}^{k-1} \left[(j+r)\alpha_{k-j} + \beta_{k-j} \right] C_j(r),\ C_k(r) = -\frac{D_k(r)}{q(r+k)} \qquad (k = 1, 2, 3, ...)$$

The C_k thus, determined are rational functions of r, and the only points where they cease to exist are the points r for which the denominator $q(r + k) = 0$ for some $k = 1, 2, 3,....$ Only two such possible points exist.

Define Φ by

$$\Phi(x, r) = c_0 x^r + x^r \sum_{k=0}^{\infty} C_k(r) x^k \qquad ....(3.2.3)$$

If the series converges for $0 < x < r_0$, then clearly

$$L(\Phi)(x, r) = c_0\, q(r) x^r,$$

since $C_k(r)$ satisfies equation 3.2.1 for every $k = 1, 2, 3....$

Thus if the function $\phi = x^r \sum_{k=0}^{\infty} C_k x^k$ is a solution of $L(y) = 0$ then r must be a root of the indicial polynomial

$$q(r) = r(r-1) + r\alpha_0 + \beta_0$$

and c_k $(k \geq 1)$ are determined uniquely in terms of r and c_0 given by equation (3.2.2), provided $q(r + k) \neq 0$ $k = 1, 2, 3,...$ Conversely if r is a root of q and if $C_k(r)$ can be determined then the function ϕ given by equation (3.2.3) is a solution of $L(y) = 0$ for any choice of c_0, provided the series in equation (3.2.3) is convergent.

Let r_1, r_2 be two roots of q and suppose $Re\, r_1 \geq Re\, r_2$. Then $q(r_1 + k) \neq 0$ for all $k = 1, 2, 3,...$ Thus, $C_k(r_1)$ exists for all $k = 1, 2, 3,...$ and for $c_0 = C_0(r) = 1$ we get a solution.

$$\phi_1(x) = \Phi(x, r_1) = x^{r_1} \sum_{k=0}^{\infty} C_k(r_1) x^k \qquad (C_0(r) = 1),$$

is a solution of $L(y) = 0$, provided the series converges.

If r_2 is a root of q distinct from r_1 and $q(r_2 + k) \neq 0$ for $k = 1, 2, 3,...$, then clearly $C_k(r_2)$ is defined for $k = 1, 2, 3,....$ and the function Φ_2 defined by

$$\Phi_2(x) = \Phi(x, r_2) = x^{r_2} \sum_{k=0}^{\infty} C_k(r_2) x^k \qquad (C_0(r_2) = 1)$$

is another solution of $L(y) = 0$, provided the series is convergent. The condition $q(r_2 + k) \neq 0$ for $k = 1, 2, ...$ is same as $r_2 + k \neq r_1$ for any $k = 1, 2, 3,....$ or $r_1 - r_2 \neq k$ i.e. $r_1 - r_2$ is not a positive integer and the result follows.

Illustration :

Consider the equation

$$L(y) = x^2 y'' + \frac{3}{2} xy' + xy = 0$$

As per theorem 3.2.1 we assume the solution ϕ of the equation $L(y) = 0$ as

$$\phi(x) = x^r \sum_{k=0}^{\infty} c_k x^k$$

$$\phi'(x) = x^{r-1} \sum_{k=0}^{\infty} (k+r) c_k x^k$$

and $$\phi''(x) = x^{r-2} \sum_{k=0}^{\infty} (k+r)(k+r-1) c_k x^k$$

$$L(\phi) = x^r \sum_{k=0}^{\infty} (k+r)(k+r-1) c_k x^k + \frac{3}{2} x^r \sum_{k=0}^{\infty} (k+r) c_k x^k + x^{r+1} \sum_{k=0}^{\infty} c_k x^k$$

$$= \left[r(r-1) + \frac{3}{2} r \right] c_0 x^r + \left\{ \left[(r+1)(r) + \frac{3}{2}(r+1) \right] c_1 + c_0 \right\} x^{r+1}$$

$$+ \left\{ \left[(r+2)(r+1) + \frac{3}{2}(r+2) \right] c_2 + c_1 \right\} x^{r+2} + \ldots.$$

$q(r) = r(r-1) + \frac{3}{2} r$ is the indicial polynomial

$$L(\phi) = q(r) c_0 x^r + [q(r+1) c_1 + c_0] x^{r+1} + [q(r+2) c_2 + c_1] x^{r+2} + \ldots..$$

$$= q(r) c_0 x^r + x^r \sum_{k=1}^{\infty} [q(r+k) c_k + c_{k-1}] x^k$$

$L(\phi) = 0$ implies $q(r) = 0$ and $q(r+k) c_k + c_{k-1} = 0$

$q(r) = r(r-1) + \frac{3}{2} r = r\left(r + \frac{1}{2} \right) = 0$ implies $r = 0, -\frac{1}{2}$

$(Re\, r_1 > Re\, r_2)$ Define $r_1 = 0,\ r_2 = -\frac{1}{2}$

$q(r+k) c_k + c_{k-1} = 0$ gives $c_k = -\frac{c_{k-1}}{q(r+k)}$, $(k = 1, 2, 3, \ldots)$

Thus, $$c_k = \left(-\frac{1}{q(r+k)} \right) \left(-\frac{1}{q(r+k-1)} \right) \left(-\frac{1}{q(r+k-2)} \right) \ldots \left(-\frac{1}{q(r+1)} \right) c_0$$

In the above expression c_{k-1} is written in terms of c_{k-2} , c_{k-2} is expressed in terms of c_{k-3} and so on.

$$c_k = \frac{(-1)^k}{q(r+k) q(r+k-1) q(r+k-2) \ldots.. q(r+1)}, \quad k = 1, 2, 3\ldots..$$

Since $r_1 = 0,\ r_2 = -\frac{1}{2},\ r_1 - r_2$ is non zero and is not an integer. Therefore we apply theorem 3.2.1. For $r = r_1 = 0$, $c_0 = 1$ we get

$$\phi_1(x) = 1 + \sum_{k=1}^{\infty} \frac{(-1)^k x^k}{q(k) q(k-1) q(k-2) \ldots.. q(1)}$$

and for $c_0 = 1$, $r = r_2 = -1/2$ we obtain another solution

$$\phi_2(x) = x^{-\frac{1}{2}} + x^{-\frac{1}{2}} \sum_{k=1}^{\infty} \frac{(-1)^k x^k}{q\left(k-\frac{1}{2}\right) q\left(k-\frac{3}{2}\right) q\left(k-\frac{5}{2}\right) \ldots\ldots q\left(\frac{1}{2}\right)}.$$

These functions ϕ_1, ϕ_2 will be solutions provided the series converge on some interval containing 0.

$$Let\ \phi_1(x) = \sum_{k=0}^{\infty} d_k(x).$$

Using the ratio test we obtain

$\left|\frac{d_{k+1}(x)}{d_k(x)}\right| = \frac{|x|}{|q(k+1)|} = \frac{|x|}{(k+1)\left(k+\frac{3}{2}\right)} \to 0$ as $k \to \infty$ provided $|x| < \infty$. Thus the series defining ϕ_1 is convergent for all finite x. The same is true for ϕ_2.

To obtain solutions for $x < 0$, all the above calculations are valid if x^r replaced by $|x|^r$, where $|x|^r = e^{r \log|x|}$

Thus two solutions which are valid for all $x \neq 0$ are

$$\phi_1(x) = 1 + \sum_{k=1}^{\infty} \frac{(-1)^k x^k}{q(k)\, q(k-1)\, q(k-2) \ldots . q(1)} \quad \text{and}$$

$$\phi_2(x) = |x|^{-\frac{1}{2}} \left[1 + \sum_{k=1}^{\infty} \frac{(-1)^k x^k}{q\left(k-\frac{1}{2}\right) q\left(k-\frac{3}{2}\right) \ldots . q\left(\frac{1}{2}\right)}\right],$$

where $|x|^{\frac{1}{2}}$ is a positive square root of $|x|$.

Thus we have seen that if the roots of indicial polynomials are distinct and the difference between these two roots is not an integer then the solutions of $L(y) = 0$ will be constructed by using power series method.

In the next theorem we prove that if the roots are identical or the difference between the roots is an integer still the power series solution exist.

Theorem 3.2.2

Consider the equation

$$L(y) = x^2 y'' + a(x)\, xy' + b(x)\, y = 0,$$

where a, b have power series expansions which are convergent for $|x| < r_0$, $r_0 > 0$. Let r_1, r_2 ($Re\ r_1 \geq Re\ r_2$) be the roots of the indicial polynomial

$$q(r) = r(r-1) + a(0) r + b(0).$$

If $r_1 = r_2$ there are two linearly independent solutions ϕ_1, ϕ_2 for $0 < |x| < r_0$ of the form

$$\phi_1(x) = |x|^{r_1} \sigma_1(x),\ \ \phi_2(x) = |x|^{r_1+1} \sigma_2(x) + (\log|x|)\, \phi_1(x),$$

where σ_1, σ_2 have power series expansions which are convergent for $|x| < r_0$ and $\sigma_1(0) \neq 0$.

If $r_1 - r_2$ is a positive integer there are two linearly independent solutions ϕ_1, ϕ_2 for $0 < |x| < r_0$ of the form

$$\phi_1(x) = |x|^{r_1} \sigma_1(x),$$

$$\phi_2(x) = |x|^{r_2} \sigma_2(x) + c(\log|x|)\phi_1(x),$$

where σ_1, σ_2 have power series expansions which are convergent for $|x| < r_0$, $\sigma_1(0) \neq 0$, $\sigma_2(0) \neq 0$, and c is a constant. It may happen that $c = 0$.

Proof :

For $x > 0$, suppose we have a solution ϕ of the form

$$\phi(x) = x^r \sum_{k=0}^{\infty} c_k x^k.$$

$$L(\phi)(x) = x^r \sum_{k=0}^{\infty} \left[(k+r)(k+r-1)c_k + \tilde{\alpha}_k + \tilde{\beta}_k \right] x^k$$

where $$\tilde{\alpha}_k = \sum_{j=0}^{k} (j+r) c_j \alpha_{k-j} \text{ and } \tilde{\beta}_k = \sum_{j=0}^{k} c_j \beta_{k-j}$$

$$a(x) = \sum_{k=0}^{\infty} \alpha_k x^k, \quad b(x) = \sum_{k=0}^{\infty} \beta_k x^k.$$

$L(\phi)(x) = 0$ implies

$$[\quad]_k = \left[(k+r)(k+r-1)c_k + \tilde{\alpha}_k + \tilde{\beta}_k \right] = 0, \quad k = 0,1,2,3,\ldots$$

$$= \left[(k+r)(k+r-1)c_k + \sum_{j=0}^{k} (j+r) c_j \alpha_{k-j} + \sum_{j=0}^{k} c_j \beta_{k-j} \right]$$

$$= \left[(k+r)(k+r-1)c_k + (k+r)\alpha_0 + \beta_0 \right] c_k$$

$$+ \sum_{j=0}^{k-1} \left[(j+r)\, \alpha_{k-j} + \beta_{k-j} \right] c_j$$

For $k = 0$ we must have

$$q(r) = r(r-1) + r\alpha_0 + \beta_0 = 0$$

Then

$$[\quad]_k = q(r+k)c_k + d_k = 0 \qquad \text{.... 3.2.4}$$

where $$d_k = \sum_{j=0}^{k-1} \left[(j+r)\, \alpha_{k-j} + \beta_{k-j} \right] c_j \qquad \text{.... 3.2.5}$$

Here, we are going to consider two cases according as the roots r_1, r_2 ($Re\, r_1 \geq Re\, r_2$) of the indicial polynomial $q(r)$ satisfy.

Case (i) $r_1 = r_2$

Case (ii) $r_1 - r_2$ is a positive integer.

Since, $Re\, r_1 \geq Re\, r_2$, $q(r_1 + k) \neq 0$ for $k = 1, 2, 3,....$ and we can solve equation (3.2.4) and (3.2.5) for c_k and d_k. Let the solutions of c_k be denoted by $C_k(r)$ and solution for d_k be denoted by $D_k(r)$. Then

$$L(\Phi)(x, r) = c_0\, q(r) x^r \qquad .....3.2.6$$

where Φ is given by

$$\Phi(x, r) = c_0\, x^r + x^r \sum_{k=0}^{\infty} C_k(r)\, x^k$$

The $C_k(r)$ are determined recursively by the formulas

$$C_0(r) = c_0 \neq 0,$$

$$q(r+k)\; C_k(r) = -D_k(r)$$

$$D_k(r) = \sum_{j=0}^{k-1} \left[(j+r)\, \alpha_{k-j} + \beta_{k-j} \right] C_j(r), \; k = 1, 2, 3, ...$$

In case (i) i.e. $r_1 = r_2,\; q(r_1) = 0,\; q'(r_1) = 0.$

On differentiating equation (3.2.6) with respect to r

We get

$$\frac{\partial}{\partial r} L(\Phi)(x, r) = L\left(\frac{\partial \Phi}{\partial r} \right)(x, r)$$

$$= C_0 \left[q'(r) + (\log x)\, q(r) \right] x^r$$

and we see that if $r = r_1 = r_2$ and $C_0 = 1$, then

$$L\left(\frac{\partial \Phi}{\partial r} \right)(x, r_1) = c_0 \left[q'(r_1) + (\log x)\, q(r_1) \right] x^{r_1}$$

$$= 0.$$

Since $L\left(\frac{\partial \Phi}{\partial r} \right)(x, r_1) = 0,\; \left(\frac{\partial \Phi}{\partial r} \right)(x, r_1)$ is a solution of $L(\phi) = 0$. Thus the term by term differentiation of equation (3.2.3) gives the second solution

$$\phi_2(x) = x^{r_1} \sum_{k=0}^{\infty} C_k'(r_1)\, x^k + (\log x)\, x^{r_1} \sum_{k=0}^{\infty} C_k(r_1)\, x^k$$

$$= x^{r_1} \sum C_k'(r_1)\, x^k + (\log x)\, \phi_1(x).$$

where ϕ_1 is the solution already obtained in (3.2.3)

$$\phi_1(x) = x^{r_1} \sum_{k=0}^{\infty} C_k(r_1)\, x^k, \quad (C_k(0) = 1)$$

Case (ii) : Suppose $r_1 = r_2 + m$, where m is a positive integer. If C_0 is given,

$$C_1(r_2),\; C_2(r_2), ..., \; C_{m-1}(r)$$

all exist as finite numbers, but since

$q(r+m)\; C_m(r) = -D_m(r),$ the coefficient of $C_m(r)$ becomes zero at $r = r_2$.

$$q(r) = (r - r_1)(r - r_2)$$

and hence,

$$\begin{aligned} q(r+m) &= (r+m-r_1)(r+m-r_2) \\ &= (r+m-r_2-m)(r+m-r_2) \\ &= (r-r_2)(r+m-r_2) \end{aligned}$$

If $D_m(r)$ also has $(r_1 - r_2)$ as a factor (i.e. $D_m(r_2) = 0$), then it will get cancel from both the sides of equation $q(r + m)\, C_m(r) = -\, D_m(r)$ and would give $C_m(r_2)$ as a finite number. Then

$$C_{m+1}(r_2),\quad C_{m+2}(r_2), \ldots..$$

all exist. In this special situation we will have a solution ϕ_2 of the form

$$\phi_2(x) = x^{r_2} \sum_{k=0}^{\infty} C_k(r_2)\, x^k \quad (C_0(r_2) = 1)$$

If we choose $C_0(r) = r - r_2$ then $D_m(r_2) = 0$, as $D_m(r)$ is linear homogeneous in $C_0(r)$, $C_1(r)$.... $C_{m-1}(r)$ and hence $D_m(r)$ has $C_0\,(r)$ as a factor.

Let

$$\psi(x,r) = x^r \sum_{k=0}^{\infty} C_k(r)\, x^k \quad (C_0(r) = (r - r_2)),$$

$$L(\psi)(x,r) = (r - r_2)\, q(r) x^r$$

Therefore $L(\psi)(x, r_2) = 0$ and

$$\psi(x) = \psi(x, r_2)$$

is the second solution of $L(y) = 0$

Since $C_0(r_2) = C_1(r_2) = \ldots.. = C_{m-1}(r_2) = 0,$ the series ψ actually starts with the m-th power in x.

To get a solution associated with r_2 differentiate

$$L(\psi)(x,r) = (r - r_2)\, q(r) x^r$$

with respect to r then

$$\begin{aligned} \frac{\partial}{\partial r}\left[L(\psi)(x,r)\right] &= L\left(\frac{\partial \psi}{\partial r}\right)(x,r) \\ &= q(r)x^r + (r - r_2)\left[q'(x) + (\log x)\, q(r)\right] x^r \end{aligned}$$

and $$L\left(\frac{\partial \psi}{\partial r}\right) = 0 \text{ at } r = r_2$$

and $$\phi_2(x) = \frac{\partial \psi}{\partial r}(x, r_2)$$

is a solution provided the series involved is convergent and

$$\phi_2(x) = x^{r_2} \sum_{k=0}^{\infty} C_k{}'(r_2)\, x^k + (\log x)\, x^{r_2} \sum_{k=0}^{\infty} C_k(r_2)\, x^k$$

where $C_0(r) = (r - r_2)$ and

$$C_0(r_2) = C_1(r_2) = C_2(r_2) = ... = C_{m-1}(r_2) = 0$$

Thus,

$$\phi_2(x) = x^{r_2} \sum_{k=0}^{\infty} C_k'(r_2)\, x^k + (\log x)\, x^{r_2} \sum_{k=m}^{\infty} C_k(r_2)\, x^k$$

$$= x^{r_2} \sum_{k=0}^{\infty} C_k'(r_2)\, x^k + (\log x)\, x^{r_2} \sum_{m=0}^{\infty} C_{k+m}(r_2)\, x^{k+m}$$

$$= x^{r_2} \sum_{k=0}^{\infty} C_k'(r_2)\, x^k + (\log x)\, x^{r_2+k} \sum_{m=0}^{\infty} C_{k+m}(r-m)\, x^m$$

$$= x^{r_2} \sum_{k=0}^{\infty} C_k'(r_2)\, x^k + (\log x)\, x^{r_1}\, C \sum_{m=0}^{\infty} C_m(r_1)\, x^m$$

$$\phi_2(x) = x^{r_2} \sum_{k=0}^{\infty} C_k'(r_2)\, x^k + (\log x) \cdot c \cdot \phi_1(x)$$

Where c is constant.

For $x < 0$, we replace x^{r_1}, x^{r_2}, $\log x$ everywhere by $|x|^{r_1}, |x|^{r_2}$, $\log |x|$ respectively and the result follows.

The method used in the theorem 3.2.2 is called the Frobenius method. The solutions ϕ_1, ϕ_2 are linearly independent. Thus, if the roots are equal or they differ by an integer then theorem 3.2.2 gives two linearly independent solutions of the differential equation

$$L(y) = x^2 y'' + x\, a(x) y' + b(x) y = 0.$$

EXAMPLES

Q. 1. Find all solutions ϕ of the form

$$\Phi(x) = |x|^r \sum_{k=0}^{\infty} C_k\, x^k \qquad (|x| > 0),$$

for the following equations.

