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Preface

It is hoped that students must learn mathematics not only to become a competent mathematicians but also skilled users of mathematics in the solution of problems in the real world especially in Engineering. They must learn how to use their mathematical knowledge in solving the problems of the real world. I believe that through the study of Linear Algebra, students will learn something about the art of applying mathematical knowledge to solve such problems. Comprehensive account of the mathematical artifact and numerous examples in this book will help the aspirants to develop an ability to use Linear Algebra.

I have a great pleasure in presenting SIM on Linear Algebra in your hands. The material of the book is the standard post-graduate syllabus of most of the Indian Universities. In this book "Linear Algebra" has been written for the use of students preparing for post-graduate examinations of Indian universities and SET/ NET aspirants. In such competitive examinations more emphasis is given on examples. Efforts have been made to put the subject matter in lucid and comprehensive manner. Various reference books by the eminent authors have been utilized in the preparation of the text and the author is gratefully indebted to them. I have streamlined the examples and exposition, making the book easier to learn oneself. It is hoped that the teachers, the students and large number of entrants to the competitive examinations will be benefited with the matter of this book.

Any constructive suggestions for the improvement of the subject matter will be highly appreciated.

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Linear Algebra

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M. Sc. (Mathematics)

Linear Algebra

Contents

Each Unit begins with the section Objectives -

Objectives are directive and indicative of :

1. What has been presented in the Unit and
2. What is expected from you
3. What you are expected to know pertaining to the specific Unit once you have completed working on the Unit.

The self check exercises with possible answers will help you to understand the Unit in the right perspective. Go through the possible answers only after you write your answers. These exercises are not to be submitted to us for evaluation. They have been provided to you as Study Tools to help keep you in the right track as you study the Unit.

UNIT - I

LINEAR ALGEBRA

Definition

Vector Space :

A non-empty set V is said to be vector space over the field F. If V is an abelian group under addition and if for every $a, b \in F$, $n_1, n_2 \in V$, such that $a \cdot n \in V$ satisfying following condition

(i) $a\left(v_1+v_2\right)=a v_1+a v_2 ; \; a \in F, \; v_1, v_2 \in V$

(ii) $(a+b) v_1=a v_1+b v_1 ; \; a, b \in F, \; v_1 \in V$

(iii) $(a b) v_1=a\left(b v_1\right) ; \; a, b \in F, \; v_1 \in V$

(iv) $1 \in F$

$\therefore 1 \cdot v_1=v_1$.

Example

Let V = F [x] over F it is a vector space usual addition and multiplication of polynomial.

Subspace :

Let $W \neq f \leq V$ and $a, b \in F$, $w_1, w_2 \in W$ wth $a w_1+b w_2 \in W$ then we called W is subspace.

Example : W is the collection of all polynomial with degree less than n is subspace of $F[x]$.

Homomorphism in vector space

If U and V are vector space over F then the mapping $T: U \rightarrow V$ is said to be a homomorphism.

If (i) $T\left(u_1+u_2\right)=T\left(u_1\right)+T\left(u_2\right)$

(ii) $T\left(a u_1\right)=a \cdot T\left(u_1\right)$ $\quad \forall u_1, u_2 \in U$ and $a \in F$

Lemma :

If V is a vector over F then

1) $a \cdot 0 = 0$

2) $0 \cdot v = 0$

3) $-a\,(v) = -(a\,v)$

4) If $v \neq 0$ and $a \cdot v = 0 \Rightarrow a = 0$, $a \in F, v \in V$.

Lemma :

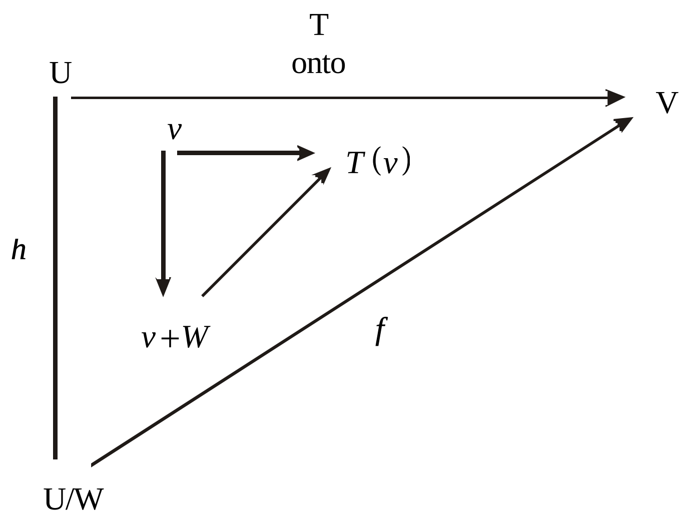
If V is a vector space over F and if W is a subspace of V then V/W is a vector space over F, where for $v_1 + W$, $v_2 + W \in V/W$

(i) $(v_1 + W) + (v_2 + W) = (v_1 + v_2) + W$

(ii) $a\,(v_1 + W) = (a\,v_1) + W$

Theorem :

If T is homomorphism of U onto V with Kernal W then V is isomorphic to V/W, conversely, if U is a vector space and W is subspace of U then there is a homomorphism of U onto U/W.