(a) $3x^2 y'' + 5xy' + 3xy = 0$

(b) $2xy'' + (1+x)y' - 2y = 0$

Test each of the series involved for convergence

Answer (a) **:** For $x > 0$ suppose we have a solution ϕ of the form

$$\phi(x) = x^r \sum_{k=0}^{\infty} c_k\, x^k, \quad c_0 \neq 0$$

then $$\phi'(x) = x^{r-1} \sum_{k=0}^{\infty} (k+r) c_k x^k \text{ and } \phi''(x) = x^{r-2} \sum_{k=0}^{\infty} (k+r)(k+r-1) c_k x^k$$

Let $L(y) = 3x^2 y'' + 5xy' + 3xy$ therefore

$$L(\phi)(x) = x^r \sum_{k=0}^{\infty} \left[3(k+r)(k+r-1) + 5(k+r)\right] c_k x^k + x^r \cdot 3 \sum_{k=0}^{\infty} c_k\, x^{k+1}$$

$$= \left[3r(r-1) + 5r\right] c_0 x^r + \left\{\left[3(r+1)r + 5(r+1)\right] c_1 + 3c_0\right\} x^{r+1}$$

$$+ \left\{\left[3(r+2)(r+1) + 5(r+2)\right] c_2 + 3c_1\right\} x^{r+2} + \ldots$$

Let $q(r) = 3r(r-1) + 5r = r(3r+2)$ then

$$L(\phi)(x,r) = q(r) c_0 x^r + \left[q(r+1) c_1 + 3c_0\right] x^{r+1} + \left[q(r+2) c_2 + 3c_1\right] x^{r+2} + \ldots.$$

$$= q(r) c_0 x^r + \sum_{k=1}^{\infty} \left[q(r+k) c_k + 3c_{k-1}\right] x^{r+k}$$

$L(\phi)(x,r) = 0$ only if $q(r) = 0$ and

$q(r+k) c_k + 3c_{k-1} = 0$ for $k = 1, 2, 3, \ldots..$

The indicial equation $q(r) = 0$ implies $r(3r+2) = 0$ that is $r = 0, -\frac{2}{3}$. Let $r_1 = 0$, $r_2 = -\frac{2}{3}$. (By choice $r_1 > r_2$)

Since $q(r) = r(3r+2)$, $q(r+k) = (r+k)(3(r+k)+2) = (r+k)(3r+3k+2)$

$q(r+k) c_k + 3c_{k-1} = 0$ gives

$$c_k = \frac{-3c_{k-1}}{q(r+k)}, \quad k = 1, 2, 3\ldots$$

$$= \frac{(-3)^k c_0}{q(r+k)\; q(r+k-1)\; q(r+k-2) \ldots..q(r+1)}$$

Case 1 : $r_1 = 0$

For $c_0 = 1$ we obtain $$c_k = \frac{(-3)^k c_0}{q(k)\; q(k-1) \ldots..q(1)}$$

$$= \frac{(-3)^k}{k(3k+2)(k-1)(3k-1)(k-2)(3k-4) \ldots..1 \cdot 5}$$

$$= \frac{(-3)^k}{k!\; 5 \cdot 8 \cdot 11 \cdots (3k-4)(3k-1)(3k+2)}$$

Thus,

$$\phi_1(x) = 1 + \sum_{k=1}^{\infty} \frac{(-3)^k x^k}{k!\; 5 \cdot 8 \cdot 11 \cdots (3k-4)(3k-1)(3k+2)}$$

Case 2 : $r_2 = -\frac{2}{3}$

For $c_0 = 1$ we obtain $c_k = \dfrac{(-3)^k c_0}{q\left(k-\frac{2}{3}\right) q\left(k-\frac{5}{3}\right) q\left(k-\frac{7}{3}\right) \ldots\ldots q\left(\frac{1}{3}\right)}$

Since $q(\text{r}) = r\,(3r + 2)$

$$c_k = \frac{(-3)^k c_0}{\left(k-\frac{2}{3}\right)(3k)\left(k-\frac{5}{3}\right)(3k-3)\left(k-\frac{8}{3}\right)(3k-6)\ \ldots\ldots \frac{1}{3}\cdot 3}$$

$$= \frac{(-3)^k c_0}{k!\,1\cdot 4\cdot 7\cdots(3k-8)(3k-5)(3k-2)}$$

Thus,

$$\phi_2(x) = x^{-\frac{2}{3}}\left[1 + \sum_{k=1}^{\infty} \frac{(-3)^k x^k}{k!\ 1\cdot 4\cdot 7\cdots\cdot(3k-8)(3k-5)(3k-2)}\right]$$

To obtain solutions for $x < 0$, we replace x^r by $|x|^r$. Thus,

$$\phi_1(x) = 1 + \sum_{k=1}^{\infty} \frac{(-3)^k x^k}{k!\ 5\cdot 8\cdot 11\cdots\cdot(3k-4)(3k-1)(3k+2)}$$

and $$\phi_2(x) = |x|^{-\frac{2}{3}}\left[1 + \sum \frac{(-3)^k x^k}{k!\ 1\cdot 4\cdot 7\cdots\cdot(3k-8)(3k-5)(3k-2)}\right]$$

These functions ϕ_1 and ϕ_2 will be solutions for $x \neq 0$, provided both the series converges on some interval containing $x = 0$.

Let $$\phi_1(x) = \sum_{k=0}^{\infty} d_k(x)$$

Using ratio test we obtain

$$\left|\frac{d_{k+1}(x)}{d_k(x)}\right| = \left|\frac{(-3)\,x}{(k+1)(3k+5)}\right| = \frac{3|x|}{(k+1)(3k+5)} \ \to\ 0$$

as $k \to \infty$ provided $|x| < \infty$. Thus, series defining ϕ_1 is convergent for all finite x.

Let $$\phi_2(x) = \sum_{k=1}^{\infty} d_k(x)$$

Using ratio test we obtain

$$\left|\frac{d_{k+1}(x)}{d_k(x)}\right| = \left|\frac{-3\,x}{3\left(k+\frac{1}{3}\right)(k+1)}\right| = \frac{3|x|}{(3k+1)(k+1)} \ \to\ 0$$

as $k \to \infty$ provided $|x| < \infty$. Thus, series defining ϕ_2 is convergent for all finite x.

Thus ϕ_1, ϕ_2 are solutions of the given equation.

(b) Suppose for $x > 0$ we have a solution ϕ of the form

$$\phi(x) = x^r \sum_{k=0}^{\infty} C_k x^k, \quad c_0 \neq 0$$

Let $\qquad L(y) = 2xy'' + (1+x)y' - 2y = 0$ then

$$L(\phi)(x,r) = \sum_{k=0}^{\infty} 2(k+r)(k+r-1)C_k x^{k+r-1} + \sum_{k=0}^{\infty} (k+r)C_k x^{k+r-1}$$

$$+ \sum_{k=0}^{\infty} (k+r)C_k x^{k+r} - 2\sum_{k=0}^{\infty} C_k x^{k+r}$$

$$= \sum_{k=0}^{\infty} \left[2(k+r)(k+r-1) + (k+r)\right] C_k x^{k+r-1} + \sum_{k=0}^{\infty} (k+r-2) C_k x^{k+r}$$

$$= \left[2r(r-1) + r\right] c_0 x^{r-1} + \left\{\left[2(r+1)(r) + (r+1)\right] c_1 + (r-2)c_0\right\} x^r$$

$$+ \left\{\left[2(r+2)(r+1) + (r+2)\right] c_2 + (r-1)c_1\right\} x^{r+1} + \cdots$$

$$= q(r) c_0 x^{r-1} + \sum_{k=1}^{\infty} \left[q(r+k) C_k + (r+k-3) C_{k-1}\right] x^{k+r}$$

The indicial equation $q(r) = 0$ implies $2r^2 - 2r + r = 0$, $r(2r-1) = 0$ gives $r = 0, \frac{1}{2}$

Let $r_1 = \frac{1}{2}$ and $r_2 = 0$.

Observe that $r_1 \neq r_2$ and $r_1 - r_2$ is not a positive integer. $L(\phi)(x, r) = 0$ if and only if $q(r) = 0$ and

$$q(r+k) C_k + (r+k-3) C_{k-1} = 0 \quad \text{or}$$

$$C_k = -\frac{(r+k-3) C_{k-1}}{q(r+k)} \quad \text{for } k = 1, 2, 3$$

Since $q(r) = r(2r-1)$ therefore $q(r+k) = (r+k)(2r+2k-1)$ and

$$C_k = +\left[-\frac{(r+k-3)}{q(r+k)}\right]\left[-\frac{(r+k-4)}{q(r+k-1)}\right]\left[-\frac{(r+k-5)}{q(r+k-2)}\right] \cdots\cdots\cdots \left[-\frac{(r-2)}{q(r+1)}\right] C_0$$

$$= \frac{(-1)^k (r+k-3)(r+k-4)(r+k-5)\cdots(r-2) C_0}{(r+k)\ (2r+2k-1)\ (r+k-1)(2r+2k-3).....(r+1)(2r+1)}$$

Case 1 : $r_1 = \frac{1}{2}, C_0 = 1$

$$C_k = \frac{(-1)^k \left(k-\frac{5}{2}\right)\left(k-\frac{7}{2}\right)\left(k-\frac{9}{2}\right)\cdots\left(-\frac{3}{2}\right)}{\left(k+\frac{1}{2}\right)(2k)\left(k-\frac{1}{2}\right)(2k-2)\left(k-\frac{3}{2}\right)(2k-4)\cdots\frac{3}{2}(2)}$$

$$= \frac{(-1)^k (2k-5)\ (2k-7)\ (2k-9)\cdots 3\ (-1)(-3)}{2^k k!(2k+1)(2k-1)(2k-3)\cdots(3)}$$

$$= \frac{(-1)^k (3)}{2^k k!(2k+1)(2k-1)(2k-3)}$$

and

$$\phi_1(x) = x^{\frac{1}{2}} \left[1 + \sum_{k=1}^{\infty} = \frac{3(-1)^k x^k}{2^k k!(2k+1)(2k-1)(2k-3)} \right].$$

Case 2 : $r_2 = 0$, $C_0 = 1$

$$C_k = \frac{(-1)^k (k-3)(k-4)(k-5)\cdots(-1)(-2)}{k(2k-1)(k-1)(2k-3)\cdots(1)}$$

for $k = 3$, $C_3 = 0$ therefore $C_k = 0$ for $k = 1, 2, 3,...$

$$C_1 = -\frac{(-2)C_0}{(1)} = 2$$

$$C_2 = -\frac{(-1)C_1}{2\cdot 3} = \frac{2}{6} = \frac{1}{3}.$$

and $$\phi_2(x) = c_0 + c_1 x + c_2 x^2$$

$$= 1 + 2x + \frac{1}{3}x^2$$

Thus, for $x \neq 0$ we get two solutions

$$\phi_1(x) = |x|^{\frac{1}{2}} \left[1 + \sum_{k=1}^{\infty} \frac{3(-1)^k x^k}{2^k k!(2k+1)(2k-1)(2k-3)} \right]$$

$$\phi_2(x) = 1 + 2x + \frac{x^2}{3}$$

Check that series in the first solution is convergent Let $\phi_1(x) = \sum_{k=0}^{\infty} d_k(x)$.

Using ratio test

$$\left| \frac{d_{k+1}}{d_k} \right| = \left| \frac{x}{2(k+1)(2k+3)} \right| = \frac{|x|}{2(k+1)(2k+3)} \to 0$$

as $k \to \infty$ if $|x| < \infty$. The series convergent for finite x.

Q. 2. Obtain two linearly independent solutions of the following equations which are valid near $x = 0$.

(a) $x^2 y'' + 3xy' + (1+x)y = 0$

(b) $x^2 y'' + 2x^2 y' - 2y = 0$

Ans. :

(a): For $x > 0$ suppose we have a solution ϕ of the form

$$\phi(x) = x^r \sum_{k=0}^{\infty} C_k x^k \quad , \quad C_0 \neq 0$$

Let $\quad L(y) = x^2 y'' + 3xy' + (1+x)y$

$$L(\phi)(x,r) = x^r \sum (k+r)(k+r-1) C_k x^k + 3x^r \sum (k+r) C_k x^k + \sum C_k x^{k+r} + x^r \sum C_k x^{k+1}$$

$$= \sum_{k=0}^{\infty} \left[(k+r)(k+r-1) + 3(k+r) + 1)\right] C_k x^{k+r} + \sum_{k=0}^{\infty} c_k x^{k+r+1}$$

$L(\phi)(x,r) = 0$ implies

$$\sum_{k=0}^{\infty} \left[(k+r)(k+r-1) + 3(k+r) + 1\right] C_k x^{k+r} + \sum_{k=0}^{\infty} C_k x^{k+r+1} = 0$$

$$\left[r(r-1) + 3r + 1\right] C_0 x^r + \left\{\left[(r+1)(r) + 3(r+1) + 1\right] C_1 + C_0\right\} x^{r+1}$$

$$+ \left\{\left[(r+2)(r+1) + 3(r+2) + 1\right] C_2 + C_1\right\} x^{r+2} + \cdots = 0$$

$q(r) = r(r-1) + 3r + 1 = 0$ is indicial equation.

$$q(r) = r^2 - r + 3r + 1 = (r+1)^2$$

$$L(\phi)(x) = q(r) C_0 x^r + \left[q(r+1) C_1 + C_0\right] x^{r+1} + \left[q(r+2) C_2 + C_1\right] q^{r+2} + \cdots = 0$$

$$= q(r) C_0 x^r + \sum_{k=1}^{\infty} \left[q(r+k) C_k + C_{k-1}\right] x^{r+k} = 0$$

$L(\phi)(x) = 0$ if and only if $q(r) = 0$ and

$$q(r+k) C_k + C_{k-1} = 0 \text{ for } k = 1, 2, 3, \ldots$$

$q(r) = 0$ implies $(r+1)^2 = 0$ that is $r = -1$ is a repeated root. Here $r = r_1 = r_2 = -1$.

$$q(r+k) C_k + C_{k-1} = 0 \text{ for } k = 1, 2, 3, \ldots$$

Since, $\quad q(r) = (r+1)^2, \ q(r+k) = (r+k+1)^2$ and

$$C_k = \frac{-C_{k-1}}{(r+k+1)^2}$$

$$= \left(\frac{-1}{(r+k+1)^2}\right)\left(\frac{-1}{(r+k)^2}\right)\left(\frac{-1}{(r+k-1)^2}\right)\left(\frac{-1}{(r+k-2)^2}\right)\cdots\left(\frac{-1}{(r+2)^2}\right)$$

$$= \frac{(-1)^k C_0}{\left[(r+2)(r+3)\cdots(r+k-2)(r+k-1)(r+k)(r+k+1)\right]^2}$$

The first solution will be constructed by substituting C_k's at $r = -1$ in the series. C_k at $r = -1$ is

$$C_k = \frac{(-1)^k C_0}{k!^2}$$

$$\phi_1(x) = x^r \sum_{k=0}^{\infty} C_k\, x^k$$

$$= x^{-1} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!^2} x^k$$

The series converges for all finite x.

Since $r = -1$ is the root of multiplicity 2, i.e. $r = r_1 = r_2 = -1$, the second solution

$$\phi_2(x) = x^{r_1} \sum_{k=0}^{\infty} C_k'(r_1)\, x^k + (\log x)\,\phi_1(x)$$

$$C_k = \frac{(-1)^k C_0}{\left[(r+2)(r+3)(r+4)\cdots(r+k+1)\right]^2}.$$

Define $D = (r + 2)\ (r + 3)\ (r + 4)\ .....\ (r + k + 1)$ then

$$C_k' = (-1)^k C_0 \frac{(-2)}{D^2}\left[\frac{1}{r+2} + \frac{1}{r+3} + \frac{1}{r+4} + \cdots + \frac{1}{r+k+1}\right]$$

$$C_k'(-1) = \frac{(-2)(-1)^k}{k!^2}\left[\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{k}\right]$$

$$\phi_2(x) = x^{-1} \sum_{k=0}^{\infty} \frac{(-2)(-1)^k}{k!^2}\left[1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{k}\right] x^k + (\log x)\,\phi_1(x)$$

To obtain solution for $x < 0$ we replace x by $|\,x\,|$.

Thus, the two solutions are

$$\phi_1(x) = |\,x\,|^{-1} \sum_{k=0}^{\infty} \frac{(-1)^k x^k}{k!^2}$$

$$\phi_2(x) = |\,x\,|^{-1} \sum_{k=0}^{\infty} \frac{(-2)(-1)^k}{k!^2}\left[1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{k}\right] x^k$$

$$+ (\log|\,x\,|)\,|\,x\,|^{-1} \sum_{k=0}^{\infty} \frac{(-1)^k x^k}{k!^2}.$$

Check that series in both the solutions converge.

(b): For $x > 0$ suppose we have a solution ϕ of the form

$$\phi(x) = x^r \sum_{k=0}^{\infty} C_k\, x^k \qquad (C_0 \neq 0)$$

Let $\quad L(y) = x^2 y'' + 2x^2 y' - 2y$ then

$$L(\phi)(x,\ r) = \sum_{k=0}^{\infty}\left[(k+r)(k+r-1)C_k x^{k+r} + 2(k+r)C_k\, x^{k+r+1} - 2C_k x^{k+r}\right]$$

$$= \sum_{k=0}^{\infty}\left[(k+r)(k+r-1) - 2\right]C_k\, x^{k+r} + \sum_{k=0}^{\infty} 2(k+r)\,C_k\, x^{k+r+1}$$

$$= [r(r-1)-2]C_0 x^r + \{[(r+1)(r)-2]C_1 + 2rC_0\}x^{r+1}$$
$$+\{[(r+2)(r+1)-2]C_2 + 2(r+1)C_1\}x^{r+2} + \cdots$$
$$= q(r)C_0 x^r + \sum_{k=1}^{\infty}[q(r+k)C_k + 2(r+k-1)C_{k-1}]x^{r+k}$$

Indicial equation is $q(r) = r(r-1)-2 = 0$ gives $r^2 - r - 2 = (r-2)(r+1) = 0$ $r = -1, 2.$ $r_1 = 2,\ r_2 = -1$ and $r_1 - r_2 = 3$ a positive integer.

If $r_1 - r_2$ is a positive integer, we try a series using the smallest root. If c_0 and c_3 both turn out to be arbitrary, we obtain the general solution by this method. Otherwise the general solution will involve a logarithm as it did in the case of equal roots. That logarithmic case is treated in theorem 3.2.2.

Let us consider the series solution as $\phi(x) = \sum_{k=0}^{\infty} c_k x^{k-1}$ $(r_2 - \text{smallest root} = -1)$

$$L(\phi) = \sum_{k=0}^{\infty}(k-1)(k-2)c_k x^{k-1} + 2\sum(k-1)c_k x^k - 2\sum c_k x^{k-1}$$
$$= \sum_{k=0}^{\infty}[(k-1)(k-2)-2]c_k x^{k-1} + 2\sum_{k=0}^{\infty}(k-1)c_k x^k$$
$$= (2-2)c_0 x^{-1} + \sum_{k=1}^{\infty}\{[(k-1)(k-2)-2]c_k + 2(k-2)c_{k-1}\}x^{k-1}$$

Since ϕ is a solution $L(\phi) = 0$ i.e.

$0.c_0 = 0$ i.e. c_0 is arbitary.

$$[(k-1)(k-2)-2]c_k + 2(k-2)c_{k-1} = 0 \qquad k = 1,2,3,\ldots$$

$k = 1$ $\qquad -2c_1 - 2c_0 = 0$ i.e. $c_1 = -c_0$

$k = 2$ $\qquad -2c_2 = 0$ i.e. $c_2 = 0$

$k = 3$ $\qquad (2-2)c_3 + 2c_2 = 0$ i.e. $0 \cdot c_3 = 0 \Rightarrow c_3$ is arbitary.

$$[(k-1)(k-2)-2]c_k + 2(k-2)c_{k-1} = 0 \qquad k = 4,5,6,\ldots$$
$$(k^2 - 3k)c_k + 2(k-2)c_{k-1} = 0$$
$$k(k-3)c_k + 2(k-2)c_{k-1} = 0$$
$$c_k = -\frac{2(k-2)c_{k-1}}{k(k-3)}, \qquad k = 4,5,6,7,\ldots.$$
$$c_k = \left(\frac{-2(k-2)}{k(k-3)}\right)\left(\frac{-2(k-3)}{(k-1)(k-4)}\right)\left(\frac{-2(k-4)}{(k-2)(k-5)}\right)\cdots\left(\frac{-2(2)}{4\cdot 1}\right)c_3$$
$$= \frac{(-2)^{k-3}(k-2)(k-3)(k-4)\cdots(2)}{k(k-3)(k-1)(k-4)(k-2)(k-5)\cdots 4\cdot 1}c_3$$

$$= \frac{(-2)^{k-3}(k-2)\,6}{k!} c_3$$

Thus, we get a solution

$$\phi(x) = c_0\, x^{-1} - c_0\, x^0 + 0 \cdot x^1 + c_3\, x^2 + \sum_{k=4}^{\infty} \frac{(-2)^{k-3}\,(k-2)\,6}{k!} c_3\, x^{k-1}$$

$$= c_0 (\, x^{-1} - 1) + c_3 \left[x^2 + \sum_{k=4}^{\infty} \frac{(-2)^{k-3}\,(k-2)\,6}{k!} x^{k-1} \right]$$

$$= c_0\, x^{-1}(1-x) + c_3 \left[x^2 + \sum_{k=1}^{\infty} \frac{(-2)^k\,(k+1)\,6}{(k+3)!} x^{k+2} \right]$$

$$= c_0\, x^{-1}(1-x) + c_3\, x^2 \left[1 + \sum_{k=1}^{\infty} \frac{(-2)^k\,(k+1)\,6}{(k+3)!} x^k \right]$$

Thus, we get two solutions

$$\phi_1(x) = x^{-1}(1-x) \text{ and } \phi_2(x) = x^2 \left[1 + \sum_{k=1}^{\infty} \frac{(-2)^k\,(k+1)\,6}{(k+3)!} x^k \right]$$

These are two solutions for $x > 0$ for $x < 0$ replace x by $|\, x \,|$ we get,

$$\phi_1(x) = |\, x \,|^{-1}\,(1-x) \text{ and } \phi_2(x) = |\, x \,|^2 \left[1 + \sum_{k=1}^{\infty} \frac{(-2)^k\,(k+1)\,6}{(k+3)!} x^k \right]$$

Check that series appearing in ϕ_2 is convergent series.