Definition

Internal Direct Sum

Let V be a vector space over F and let U_1, U_2, ..., U_n be subspace of V, V is said to be the internal direct sum of U_1, U_2, ..., U_n. If every element $v \in V$ can be written in one and only one as $v = u_1 + u_2 + ... + u_n$ where $u_i \in U_i$.

Definition

External Direct Product

Any finite number of vector spaces over F, V_1, V_2, ..., V_n. Consider the set V of all order n tuples (v_1, v_2, ..., v_n) where $v_i \in V_i$, V is called external direct sum V_1, V_2, ..., V_n.

1) Let $v = u + w$

$$= (u_1, u_2, ..., u_m) + (v_1, v_2, ..., v_m)$$

$$= (u_1 + v_1, u_2 + v_2, ..., u_m + v_m),\ u, w \in V$$

2) $a v = a\,(u_1, u_2, ..., u_m)$

$$= (a u_1, a u_2, ..., a u_m)$$

Theorem

If V is the internal direct sum of $U_1, U_2, ..., U_n$ then V is isomorphic to the external direct sum of $U_1, U_2, ..., U_n$.

$$\left[v = u_1 + u_2 + ... + u_n \rightarrow (u_1, u_2, ..., u_n)\right]$$

Linear Independent and Basis

Linear Combination

If V be a vector space over F and $v_1, v_2, ..., v_n \in V$ then any element of the form, $a_1 v_1 + a_2 v_2 + ... + a_n v_n$ where $a_i \in F$ is a linear combination over F of $v_1, v_2, ..., v_n$.

Linear Span

If S be a non-empty subset of vector space V then L (S) the Linear Span of S is the set of all linear combinations of elements of S.

Lemma :

L (S) is subspace of V. If S, T are subsets of V then

1) $S \subset T \Rightarrow L(S) \subset L(T)$

2) $L(L(S)) = L(S)$

3) $L(S \cup T) = L(S) + L(T)$

Finite Dimensional

The vector space V is said to be finite dimensional if there is a finite subset S in V such that $V = L(S)$.

Linear Dependent Set :

If V is vector space and if $v_1, v_2, ..., v_n$ are in V. We say that they are linearly dependent over F if there exist elements $a_1, a_2, ..., a_n$ in F not all of them zero such that,

$$a_1 v_1 + a_2 v_2 + ... + a_n v_n = 0$$

Linearly Independent Set :

If V is vector space and if $v_1, v_2, ..., v_n$ are in V we say that they are linearly independent in F all are zero such that $a_1 v_1 + a_2 v_2 + ... + a_n v_n = 0$

Lemma

If $v_1, v_2, ..., v_n \in V$ are linearly independent then every element in their linear span have a unique representation in the form $a_1 v_1 + a_2 v_2 + ... + a_n v_n$ with a_i 's $\in F$.

Theorem

If $v_1, v_2, ..., v_n$ are in V then either they are linearly independent or some v_k is linear combination of preceeding once $v_1, v_2, ..., v_{k-1}$.

Carollary :

If $v_1, v_2, ..., v_n$ in V have W as a linear span and if $v_1, v_2, ..., v_k$ are linearly independent then we can find a subset of $v_1, v_2, ..., v_n$ of the form $v_1, v_2, ..., v_k$, $v_{i1}, v_{i2}, ..., v_{ir}$ consisting of linearly independent elements whose linear span is also W.

Carollary :

If V is a finite dimensional vector space then it contains a finite set $v_1, v_2, ..., v_n$ of linearly independent elements whose linear span is V.

Basis :

A subset S of a vector space V is called a basis of V if S consist of linearly independent elements and V = L (S).

Carollary :

If V is a finite dimensional vector space and if $u_1, u_2, ..., u_m$ span V then some subset of $u_1, u_2, ..., u_m$ forms a basis of V..

Lemma :

If $v_1, v_2, ..., v_n$ is a basis of V over F and if $w_1, w_2, ..., w_m$ in V are linearly independent over F, then $m \le n$.

Carollary :

If V is finite dimensional over F then any two basis of V have the same number of elements.

Carollary :

$F^{(n)}$ is isomorphic to $F^{(m)}$ if and only if n = m. (by above carollary). (Two vector spaces are isomorphic if and only if dimension is same).

Carollary :

If V is finite dimensional over F then V is isomorphic to $F^{(n)}$ for a unique integer n. (integer n depends on dimensions of V). (No. of elements in the basis : dimension).

Carollary :

Any two finite dimensional vector spaces over F of the Same dimension are isomorphic.

Lemma :

If V is finite dimensional over F and if $u_1, u_2, ..., u_m \in V$ are linearly independnt then we can find vectors $u_{m+1}, u_{m+2}, ..., u_{m+r} \in V$ such that $u_1, u_2, ..., u_m, u_{m+1}, u_{m+r}$ form a basis of W..

Lemma :

If V is finite dimensional and if W is a subspace of V then W is finite dimensional, $\dim W \leq \dim V$ and $\dim \frac{V}{W} = \dim(V) - \dim(W)$.

Carollary :

If A and B are finite dimensional subspaces of a vector space V then A + B is finite dimesnional and $\dim(A+B) = \dim A + \dim B - \dim(A \cap B)$.