EXERCISE

1. Compute indicial polynomials and their roots for the following equations.

(a) $x^2 y'' + (x + x^2) y' - y = 0$

(b) $x^2 y'' + x\, y' + \left(x^2 - \frac{1}{4} \right) y = 0$

(c) $4x^2 y'' + (4x^4 - 5x) y' + (x^2 + 2) y = 0$

(d) $x^2 y'' + (x - 3x^2) y' + e^x y = 0$

2. Find a solutions ϕ of the form

$$\phi(x) = x^r \sum_{k=0}^{\infty} C_k\, x^k \;, \quad (x > 0)$$

for the following equations.

(a) $2x^2y'' + (x^2 - x)y' + y = 0$

(b) $x^2y'' + (x - x^2)y' + y = 0$

3. For each equation obtain two linearly independent solutions valid near origin

(a) $2x(x-1)y'' + 3(x-1)y' - y = 0$

(b) $2xy'' + 5(1+2x)y' + 5y = 0$

(c) $3xy'' + (2-x)y' - 2y = 0$

(d) $2xy'' + (1-2x^2)y' - 4xy = 0$

4. Consider the following equation near $x = 0$

(a) $2x^2y'' + (5x + x^2)y' + (x^2 - 2)y = 0$

(b) $4x^2y'' - 4xe^x y' + 3(\cos x)y = 0$

Compute the roots r_1, r_2 of the indicial equation for each relative to $x = 0$.

5. Obtain two linearly independent solutions of the following equations which are valid near $x = 0$.

(a) $x^2y'' - 2x(x+1)y' + 2(x+1)y = 0$

(b) $x^2y'' - 2x^2y' + (4x - 2)y = 0$

(c) $xy'' - (4+x)y' + 2y = 0$

(d) $x^2y'' + 2x(x-2)y' + 2(2-3x)y = 0$.

Answers :

1. (a) $q(r) = r^2 - 1 \quad ; \quad r_1 = 1, \; r_2 = -1$

(b) $q(r) = r^2 - \frac{1}{4} \quad ; \quad r_1 = \frac{1}{2}, \; r_2 = -\frac{1}{2}$

(c) $q(r) = r^2 - \frac{9}{4}r + \frac{1}{2} \quad ; \quad r_1 = 2, \; r_2 = \frac{1}{4}$

(d) $q(r) = r^2 + 1 \quad ; \quad r_1 = i, \; r_2 = -i$

2. (a) $y_1(x) = x\sum_{k=0}^{\infty} \frac{(-1)^k x^k}{1\cdot 3\cdot 5\cdot 7\cdots(2k+1)}$, $\qquad y_2(x) = x^{1/2}\sum_{k=0}^{\infty} \frac{(-1)^k x^k}{2^k k!} = x^{1/2}e^{-\frac{x}{2}}$

(b) $y_1(x) = x^i \sum_{k=0}^{\infty} \frac{i(1+i)\cdots(k-1+i)}{(1+2i)(2+2i)\cdots(k+2i)k!}x^k; \qquad y_2(x) = x^{-i}\sum_{k=0}^{\infty} \frac{(-i)(1-i)\cdots(k-1-i)}{k!\,(1+2i)(2-2i)\cdots(k-2i)}x^K$

3. **(a)** $y_1 = 1 - \sum\limits_{n=1}^{\infty} \frac{x^n}{4n^2 - 1}$; $y_2 = x^{-1/2} - x^{1/2}$

(b) $y_1 = 1 + \sum \frac{3(-5)\,x^n}{n!(2n+1)(2n+3)}$; $y_2 = x^{-\frac{3}{2}} - 10x^{-1/2}$

(c) $y_1 = \sum\limits_{n=0}^{\infty} \frac{(3n+4)\,x^{n+\frac{1}{3}}}{4 \cdot 3^n n!}$; $y_2 = 1 + \sum\limits_{n=1}^{\infty} \frac{(n+1)\,x^n}{2 \cdot 5 \cdot 8 \cdots (3n-1)}$

(d) $y_1 = \sum\limits_{n=0}^{\infty} \frac{x^{2k+1}}{2^k k!}$; $y_2 = 1 + \sum\limits_{n=1}^{\infty} \frac{2^k x^{2k}}{3 \cdot 7 \cdot 11 \cdots (4k-1)}$

4. **(a)** $r_1 = \frac{1}{2}$; $r_2 = -2$

(b) $r_1 = \frac{3}{2}$; $r_2 = \frac{1}{2}$

5. **(a)** $y_1(x) = x$; $y_2(x) = x(e^{2x} - 1)$

(b) $y_1(x) = x^2$; $y_2(x) = x^{-1}\left[1 + 3x + 6x^2 - 3\sum\limits_{k=4}^{\infty} \frac{2^k x^k}{(k-3)k!}\right] - 4x^2 \log|x|$

(c) $y_1 = 1 + \frac{1}{2}x + \frac{1}{12}x^2$; $y_2 = x^5 + \sum\limits_{n=6}^{\infty} \frac{60\,x^n}{(n-5)!\,n\,(n-1)(n-2)}$

(d) $y_1 = x - 2x^2 + 2x^3$; $y_2 = x^4 + \sum\limits_{n=4}^{\infty} \frac{6(-2)^{n-3} x^{n+1}}{n!}$

Unit 3 : The Bessel equation

If α is a constant, *Re* $\alpha \geq 0$ the Bessel equation of order α is the equation

$$x^2 y'' + xy' + (x^2 - \alpha^2) y = 0$$

This has the form

$$y'' + \frac{1}{x} y' + \frac{x^2 - \alpha^2}{x^2} y = 0$$

where $p(x) = \frac{1}{x}$ and $q(x) = \frac{x^2 - \alpha^2}{x^2}$. $x = 0$ is a singular point. Since the denominator of $p(x)$ does not contain x to a power higher than one and the denominator of $q(x)$ (i.e. x^2) does not contain the factor x to a power higher then 2, $x = 0$ is a regular singular point. Therefore the power series solution ϕ will have the form

$$\phi(x) = x^r \sum_{k=0}^{\infty} C_k\, x^k$$

Let $\qquad L(y) = x^2 y'' + xy' + (x^2 - \alpha^2) y.$

$$L(\phi)(x, r) = \sum (k+r)(k+r-1) C_k\, x^{k+r} + \sum (k+r) C_k\, x^{k+r} + (x^2 - \alpha^2) \sum C_k\, x^{k+r}$$

$$= \sum_{k=0}^{\infty} \left[(k+r)(k+r-1) + (k+r) - \alpha^2\right] C_k\, x^{k+r} + \sum_{k=0}^{\infty} C_k\, x^{k+r+2}$$

$$= \left[r(r-1) + r - \alpha^2\right] C_0\, x^r + \left[(r+1)r + (r+1) - \alpha^2\right] C_1 x^{r+1}$$

$$+ \sum_{k=2}^{\infty} \left\{\left[(k+r)^2 - \alpha^2\right] C_k + C_{k-2}\right\} x^{k+r}$$

The indicial equation is

$$q(r) = r(r-1) + r - \alpha^2 = 0$$

$$q(r) = r^2 - \alpha^2 = 0.$$

The indicial polynomial $q(r)$ has two roots $r_1 = \alpha$ and $r_2 = -\alpha$. We shall construct solutions for $x > 0$. We consider three cases namely $\alpha = 0$, 2α is not a positive integer and 2α is a positive integer.

Case 1 : $\alpha = 0$

Since the roots are both equal to zero by theorem 3.2.2, there are two solutions ϕ_1, ϕ_2 of the form

$$\phi_1(x) = \sigma_1(x) \text{ and } \phi_2(x) = x\sigma_2(x) + (\log x)\phi_1(x),$$

Where $\sigma_1(x)$, $\sigma_2(x)$ have power series expansions which converge for all finite x. Since $\alpha = 0$,

$$L(y) = x^2 y'' + x\, y' + x^2 y$$

Suppose $\sigma_1(x) = x^{\alpha} \sum_{k=0}^{\infty} C_k\, x^k = \sum_{k=0}^{\infty} C_k\, x^k \quad (C_0 \neq 0)$ be a solution of $L(y) = 0$. Then

$$L(\sigma_1)(x) = \sum_{k=2}^{\infty} k(k-1) C_k\, x^k + \sum_{k=1}^{\infty} k\, C_k\, x^k + \sum_{k=0}^{\infty} C_k\, x^{k+2}$$

$$= \sum_{k=2}^{\infty} k(k-1) C_k\, x^k + C_1 x + \sum_{k=2}^{\infty} k\, C_k\, x^k + \sum_{k=2}^{\infty} C_{k-2}\, x^k$$

$$= C_1 x + \sum_{k=2}^{\infty} \left\{\left[k(k-1) + k\right] C_k + C_{k-2}\right\} x^k$$

Since σ_1 is a solution $L(\sigma_1) = 0$ for all x. Therefore $C_1 = 0$ and $\left[k(k-1) + k\right] C_k + C_{k-2} = 0$, $k = 2, 3, 4$

Thus, $C_1 = 0$ and

$$k^2 C_k = -C_{k-2} \text{ for } k = 2,3,4,.....$$

The recurrance relation becomes

$$C_k = -\frac{C_{k-2}}{k^2}, \quad k = 2,3,4,.....$$

Since $\qquad C_1 = 0,\ C_3 = C_5 = C_7 = ... = C_{2n+1} = ... = 0.$

The choice $C_0 = 1$ implies

$$C_2 = -\frac{C_0}{2^2} = -\frac{1}{2^2},\ C_4 = -\frac{C_2}{4^2} = \left(-\frac{1}{4^2}\right)\left(-\frac{1}{2^2}\right) = \frac{(-1)^2}{2^2 \cdot 4^2},.......$$

In general

$$C_{2m} = \frac{(-1)^m}{2^2 \cdot 4^2 \cdot 6^2 \cdots (2m)^2} = \frac{(-1)^m}{2^{2m}\, m!^2}, \quad m = 1,2,3,.....$$

Thus σ_1 contains only even powers of x and we get

$$\sigma_1(x) = \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{2^{2m}\, m!^2}$$

The function defined by this series is called the Bessel function of zero order of the first kind and is denoted by J_0. Thus

$$J_0(x) = \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{2^{2m}\, m!^2}$$

Let $\qquad J_0(x) = \sum_{k=0}^{\infty} d_k(x)$

Using ratio test

$$\left|\frac{d_{k+1}}{d_k}\right| = \left|\frac{x^2}{2^2 (k+1)^2}\right| = \frac{x^2}{4(k+1)^2} \to 0$$

as $k \to \infty$ if $|x| < \infty$. Thus, the series converges for $|x| < \infty$ and $J_0(x)$ is the first solution of Bessel equation with $\alpha = 0$.

Now we determine a second solution ϕ_2 for the Bessel equation of order zero (i.e. $\alpha = 0$). Let $\phi_1(x) = J_0(x)$ then the solution ϕ_2 has the form

$$\phi_2(x) = x\sigma_2(x) + (\log x)\phi_1(x)$$

$$= \sum_{k=0}^{\infty} c_k x^k + (\log x)\phi_1(x), \qquad (C_0 = 0).$$

Since ϕ_2 is second solution

$$L(\phi_2)(x) = x^2\phi_2'' + x\phi_2' + x^2\phi_2 = 0$$

$$\phi_2'(x) = \sum_{k=1}^{\infty} k c_k x^{k-1} + \frac{\phi_1(x)}{x} + (\log x)\phi_1'(x)$$

$$\phi_2''(x) = \sum_{k=2}^{\infty} k(k-1)c_k\, x^{k-2} + \frac{2\phi_1'(x)}{x} - \frac{\phi_1}{x^2} + (\log x)\phi_1''$$

$$L(\phi_2)(x) = \sum_{k=2}^{\infty} k(k-1)c_k\, x^k + 2x\phi_1'(x) - \phi_1(x)$$

$$+x^2(\log x)\phi_1'' + \sum_{k=1}^{\infty} k\, c_k\, x^k + \phi_1(x) + x(\log x)\phi_1'(x)$$

$$+x^2 \sum_{k=0}^{\infty} c_k\, x^k + x^2\, (\log x)\phi_1(x)$$

$$= \sum_{k=2}^{\infty} k(k-1)c_k\, x^k + \sum_{k=1}^{\infty} k\, c_k\, x^k + \sum_{k=0}^{\infty} c_k\, x^{k+2} + 2x\phi_1'(x)$$

$$+(\log x)(x^2\phi_1'' + x\phi_1' + x^2\phi_1)$$

Since $\quad L(\phi_1) = x^2\phi_1'' + x\phi_1' + \phi_1 = 0,$

$$L(\phi_2)(x) = \sum_{k=2}^{\infty} k(k-1)c_k\, x^k + c_1\, x + \sum_{k=2}^{\infty} k\, c_k\, x^k + \sum_{k=2}^{\infty} c_{k-2}\, x^k + 2x\phi_1'(x)$$

Since $L(\phi_2)$ should be zero and $\phi_1'(x) = J_0'(x),$

$$\phi_1'(x) = J_0'(x) = \sum \frac{(-1)^m\, 2m\, x^{2m-1}}{2^{2m}(m!)^2},$$

Thus $\quad \displaystyle\sum_{k=2}^{\infty} k(k-1)c_k\, x^k + c_1\, x + \sum_{k=2}^{\infty} k\, c_k\, x^k + \sum_{k=2}^{\infty} c_{k-2}\, x^k = -2\sum \frac{(-1)^m\, 2m\, x^{2m}}{2^{2m}(m!)^2}$

$$c_1 x + \sum_{k=2}^{\infty} \left\{[k(k-1)+k]c_k + c_{k-2}\right\} x^k = -2\sum \frac{(-1)^m\, 2m\, x^{2m}}{2^{2m}(m!)^2}$$

Since the series on right has only even powers of x, all odd terms on the left hand side should be zero $c_1 = c_3 = c_5 = c_7 = ... = c_{2n+1} = ...= 0$.

The relation for the other coefficients that is for k even (let k = 2m) is

$$[2m(2m-1)+2m]c_{2m} + c_{2m-2} = -2\ \frac{(-1)^m\, 2m}{2^{2m}(m!)^2}$$

$$[4m^2]c_{2m} + c_{2m-2} = \frac{(-1)^{m+1} m}{2^{2m-2}(m!)^2},\quad m = 1, 2, 3, 4, ...$$

$$c_{2m} = \frac{1}{(2m)^2}\left[\frac{(-1)^{m+1} m}{2^{2m-2}(m!)^2} - c_{2m-2}\right],\quad m = 1, 2, 3, 4....$$

$$c_0 = 0$$

$$c_2 = \frac{1}{2^2}, \ c_4 = \frac{1}{4^2}\left[-\frac{1}{2.2^2} - \frac{1}{2^2}\right] = -\frac{1}{2^2 4^2}\left(\frac{1}{2}+1\right)$$

$$c_6 = \frac{1}{6^2}\left[\frac{1}{2^2\,4^2\,3} + \frac{1}{2^2\,4^2}\left(\frac{1}{2}+1\right)\right] = +\frac{1}{2^2\,4^2\,6^2}\left(\frac{1}{3}+\frac{1}{2}+1\right)$$

and it can be shown by induction that

$$c_{2m} = \frac{(-1)^m}{2^{2m}(m!)^2}\left(\frac{1}{m}+\frac{1}{m-1}+\frac{1}{m-2}+\cdots+\frac{1}{3}+\frac{1}{2}+1\right), \ m = 1, 2, 3, \ldots.$$

The solution thus determined is called a Bessel function of zero order of second kind, and is denoted by K_0. Hence,

$$K_0(x) = -\sum_{m=1}^{\infty} \frac{(-1)^m}{(m!)^2}\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{m}\right)\left(\frac{x}{2}\right)^{2m} + (\log x)\ J_0(x)$$

Using the ratio test we can check that the series on the right is convergent for all finite values of x.

Now we compute solution for Bessel equation of order α, where $\alpha \neq 0$ and $Re\ \alpha \geq 0$.

$$L(y) = x^2 y'' + x\,y' + (x^2 - \alpha^2)y = 0$$

Let $x > 0$. The roots of indicial equation are

$$r_1 = \alpha, \ r_2 = -\alpha.$$

Let us find out the solution corresponding to $r_1 = \alpha$. A solution $\phi_1(x)$ has the form

$$\phi_1(x) = x^{\alpha} \sum_{k=0}^{\infty} c_k\, x^k, \qquad (c_0 \neq 0)$$

Then

$$L(\phi_1)(x) = \sum_{k=0}^{\infty} (k+\alpha)(k+\alpha-1)c_k\, x^{k+\alpha} + \sum_{k=0}^{\infty} (k+\alpha)c_k\, x^{k+\alpha}$$

$$+ \sum_{k=0}^{\infty} c_k\, x^{k+\alpha+2} - \alpha^2 \sum_{k=0}^{\infty} c_k\, x^{k+\alpha}$$

$$= \sum_{k=0}^{\infty}\left[(k+\alpha)(k+\alpha-1)+(k+\alpha)-\alpha^2\right]c_k\, x^{k+\alpha} + \sum_{k=0}^{\infty} c_k\, x^{k+\alpha+2}$$

$$= \sum_{k=0}^{\infty}\left[(k+\alpha)^2-\alpha^2\right]c_k\, x^{k+\alpha} + \sum_{k=0}^{\infty} c_k\, x^{k+\alpha+2}$$

$$= 0\cdot c_0\, x^{\alpha} + \left[(\alpha+1)^2-\alpha^2\right]c_1\, x^{\alpha+1} + \sum_{k=2}^{\infty}\left[(k+\alpha)^2-\alpha^2\right]c_k\, x^{k+\alpha}$$

$$+ \sum_{k=2}^{\infty} c_{k-2}\, x^{k+\alpha}$$

$$= 0\cdot c_0\, x^{\alpha} + \left[(\alpha+1)^2 - \alpha^2\right] c_1\, x^{\alpha+1} + \sum_{k=2}^{\infty}\left[\left((k+\alpha^2)-\alpha^2\right)c_k + c_{k-2}\right] x^{k+\alpha}$$

Thus, $L(\phi_1)(x) = 0$ implies

$$0\cdot c_0 = 0,\ c_0 \text{ is arbitrary.}$$

$$c_1 = 0,$$

$$\left[(\alpha+k)^2 - \alpha^2\right] c_k + c_{k-2} = 0 \quad k = 2, 3, 4, ...$$

Since $(\alpha+k)^2 - \alpha^2 = k(2\alpha+k) \neq 0$ for $k = 2, 3, ...,$

$$k(2\alpha+k)c_k + c_{k-2} = 0 \text{ gives}$$

$$c_k = -\frac{c_{k-2}}{k(2\alpha+k)}.$$

Since $c_1 = 0,\ c_{2k+1} = 0$ for $k = 0, 1, 2, 3, ...$

that is all odd terms are zero.

We find

$$c_2 = -\frac{c_0}{2(2\alpha+2)} = \frac{-c_0}{2^2(\alpha+1)}$$

$$c_4 = -\frac{c_2}{4(2\alpha+4)} = \frac{c_0}{2^2(\alpha+1)(\alpha+2)}$$

$$c_6 = \frac{-c_4}{6(2\alpha+6)} = \frac{-c_4}{12(\alpha+3)} = \frac{-c_4}{2\cdot 3!(\alpha+3)} = \frac{-c_0}{2^6\cdot 3!(\alpha+1)(\alpha+2)(\alpha+3)}$$

$$c_8 = \frac{-c_6}{8(2\alpha+8)} = \frac{-c_0}{2^8\, 4!(\alpha+1)(\alpha+2)(\alpha+3)(\alpha+4)}$$

In general,

$$c_{2m} = \frac{(-1)^m c_0}{2^{2m}\, m!(\alpha+1)(\alpha+2)(\alpha+3)\cdots(\alpha+m)}.$$

Thus the solution ϕ_1 becomes

$$\phi_1(x) = x^{\alpha}\sum_{k=0}^{\infty} c_k\, x^k \qquad (c_0 \neq 0)$$

$$= c_0 x^{\alpha} + x^{\alpha}\sum_{m=1}^{\infty}\frac{(-1)^m x^{2m}}{2^{2m}\, m!(\alpha+1)(\alpha+2)\cdots(\alpha+m)}$$

for $\alpha = 0$ and $c_0 = 1$, $\phi_1(x)$ becomes $J_0(x)$. Before going for the second solution let us define gamma function and study some properties of gamma function.

Definition 3.3.1

The gamma function is defined by

$$\Gamma(z)=\int_0^\infty e^{-x}\,x^{z-1}dx, \qquad (\operatorname{Re} z>0)$$

Lemma 1 : $\Gamma(z+1)=z\Gamma(z)$

Proof : $\Gamma(z+1)=\int_0^\infty e^{-x}\,x^{z}dx$

$$=\lim_{T\to\infty}\int_0^\infty e^{-x}\,x^{z}dx$$

$$=\lim_{T\to\infty}\left[x^z\frac{e^{-x}}{-1}\Big|_{x=0}^{T}-\int_0^T -e^{-x}\cdot ze^{z-1}dx\right]$$

$$=\lim_{T\to\infty} z\int_0^T e^{-x}x^{z-1}dx$$

$$=z\int_0^\infty e^{-x}x^{z-1}dx$$

$$=z\ \Gamma(z)$$

Observe that $\lim_{T\to\infty} x^z e^{-x}\Big|_0^T=\lim_{T\to\infty} T^z\cdot e^{-T}-0=0$

By definition 3.3.1

$$\Gamma(1)=\int_0^\infty e^{-x}dx=1$$

Thus, if z is a positive integer n,

$$\Gamma(n+1)=n\Gamma(n)=n(n-1)(n-2)$$
$$=(n)(n-1)(n-2)(n-3)\cdots\Gamma(1)$$
$$=n\,!$$

Thus, gamma function is an extension of the factorial function to numbers which are not integers.

Suppose Re $z<0$ and z is not a negative integer then there is a natural number N such that

$$-N<\operatorname{Re} z<-N+1$$

But then Re $(z+N)>0$ and therefore we can define

$$\Gamma(z+N)=(z+N-1)(z+N-2)\cdots(z+1)\,z\,\Gamma(z).\ \text{Then}$$

$$\Gamma(z)=\frac{\Gamma(z+N)}{(z+N-1)(z+N-2)(z+N-3)\cdots(z+1)\,z}, \quad (\operatorname{Re} z<0)$$

The gamma function is not defined at 0, –1, –2, –3,.......

We have a solution $\phi_1(x)$ as

$$\phi_1(x) = c_0 x^{\alpha} + c_0 x^{\alpha} \sum_{m=1}^{\infty} \frac{(-1)^m x^{2m}}{2^{2m}\, m!(\alpha+1)(\alpha+2)\cdots(\alpha+m)}$$

Now choose $c_0 = \dfrac{1}{2^{\alpha}\Gamma(\alpha+1)}$. we obtain a solution of the Bessel equation of order α which is denoted by J_α and is called the Bessel function of order α of the first kind.

$$J_\alpha(x) = \left(\frac{x}{2}\right)^{\alpha} \sum_{m=0}^{\infty} \frac{(-1)^m}{m!(\alpha+m)(\alpha+m-1)\cdots(\alpha+1)\Gamma(\alpha+1)} \left(\frac{x}{2}\right)^{2m}$$

$$= \left(\frac{x}{2}\right)^{\alpha} \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\,\Gamma(\alpha+m+1)} \left(\frac{x}{2}\right)^{2m}, \quad (\operatorname{Re} z > 0)$$

Observe that this formula for J_α reduces to J_0 when $\alpha = 0$ as $\Gamma(m + 1) = m!$. $J_\alpha(x)$ is one solution of Bessel equation with $\alpha \neq 0$ and Re $\alpha \geq 0$.

To determine second solution we have to consider two situations. Either 2α is not a positive integer or 2α is a positive integer. We determine second solution for both the situations.

Case 2 : 2α is not a positive integer

If 2α is not a positive integer there is another solution $\phi_2(x)$ corresponding to the root $r_2 = -\alpha$ of the form

$$\phi_2(x) = x^{-\alpha} \sum_{k=0}^{\infty} c_k x^k.$$

On repeating the same calculations we have carried out for the root α, (replace α by $-\alpha$ everywhere)

We get the second solution

$$J_{-\alpha}(x) = \left(\frac{x}{2}\right)^{-\alpha} \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\,\Gamma(m-\alpha+1)} \left(\frac{x}{2}\right)^{2m}.$$

Observe that $\Gamma(m - \alpha + 1)$ exists for all $m = 0, 1, 2, 3, ...$ since α is not a positive integer.

Case 3 : 2α is a positive integer

(a) α is not a positive integer .

If α is not a positive integer $\Gamma(m - \alpha + 1)$ exists and the function $J_{-\alpha}(x)$ is the second solution of the Bessel equation.

(b) α is a positive integer.

Suppose $\alpha = n$. According to theorem 3.2.2 there is a solution ϕ_2 of the form

$$\phi_2(x) = x^{-n} \sum_{k=0}^{\infty} c_k x^k + c(\log x) J_n(x).$$

$$L(\phi_2)(x) = x^2 \phi_2''(x) + x\phi_2'(x) + (x^2 - n^2)\phi_2(x) = 0 \text{ implies}$$

$$x^2\left[\sum_{k=0}^{\infty}(k-n)(k-n-1)c_k x^{k-n-2}+\frac{2c}{x}J_n'-\frac{c}{x^2}J_n+(c\log x)J_n''\right]$$

$$+\ x\left[\sum_{k=0}^{\infty}(k-n)c_k\,x^{k-n-1}+\frac{c}{x}J_n+(c\ \log\ x)J_n'\right]$$

$$+\ (x^2-n^2)\left[\sum_{k=0}^{\infty}c_k\,x^{k-n}+(c\ \log\ x)J_n(x)\right]=0.$$

Since $J_n(x)$ is the first solution corresponding to $\alpha=n,\ x^2J_n''+x\ J_n'+(x^2-n^2)\ \ J_n(x)=0,$ and above equation gets reduce to

$$\sum_{k=0}^{\infty}(k-n)(k-n-1)c_k\,x^{k-n}+2c\,x\,J_n'-c\,J_n+\sum_{k=0}^{\infty}(k-n)c_k x^{k-n}+c\,J_n$$

$$+\sum_{k=0}^{\infty}c_k\,x^{k-n+2}-n^2\sum_{k=0}^{\infty}c_k\,x^{k-n}=0$$

Therefore

$$\sum_{k=0}^{\infty}\left[(k-n)(k-n-1)+(k-n)-n^2\right]c_k\,x^{k-n}+\sum_{k=0}^{\infty}c_k x^{k-n+2}+2\,c\,x\,J_n'(x)=0.$$

or

$$\sum_{k=0}^{\infty}\left[(k-n)^2-n^2\right]c_k\,x^{k-n}+\sum_{k=0}^{\infty}c_k x^{k-n+2}+2\,c\,x\,J_n'(x)=0$$

that is

$$0\cdot c_0\,x^{-n}+\left[(1-n)^2-n^2\right]c\,x^{1-n}+\sum_{k=2}^{\infty}\left[(k-n)^2-n^2\right]c_k\,x^{k-n}+\sum_{k=0}^{\infty}c_k\,x^{k-n+2}$$

$$+2\,c\,x\,J_n'(x)=0.$$

Since $\displaystyle\sum_{k=0}^{\infty}c_k\,x^{k-n+2}=c_0\,x^{-n+2}+c_1x^{1-n+2}+c_2\,x^{2-n+2}+c_3\,x^{3-n+2}+\cdots$

$$=\sum_{k=2}^{\infty}c_{k-2}\,x^{k-n},\ \text{we get}$$

$$0\cdot c_0x^{-n}+\left[(1-n)^2-n^2\right]c_1\,x^{1-n}+\sum_{k=2}^{\infty}\left\{\left[(k-n)^2-n^2\right]c_k+c_{k-2}\right\}x^{k-n}+c\,x\,J_n'=0.$$

On multiplying by x^n we have

$$0\cdot c_0+(1-2n)c_1\,x+\sum_{k=2}^{\infty}\left[k\,(k-2n)\,c_k+c_{k-2}\right]x^k\ =-2\,c\,x\,J_n'(x)\cdot x^n \quad \text{.............. (3.3.1)}$$

Since the first solution is $J_\alpha\ (x)$ with $\alpha=n$ and for $\alpha=n$, $\Gamma\,(m+\alpha+1)=\Gamma\,(m+n+1)=(m+n)\,!$ we have

$$J_n(x)=\sum_{m=0}^{\infty}\frac{(-1)^m}{m!\,(m+n)!}\left(\frac{x}{2}\right)^{2m+n}$$

Therefore

$$J_n'(x) = \sum \frac{(2m+n)(-1)^m}{m!\,(m+n)!}\left(\frac{x}{2}\right)^{2m+n-1}\left(\frac{1}{2}\right)$$

Thus equation (3.3.1) becomes

$$0\cdot c_0 + (1-2n)c_1 x + \sum_{k=2}^{\infty}\left[k(k-2n)c_n + c_{k-2}\right]x^k = -c\sum_{m=0}^{\infty}\frac{(2m+n)(-1)^m x^{2m+2n}}{m!\,(m+n)!\,2^{2m+n-1}}$$

.... (3.3.2)

The series on the right side begin with x^{2n} and since n is positive integer, the right side do not contain any odd terms. Therefore $c_1 = 0$ $c_{2k+1} = 0$ for $k = 1, 2, 3,....$ and if $n > 1$ then

$$k(k-2n)c_k + c_{k-2} = 0, \text{ for } k = 2,3,4,....,2n-1.$$

Since $\quad c_1 = 0,\ c_3 = c_5 = c_7 = ... = c_{2n-1} = 0.$

Whereas $\quad c_k = \dfrac{+c_{k-2}}{k(2n-k)}$ gives

$$c_2 = \frac{c_0}{2^2(n-1)},\quad c_4 = \frac{c_2}{4\cdot 2(n-2)} = \frac{c_0}{2^4\cdot 2\cdot(n-2)(n-1)}$$

$$c_6 = \frac{c_4}{6\cdot 2(n-3)} = \frac{c_0}{3!2^6(n-3)(n-2)(n-1)}$$

in general $\quad c_{2j} = \dfrac{c_0}{2^{2j}\, j!(n-1)(n-2)(n-3)\cdots(n-j)},\quad j = 1,2,3,.....,n-1 \qquad$ ...(3.3.3)

In particular

$$c_{2n-2} = \frac{c_0}{2^{2n-2}(n-1)!(n-1)!}$$

On comparing the coefficients of x^{2n} in equation (3.3.2) we get

$$c_{2n-2} = -c\frac{n}{n!2^{n-1}} = -\frac{c}{(n-1)!2^{n-1}}$$

Thus $\quad c_{2n-2} = \dfrac{c_0}{2^{2n-2}(n-1)!^2} = -\dfrac{c}{2^{n-1}(n-1)!}$

and therefore $\quad c = -\dfrac{c_0}{2^{n-1}(n-1)!}. \qquad$ .....(3.3.4)

Since c_{2n-2} is used to find c, c_{2n} remains undetermined, but the remaining coefficients c_{2n+2}, c_{2n+4}, c_{2n+6},..... can be obtained by comparing the coefficients of $x^{2(n+j)}$ in equation (3.3.2).

$$(2n+2j)(2n+2j-2n)c_{2n+2j} + c_{2n+2j-2} = -c\frac{(-1)^j(2j+n)}{j!(n+j)!}\frac{1}{2^{2j+n-1}} \quad j = 1,2,3.....$$

$$4j(n+j)\, c_{2n+2j} + c_{2n+2j-2} = -c\, \frac{(-1)^j\,(2j+n)}{j!(n+j)!} \frac{1}{2^{2j+n-1}} \qquad j = 1, 2, 3.....$$

for $j = 1$ we have

$$4(n+1)\, c_{2n+2} + c_{2n} = -c\, \frac{(-1)(n+2)}{(n+1)!\, 2^{n+1}}$$

$$c_{2n+2} = \frac{c}{4 \cdot 2^{n+1} \cdot (n+1)!}\left(1 + \frac{1}{n+1}\right) - \frac{c_{2n}}{4(n+1)}$$

Choose $\dfrac{c_{2n}}{4(n+1)} = \dfrac{-c}{4 \cdot 2^{n+1} \cdot (n+1)!} \left(1 + \dfrac{1}{2} + \dfrac{1}{3} + \dfrac{1}{4} + \cdots + \dfrac{1}{n}\right)$

i.e. $\qquad c_{2n} = \dfrac{-c}{2^{n+1} n!} \left(1 + \dfrac{1}{2} + \dfrac{1}{3} + \dfrac{1}{4} + \cdots + \dfrac{1}{n}\right)$

Then $\qquad c_{2n+2} = \dfrac{+c}{4 \cdot 2^{n+1} (n+1)!} \left(1 + 1 + \dfrac{1}{2} + \dfrac{1}{3} + \dfrac{1}{4} + \cdots + \dfrac{1}{n} + \dfrac{1}{n+1}\right)$

for $j = 2$ we have

$$4 \cdot 2(n+2) c_{2n+4} = -\frac{c(-1)^2\,(n+4)}{2!(n+2)! 2^{n+3}} - c_{2n+2}$$

$$c_{2n+4} = \frac{-c}{4 \cdot 2 \cdot 2!(n+2)!\, 2^{n+3}}\left(1 + \frac{2}{n+2}\right) - \frac{1}{4 \cdot 2(n+2)} \quad \frac{c}{4 \cdot 2^{n+1}(n+1)!}\left(1 + 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n+1}\right)$$

$$= \frac{-c}{4^2 (n+2)!\, 2^{n+2}}\left(\frac{1}{2} + \frac{1}{n+2} + 1 + 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} + \frac{1}{n+1}\right)$$

$$= \frac{-c}{4^2 (n+2)!\, 2^{n+2}}\left[\left(1 + \frac{1}{2}\right) + \left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} + \frac{1}{n+1}\right)\right]$$

$$= \frac{-(-1)^2\, c}{2 \cdot 2!(n+2)!\, 2^{n+4}}\left[\left(1 + \frac{1}{2}\right) + \left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} + \frac{1}{n+1}\right)\right]$$

It can be shown by induction that

$$c_{2n+2m} = \frac{-(-1)^m\, c}{2 \cdot m!(n+m)!\, 2^{n+2m}}\left[\left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{m}\right) + \left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n+m}\right)\right],$$

$$m = 1, 2, 3, ...$$

Finally we get a solution ϕ_2

$$\phi_2(x) = x^{-n} \sum_{k=0}^{\infty} c_k\, x^k + (c \log x)\; J_n(x)$$

$$= x^{-n} \sum_{k=0}^{2n-1} c_k\, x^k + x^{-n} c_{2n-1} x^{2n-1} + x^{-n} c_{2n}\, x^{2n} + x^{-n} \sum_{k=2n-1}^{\infty} c_k\, x^k + (c \log x) J_n(x)$$

Since all odd terms $c_{2k+1} = 0$, $k = 1, 2, 3,.....$, we get,

$$\phi_2(x) = x^{-n} c_0 + x^{-n} \sum_{j=1}^{n-1} \frac{x^{2j} c_0}{2^{2j}\, j!(n-1)(n-2)\cdots(n-j)} - \frac{c}{2^{n+1} n!}\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}\right)x^n$$

$$-\sum_{m=1}^{\infty} \frac{(-1)^m c\, x^{2m+n}}{2\cdot m!(n+m)!\, 2^{n+2m}} \left[\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{m}\right)+\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n+m}\right)\right]$$

$$+(c\log x)\ J_n(x).$$

Where c_0 and c are constants related by equation (3.3.4) when $c = 1$, the resulting solution ϕ_2 is often denoted by K_n. If $c = 1$ then $c_0 = -2^{n-1}(n-1)!$ and

$$\phi_2(x) = -x^{-n} 2^{n-1}(n-1)! + x^{-n} \sum_{j=1}^{n-1} \frac{(-1)\, 2^{n-1}(n-1)!\, x^{2j}}{2^{2j}\, j!(n-1)(n-2)\cdots(n-j)}$$

$$-\frac{x^n}{2^{n+1} n!}\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}\right) - \sum_{m=1}^{n-1} \frac{(-1)^m}{2\cdot m!(n+m)!}\left(\frac{x}{2}\right)^{n+2m}$$

$$\left[\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{m}\right)+\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n+m}\right)\right] + (c\log x)\ J_n(x)$$

$$= -\left(\frac{x}{2}\right)^{-n} \frac{1}{2}(n-1)! + \left(\frac{x}{2}\right)^{-n} \cdot \frac{1}{2} \sum_{j=1}^{n-1} \frac{-(n-j-1)!}{j!}\left(\frac{x}{2}\right)^{2j}$$

$$-\frac{x^n}{2^{n+1} n!}\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}\right) - \sum_{m=1}^{\infty} \frac{(-1)^m}{2\cdot m!\,(n+m)!}\left(\frac{x}{2}\right)^{n+2m}$$

$$\left[\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{m}\right)+\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n+m}\right)\right] + (c\log x) J_n.$$

The function ϕ_2 when $c = 1$ is denoted by K_n . Thus

$$K_n(x) = -\frac{1}{2}\left(\frac{x}{2}\right)^{-n} \sum_{j=0}^{n-1} \frac{(n-j-1)!}{j!}\left(\frac{x}{2}\right)^{2j} - \frac{1}{2}\cdot\frac{1}{n!}\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}\right)\left(\frac{x}{2}\right)^n$$

$$-\frac{1}{2}\left(\frac{x}{2}\right)^n \sum_{m=1}^{\infty} \frac{(-1)^m}{m!(n+m)!}\left(\frac{x}{2}\right)^{2m}\left[\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{m}\right)+\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{m+n}\right)\right]$$

$$+\ (\log x) J_n(x). \quad \text{.......... (3.3.5)}$$

The function K_n is called a Bessel function of order n of second kind.

In this section we have derived all kinds of Bessel functions. We list all these functions here.