Dual Space

Lemma :

Hom (V, W) is a vector space over F under the operation $(T+S)(v) = T(v) + S(v)$.

$T(av) = aT(v)$, $a \in F$ and $v \in V$.

Proof : Let V and W be vector spaces over F and consider a collection of homomorphisms form V to W.

As, (Hom (V, W), +)

(i) Let $(T+S)(v) = T(v) + S(v)$ and

$T(av) = aT(v)$; where $a \in F$ and $v \in V$.

Take $v_1, v_2 \in V$ and $T, S \in$ Hom (V, **W)**

$\Rightarrow (T+S)(v_1 + v_2) = T(v_1 + v_2) + S(v_1 + v_2)$

$= T(v_1) + T(v_2) + S(v_1) + S(v_2)$

$$= T(v_1) + S(v_1) + T(v_2) + S(v_2)$$

$$= (T+S)(v_1) + (T+S)(v_2)$$

$$\Rightarrow T + S \in Hom(V, W)$$

(ii) Scalar Multiplication

Let $(T+S)(av) = T(av) + S(av)$; $a \in F$ and $v \in V$.

$$= aT(v) + aS(v) \quad \ldots\ldots \because T, S \in Hom(V, W)$$

$$= a(T(v) + S(v))$$

$$= a(T+S)(v)$$

(iii) Associative Property

Let T$_1$, T_2, $T_3 \in$ Hom (V, W) and $v \in V$.

$$\therefore ((T_1 + T_2) + T_3)(v) = (T_1 + T_2)(v) + T_3(v)$$

$$= T_1(v) + T_2(v) + T_3(v)$$

$$= T_1(v) + (T_2 + T_3)(v)$$

$$= (T_1 + (T_2 + T_3))(v)$$

Now,

$$(T_1 - T_2)(v_1 + v_2) = (T_1 + (-T_2))(v_1 + v_2)$$

$$= T_1(v_1 + v_2) + (-T_2)(v_1 + v_2)$$

$$= T_1(v_1) + T_1(v_2) + (-T_2)(v_1) + (-T_2)(v_2)$$

$$= T_1(v_1) + T_1(v_2) + (-1)T_2(v_1) + (-1)T_2(v_2)$$

$$= T_1(v_1) + T_1(v_2) - T_2(v_1) - T_2(v_2)$$

$$= T_1(v_1) - T_2(v_1) + T_1(v_2) - T_2(v_2)$$

$$= (T_1 + (-T_2))(v_1) + (T_1 + (-T_2))(v_2)$$

$$= (T_1 - T_2)(v_1) + (T_1 - T_2)(v_2)$$

$\Rightarrow T_1 - T_2 \in Hom(V,W)$

Also

$(T_1 - T_2)(av) = T_1(av) + (-T_2)(av)$

$= aT_1(v) + (-1)T_2(av)$

$= aT_1(v) - aT_2(v)$

$= a(T_1 - T_2)v$

$\therefore (Hom(V,W),+)$ is a group.

Let $(T_1 + T_2)(v) = T_1(v) + T_2(v)$

$= T_2(v) + T_1(v)$ $\because T_1(v),\ T_2(v) \in W$ and W is a vector S.

$= (T_2 + T_1)(v)$

Hence, $(Hom(V,W),+)$ is abelian.

Let $l \in F,\ T \in Hom(V,W)$ and $a,b \in F$.

$\Rightarrow (lT)(av_1 + bv_2) = T(l(av_1 + bv_2))$ .... by linearity of T.

$= T(lav_1) + T(lav_2)$

$= laT(v_1) + lbT(v_2)$

$= a(lT)(v_1) + b(lT)(v_2)$

$lT \in Hom(V,W)$

Scalar multiplication distrubute over addition.

$l(T_1 + T_2)(v) = l[(T_1 + T_2)(v)]$

$= l[T_1(v) + T_2(v)]$

$= (lT_1)(v) + (lT_2)(v)$ .... W is a V. S. and $v_1, v_2 \in V$.

$= ((lT_1) + (lT_2))(v)$

Vector multiplication distrubute over scalar addition

$$(l+b)T_1(v)=(l+b)(T_1(v))$$

$$= lT_1(v)+bT_1(v) \text{ .... W is a Vector space}$$

$$=(lT_1)(v)+(bT_1)(v) \text{ .... } lT_1, bl_1 \in Hom(V,W)$$

$$=(lT_1+bT_1)(v)$$

$$(lb)T(v)=l(bT(v)) \qquad \text{.....} T(v)\in W$$

$$=b(lT(v))$$

Identity w.r.t. multiplication

$$1\cdot T(v)=T(1\cdot v)=1\cdot T(v)$$

$$=T(v)$$

$\therefore$ Hom (V, W) is a vector space over F.

Hence the proof.

Theorem :

If V and W are of dimensions m and n respectively over F, then Hom (V, W) is of dimension mn over F.

Proof :

We prove the theorem by exhibiting a basis of Hom (V, W) over F consisting m, n elements. Let $v_1, v_2, ..., v_m$ be a basis of V over F and $w_1, w_2, ..., w_n$ be a basis of W over F.