(1) Bessel function of zero order of the first kind denoted by $J_0(x)$ and defined by

$$J_0(x) = \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{2^{2m}\, m!^2}$$

(2) Bessel function of zero order of second kind denoted by $K_0(x)$ is

$$K_0(x) = -\sum_{m=1}^{\infty} \frac{(-1)^m}{m!^2}\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{m}\right)\left(\frac{x}{2}\right)^{2m} + (\log x)\ J_0(x)$$

(3) Bessel function of order α of first kind denoted by $J_\alpha(x)$ is defined by

$$J_\alpha(x) = \left(\frac{x}{2}\right)^{\alpha} \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\Gamma(\alpha+m+1)} \left(\frac{x}{2}\right)^{2m}, \ (\text{Re}\,\alpha > 0)$$

(4) Bessel function of order n of second kind is defined by equation (3.3.5)

$J_0(x)$ is a solution of Bessel equation with $\alpha = 0$ $K_0(x)$ is a second solution of Bessel equation with $\alpha = 0$ obtained according to theorem 3.2.2 where the roots of indicial equation $r_1 = r_2 = \alpha = 0$. $J_\alpha(x)$ is the first solution of Bessel equation where 2α is not a positive integer and $K_n(x)$ is the second solution of Bessel equation where $\alpha = n$ a positive integer.

Depending upon the situation choose α and then find the required Bessel function.

EXAMPLES

Ex. 1. Suppose ϕ is any solution of $x^2y'' + xy' + x^2y = 0$ for $x > 0$ and let $\psi(x) = x^{\frac{1}{2}}\phi(x)$.

show that ψ satisfies the equation $x^2y'' + (x^2 + \frac{1}{4})y = 0$ for $x > 0$.

Ans. : Since ϕ is a solution, $x^2\phi'' + x\phi' + x^2\phi = 0$

Let $\psi(x) = x^{\frac{1}{2}}\phi(x)$ then $\psi'(x) = \frac{1}{2}x^{-\frac{1}{2}}\phi + x^{\frac{1}{2}}\phi'$

and $\psi''(x) = -\frac{1}{4}x^{-\frac{3}{2}}\phi + x^{-\frac{1}{2}}\phi' + x^{\frac{1}{2}}\phi''$

$$x^2\psi''(x) = -\frac{1}{4}x^{\frac{1}{2}}\phi + x^{\frac{3}{2}}\phi' + x^{\frac{5}{2}}\phi''$$

$$= -\frac{1}{4}x^{\frac{1}{2}}\phi + x^{\frac{1}{2}}(x\phi' + x^2\phi'')$$

$$= -\frac{1}{4}x^{\frac{1}{2}}\phi + x^{\frac{1}{2}}(-x^2\phi) \qquad (\text{Since } x^2\phi'' + x\phi' + x^2\phi = 0)$$

$$= -(\frac{1}{4} + x^2)\,x^{\frac{1}{2}}\phi = -(\frac{1}{4} + x^2)\psi(x)$$

Thus, $x^2\psi''(x) + (x^2 + \frac{1}{4})\psi(x) = 0$

and ψ satisfies the equation $x^2y'' + (x^2 + \frac{1}{4})y = 0$ for $x > 0$.

Ex. 2. Let ϕ be a real valued non-trivial solution of $y'' + \alpha(x)y = 0$ on $a < x <$ b

Let ψ be a real valued non-trivial solution of $y'' + \beta(x)y = 0$ on $a < x <$ b

Here α, β are real valued continuous functions.

Suppose that $\beta(x) > \alpha(x), (a < x < \text{b})$ Show that if x_1 and x_2 are successive zeros of ϕ on $a < x <$ b, then ψ must vanish at sone point r_1, $x_1 < r_1 < x_2$.

Ans. : Suppose $\psi(x) \neq 0$ for $x_1 < x < x_2$ then either $\psi(x) > 0 \quad \forall x \in (x_1, x_2)$ or $\psi(x) < 0$ $\forall x \in (x_1, x_2)$ suppose $\psi(x) > 0$ for $x_1 < x < x_2$.

Since x_1 and x_2 are successive zeros either $\phi(x) > 0$ on (x_1, x_2) or $\phi(x) < 0$ on (x_1, x_2) suppose $\phi(x) > 0$ on (x_1, x_2). Then

$$(\psi\phi' - \phi\psi')' = \psi\phi'' - \phi\psi'' = -\alpha(x)\psi\phi + \beta\psi\phi$$
$$= (\beta - \alpha)\phi\psi$$

[Since ϕ is solution of $y'' + \alpha y = 0$, $\phi'' + \alpha(x)\phi = 0$ similarly $\psi''(x) + \beta(x)\psi = 0$]

Thus, $(\psi\phi' - \phi\psi')' = (\beta - \alpha)\phi\psi > 0 \ (\beta > \alpha,\ \phi, \psi > 0)$ Integration of above inequality between x_1 and x_2 gives

$$[\psi(x_2)\phi'(x_2) - \phi(x_2)\psi'(x_2)] - [\psi(x_1)\phi'(x_1) - \phi(x_1)\psi'(x_1)] > 0$$

But x_1 and x_2 are zeros of ϕ therefore $\phi(x_1) = \phi(x_2) = 0$ and above inequality becomes

$$\psi(x_2)\phi'(x_2) - \psi(x_1)\phi'(x_1) > 0.$$

Since $\phi(x) > 0$ for $x_1 < x < x_2$ and $\phi(x_1) = 0$, $\phi(x_1 - h) < 0$ for $h > 0$. Therefore

$$\phi'(x_1) = \lim_{h \to 0} \frac{\phi(x_1) - \phi(x_1 - h)}{h} > 0$$

Similarly $\phi'(x_2) < 0$

Let $\phi'(x_2) = -L_1$ and $\phi'(x_1) = L_2$ then $L_1, L_2 > 0$, $-\psi(x_2)L_1 - \psi(x_1)L_2 > 0$

i.e. $\qquad L_1\psi(x_2) + L_2\psi(x_1) < 0$

But $\psi(x) > 0$ for $x_1 < x < x_2$ and $L_1, L_2 > 0$.

This is a contradiction to our assumption that $\psi(x) > 0$ for $x_1 < x < x_2$. Therefore ψ takes both positive and negative values in the interval (x_1, x_2) and hence $\exists$ there exists $r_1 \in (x_1, x_2)$ such that $\psi(r_1) = 0$.

Ex. 3. Show that J_0 has an infinity of positive zeros.

Ans. : $J_0(x)$ is a solution of differential equation $x^2y'' + xy' + x^2y = 0$

If $\psi(x) = x^{\frac{1}{2}} J_0(x)$ then by example 1, ψ satisfies

$$y'' + \left[1 + \frac{1}{4x^2}\right]y = 0, \qquad (x > 0)$$

The function satisfies $f(x) = \sin x$ satisfies $y'' + y = 0$

Since $1 + \frac{1}{4x^2} > 1$ and $\sin x = 0$ has infinitely many zeroes $x = n\pi,\ n = 0, 1, 2, 3, \ldots.$

By above example $\left[\beta(x)=1+\frac{1}{4x^2} \text{ and } \alpha(x)=1\right]$

$\psi(x)=x^{\frac{1}{2}}J_0(x)$ has a zero between $n\pi$ and $(n+1)\,\pi$ for $n = 0, 1, 2,.....$ Thus, $J_0\,(x)$ has infinite number of positive zeros.

Ex. 4. (a) If $\lambda > 0$ and $\phi_\lambda(x)=x^{\frac{1}{2}}J_0(\lambda x)$ **shows that** $\phi_\lambda'' + \frac{1}{4x^2}\phi_\lambda = -\lambda^2\phi_\lambda.$

(b) If λ, μ are positive constants, show that

$$(\lambda^2-\mu^2)\int_0^1 \phi_\lambda(x)\phi_\mu(x)\,dx = \phi_\lambda(1)\phi_\mu'(1) - \phi_\mu(1)\phi_\lambda'(1).$$

(c) If $\lambda \neq \mu$ and $J_0(\lambda)=0,\ J_0(\mu)=0,$ **show that**

$$\int_0^1 \phi_\lambda(x)\phi_\mu(x)\,dx = \int_0^1 x\,J_0(\lambda x)\;J_0(\mu x)\,dx = 0$$

Ans. (a) : $J_0\,(x)$ is solution of $x^2y''+xy'+x^2y=0$ therefore $J_0\,(\lambda x)$ is solution of $\lambda^2x^2\ddot{y}+\lambda x\,\dot{y}+\lambda^2x^2y=0$ where '·' represents differentiation with respect to λx.

If $\quad \phi_\lambda(x)=x^{\frac{1}{2}}J_0(\lambda x)$ then $\phi_\lambda'(x)=\frac{1}{2}x^{-\frac{1}{2}}J_0(\lambda x)+x^{\frac{1}{2}}\lambda\cdot J_0'(\lambda x)$

$$\phi_\lambda''(x)=-\frac{1}{4}x^{-\frac{3}{2}}J_0(\lambda x)+x^{-\frac{1}{2}}\lambda\,J_0'(\lambda x)+x^{\frac{1}{2}}\lambda^2J_0''(\lambda x)$$

$$x^2\phi_\lambda''(x)=-\frac{1}{4}x^{\frac{1}{2}}J_0(\lambda x)+x^{\frac{1}{2}}\left[\lambda^2x^2\,J_0''(\lambda x)+\lambda xJ_0''(\lambda x)\right]$$

$$=-\frac{1}{4}x^{\frac{1}{2}}J_0(\lambda x)+x^{\frac{1}{2}}\left[-\lambda^2x^2\,J_0(\lambda x)\right]$$

$$=-x^{\frac{1}{2}}J_0(\lambda x)\left[\frac{1}{4}+\lambda^2x^2\right]$$

Therefore $\quad x^2\phi_\lambda''(x)+\left(\frac{1}{4}+\lambda^2x^2\right)\phi_\lambda(x)=0.$

$$\phi_\lambda''(x)+\frac{1}{4x^2}\phi_\lambda(x)=-\lambda^2\phi_\lambda$$

Ans. (b) : $\qquad \phi_\mu''(x)+\frac{1}{4x^2}\phi_\mu(x)=-\mu^2\phi_\mu$

$$\phi_\lambda''(x)+\frac{1}{4x^2}\phi_\lambda(x)=-\lambda^2\phi_\lambda.$$

Multiply first equation by ϕ_λ and second equation by ϕ_μ and subtract these equations.

$$\phi_\mu''\phi_\lambda - \phi_\lambda''\phi_\mu = -\mu^2\phi_\mu\phi_\lambda + \lambda^2\phi_\mu\phi_\lambda = (\lambda^2-\mu^2)\phi_\lambda\phi_\mu.$$

Thus, $(\lambda^2-\mu^2)\phi_\lambda\phi_\mu = \left(\phi_\mu'\phi_\lambda - \phi_\lambda'\phi_\mu\right)'$

Integrate above equation between 0 to 1. Since

$$\phi_\lambda(x) = x^{\frac{1}{2}}J_0(\lambda x),\ \ \phi_\lambda(0)=0 \text{ and } \phi_\mu(x) = x^{\frac{1}{2}}J_0(\mu x),\ \ \phi_\mu(0)=0.$$

Therefore

$$(\lambda^2-\mu^2)\int_0^1 \phi_\lambda(x)\phi_\mu(x)\,dx = \phi_\mu'(1)\phi_\lambda(1) - \phi_\lambda'(1)\phi_\mu(1)$$

Ans. (c) : $(\lambda^2-\mu^2)\int_0^1 \phi_\lambda(x)\phi_\mu(x)\,dx = \phi_\mu'(1)\phi_\lambda(1) - \phi_\lambda'(1)\phi_\mu(1)$

Since, $J_0(\lambda)=0,\ \phi_\lambda(1)=0$ and $J_0(\mu)=0 \Rightarrow \phi_\mu(1)=0.$

$$\therefore \qquad (\lambda^2-\mu^2)\int_0^1 \phi_\lambda(x)\phi_\mu(x)\,dx = 0$$

i.e. $\int_0^1 \phi_\lambda(x)\phi_\mu(x)\,dx = \int_0^1 x^{\frac{1}{2}}J_0(\lambda x)x^{\frac{1}{2}}J_0(\mu x)dx = 0$

i.e. $\int_0^1 x\,J_0(\lambda x)\,J_0(\mu x)\,dx = 0.$

Ex. 5. Show that $J_0'(x) = -J_1(x)$.

Ans. : $J_0(x) = \sum_{m=0}^{\infty}\frac{(-1)^m}{m!^2}\left(\frac{x}{2}\right)^{2m}$

$$J_0'(x) = \sum_{m=1}^{\infty}\frac{(-1)^m}{m!^2}\,\frac{2m\,x^{2m-1}}{2^{2m}}$$

$$= \sum_{m=1}^{\infty}\frac{(-1)^m}{m!^2}\,\frac{m\,x^{2m-1}}{2^{2m-1}}$$

$$= \sum_{m=0}^{\infty}\frac{(-1)^{m+1}}{(m+1)!^2}\,\frac{(m+1)\,x^{2m+1}}{2^{2m+1}} \qquad \text{(Replace } m \text{ by } m+1)$$

$$= -\left(\frac{x}{2}\right)\sum_{m=0}^{\infty}\frac{(-1)^m}{m!(m+1)!}\left(\frac{x}{2}\right)^{2m}$$

$$= -J_1(x)$$

Ex. 6. Define, $\frac{1}{\Gamma(k)}$ when *k* is a non-positive integer to be zero. Show that if *n* is a positive integer the formula for $J_{-n}(x)$ gives.

$$J_{-n}(x) = (-1)^n J_n(x)$$

Ans. :

$$J_{-n}(x) = \left(\frac{x}{2}\right)^{-n} \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\,\Gamma(m-n+1)} \left(\frac{x}{2}\right)^{2m}$$

$$= \left(\frac{x}{2}\right)^{-n} \sum_{m=n}^{\infty} \frac{(-1)^m}{m!\,(m-n)!} \left(\frac{x}{2}\right)^{2m} \qquad \left(\text{As } \frac{1}{\Gamma(k)} = 0 \text{ for } k \le 0\right)$$

$$= \left(\frac{x}{2}\right)^{-n} \sum_{m=0}^{\infty} \frac{(-1)^{m+n}}{(m+n)!\,m!} \left(\frac{x}{2}\right)^{2m+2m} \qquad (\text{Replace } m \text{ by } m+n)$$

$$= (-1)^n \left(\frac{x}{2}\right)^{n} \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\,(m-n)!} \left(\frac{x}{2}\right)^{2m}$$

$$= (-1)^n \; J_n(x).$$

Ex. 7. Show that

(a) $(x^{\alpha} J_{\alpha})'(x) = x^{\alpha} J_{\alpha-1}(x)$

(b) $(x^{-\alpha} J_{\alpha})'(x) = -x^{-\alpha} J_{\alpha+1}(x)$

Ans. (a) :

$$J_{\alpha}(x) = \left(\frac{x}{2}\right)^{\alpha} \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\,\Gamma(m+\alpha+1)} \left(\frac{x}{2}\right)^{2m}$$

$$\therefore \quad x^{\alpha} J_{\alpha}(x) = \frac{x^{2\alpha}}{2^{\alpha}} \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\,\Gamma(m+\alpha+1)} \left(\frac{x}{2}\right)^{2m}$$

$$= \frac{1}{2^{\alpha}} \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\,\Gamma(m+\alpha+1)} \frac{x^{2m+2\alpha}}{2^{2m}}$$

$$(x^{\alpha} J_{\alpha})' = \frac{1}{2^{\alpha}} \sum_{m=0}^{\infty} \frac{(-1)^m (2m+2\alpha)}{m!\,\Gamma(m+\alpha+1)} \frac{x^{2m+2\alpha-1}}{2^{2m}}$$

$$= x^{\alpha} \left(\frac{x}{2}\right)^{\alpha-1} \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\,\Gamma(m+\alpha)} \left(\frac{x}{2}\right)^{2m}$$

$$= x^{\alpha} J_{\alpha-1}(x)$$

Ans. (b) :

$$J_{\alpha}(x) = \left(\frac{x}{2}\right)^{\alpha} \sum \frac{(-1)^m}{m!\,\Gamma(m+\alpha+1)} \left(\frac{x}{2}\right)^{2m}$$

$$x^{-\alpha} J_\alpha = \frac{1}{2^\alpha} \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\,\Gamma(m+\alpha+1)} \left(\frac{x}{2}\right)^{2m}$$

$$\left(x^{-\alpha} J_\alpha\right)' = \frac{1}{2^\alpha} \sum_{m=1}^{\infty} \frac{(-1)^m}{m!\,\Gamma(m+\alpha+1)} \frac{2m\,x^{2m-1}}{2^{2m}}$$

$$= \frac{1}{2^\alpha} \sum_{m=0}^{\infty} \frac{(-1)^{m+1}\,2(m+1)}{(m+1)!\,\Gamma(m+\alpha+2)} \frac{x^{2m+1}}{2^{2m+2}}$$

$$= -\frac{x}{2^{\alpha+1}} \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\,\Gamma(m+\alpha+2)} \frac{x^{2m}}{2^m}$$

But $\quad J_{\alpha+1}(x) = \left(\frac{x}{2}\right)^{\alpha+1} \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\,\Gamma(m+\alpha+2)} \left(\frac{x}{2}\right)^{2m}$

$\therefore \quad -x^{-\alpha} J_{\alpha+1} = -\frac{x}{2^{\alpha+1}} \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\,\Gamma(m+\alpha+2)} \left(\frac{x}{2}\right)^{2m}$

Thus $\left(x^{-\alpha} J_\alpha\right)' = -x^{-\alpha} J_{\alpha+1}$

8. Show that

(a) $\quad J_{\alpha-1}(x) - J_{\alpha+1}(x) = 2J_\alpha{}'(x)$

(b) $\quad J_{\alpha-1}(x) + J_{\alpha+1}(x) = 2\alpha\, x^{-1} J_\alpha(x)$

Ans. $\quad x^\alpha J_{\alpha-1}(x) = (x^\alpha J_\alpha)' = \alpha\, x^{\alpha-1} J_\alpha + x^\alpha J'_\alpha \qquad$ .... (i)

$-x^{-\alpha} J_{\alpha+1}(x) = (x^{-\alpha} J_\alpha)' = -\alpha\, x^{-\alpha-1} J_\alpha + x^{-\alpha} J'_\alpha \qquad$ .... (ii)

(a) Multiply equation (i) by $x^{-\alpha}$ and equation (ii) by x^α and add.

$$J_{\alpha-1}(x) - J_{\alpha+1}(x) = (\alpha\, x^{-1} - \alpha\, x^{-1}) J_\alpha + J_\alpha{}' + J_\alpha{}' = 2\,J_\alpha{}'(x)$$

Thus, $J_{\alpha-1}(x) - J_{\alpha+1}(x) = 2J_\alpha{}'(x)$

(b) Multiply equation (i) by $x^{-\alpha}$ and equation (ii) by x^α and subtract.

$$J_{\alpha-1}(x) + J_{\alpha+1}(x) = 2\alpha\, x^{-1} J_\alpha(x)$$

Thus, using results of example 7 we have proved the required result.