Define $T_{ij}: V \to W$ as $T_{ij}(v)=a_i w_j$ and $T_{ij}(v_k)=w_j$; for i = k.

$= 0$; for $i \neq k$.

We claim that $T_{ij} \in Hom(V,W)$ and $\{T_{ij} \mid i=1,2,....,m,\ j=1,2,....,n\}$ is linearly independent and spans Hom (V, W).

Let $v, u \in V$.

$\therefore v=a_1v_1+a_2v_2+.....+a_mv_m$ and

$u=b_1v_1+b_2v_2+.....+b_mv_m$, $a_1, a_2, ..., a_m$, $b_1, b_2, ..., b_m \in F$.

Now,

$$T_{ij}(av+bu)=T_{ij}(a a_1 v_1 + a a_2 v_2 + ... + a a_m v_m + b b_1 v_1 + b b_2 v_2 + ... + b b_m v_m)$$

$$= T_{ij}(a a_1 + b b_1) v_1 + ... + (a a_m + b b_m) v_m$$

$$= 0 + ... + (a a_i + b b_i) w_j + 0 + ... + 0$$

$$= (a a_i + b b_i) w_j$$

$$= a\,(0 w_1 + ... + a_i w_j + 0 \cdot w_{j+1} + ... + 0 \cdot w_m)$$

$$+ b\,(0 w_1 + ... + b_i w_j + 0 \cdot w_j + ... + 0 \cdot w_n)$$

$$= a T_{ij}(v) + b T_{ij}(u)$$

$\Rightarrow T_{ij} \in Hom(V,W)$.

Let $S \in Hom(V,W)$ and $v_1 \in V$.

$\therefore S(v_1) \in W$.

Hence, $S(v_1) = a_{11} w_1 + a_{12} w_2 + ... + a_{1n} w_n$.

For some $a_{11}, a_{12}, ..., a_{1n} \in F$.

In fact, $S(v_i) = a_{i1} w_1 + a_{i2} w_2 + ... + a_{in} w_n$, i = 1, 2, ..., m.

Consider

$$S_0 = a_{11} T_{11} + a_{12} T_{12} + ... + a_{1n} T_n + a_{21} T_{21} + ... + a_{2n} T_{2n}$$

$$+ ...... + a\, m_1 T m_1 + ... + a_{mn} T_{mn}$$

Let us compute $S_0(v_i)$.

$$\therefore S_0(v_i) = [a_{11} T_{11} + ... + a_{1n} T_{1n} + ... + a\, m_1 T m_1 + ... + a_{mn} T_{mn}](v_i)$$

$$= 0 + ... + 0 + a_{i1} w_1 + a_{i2} w_2 + ... + a_{in} w_n + 0 + ... + 0$$

$$\left(\because T_{ij}(v_k) = \begin{matrix} w_j & i = k \\ 0 & i \neq k \end{matrix} \right)$$

$$= S(v_i)$$

Thus, the Homomorphisms S_0 and S agree on the basis of V.

$\therefore S_0 = S$

However, S_0 is linear combination of T_{ij} whence S must be the same linear combination.

Thus, the set B spans Hom (V, W).

Now, we will show B is linearly independent

Suppose,

$$\left(b_{11}T_{11} + b_{12}T_{12} + ... + b_{1n}T_{1n} + ... + b_{m1}T_{m1} + ... + b_{mn}T_{mn}\right) = 0 \,;$$

where $B_{ij} \in F$.

Apply this on a basis vector v_i of V.

$$\Rightarrow \left(b_{11}T_{11} + b_{12}T_{12} + ... + b_{1n}T_{1n} + ... + b_{m1}T_{m1} + ... + b_{mn}T_{mn}\right)\left(v_i\right) = 0\left(v_i\right) = 0$$

$$\Rightarrow 0 + ... + 0 + b_{i1}w_i + b_{i2}w_2 + ... + b_{in}w_n + 0 + ... + 0 = 0$$

$$\Rightarrow b_{i1} = b_{i2} = ...b_{in} \,.$$

Since $w_i\,'s$ are basis elements of W.

This implies $b_{ij} = 0$ for all i and j.

Thus, b is linearly independent over F and forms a basis of Hom (V, W) over F.

$\therefore$ dim Hom (V, W) is mn.

Hence the proof.

Corollary :

1. If dim V = m then dim Hom (V, V) = m^2.

   Proof : Replace W by V and n by m.

2. If dim V = m then dim (Hom (V, F)) = m.

   Proof : As F a vector space is of dimension one over F.

Note : If V is finite dimensional over F. It is isomorphic to Hom (V, F).

Dual Space :

If V is a vector space then it’s dual space is Hom (V. F). It is denoted by $\hat{V}$. The elements of $\hat{V}$ will be called a linear functional on V into F.

Problem : Show that $\hat{v}_1, \hat{v}_2, ..., \hat{v}_n$ is a basis of $\hat{V}$, for $v_1, v_2, ..., v_n$ is basis of V and

Solution : $\hat{v}_i(v_j) = 1$ if i = j ... (1)

$= 0$ if i$\neq$ j.