Ex. 9. Show that $K_0{}'(x) = -K_1(x)$

Ans. : $K_0(x) = -\sum_{m=1}^{\infty} \frac{(-1)^m}{m!^2} \left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{m}\right) \left(\frac{x}{2}\right)^{2m} + \log x\, J_0(x)$

$$K_0'(x) = -\sum_{m=1}^{\infty} \frac{(-1)^m}{m!^2}\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{m}\right)\frac{2m\,x^{2m-1}}{2^{2m}} + \frac{1}{x}J_0(x) + \log x\, J_0'(x)$$

$$= -\sum_{m=1}^{\infty} \frac{(-1)^m}{m!^2}\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{m}\right)\frac{m\,x^{2m-1}}{2^{2m-1}} + \frac{1}{x}\left(\sum_{m=0}^{\infty}\frac{(-1)^m}{m!^2}\left(\frac{x}{2}\right)^{2m}\right) + \log x\, J_0'$$

$$= \frac{x}{2} - \sum_{m=2}^{\infty} \frac{(-1)^m}{m!^2}\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{m}\right)\frac{m\,x^{2m-1}}{2^{2m-1}} + \frac{1}{x}\left(1-\frac{x^2}{4}\right) + \sum_{m=2}^{\infty}\frac{(-1)^m}{m!^2}\frac{x^{2m-1}}{2^{2m}}$$

$$+ \log x\, J_0'(x)$$

$$= \frac{1}{x} + \frac{x}{4} + \sum_{m=2}^{\infty}\frac{(-1)^m}{m!^2}\frac{x^{2m-1}}{2^{2m-1}}\left[\frac{1}{2} - m\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{m}\right)\right] + \log x\, J_0'(x)$$

$$= \frac{1}{x} + \frac{x}{4} + \left(\frac{x}{2}\right)\sum_{m=2}^{\infty}\frac{(-1)^m x^{2m-2}}{m!^2\, 2^{2m-2}}\left[-\frac{1}{2} - m\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{m-1}\right)\right] + \log x\, J_0'$$

Replace m by $m + 1$ and use the result $J_0' = -J_1$ then

$$K_0'(x) = \frac{1}{x} + \frac{x}{4} + \left(\frac{x}{2}\right)\sum_{m=1}^{\infty}\frac{(-1)^{m+1}}{(m+1)!m!}\left(\frac{x}{2}\right)^{2m}\left[-\frac{1}{2(m+1)} - \frac{(m+1)}{(m+1)}\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{m}\right)\right]$$

$$- \log x \cdot J_1(x)$$

$$= \frac{1}{x} + \frac{x}{4} + \frac{1}{2}\left(\frac{x}{2}\right)\sum \frac{(-1)^m}{m!(m+1)!}\left(\frac{x}{2}\right)^{2m}\left[\frac{1}{m+1} + 2\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{m}\right)\right]$$

$$- \log x \cdot J_1(x)$$

$$= -K_1(x)$$

Thus, $K_0'(x) = -K_1(x)$

EXAMPLES

1. Let ϕ be any solution for $x > 0$ of the Bessel equation of order α

$$x^2 y'' + xy' + (x^2 - \alpha^2)y = 0$$

Put $\psi(x) = x^{\frac{1}{2}}\phi(x)$. show that ψ satisfies equation

$$y'' + \left[1 + \frac{\frac{1}{4} - \alpha^2}{x^2}\right]y = 0$$

2. Show that if $\alpha > 0$ then J_α has an infinite number of positive zeros.

3. Show that J_0' satisfies the Bessel equation of order one

$$x^2 y'' + x y' + (x^2 - 1) y = 0.$$

4. For a fixed $\alpha > 0$ and $\lambda > 0$ let $\phi_\lambda(x) = x^{\frac{1}{2}} J_\alpha(\lambda x)$ show that

$$\phi_\lambda'' + \left[\frac{\frac{1}{4} - \alpha^2}{x^2}\right]\phi_\lambda = -\lambda^2 \phi_\lambda$$

5. If λ, μ are positive show that

$$(\lambda^2 - \mu^2)\int_0^1 \phi_\lambda(x)\phi_\mu(x)dx = \phi_\lambda(1)\phi_\mu'(1) - \phi_\mu(1)\phi_\lambda'(1)$$

Unit 4 : Regular singular points at infinity

At the beginning of chapter 3 we have defined singular points of linear differential equation of order n on the domain $|x| < \infty$. In unit 2 of chapter 3 we have discussed the power series solutions of second order differential equation with regular singular points. These singular points lie in a finite plane $|x| < \infty$. Often it is necessary to investigate solution of the differential equation for large values of $|x|$. A simple way of doing this is to change the independent variable by its reciprocal $x = \frac{1}{t}$ and study the solution of the resulting equation near $t = 0$. If the resulting equation possesses the regular singular point of $t = 0$. We say that the original equation has a regular singular point at infinity. The results on analytic solution and equations with regular singular point at $t = 0$ can be applied to the transformed equation. Analysis of equation at $t = 0$ gives the analysis of given equation for infinite x.

Let us consider the second order differential equation

$$L(y) = y'' + a_1(x)y' + a_2(x)y = 0$$

for large values of $|x|$.

Suppose ϕ is a solution of $L(y) = 0$ for $|x| > r_0$ for some $r_0 > 0$.

Define $t = \frac{1}{x}$ and let $\tilde{\phi}(t) = \phi\left(\frac{1}{t}\right)$, $\tilde{a}_1(t) = a_1\left(\frac{1}{t}\right)$, $\tilde{a}_2(t) = a_2\left(\frac{1}{t}\right)$. These functions will exist for $|t| < \frac{1}{x_0}$.

$$\frac{d\tilde{\phi}(t)}{dt} = \frac{d}{dx}\phi(x)\cdot\frac{dx}{dt} = \phi'(x)\left(\frac{-1}{t^2}\right) \Rightarrow \phi'(x) = -t^2\frac{d\tilde{\phi}}{dt}.$$

$$\frac{d^2\tilde{\phi}(t)}{dt^2} = \left(-\frac{1}{t^2}\right)\frac{d}{dx}(\phi'(x))\frac{dx}{dt} + \phi'(x)\frac{d}{dt}\left(\frac{-1}{t^2}\right)$$

$$= \frac{1}{t^4}\phi''(x) + \frac{2}{t^3}\phi'(x)$$

Therefore

$$\phi''(x) = t^4 \frac{d^2\tilde{\phi}(t)}{dt^2} - 2t\,\phi'(x)$$

$$= t^4 \frac{d^2\tilde{\phi}}{dt^2} - 2t\left(-t^2 \frac{d\tilde{\phi}}{dt}\right)$$

and

$$L(\phi) = \tilde{L}(\tilde{\phi}) = t^4\tilde{\phi}''(t) + 2t^3\tilde{\phi}'(t) + \tilde{a}_1(t)(-t^2)\tilde{\phi}'(t) + \tilde{a}_2(t)\tilde{\phi}(t)$$

$$= t^4\tilde{\phi}''(t) + \left[2t^3 - \tilde{a}_1(t)t^2\right]\tilde{\phi}'(t) + \tilde{a}_2(t)\tilde{\phi}(t)$$

$L(\phi) = 0$ gives $\tilde{L}(\tilde{\phi}) = 0.$ Thus, $\tilde{\phi}$ satisfies

$$\tilde{L}(y) = t^4 y'' + \left[2t^3 - \tilde{a}_1(t)t^2\right]y' + \tilde{a}_2(t)y = 0 \qquad \textbf{.......(3.4.1)}$$

Where the prime denotes differentiation with respect to *t*.

Convertsely if $\tilde{\phi}$ satisfies $\tilde{L}(y) = 0$ the function ϕ will satisfy $L(y) = 0$. The equation (3.4.1) is called the induce equation associated with $L(y) = 0$ and the substitution $x = \frac{1}{t}$.

Definition 3.4.1 :

We say that infinity is a regular singular point for $L(y) = y''(x) + a_1(x)y'(x) + a_2(x)y(x) = 0$ if $t = 0$ is a regular singular point of

$$\tilde{L}(y) = t^4 y''(t) + \left[2t^3 - t^2\tilde{a}_1(t)\right]y'(t) + \tilde{a}_2(t)\,y(t) = 0$$

$\tilde{L}(y)$ is equivalent to the equation

$$y''(t) + \frac{2t - \tilde{a}_1(t)}{t^2}y'(t) + \frac{\tilde{a}_2(t)}{t^4}y(t) = 0$$

On comparing this equation with the equation in definition 3.1.3(b) we see that

$$p(t) = \frac{2t - \tilde{a}_1(t)}{t^2} \quad \text{and} \quad q(t) = \frac{\tilde{a}_2(t)}{t^4}$$

If $\tilde{a}_1(t) = t\sum_{k=0}^{\infty}\alpha_k t^k$ and $\tilde{a}_2(t) = t^2\sum_{k=0}^{\infty}\beta_k t^k$ where the series converge for $|t| < \frac{1}{r_0}$, $r_0 > 0$, then the denomiantor of $p(t)$ will not contain a factor *t* to a power higher than one and the denominator of $q(t)$ will not contain a factor *t* to a power higher than two. By definition 3.1.3(b) $t = 0$ is a regular singular point of $\tilde{L}(y) = 0$ and therefore infinity is a regular singular point of $L(y) = 0$.

EXAMPLES

Ex. 1. Check whether infinity is regular singular point of $x^2y'' + a\,x\,y' + b\,y = 0$, where *a*, *b* are constants.

Ans. : Put $x = \frac{1}{t}$.

$$y'(x)=\frac{dy}{dt}\cdot\frac{dt}{dx}=-t^2\dot{y}(t) \text{ where .dot (.) represents differentiation with respect to } t.$$

$$y''(x)=\frac{d}{dx}\left[y(x)\right]=\frac{d}{dt}\left[y'(x)\right]\frac{dt}{dx}=\left[-t^2\ddot{y}(t)-2t\,\dot{y}(t)\right](-t^2)$$

$$=t^4\ddot{y}(t)+2t^3\dot{y}(t)$$

$$x^2y''(x)+axy'+by=\frac{1}{t^2}\left[t^4\ddot{y}(t)+2t^3\dot{y}(t)\right]+a\cdot\frac{1}{t}\left[-t^2\dot{y}(t)\right]+by$$

$$=t^2\ddot{y}(t)+\left[2-a\right]t\,\dot{y}(t)+b\,y$$

$L\left[y(x)\right]=0$ implies

$$\ddot{y}(t)+\frac{2-a}{t}y(t)+\frac{b}{t^2}y(t)=0.$$

This equation is of form $y''+p\,y'+q\,y=0$. Since denomination of p contains a factor t^r, $r \not> 1$ and denominator of q contain a factor t^r, $r \not> 2$, $t=0$ is a regular singular point. Thus, infinity is a regular singular point of the given differential equation.

Ex. 2. Show that infinity is not a regular singular point for the equation

$$y''+a\,y'+b\,y=0$$

where a, b are constants, not both zero.

Ans. : $y'(x)=-t^2\,\dot{y}(t)$

$$y''(x)=t^4\ddot{y}(t)+2t^3\dot{y}(t)$$

and $y''(x)+a\,y'(x)+b\,y(x)=0$ gives

$$t^4\ddot{y}(t)+2t^3\dot{y}(t)-a\,t^2\dot{y}(t)+b\,y(t)=0.$$

Therefore

$$\ddot{y}(t)+\frac{2t-a}{t^2}\dot{y}(t)+\frac{b}{t^4}y(t)=0$$

Here $p(t)=\dfrac{2t-a}{t^2}$ and $q(t)=\dfrac{b}{t^4}$

[If $a=b=0$ then $p(t)=\dfrac{2}{t}$ and $q(t)=0$. Since denominator of $p(t)$ contain a factor t^r, $r \not> 1$ and denominator of $q(t)$ contains a factor t^r, $r=0 \not> 2$ $t=0$ is a regular singular point and infinity is regular singular point of the equation.]

Since either a or b is non-zero, $p(t)=\dfrac{2t-a}{t^2}$ contains the determinator t^r with $r = 2 > 1$ or $q(t)=\dfrac{b}{t^4}$ contains the denominator t^r with $r = 4 > 2$. Therefore $t = 0$ is an irregular singular point of the transformed equation and infinity is an irregular singular point of the given equation.

Ex. 3. Show that infinity is a regular singular point for the Legendre equation

$$(1-x^2)y'' - 2xy' + \alpha(\alpha+1)y = 0$$

where a, b are constants, not both zero.

Ans. : $$y'(x) = -t^2\dot{y}(t),\ y''(x) = t^4\ddot{y}(t) + 2t^3\dot{y}(t),\ x = \frac{1}{t}$$

$\therefore$ Legendre equation becomes,

$$\left(1-\frac{1}{t^2}\right)\left[t^4\ddot{y} + 2t^3\dot{y}\right] - 2\frac{1}{t}\left(-t^2\dot{y}\right) + \alpha(\alpha+1)y = 0$$

$$(t^2-1)t^2\ddot{y} + 2(t^2-1)t\,\dot{y} + 2t\,\dot{y} + \alpha(\alpha+1)y = 0$$

$$\ddot{y} + \frac{2t}{(t^2-1)}\dot{y} + \frac{\alpha(\alpha+1)}{t^2(t^2-1)}y = 0$$

Here $p(t) = \dfrac{2t}{(t^2-1)}$ contains a factor t^r in the denominator with $r = 0 \not> 1$ and $q(t) = \dfrac{\alpha(\alpha+1)}{t^2(t^2-1)}$ contains a factor t^r in the denominator with $r = 2 \not> 2$.

Therefore by definition 3.1.3(b) $t = 0$ is a regular singular point of the transformed equation and infinity is a regular singular point of a given equation.

4. Find two linearly independent solutions of the equation $(1-x^2)y'' - 2xy' + 2y = 0$ **of the form** $x^{-r}\sum_{k=0}^{\infty} c_k x^{-k}$ **valid for | x | > 1**

Ans. : Put $x = \frac{1}{t}$ then

$$y'(x) = -t^2\dot{y}(t),\ y''(x) = t^4\ddot{y} + 2t^3\dot{y}(t)$$

Given equation becomes

$$\left(1-\frac{1}{t^2}\right)\left[t^4\ddot{y} + 2t^3\dot{y}\right] - 2\frac{1}{t}\left(-t^2\dot{y}\right) + 2y = 0$$

or $$(t^2-1)\left[t^2\ddot{y} + 2t\,\dot{y}\right] + 2t\dot{y} + 2y = 0$$

$$L(y) = t^2(t^2-1)\ddot{y} + 2t^3\dot{y} + 2y = 0$$

From example 2 we observe that $x = 0$ is a regular singular point of $L(y) = 0$.

Let ϕ be a solution of $L(y) = 0$ of the form

$$\phi(t) = t^r \sum_{k=0}^{\infty} c_k t^k$$

$$L(\phi) = t^2(t^2-1)\sum_{k=0}^{\infty}(k+r)(k+r-1)\,c_k\,t^{k+r-2} + 2t^3\sum_{k=0}^{\infty}(k+r)t^{k+r-1} + 2\sum_{k=0}^{\infty}c_k\,t^{k+r}$$

$$= \sum_{k=0}^{\infty}(k+r)(k+r-1)\,c_k\,t^{k+r+2} - \sum_{k=0}^{\infty}(k+r)(k+r-1)c_k\,t^{k+r}$$

$$+2\sum_{k=1}^{\infty}(k+r)t^{k+r+2} + 2\sum_{k=0}^{\infty}c_k\,t^{k+r}$$

$$= \sum_{k=0}^{\infty}\left[(k+r)(k+r-1)+2(k+r)\right]c_k\,t^{k+r+2}$$

$$-\sum_{k=0}^{\infty}\left[(k+r)(k+r-1)-2\right]c_k\,t^{k+r}$$

$$= \sum_{k=0}^{\infty}\left[(k+r)(k+r+1)\right]c_k\,t^{k+r+2} - (r(r-1)-2)c_0\,t^r - \big((r+1)(r)-2\big)c_1 t^{r+1}$$

$$-\sum_{k=2}^{\infty}\left[(k+r)(k+r-1)-2\right]c_k\,t^{k+r}$$

$$= -\left[r^2-r-2\right]c_0\,t^r - \left[r^2+r-2\right]c_1\,t^{r+1} - \sum_{k=2}^{\infty}\left[(k+r)(k+r-1)-2\right]c_k\,t^{k+r}$$

$$+\sum_{k=2}^{\infty}\left[(k+r-2)(k+r-1)c_{k-2}\right]t^{k+r}$$

$$= -\left(r^2-r-2\right)c_0\,t^r - \left(r^2+r-2\right)c_1\,t^{r+1}$$

$$-\sum_{k=2}^{\infty}\left\{\left[(k+r)(k+r-1)-2\right]c_k - \left[(k+r-2)(k+r-1)\right]c_{k-2}\right\}t^{k+r}$$

The indicial equation is

$q(r) = r^2 - r - 2 = 0$ gives $r = -1, 2$ Since $r_1 - r_2 = 2 + 1 = 3$ a positive integer we try a series solution using the smallest root, $r = -1$.

At $r = -1$, $L(\phi) = 0$ implies

$0.c_0 = 0$, $2c_1 = 0$ and

$\left[(k-1)(k-2)-2\right]c_k - (k-3)(k-2)c_{k-2} = 0 \quad k = 2, 3, 4, 5, ...$

$k = 2,\ -2c_2 = 0$ gives $c_2 = 0$

$k = 3,\ 0.c_3 = 0$ that is c_3 is orbitrary

Thus, c_0 and c_3 are arbitrary whereas $c_1 = c_2 = 0$.

Since $c_2 = 0$, all even terms $c_{2k} = 0$, $k = 1, 2, 3, ...$ and

$$c_k = \frac{(k-3)(k-2)}{(k-1)(k-2)-2} c_{k-2} \quad k = 4, 5, ...$$

In particular

$$c_{2k+1} = \frac{(2k-2)(2k-1)}{2k(2k-1)-2} c_{2k-1} \quad k = 2, 3, 4, ...$$

$$= \frac{(2k-2)(2k-1)}{(2k-2)(2k+1)} c_{2k-1}$$

$$= \frac{2k-1}{2k+1} c_{2k-1}, \ k = 2, 3, 4, ...$$

$$c_{2k+1} = \left(\frac{2k-1}{2k+1}\right)\left(\frac{2k-3}{2k-1}\right)\left(\frac{2k-5}{2k-3}\right)\left(\frac{2k-7}{2k-5}\right) \cdots \frac{1}{3} c_3$$

$$= \frac{c_3}{2k+1}$$

Thus we get a solution

$$\phi(t) = c_0 t^{-1} + c_3 t^2 + c_5 t^4 + c_7 t^6 + ...$$

$$= c_0 t^{-1} + \sum_{k=1}^{\infty} c_{2k+1} t^{2k}$$

$$= c_0 t^{-1} + \sum_{k=1}^{\infty} \frac{c_3}{(2k+1)} t^{2k}$$

$$= c_0 t^{-1} + c_3 \sum_{k=1}^{\infty} \frac{t^{2k}}{(2k+1)}$$

Thus, we get two solutions

$$\phi_1(t) = t^{-1} \text{ and } \phi_2(t) = \sum_{k=1}^{\infty} \frac{t^{2k}}{2k+1}$$

Let $\phi_2(t) = \sum_{k=1}^{\infty} d_k(t).$

By ratio test

$$\left|\frac{d_{k+1}(t)}{d_k(t)}\right| = \left|\frac{t^{2k+2}}{2k+3} \times \frac{2k+1}{t^{2k}}\right| = \left|\frac{(2k+1)t^2}{(2k+3)}\right|$$

Since $\underset{k \to \infty}{Lt} \left|\frac{2k+1}{2k+3}\right| = 1$

The series converger for $|t| < 1$.

But $x = \frac{1}{t}$, therefore

$\phi_1(x) = x$ and $\phi_2(x) = \sum_{k=1}^{\infty} \frac{x^{-2k}}{2k+1}$ are two solutions of given equation. Second converges for $|x| > 1$.

Ex. 5. For each equation locate and classify all its singular points.

(a) $x^3(x-1)y'' + (x-1)y' + 4xy = 0$

(b) $x^2(x^2-4)y'' + 2x^3y' + 3y = 0$

(c) $y'' + xy = 0$

(d) $x^2(x-4)^2 y'' + 3xy' - (x-4)y = 0$

Ans. : In chapter 3 Unit I, example 3, we have classified all its singular points in a finite plane. It remains to check whether infinity is a singular point and whether it is a regular singular point.