Consider $a_1\hat{v}_1 + a_2\hat{v}_2 + ... + a_n\hat{v}_n = 0$ for $a_1, a_2, ..., a_n \in F$

$\therefore (a_1\hat{v}_1 + ... + a_n\hat{v}_n)(v_i) = 0 \Rightarrow a_i = 0$ it is true $\forall i$

$\Rightarrow \hat{v}_1, ..., \hat{v}_n$ are linearly independent.

$\dim \hat{V} = \dim V \Rightarrow \hat{v}_1, \hat{v}_2, ..., \hat{v}_n$ is basis of V..

Lemma : If V is finite dimensional and $v \neq 0$ in V then there is an element $f \in \hat{V}$ such that $f(v) \neq 0$.

Proof : Let V is finite dimensional vector space over F and let $v_1, v_2, ..., v_n$ be a basis of V..

Let $\hat{v}_i \in \hat{V}$ defined by $\hat{v}_i(v_j) = 1$ if i = j

$= 0$ if i$\neq$ j.

Consider, $\hat{v}_i(v) = 0$ for $v \neq 0$ in V, $\therefore v = \sum_{i=1}^{n} a_i v_i$, $a_i \in F$.

$\therefore \hat{v}_i(a_1v_1 + a_2v_n + ... + a_nv_n) = 0$

$\Rightarrow a_i = 0$ ...... by definition of $\hat{v}_i$.

$\Rightarrow$ All a_i's used in the representation of v are zero.

Hence v = 0, a contradiction.

Thus, $\therefore \hat{v}_i(v) \neq 0$.

$\Rightarrow \hat{v}_i = f \in \hat{V}$ such that $f(v) \neq 0$.

Hence the proof.

Definition :

Let the functional on $\hat{V}$ into F, $T_{v_0}(f) = f(v_0)$ for $f \in \hat{V}$.

with $T_{v_0} = (f + g)(v_0)$

$= f(v_0) + g(v_0)$ ... Hom (V, F)

$= T_{v_0}(f) + T_{v_0}(g)$

$T_{v_0}(l\,f) = l\,f(v_0)$

$l \cdot T_{v_0}(f)$

T_{v_0} is in dual of $\hat{V}$ it is called **Second Dual** of V. It is denoted by $\hat{\hat{V}}$.

Leema : If V is finite dimensional then there is an isomorphism of V onto $\hat{\hat{V}}$.

Proof : Let V is finite dimensional vector space. Define the map $y : V \to \hat{\hat{V}}$ by $y(v) = T_v$ for every $v \in V$.

We will show y is well-defined, one-one, onto, homomorphism.

Let $u, v \in V$. Let $u = v$.

$\Leftrightarrow f(u) = f(v)$ ... $f \in \hat{V}$

$\Leftrightarrow T_u = T_v$

$\Leftrightarrow y(u) = y(v)$

$\therefore$ Thus y is well-defined and one-one. Now, consider $u, v \in V$.

$y(u+v) = T_{u+v}$

but $T_{u+v} = f(u+v)$

$= f(u) + f(v) \quad = T_u(f) + T_v(f)$

$= (T_u + T_v)(f)$

$\Rightarrow T_{u+v} = T_u + T_v$

$\therefore y(u+v) = T_u + T_v$

$= y(u) + y(v)$

$\therefore y(u+v) = y(u) + y(v)$

Let a be any element.

$y(av) = T_{av} = f(av) = a\,f(v)$

$= a \cdot T_v$ ...... Vector space.

$= a \cdot y(v)$

$\therefore y$ is homomorphism.

Annihilator :

If W is a subspace of V then the annihilator of W

$A(W)=\left\{ f \in \hat{V} \mid f\ (w)=0; \forall\ w \in W \right\}$

EXERCISE ....

1. Show that $A(W)$ is subspace of $\hat{V}$.

Proof : Let $f, g \in A(w)$ and $a, b \in F$; w is arbitrary element in **W.**

Claim : $a\ f + b\,g \in A(W)$

$f\ (w)=0=g\ (w)$, $\forall w \in W$.

$\Rightarrow (a\ f + b\,g)(w) = a\ f\ (w) + b\ g\ (w)$

$= a \cdot 0 + b \cdot 0$

$= 0$

$\Rightarrow a\ f + b\,g \in A(W)$

Hence the proof.

Note :

1. If $W=\{0\}$ is the 0 subspace of V then $A(W)=\hat{V}$.
2. If W = V then $A(W)=\{0\}$.
3. If V is finite dimensional vector space and W contains a non-zero vector and also W is a proper subspace then $A(W)$ is non-trivial, proper subspace of $\hat{V}$.

Lemma :

If V is finite dimensional vector space over F and W is a subspace of V then $\hat{W}$ is isomorphic to $\hat{V}/A(W)$ and $\dim(A(W)) = \dim V - \dim W$.

Proof : Let W be a subspace of V where V is finite dimensional.

If $f \in \hat{V}$, let $\overline{f}$ be the restriction of f to W and is defined on W.

As, $f\ (w) = \overline{f}\ (w)$ for every $w \in W$.

$\therefore \overline{f} \in \hat{W}$; since $f \in \hat{V}$.

Now, consider the mapping $T : \hat{V} \to \hat{W}$; defines as

$T(f) = \overline{f}$; for $f \in \hat{V}$.