(a) $a_0(x) = x^3(x-1) = 0$ gives $x = 0$, $x = 1$ are singularities $x = 0$ is irregular singular point whereas $x = 1$ is a regular singular point.

put $x = \frac{1}{t}$ then $y'(x) = \frac{dy}{dt} \cdot \frac{dt}{dx} = \dot{y}(t)(-t^2)$ and $y''(x) = \frac{d}{dt}\left(\dot{y}(t)(-t^2)\right)\frac{dt}{dx}$

$= \left[-t^2\ddot{y}(t) - 2t\,\dot{y}(t)\right]\left[-t^2\right]$ so $y''(x) = t^4\ddot{y} + 2t^3\dot{y}$.

$$L(y) = x^3(x-1)y'' + (x-1)y' + 4xy$$

$$= \frac{1}{t^3}\left(\frac{1}{t} - 1\right)\left[t^4\ddot{y} + 2t^3\dot{y}\right] + \left(\frac{1}{t} - 1\right)(-t^2)\dot{y} + 4\frac{1}{t}y$$

$$= (1-t)\,\ddot{y} + \left(\frac{2}{t} - 2 - t + t^2\right)\dot{y} + \frac{4}{t}y$$

$$L(y) = (1-t)\,\ddot{y} + \frac{2-(2+t)t+t^3}{t}\,\dot{y} + \frac{4}{t}y$$

$L(y) = 0$ can be put in the form

$$\ddot{y} + \frac{(t-1)(t^2-2)}{(1-t)t}\,\dot{y} + \frac{4}{t(1-t)}\,y = 0$$

or
$$\ddot{y} + \frac{t^2-2}{t}\,\dot{y} + \frac{4}{t(1-t)}\,y = 0$$

This equation is of the type $y'' + p(t)y' + Q(t)\,y = 0$ where $p(t) = \frac{t^2-2}{t}$ and $Q(t) = \frac{4}{t(1-t)}$

Since the denominator of $p(t)$ contains a factor t^r, for $r = 1 \not> 1$ and $q(t)$ contains a denominator t^r, for $r = 1 \not> 2$, $t = 0$ is a regular singular point.

Thus, $x = 1$ and infinity are regular singular whereas $x = 0$ is irregular singular point.

(b) Put $x=\frac{1}{t}$ then $y'(x)=-t^2\dot{y}$ and $y''(x)=t^4\ddot{y}+2t^3\dot{y}$

$$L(y)=x^2(x^2-4)y''+2x^3y'+3y$$

$$=\frac{1}{t^2}\left(\frac{1}{t^2}-4\right)\left[t^4\ddot{y}+2t^3\dot{y}\right]+\frac{2}{t^3}\,(-t^2\dot{y})+3y$$

$$=(1-4t^2)\,\ddot{y}+\left(\frac{2}{t}-8t-\frac{2}{t}\right)\dot{y}+3y$$

$$=(1-4t^2)\,\ddot{y}-8t\,\dot{y}+3y$$

Since t = 0 is not a singular point of $(1-4t^2)\ddot{y}-8t\,\dot{y}+3y=0,$ infinity is not a singular point of the given equation.

(c) $y''+x\,y=(t^4\ddot{y}+2t^3\dot{y})+\frac{1}{t}\,y.$

Therefore $L(y)=t^4\ddot{y}+2t^3\dot{y}+\frac{1}{t}\,y=0$ can be written in the form $\ddot{y}+p(t)\dot{y}+q(t)y=0$ where $p(t)=\frac{2}{t}$ and $q(t)=\frac{1}{t^5}.$

Here t = 0 is a singular point but since the denominator of $q(t)$ contains a factor t^r, r = 5 > 2, t = 0 is not a regular singular point.

Since t = 0 is irregular singular point infinity is irregular singular point of the equation $y''+xy=0.$

(d) x = 0 is regular singular point and x = 4 is irregular singular point.

Put $x=\frac{1}{t}$ then $y'=-t^2\dot{y},\;\; y''=t^4\ddot{y}+2t^3\dot{y}$

$$L(y)=x^2(x-4)^2y''+x\,y'-(x-4)y$$

$$=\frac{1}{t^2}\left(\frac{1}{t}-4\right)^2\left[t^4\ddot{y}+2t^3\dot{y}\right]+\frac{1}{t}[-t^2\dot{y}]-\left(\frac{1}{t}-4\right)y$$

$$=(4t-1)^2\,\ddot{y}+\left[\frac{2(4t-1)^2}{t}-t\right]\dot{y}+\left(\frac{4t-1}{t}\right)y$$

$$=(4t-1)^2\,\ddot{y}+\left[\frac{31t^2-16t+1}{t}\right]\dot{y}+\left(\frac{4t-1}{t}\right)y$$

$L(y)$ = 0 can be written in the form $\ddot{y}+p(t)\dot{y}+q(t)y=0$ where $p(t)=\frac{31t^2-16t+1}{(4t-1)^2\cdot t}$ and

$$q(t) = \frac{1}{t(4t-1)}$$

Since $t = 0$ is a singularity of $p(t)$ and $q(y)$ and is a simple pole by definition 3.1.3(a) $t = 0$ is a regular singular point and infinity is regular singular point of the given equation.

Chapter 4

Existence and Uniqueness of Solutions to First Order Equations

Contents :

Introduction :

In the last three chapters we have seen the methods of finding a solution to the given linear differential equations. For linear differential equation with constant coefficients there is a method to find all the solutions whereas for linear equations with variable coefficients, there are very few types of equations whose solutions can be expressed in terms of elementary functions and therefore we go for power series solutions. All the equations considered so far were linear differential equations.

In this chapter we consider the general first order equation $y' = f(x, y)$ where f is some continuous function (need not be linear in y) Only in special cases it is possible to find explicit analytic expressions for the solutions of $y' = f(x, y)$.

Our main purpose in this chapter is to prove that a wide class of initial value problems

$$y' = f(x, y), \quad y(x_0) = y_0$$

has a solution. Though it may not be possible to find out the exact solution, it is feasible to construct a sequence of approximate solutions that may converge to the exact solution.

Unit 1 : Methods of successive approximations

In this unit we study the general problem of finding solutions of the equation

$$y' = f(x, y), \quad y(x_0) = y_0 \qquad \text{.......... (4.1.1)}$$

Where f is any continuous real valued function defined on some rectangle

$$R = \{(x, y) / |x - x_0| \le a, |y - y_0| \le b, a, b > 0\}$$

in the real (x, y) plane.

A function ϕ is a solution of equation (4.1.1) if $\phi(x_0) = y_0$ and $\phi'(x) = f\left(x, \phi(x)\right)$.

Theorem 4.1.1

A function ϕ is a solution of the initial value problem (4.1.1) on an interval I if and only if it is a solution of the integral equation

$$y = y_0 + \int_{x_0}^{x} f(t, y)\, dt \text{ on I} \qquad \text{.......... (4.1.2)}$$

Proof : Suppose ϕ is a solution of the initial values problem on I. Then

$$\phi'(t) = f\left(t, \phi(t)\right) \text{ and } \phi(x_0) = y_0$$

Since ϕ is continuous on I and f is continuous on R, the function F defined by

$$F(t) = f(t, \phi(t))$$

is continuous on I.

$$\phi'(t) = f(t,\ \phi(t)) \text{ and } \phi(x_0) = y_0$$

On integrating above equation between x_0 and x we get

$$\int_{x_0}^{x} \phi'(t)\, dt = \int_{x_0}^{x} f\left(t, \phi(t)\right) dt$$

$$\phi(x) - \phi(x_0) = \int_{x_0}^{x} f\left(t, \phi(t)\right) dt$$

or $$\phi(x) = y_0 + \int_{x_0}^{x} f\left(t, \phi(t)\right) dt \qquad (\text{as } \phi(x_0) = y_0)$$

Thus ϕ is solution of (4.1.2)

Conversely suppose ϕ satisfies (4.1.2) on I that is

$$\phi(x) = y_0 + \int_{x_0}^{x} f(t, \phi(t))\, dt$$

Differentiate this equation with respect to x and use the fundamental theorem of integral calculus. The integral equation becomes

$$\phi'(x) = f(x, \phi(x)) \text{ for all } x \in I.$$

From (4.1.2) it is obvious that $\phi(x_0) = y_0$.

Thus ϕ is a solution of equation (4.1.1).

Successive approximate solutions

As a first approximation to a solution defined

$$\phi_0(x) = y_0.$$

Then ϕ_0 satisfies an initial condition but does not in general satisfy the differential equation.

Since ϕ_0 is a first approximate solution, substitute $y = \phi_0$ in equation (4.1.2) to generate second approximate solution. Call this solution as ϕ_1 then

$$\phi_1(x) = y_0 + \int_{x_0}^{x} f(t, \phi_0(t))\, dt.$$

Clearly $\phi_1(x_0) = y_0$. Therefore ϕ_1 satisfies initial condition.

If we continue the process and define successively

$$\phi_0(x) = y_0,$$

$$\phi_{k+1}(x) = y_0 + \int_{x_0}^{x} f(t, \phi_k(t))\, dt \qquad (k = 0, 1, 2, ...) \qquad .... (4.1.3)$$

We get a sequence of functions $\{\phi_k\}_{k=0}^{\infty}$. If this sequence converges then it may happen that the limit function will turn out to be the solution of differential equation (4.1.1).

We now show that there is an interval I containing x_0 where all the functions $\phi_k,\ k = 1, 2, ...$ exist. Since f is continuous on a compact set R, it is bounded on R, that is there exists a constant $M > 0$ such that

$$|f(x, y)| \le M \qquad \text{for all } (x, y) \in R.$$

Theorem 4.1.2

The successive approximations defined by (4.1.3) exist and are continuous on

$$I = \{x \,/\, |x - x_0| \le \alpha \text{ where } \alpha = \min\{a, b/M\}\}$$

and for $x \in I,\ (x, \phi_k(x)) \in R$.

The function ϕ_k satisfy

$$|\phi_k(x) - y_0| \le M\, |x - x_0| \qquad \text{for all } x \text{ in } I$$

Proof : We will prove this result by mathematical induction,

(i) Clearly $\phi_0(x) = y_0$ is continuous on I and

$$|\phi_0(x) - y_0| = 0$$

Thus the theorem is true for $k = 0$.

(ii) $\phi_1(x) = y_0 + \int_{x_0}^{x} f(t, \phi_0(t))\, dt$

$$= y_0 + \int_{x_0}^{x} f(t, y_0)\, dt$$

Since f is continuous and continuous function is integrable, $\phi_1(x)$ exist.

$$|\phi_1(x) - y_0| = \left| \int_{x_0}^{x} f(t,\ y_0) dt \right| \le \int_{x_0}^{x} |f(t,\ y_0)| dt \le M\, |x - x_0|$$

Therefore $|\phi_1(x) - y_0| \le M\, |x - x_0|$

Since f is continuous on R the function F_0 defined by

$$F_0(t) = f(t,\ y_0)$$

is continuous on I. Therefore ϕ_1 defined by

$$\phi_1(x) = y_0 + \int_{x_0}^{x} F_0(t)dt$$

is continuous on I.

The theorem is true for $k = 1$.

(iii) Assume that the theorem is true for ϕ_k

(iv) To prove the result for ϕ_{k+1}

We know that $(t, \phi_k(t)) \in R$ for $t \in I$.

Since f is continuous on R and ϕ_k is continuous on I,

$$F_k(t) = f(t, \phi_k(t))$$

exist for $t \in I$ and F_k is continuous. The function ϕ_{k+1} given by

$$\phi_{k+1}(x) = y_0 + \int_{x_0}^{x} f(t, \phi_k)dt = y_0 + \int_{x_0}^{x} F_k(t)dt$$

exists and is continuous function on I.

$$|\phi_{k+1}(x) - y_0| = \left|\int_{x_0}^{x} F_k(t)dt\right| \le \int_{x_0}^{x} |F_k(t)|dt \le M\,|x - x_0|$$

$$\left(\text{Since } |F_k(t)| = |f(t, \phi_k)| \le M\right)$$

Thus ϕ_{k+1} exist is continuous and satisfies the required inequality.

Definition : Let f be a function defined for (x, y) in a set S. We say f satisfies a Lipschitz condition on S if there exists a constant $K > 0$ such that

$$|f(x, y_1) - f(x, y_2)| \le K\,|y_1 - y_2|$$

for all $(x, y_1), (x, y_2) \in S$. The constant K is called Lipschitz constant.

Theorem 4.1.3 :

Suppose S is either a rectangle

$$|x - x_0| \le a, \ |y - y_0| \le b \qquad (a, b > 0);$$

or a strip

$$|x - x_0| \le \alpha, \ |y| < \infty \qquad (a > 0)$$

and that f is real valued function defined on S.

Such that $\dfrac{\partial f}{\partial y}$ exists, is continuous on S and

$\left|\dfrac{\partial f}{\partial y}(x, y)\right| \le K,$ for $(x, y) \in S$ and for some $K > 0$. Then f satisfies a Lipschitz condition on S with Lipschitz constant K.

Proof : $f(x, y_1) - f(x, y_2) = \displaystyle\int_{y_2}^{y_1} \frac{\partial f}{\partial y}(x, t)\,dt$

Therefore

$$\left| f(x, y_1) - f(x, y_2) \right| = \left| \int_{y_2}^{y_1} \frac{\partial f}{\partial y}(x, t)\, dt \right|$$

$$\leq \left| \int_{y_2}^{y_1} \left| \frac{\partial f}{\partial y}(x, t) \right| dt \right|$$

$$\leq \left| \int_{y_2}^{y_1} K\, dt \right|$$

$$\leq K \left| y_1 - y_2 \right|$$

Thus, $\left| f(x, y_1) - f(x, y_2) \right| \leq K \mid y_1 - y_2 \mid$ for all (x, y_1) (x, y_2) in S.

EXAMPLES

1. Consider the initial value problem

$$y' = 3y + 1, \quad y(0) = 2.$$

(a) Show that all the successive approximations $\phi_0, \phi_1, \phi_2, \ldots\ldots$ exist for all real x.

(b) Compute the first four approximations $\phi_0, \phi_1, \phi_2, \phi_3$ to the solution.

(c) Compute exact solution.

(d) Compare exact and approximate solution.

Answer :

(a) We will prove this result by induction on k.

$k = 0$,

$$\phi_0(x) = y_0 = 2$$

ϕ_0 exist and is continuous.

Assume that ϕ_K exist and is continuous.

$$\phi_{k+1} = y_0 + \int_{x_0}^{x} f(t, \phi_k(t))\, dt$$

$$= y_0 + \int_{x_0}^{x} \left[3\phi_k(t) + 1\right] dt$$

$$= y_0 + 3\int_{x_0}^{x} \phi_k(t)\, dt + (x - x_0)$$

Since ϕ_k is continuous, ϕ_K is integrable.

Therefore ϕ_{k+1} exist and is continuous.

Thus, $\phi_0, \phi_1, \phi_2, \ldots\ldots$ exist for all real x.

(b) $\phi_0(x) = 2$

$$\phi_1(x) = y_0 + \int_{x_0}^{x} f(t, \phi_0(t))\, d\, t$$

Here $f(t, y) = 3y + 1$. Therefore

$$\phi_1(x) = 2 + \int_0^x [3\,\phi_0 + 1]\, d\, t$$

$$= 2 + \int_0^x 7dt = 2 + 7x$$

$$\phi_2(x) = 2 + \int_0^x [3\,\phi_1 + 1]\, d\, t$$

$$= 2 + \int_0^x [3\,(2 + 7t) + 1]\, d\, t$$

$$= 2 + \int_0^x (21t + 7)\, d\, t$$

$$= 2 + \frac{21x^2}{2} + 7x = 2 + 7x + \frac{21x^2}{2}.$$

$$\phi_3(x) = 2 + \int_0^x [3\,\phi_2(t) + 1]\, d\, t$$

$$= 2 + \int_0^x \left[3\left(2 + 7t + \frac{21}{2}t^2 \right) + 1 \right] d\, t$$

$$= 2 + \int_0^x \left[7 + 21t + \frac{63}{2}t^2 \right] d\, t$$

$$= 2 + 7x + 21\frac{x^2}{2} + \frac{63}{2}\,\frac{x^3}{3}$$

$$= 2 + 7x + \frac{21}{2}x^2 + \frac{21}{2}x^3.$$

(c) $y' - 3y = 1$

$$y = e^{3x}\left[\int 1e^{-3x} d\, x + c_1 \right]$$

$$= e^{3x}\left[\frac{e^{-3x}}{-3} + c_1 \right]$$

$$= -\frac{1}{3} + c_1\, e^{3x}$$

Since at $x = 0$, $y = 2$, $2 = -\frac{1}{3} + c_1 e^0$ i.e. $c_1 = \frac{7}{3}$. and

$$y(x) = -\frac{1}{3} + \frac{7}{3} e^{3x}$$

$$= -\frac{1}{3} + \frac{7}{3}\left[1 + 3x + \frac{(3x)^2}{2} + \cdots\right]$$

$$= 2 + 7x + \frac{21}{2}x^2 + \frac{63}{8}x^4 + \cdots$$

(d) $\phi_0, \phi_1, \phi_2, \phi_3$ are respectively first, first 2, first 3 and first 4 terms of the series solution $y = -\frac{1}{3} + \frac{7}{3} e^{3x}$.

2. For each of the following problems compute the first four successive approximations $\phi_0, \phi_1, \phi_2, \phi_3$.

(a) $y' = x^2 + y^2,\ y(0) = 0$ **(b)** $y' = 1 + xy,\ y(0) = 1$

(c) $y' = y^2,\ y(0) = 1$

Answers :

(a) $\phi_0(x) = y_0 = 0,\ f(x, y) = x^2 + y^2$

$$\phi_1(x) = y_0 + \int_{x_0}^{x} f\left(t, \phi_0(t)\right) dt$$

$$= 0 + \int_0^x \left(t^2 + 0^2\right) dt$$

$$= \frac{x^3}{3}$$

$$\phi_2(x) = y_0 + \int_{x_0}^{x} f\left(t, \phi_1(t)\right) dt$$

$$= 0 + \int_0^x f\left(t, \frac{t^3}{3}\right) dt$$

$$= \int_0^x \left[t^2 + \left(\frac{t^3}{3}\right)^2\right] dt$$

$$= \frac{x^3}{3} + \frac{x^7}{63}$$

$$\phi_3(x) = \int_0^x \left[t^2 + \left(\frac{t^3}{3} + \frac{t^7}{63}\right)^2\right] dt$$

$$= \int_0^x \left[t^2 + \frac{t^6}{9} + \frac{t^{14}}{63^2} + \frac{2t^{10}}{3\times 63} \right] dt$$

$$= \frac{x^3}{3} + \frac{x^7}{7.9} + \frac{x^{15}}{15\times 63\times 63} + \frac{2x^{11}}{11\times 3\times 63}$$

$$= \frac{x^3}{3} + \frac{x^7}{7.9} + \frac{2x^{11}}{11.3.68} + \frac{x^{15}}{15.63.63}$$

(b) $\phi_0(x) = 1 \qquad f(x, y) = 1 + xy$

$$\phi_1(x) = y_0 + \int_{x_0}^{x} (1 + t\phi_0)\, dt$$

$$= 1 + \int_0^x [1 + t]\, dt$$

$$= 1 + x + \frac{x^2}{2}$$

$$\phi_2(x) = 1 + \int_0^x \left[1 + t\left(1 + t + \frac{t^2}{2} \right) \right] dt$$

$$= 1 + \int_0^x \left[1 + t + t^2 + \frac{t^3}{2} \right] dt$$

$$= 1 + x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{8}$$

$$\phi_3(x) = 1 + \int_0^x \left[1 + t\left(1 + t + \frac{t^2}{2} + \frac{t^3}{3} + \frac{t^4}{8} \right) \right] dt$$

$$= 1 + \int_0^x \left[1 + t + t^2 + \frac{t^3}{2} + \frac{t^4}{3} + \frac{t^5}{8} \right] dt$$

$$= 1 + x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{8} + \frac{x^5}{15} + \frac{x^6}{48}$$

(c) $\phi_0(x) = y_0 = 1 \qquad f(x, y) = y^2$

$$\phi_1(x) = 1 + \int_0^x [(1)^2]\, dt$$

$$= 1 + x$$

$$\phi_2(x) = 1 + \int_0^x [1 + t]^2\, dt$$

$$= 1 + x + \frac{2x^2}{2} + \frac{x^3}{3}$$

$$= 1 + x + x^2 + \frac{x^3}{3}$$

$$\phi_3(x) = 1 + \int_0^x \left[1 + t + t^2 + \frac{t^3}{3}\right]^2 dt$$

$$= 1 + \int_0^x \left[1 + t^2 + t^4 + \frac{t^6}{9} + 2\left(t + t^2 + \frac{t^3}{3} + t^3 + \frac{t^4}{3} + \frac{t^5}{3}\right)\right] dt$$

$$= 1 + x + \frac{x^3}{3} + \frac{x^5}{5} + \frac{x^7}{7 \cdot 9} + 2\left(\frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{12} + \frac{x^4}{4} + \frac{x^5}{15} + \frac{x^6}{18}\right)$$

$$\phi_3(x) = 1 + x + x^2 + x^3 + \frac{2}{3}x^4 + \frac{1}{3}x^5 + \frac{x^6}{9} + \frac{x^7}{63}$$

3. Consider the problem

$$y' = x^2 + y^2 \qquad y(0) = 0$$

on $R : |x| \leq 1, \quad |y| \leq 1$

(a) Compute an upper bound M for $f(x, y) = x^2 + y^2$ on R

(b) On what interval containing $x = 0$ will all the sucdessive approximations exist and be such that their graphs are in R.