Let $f, g \in \hat{V}$ such that $f = g$.

$\Leftrightarrow f(v) = g(v)$; for every $v \in V$

$\Leftrightarrow$ i.e. $f(v) = g(v)$; for every $v \in W \subseteq V$

$\Leftrightarrow \overline{f}(v) = \overline{g}(v)$; for $v \in W$

$\Leftrightarrow \overline{f} = \overline{g}$

$\Leftrightarrow T(f) = T(g)$ ...... by definition of T.

Hence T is well-derined and one-one.

For Homomosphism of T,

Consider,

$$T(f+g) = \overline{f+g}$$

$$\therefore \left(\overline{f+g}\right)(v) = \left(\overline{f} + \overline{g}\right)(v)$$

$$= \overline{f}(v) + \overline{g}(v)$$

$$= \left(\overline{f} + \overline{g}\right)(v)$$

$$\therefore T(f+g) = \overline{f} + \overline{g}$$

$$= T(f) + T(g)$$

and $T(l\,f) = l\,\overline{f} = l \cdot \overline{f} = l \cdot T(f)$ ..... since l is scalar.

$\therefore$ T is Homomorphism.

Now, we will show that T is onto.

i.e. for a given any element $h \in \hat{W}$. Then h is the restriction of some $f \in \hat{V}$.

i.e. $\overline{f} = h$

We know that, “if $w_1, w_2, ..., w_m$ is a basis of W, subspace of V..

Then it can be expanded to a basis of V of the form $\{w_1, ..., w_m w_{m+1}, ...w_n\}$; whose dim (V) = n.

Let W_1 be the subspace of V spanned by $\{w_{m+1}, w_{m+2}, ..., w_n\}$.

Thus, $V = W \oplus W_1$ ($\because \{w_{m+1}, ..., w_n\}$ does not belong to $\{w_1, ..., w_n\}$ so $W \cap W_1 = f$ and $W \cup W_1 =$ whole space)

Any element of V is represented as $v = w + w_1$; $w \in W$ and $w_1 \in W_1$.

For $h \in \hat{W}$, define $f \in \hat{V}$ as $f(v) = h(w)$.

$\Rightarrow f(w + w_1) = h(w)$

$\Rightarrow f(w) + f(w_1) = h(w)$

$\Rightarrow f(v) = h(w)$

$\therefore f \in \hat{V}$ we have $\bar{f} = h$.

$\Rightarrow \bar{f}(w) = h(w)$

Thus, $T(f) = h$ and so T maps $\hat{V}$ onto $\hat{W}$.

Consider,

$$\ker T = \{f \in \hat{V} \mid T(f) = 0\}$$

$$= \{f \in \hat{V} \mid \bar{f} = 0\}$$

$$= \{f \in \hat{V} \mid \bar{f}(w) = 0 \forall w \in W\}$$

$$= \{f \in \hat{V} \mid f(w) = 0 \forall w \in W\} \qquad \ldots f(w) = \bar{f}(w) \text{ every } w \in W.$$

$$= A(W)$$

Thus, by fundamental theorem of isomorphism (algebra).

$$\hat{W} \cong \frac{\hat{V}}{A(W)}$$

In particular they have the same dimensions $\dim \hat{W} = \dim\left(\frac{\hat{V}}{A(W)}\right)$, Also, we know $\dim V = \dim \hat{V}$ and $\dim W = \dim \hat{W}$ ·

$\therefore$ Above expression become,

$$\therefore \dim W = \dim\left(\frac{V}{A(W)}\right)$$

$$\dim W = \dim V - \dim A(W)$$

Hence the proof.

Theorem :

If V be a finite dimensional vector space over the field F. Let W be a subspace of V, then

$$\dim W + \dim A(W) = \dim V$$

Proof :

If W is the 0 subspace of V then $A(W) = \hat{V}$.

$$\therefore \dim(A(W)) = \dim \hat{V}$$

$$= \dim V.$$

Similarly, the result is obvious when W = V.

Let us suppose that W is proper subspace of V and dim W = m, dim V = n with 0 < m < n.

Let $B_1 = \{w_1, w_2, ..., w_m\}$ be as basis for W. Since, B_1 is linearly independent subset of V also.

$\therefore$ It can be extended to form a basis for V..

Let $B = \{w_1, w_2, ..., w_m, w_{m+1}, ..., w_n\}$ be a basis for V..

Let $\hat{B} = \{f_1, f_2, ..., f_m, f_{m+1}, ..., f_n\}$ be a dual basis of V..

Then $\hat{B}$ is a basis for $\hat{V}$ such that $f_i(w_j) = d_{ij} = 0$; if $i \neq j$

=1 ; if i = j.

We claim that $S = \{f_{m+1}, f_{m+2}, ..., f_n\}$ is a basis of $A(W)$.

Since $S \subset \hat{B}$. Therefore S is linearly independent because $\hat{B}$ is linearly independent.

Therefore, S is basis of $A(W)$ if $L(S) = A(W)$.

Let $f \in A(W)$, $f \in \hat{V}$.

So, let $f=\sum_{i=1}^{n} a_i f_i$; $a_i \in F$ ..... (1)

Now, $f \in A(W)$.

$\Rightarrow f(w)=0 \qquad \forall w \in A(W)$.

$\Rightarrow f(w_j)=0$ for each j = 1, 2, ...,m.

$$\therefore \sum_{i=1}^{n} a_i f_i(w_j)=0$$

$$\Rightarrow a_1 f_1(w_j)+a_2 f_2(w_j)+......+a_{j-1} f_{j-1}(w_j)+a_i f_i(w_j)+.....+a_n f_n(w_j)=0$$

$$\Rightarrow a_j=0 \qquad \left[\begin{array}{r}\because f_i(w_j)=0,\ i\neq j\\ =1,\ i=j\end{array}\right]$$

Putting, $a_1=a_2=......=a_m=0$ in (1)

$$f=\sum_{i=m+1}^{n} a_i f_i$$

$\Rightarrow f \in L(S)$

$\therefore A(W)$ contained in $L(S)$.

Let $g \in L(S)$.

$$\therefore g=\sum_{i=m+1}^{n} B_i f_i$$

Let $w \in W$.

$$w=\sum_{j=1}^{m} g_j w_j$$

$$\Rightarrow g(w)=\left(\sum_{i=m+1}^{n} b_i f_i\right)\left(\sum_{j=1}^{m} g_j w_j\right)$$

$$=\sum_{j=1}^{m} g_j\left(\sum_{i=m+1}^{n} b_i f_i\right)(w_j)$$

$$= \sum_{j=1}^{m} g_j \left(b_{m+1} + f_{m+1}(w_j) + \ldots\ldots + b_n f_n(w_j) \right)$$

$$= g_1 b_{m+1} f_{m+1}(w_1) + \ldots\ldots + g_m b_m f_{m+1}(w_j) +$$

$$g_1 b_{m+2} f_{m+2}(w_1) + \ldots\ldots + g_m b_{m+2} f_{m+2}(w_m)$$

$$+ \ldots\ldots + g_1 b_n f_n(w_1) + \ldots\ldots + g_m b_n f_n(w_n)$$

$$= 0$$

$$\Rightarrow g(w) = 0$$

Hence, $g \in A(W)$.

Therefore, $L(S) \subseteq A(W)$

$\Rightarrow$ we have $L(S) = A(W)$

$$\dim \hat{V} = \dim A(A(W)) + \dim A(W)$$

$$\therefore \dim(A(W)) = n - m$$

$$= \dim V - \dim W$$

$$\Rightarrow \dim V = \dim W + \dim A(W)$$

Anihilator of Anihilator

Let V be a vector space over F if any subset of V then A (S) is subspace of $\hat{V}$ and by definition of annihilator.

$$A(A(S)) = \{ L \in \hat{V} \mid L(f) = 0, \forall f \in A(S) \}$$

Example :

1) Show that $A(A(S))$ is subspace of $\hat{V}$.

Note :

If V is finite dimensional vector space then we have identity $\hat{\hat{V}}$ with V through the isomorphism $v \to L_v$.

Therefore, we may regard $A(A(S))$ as subsapce of V..

$A(A(S))=\{v\in V \mid f(v)=0 \forall f\in A(S)\}$

Corollary :

If W is subspace of V finite dimensional vector space then $A(A(W))=W$.

Proof : We have,

$A(W)=\{f\in\hat{V} \mid f(w)=0, \forall w\in W\}$ .... (1)

$A(A(W))=\{v\in V \mid f(v)=0, \forall f\in A(W)\}$ .... (2)

Let $w\in W$. Then by (1) $f(w)=0$, $\forall w\in W\subseteq V$.

There from equation (2);

$f(w)=0$, $\forall f\in A(W)$.

Therefore we have, $A(A(W))$

Hence, $W\subseteq A(A(W))$

Let $v\in A(A(W))$, $\forall v\in V$.

$\Rightarrow f(v)=0$, $\forall f\in A(W)$

$\Rightarrow f(v)=0$, $\forall f\in A(W)$ for $v\in W$

$\Rightarrow A(A(W))\subseteq W$.

$\therefore A(A(W))=W$

$\dim\hat{V}=\dim A(A(W))+\dim A(W)$

$\dim V=\dim A(A(W))+\dim(V)-\dim W$

$\Rightarrow \dim A(A(W))+\dim(W)\Rightarrow A(A(W))=W$. Hence the required.

Problem : Let V be finite dimensional vector space over the field F. If S is subset of V prove that $A(S)=A(L(S))$ where $L(S)$ is linear span of S.

Solution : Let V be a finite dimensional and S is any subset of V.

We know that $S\subseteq L(S)$.

Therefore, $A(L(S)) \subseteq A(S)$ ...... (1)

$(W \subseteq V$ and $W = \{0\} \subset V \Rightarrow A(W) = \hat{V}$, $A(V) = \{0\}$, $\therefore \hat{V} \supset \{0\}$)

Now, let $f \in A(S)$ then $f(s) = 0$ for all $s \in S$.

If u is any element of $L(S)$ then $u = \sum_{i=1}^{n} a_i s_i \in L(S)$.