Answers :

(a)

$$M = \sup_R f(x, y)$$

$$= \sup_{|x| \leq 1, |y| \leq 1} (x^2 + y^2)$$

$$= 2$$

(b) By theorem 4.1.2

$$I = \left\{ x \,/\, |x - x_0| \leq \alpha \text{ where } \alpha = \min\left(a, \frac{b}{M}\right)\right\}$$

Here, $x_0 = 0, \; y_0 = 0, \; a = b = 1$ and $M = 2$

$$\therefore \quad I = \left\{ x \,/\, |x| \leq \alpha \text{ where } \alpha = \min\left(1, \frac{1}{2}\right)\right\}$$

$$\therefore \quad I = \left\{ x \,/\, |x| \leq \frac{1}{2}\right\}.$$

4. By computing appropriate Lipschitz constants show that the following functions satisfy Lipschilz conditions on the set *S*.

(a) $f(x,\ y) = 4x^2 + y^2$ on $S = \{(x,y) / |x| \leq 1,\ |y| \leq 1\}$

(b) $f(x,\ y) = x^2 \cos^2 y + y \sin^2 x$ on $S = \{(x,y) / |x| \leq 1,\ |y| < \infty\}$

Answers :

(a) $$f(x,y) = 4x^2 + y^2$$

$$|f(x,y_1) - f(x,y_2)| = \left|\left[4x^2 + y_1^2\right] - \left[4x^2 + y_2^2\right]\right|$$

$$= \left| y_1^2 - y_2^2 \right|$$

$$= |y_1 + y_2|\ |y_1 - y_2|$$

But $|y| \leq 1$ $\therefore$ $|y_1| \leq 1$ and $|y_2| \leq 1$

$$|f(x,y_1) - (f(x,y_2)| \leq (|y_1| + |y_2|)\ |y_1 - y_2|$$

$$\leq 2\ |y_1 - y_2|$$

Therefore Lipschitz constant $K = 2$.

(b) $$f(x,y) = x^2 \cos^2 y + y \sin^2 x$$

$$|f(x,y_1) - f(x,y_2)| = \left|\left[x^2 \cos^2 y_1 + y_1 \sin^2 x\right] - \left[x^2 \cos^2 y_2 + y_2 \sin^2 x\right]\right|$$

$$= \left| x^2 \left(\cos^2 y_1 - \cos^2 y_2\right) + \sin^2 x\,(y_1 - y_2) \right|$$

$$\leq |x|^2 \left|\cos^2 y_1 - \cos^2 y_2\right| + \left|\sin^2 x\right|\ |y_1 - y_2|$$

$$\leq \left|\cos^2 y_1 - \cos^2 y_2\right| + |y_1 - y_2|$$

By mean value theorem $f(b) - f(a) = f'(c)(b - a)$

$$\cos^2 y_1 - \cos^2 y_2 = -2 \cos y\ \sin y\ (y_1 - y_2)$$

Therefore

$$|f(x,y_1) - f(x,y_2)| \leq |2\ \cos y\ \sin y|\,|y_1 - y_2| + |y_1 - y_2|$$

$$\leq 3|y_1 - y_2|$$

Therefore $k = 3$ is a Lipschitz constant.

5. (a) Show that the function *f* given by

$$f(x,y) = x^2\ |y|$$

Satisfies Lipschitz condition on $R = \{(x,y) \mid |x| \leq 1,\ |y| \leq 1\}$

(b) Show that $\dfrac{\partial f}{\partial y}$ does not exist at $(x, 0)$ if $x \neq 0$.

Answer :

(a) $\left|f(x,y_1)-f(x,y_2)\right| = \left|x^2 |y_1| - x^2 |y_2|\right|$

$$\leq |x^2| \, |y_1 - y_2|$$

$$\leq 1 |y_1 - y_2|$$

Thus, function satisfies Lipschitz condition with Lipschitz constant $k = 1$.

(b) Since $|y|$ is not differentiable at $y = 0$, $\frac{\partial f}{\partial y}$ do not exist at $(x, 0)$ unless $x = 0$ if $x = 0$ then the function itself is zero.

EXERCISE

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1. Compute Lipschitz constant for the following functions.

(a) $f(x,y) = a(x)y^2 + b(x)y + c(x)$ on $S = \{|x| \leq 1, |y| < 2\}$ (a, b, c are continuous functions on $|x| \leq 1$)

(b) $f(x,y) = a(x)y + b(x)$ on $S = \{(x,y)/|x| \leq 1, |y| < \infty\}$ (a, b are continuous functions on $|x| \leq 1$)

(c) $f(x,y) = x^3 e^{-xy^2}$ on $S = \{(x,y)/0 \leq x \leq a, |y| < \infty\}$

2. (a) Show that the function *f* given by

$$f(x, y) = y^{\frac{1}{2}}$$

does not satisfy Lipschitz condition on

$$S = \{(x,y)/|x| \leq 1, 0 \leq y \leq 1\}$$

(b) Show that this *f* satisfies a Lipschitz condition on any rectangle *R* of the form

$$R = \{(x,y)/|x| \leq a, b \leq y \leq c, a,b,c > 0\}$$

3. Show that the function *f* given by

$$f(x, y) = 0, \text{ if } x = 0, |y| \leq 1$$

$$= 2x, \text{ if } 0 < |x| \leq 1, -1 \leq y < 0$$

$$= 2x - \frac{4y}{x}, \text{ if } 0 < |x| \leq 1, 0 \leq y \leq x^2$$

$$= -2x \text{ if } 0 < |x| \leq 1, x^2 \leq y \leq 1$$

does not satisfy a Lipschitz condition on $R = \{(x,y)/|x| \leq 1, |y| \leq 1\}$.

4. Determine the bound for the function given by $f(x, y) = 1 - 2xy$

on $S = \left\{(x,y)/|x| \leq \frac{1}{2}, |y| \leq 1\right\}$.

Unit 2 : Convergence of successive approximations

In the last unit we have found the successive approximate solutions to a differential equation (4.1.1). In this unit let us prove that this sequence of successive approximate solutions actually converges to the exact solution of differential equation (4.1.1).

Theorem 4.2.1 : (Existence Theorem)

Let f be a continuous real valued function on the rectangle

$$R=\{(x,y)/|x-x_0|\le a, |y-y_0|\le b, a>0, b>0\}$$

and let $|f(x, y)|\le M$ for all $(x, y)\in R$.

Suppose f satisfies a Lipschitz condition with Lipschitz constant K in R. Then the successive approximations.

$$\phi(x_0)=y_0,\quad \phi_{k+1}(x)=y_0+\int_{x_0}^{x} f(t,\phi_k(t))\,dt, \qquad k=0,1,2,3,.....,$$

Converge on the interval $I=\{x/|x-x_0|\le\alpha\}$ where $\alpha=\min\left\{a, \frac{b}{M}\right\}$ to a solution ϕ of the initial value problem (4.1.1)

$$y'=f(x, y),\quad y(x_0)=y_0 \text{ on } I$$

Proof (a) : Convergnce of $\{\phi_k\}$

Since the function ϕ_k can be written as

$$\phi_k(x)=\phi_0+(\phi_1-\phi_0)+(\phi_2-\phi_1)+(\phi_3-\phi_2)\cdots+(\phi_k-\phi_{k-1})$$

$$\phi_k(x)=\phi_0(x)+\sum_{p=1}^{k}\left[\phi_p(x)-\phi_{p-1}(x)\right]$$

The sequence ϕ_k converges, that is $\lim_{k\to\infty}\phi_k$ exists if and only if the series $\phi_0(x)+\sum_{p=1}^{\infty}\left[\phi_p(x)-\phi_{p-1}(x)\right]$ is a convergent series.

By theorem 4.1.2 the functions ϕ_p all exist, each is continuous on I and $(x,\phi_p(x))\in R$ for x in I.

Moreover $|\phi_1(x)-\phi_0(x)|\le M|x-x_0|$ for x in I

$$\phi_2(x)-\phi_1(x)=\left[y_0+\int_{x_0}^{x} f(t,\phi_1(t))\,dt\right]-\left[y_0+\int_{x_0}^{x} f(t,\phi_0(t))\,dt\right]$$

$$=\int_{x_0}^{x}\left[f(t,\phi_1(t))-f(t,\phi_0(t))\right]dt$$

Therefore $$\left|\phi_2(x)-\phi_1(x)\right| \le \left|\int_{x_0}^{x} \left| f(t,\phi_1)-f(t,\phi_0)\right| dt\right|$$

Since f satisfies Lipschitz condition with constant K

$$\left| f(t,\phi_1)-f(t,\phi_2)\right| \le K\left|\phi_1-\phi_2\right| \text{ and we have}$$

$$\left|\phi_2(x)-\phi_1(x)\right| \le K\left|\int_{x_0}^{x} \left|\phi_1(t)-\phi_0(t)\right| dt\right|$$

But $\left|\phi_1(x)-\phi_0(x)\right| \le M\left|x-x_0\right|$ for x in I

Therefore $$\left|\phi_2(x)-\phi_1(x)\right| \le K\int_{x_0}^{x} M\left|t-x_0\right| dt$$

and $$\left|\phi_2(x)-\phi_1(x)\right| \le KM\frac{(x-x_0)^2}{2}.$$

By mathematical induction we will prove that

$$\left|\phi_p(x)-\phi_{p-1}(x)\right| \le MK^{p-1}\frac{|x-x_0|^p}{p!} \text{ for every } x \text{ in I............} \qquad (4.2.1)$$

We have seen that this inequality is true for $p = 1$ and $p = 2$. Let us assume the result for $p = m$ and we will prove it for $p = m + 1$.

Without loss of generality assume that $x \ge x_0$.

By definition of ϕ_{m+1} and ϕ_m we get

$$\phi_{m+1}(x)-\phi_m(x)=\left[y_0+\int_{x_0}^{x} f(t,\phi_m(t))\,dt\right]-\left[y_0+\int_{x_0}^{x} f(t,\phi_{m-1})\,dt\right]$$

$$=\int_{x_0}^{x}\left[\, f(t,\phi_m(t))-f(t,\phi_{m-1}(t))\,\right] dt$$

Thus, $$\left|\phi_{m+1}(x)-\phi_m(x)\right| \le \int_{x_0}^{x}\left| f(t,\phi_m(t))-f(t,\phi_{m-1}(t))\right| dt$$

Since f satisfies Lipschitz condition we get

$$\left|\phi_{m+1}(x)-\phi_m(x)\right| \le K\int_{x_0}^{x}\left|\phi_m(t)-\phi_{m-1}(t)\right| dt$$

But $$\left|\phi_m(t)-\phi_{m-1}(t)\right| \le MK^{m-1}\frac{\left|t-x_0\right|^m}{m!}.$$

Therefore $$\left|\phi_{m+1}(x)-\phi_m(x)\right| \le \frac{MK^m}{m!}\int_{x_0}^{x}\left|t-x_0\right|^m dt$$

$$\le \frac{MK^m}{m!}\frac{(x-x_0)^{m+1}}{m+1}=\frac{M}{K}\frac{(K\,|x-x_0|)^{m+1}}{(m+1)!}$$

Thus by induction the inequality (4.2.1) is true for p = 1, 2, 3,.....

Since $$\left|\phi_p(x)-\phi_{p-1}(x)\right| \le \frac{M}{K}\;\frac{K^p\,|x-x_0|^p}{p!},$$

$$\sum_{p=1}^{k}\left|\phi_p(x)-\phi_{p-1}(x)\right| \le \frac{M}{K}\sum_{p=1}^{k}\frac{K^p\,|x-x_0|^p}{p!} \le \frac{M}{K}\left(e^{k|x-x_0|}-1\right)$$

And by weierstrass M-test, left hand series is uniformly convergent. Therefore the series

$$\phi_0(x)+\sum_{p=1}^{\infty}\left[\phi_p(x)-\phi_{p-1}(x)\right]$$

is absolutely convergent on I. Let $\phi(x)$ be a limit function of the series. Then

$$\underset{k\to\infty}{Lim}\ \phi_k(x)=\underset{k\to\infty}{Lt}\left\{\phi_0(x)+\sum_{p=1}^{k}\left[\phi_p(x)-\phi_{p-1}(x)\right]\right\}$$

$$=\phi_0(x)+\sum_{p=1}^{\infty}\left(\phi_p(x)-\phi_{p-1}(x)\right)$$

$$=\phi(x)$$

Thus the sequence $\{\phi_k\}$ of successive approximations is a convergent sequence.

(b) Properties of limit function ϕ.

The limit function ϕ is a continuous function on I.

$$\left|\phi_{k+1}(x_1)-\phi_{k+1}(x_2)\right| = \left\|\left[y_0+\int_{x_0}^{x_1} f(t,\,\phi_k(t)\,dt\right]-\left[y_0+\int_{x_0}^{x_2} f(t,\,\phi_k(t)\,dt\right]\right\|$$

$$=\left|\int_{x_0}^{x_1}[\,f(t,\,\phi_k(t)]\,dt-\int_{x_0}^{x_2}[\,f(t,\,\phi_k(t)]\,dt\right|$$

$$=\left|\int_{x_1}^{x_2} f(t,\,\phi_k(t)\,dt\right|$$

Since f is bounded by M, that is,

$$|\,f(x,\,y)|\le M \qquad \text{for } (x,\,y)\in R,$$

$$\left|\phi_{k+1}(x_1)-\phi_{k+1}(x_2)\right|\le M\,|\,x_1-x_2\,| \qquad \forall x_1,x_2\in I$$

By taking limit as $k\to\infty$ we get

$$\left|\phi(x_1)-\phi(x_2)\right|\le M\,|\,x_1-x_2\,|$$

Therefore as $x_2\to x_1,\ \phi(x_2)\to\phi(x_1),$ that is, ϕ is continuous on I.

In particular

$$\left|\phi(x)-\phi(x_0)\right|\le M\,|\,x-x_0\,|, \qquad \forall x\in I$$

Since $x \in R$, $|x - x_0| \leq \alpha = \min\left\{a, \frac{b}{M}\right\}$ and $|x - x_0| \leq \frac{b}{M}$ implies $M|x - x_0| \leq b$.

Therefore $|\phi(x) - \phi(x_0)| \leq M|x - x_0| \leq b$

Thus, $x \in I$ and $|\phi(x) - \phi(x_0)| \leq b$ implies $(x, \phi(x)) \in R$

(c) Bounds for $|\phi(x) - \phi_k(x)|$

We have $\phi(x) = \phi_0(x) + \sum_{p=1}^{\infty}\left[\phi_p(x) - \phi_{p-1}(x)\right]$

and $$\phi_k(x) = \phi_0(x) + \sum_{p=1}^{k}\left[\phi_p(x) - \phi_{p-1}(x)\right]$$

Therefore

$$\phi(x) - \phi_k(x) = \sum_{p=k+1}^{\infty}\left[\phi_p(x) - \phi_{p-1}(x)\right]$$

$$\left|\phi(x) - \phi_k(x)\right| \leq \sum_{p=k+1}^{\infty}|\phi_p(x) - \phi_{p-1}(x)|$$

But $\left|\phi_p(x) - \phi_{p-1}(x)\right| \leq \frac{M}{K}\frac{K^p|x - x_0|^p}{p!}$ and $|x - x_0| < \alpha$

Therefore $\left|\phi_p(x) - \phi_{p-1}(x)\right| \leq \frac{M}{K}\frac{K^p\alpha^p}{p!}$

Thus, $$\left|\phi(x) - \phi_k(x)\right| \leq \sum_{p=k+1}^{\infty}\frac{M}{K}\frac{K^p\alpha^p}{p!}$$

$$\leq \frac{M}{k}\frac{(K\alpha)^{k+1}}{(k+1)!}\sum_{p=0}^{\infty}\frac{(K\alpha)^p}{p!}$$

$$\leq \frac{M}{K}\frac{(K\alpha)^{k+1}}{(k+1)!}e^{k\alpha}$$

for every k we have

$$\left|\phi(x) - \phi_k(x)\right| \leq \frac{M}{K}\frac{(K\alpha)^{k+1}}{(k+1)!}e^{K\alpha}.$$

(d) The limit ϕ is a solution

We must show that

$$\phi(x) = y_0 + \int_{x_0}^{x} f(t, \phi(t))\,dt, \qquad \text{for all } x \text{ in } I.$$

Since ϕ is continuous on I and f is continuous on R, the function F defined by

$F(t) = f(t, \phi(t))$ is continuous on I and therefore is integrable.

Thus, $y_0 + \int_{x_0}^{x} f(t, \phi(t))\, dt$ is we defined

Now $\quad \phi_{k+1}(x) = y_0 + \int_{x_0}^{x} f(t, \phi_k(t))\, dt$

Taking limit on both sides we get

$$\phi(x) = \underset{k\to\infty}{Lt}\ \phi_{k+1}(x) = y_0 + \underset{k\to\infty}{Lt} \int_{x_0}^{x} f(t, \phi_k(t))\, dt$$

Therefore it is sufficient to prove that

$$\underset{k\to\infty}{Lt} \int_{x_0}^{x} f(t, \phi_k(t))\, dt = \int_{x_0}^{x} f(t, \phi(t))\, dt$$

$$\left| \int_{x_0}^{x} f(t, \phi_k(t))\, dt - \int_{x_0}^{x} f(t, \phi(t))\, dt \right|$$

$$\leq \left| \int_{x_0}^{x} \left| f(t, \phi_k(t)) - f(t, \phi(t)) \right| dt \right|$$

$$\leq K \left| \int_{x_0}^{x} \left| \phi_k(t) - \phi(t) \right| dt \right|$$

But by (c)

$$\left| \phi_k(t) - \phi(t) \right| \leq \frac{M}{K} \frac{(K\alpha)^{k+1}}{(k+1)!} e^{K\alpha}$$

Therefore $\quad \left| \int_{x_0}^{x} f(t, \phi_k(t))\, dt - \int_{x_0}^{x} f(t, \phi(t))\, dt \right| \leq M \dfrac{(K\alpha)^{k+1}}{(k+1)!} e^{K\alpha} \,|\, x - x_0 \,|$

Since $\quad \dfrac{(K\alpha)^{k+1}}{(k+1)!} \to 0$ as $k \to \infty$,

$\int_{x_0}^{x} f(t, \phi_k(t))\, dt \to \int_{x_0}^{x} f(t, \phi(t))\, dt$ that is

$$\underset{k\to\infty}{Lt} \int_{x_0}^{x} f(t, \phi_k(t))\, dt = \int_{x_0}^{x} f(t, \phi(t))\, dt$$

And $\quad \phi(x) = y_0 + \int_{x_0}^{x} f(t, \phi(t))\, dt$

Thus ϕ is a solution of the initial value problem $y' = f(x, y), \quad y(x_0) = y_0$ on I.

In theorem 4.2.1 we have shown the existence of solution of initial value problem 4.1.1. The solution thus obtained is a uniques solution.

Picard-Lindel of theorem states that if f is continuous function and satisfies Lipschitz condition on R, then the successive approximations ϕ_k exist on $|x - x_0| \leq \alpha$, ϕ_k's are continuous and converge uniformaly on the interval I to a unique solution passing through $(x_0, y_0) \in R$.

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