Consider, $f(u) = f\left(\sum_{i=1}^{n} a_i S_i\right)$

$$= \sum_{i=1}^{n} a_i f(S_i)$$

$$= \sum_{i=1}^{n} a_i \cdot 0; \quad S_i \in S$$

$$= 0$$

$\Rightarrow f \in A(L(S))$ .... (2)

$\therefore A(S) \subseteq A(L(S))$

$\Rightarrow A(L(S)) = A(S)$

Hence the result.

Problem : Let V be finite dimensional vector space over F. If S is any subset of V then prove that $A(A(S)) = L(S)$.

Solution :

By previous $A(S) = A(L(S))$. By taking annihilator on both sides,

$\Rightarrow A(A(S)) = L(S)$ .... $L(S)$ is subspace of V..

Problem : Let W_1 and W_2 be subspaces of V which is finite dimensional. Describe $A(W_1 + W_2)$ in terms of $A(W_1)$ and $A(W_2)$.

Solution :

Let W_1 and W_2 be two subspaces of W..

We have, $W_1 \subseteq W_1 + W_2$ and $W_2 \subseteq W_1 + W_2$.

Since, $A(W_1 + W_2) \subseteq A(W_1)$ and $A(W_1 + W_2) \subseteq A(W_2)$.

$\Rightarrow A(W_1 + W_2) \subseteq A(W_1) \cap A(W_2)$

Conversely let, $f \in A(W_1) \cap A(W_2)$

$\Rightarrow f \in A(W_1)$ and $f \in A(W_2)$

$\therefore f(w_1) = 0$, $f(w_2) = 0$. $\forall w_1 \in W_1$, $w_2 \in W_2$.

Let any $w \in W$. Thus $w \in W$ is represented as $w = w_1 + w_2$.

$$\Rightarrow f(w) = f(w_1 + w_2)$$

$$= f(w_1) + f(w_2)$$

$$= 0$$

$\Rightarrow f(w) = 0$

$\Rightarrow f \in A(W_1 + W_2)$

$\Rightarrow A(W_1) \cap A(W_2) \subseteq A(W_1 + W_2)$

$\therefore A(W_1 + W_2) = A(W_1) \cap A(W_2)$

Hence, the result.

Problem : If W_1 and W_2 be subspaces of finite dimensional vector spaces.

Describe $A(W_1 \cap W_2)$ in terms of $A(W_1) + A(W_2)$.

Solution : By using previous exercise by replacing V by $\hat{V}$, W_1 by $A(W_1)$, W_2 by $A(W_2)$ we get;

$$A\big(A(W_1) + A(W_2)\big) = A\big(A(W_1)\big) \cap A\big(A(W_2)\big)$$

$$= W_1 \cap W_2$$

$$\Rightarrow A\Big(A\big(A(W_1) + A(W_2)\big)\Big) = A(W_1 \cap W_2)$$

$\Rightarrow A(W_1) + A(W_2) = A(W_1 \cap W_2)$ .... $A(W_1) + A(W_2)$ is a subspace of $\hat{V}$.

Hence, the result.

System of Linear Homogenous Equation

Theorem : If the system of homogeneous linear equations

$$a_{1n}x_1 + a_{12}x_2 + ... + a_{1n}x_n = 0$$

$$a_{21}x_1 + a_{22}x_2 + ... + a_{2n}x_n = 0$$

$$\vdots$$

$$a_{m1}x_1 + a_{m2}x_2 + ... + a_{mn}x_n = 0 \qquad \text{where } a_{ij} \in F$$

is of rank *r* then there are *n* – *r* linearly independent solutions in $F^{(n)}$.

Proof :

Consier, the system of m equations and n unknowns.

$$a_{11}x_1 + a_{12}x_2 + ... + a_{1n}x_n = 0$$

$$\vdots$$

$$a_{m1}x_1 + a_{m2}x_2 + ... + a_{mn}x_n = 0 \qquad \text{where } a_{ij} \in F$$

Now, we find how many linearly independent solutions $(x_1, x_2, ..., x_n)$ in $F^{(n)}$.

Let U be the subspace generated by m vectors with

$$(a_{11}, a_{12}, ..., a_{1n})$$

$$(a_{21}, a_{22}, ..., a_{2n})$$

$$\vdots$$

$(a_{m1}, a_{m2}, ..., a_{mn})$ and supposed that U is of dimensions *r*.

Let $v_1 = (1,0,....,0)$, $v_2 = (0,1,....,0)$, ..., $v_n = (0,0,0,....,0,1)$ be a basis for $F^{(n)}$.

$\therefore \hat{v}_1, \hat{v}_2, ..., \hat{v}_n$ be it's dual basis of $\hat{F}^{(n)}$, any small $f \in \hat{F}^{(n)}$ can be expressed as a linear combination of $\hat{v}_i$'s.

$$f = \sum_{i=1}^{n} x_i \hat{v}_i \; ; \quad x_i \in F$$

$\therefore$ For $(a_{11}, a_{12}, ..., a_{1n}) \in U$.

We have, $f(a_{11}, a_{12}, ..., a_{1n}) = f(a_{11}v_1 + a_{12}v_2 + ... + a_{1n}v_n)$

$(\because a_{11}(1,0,....,0) + a_{12}(0,1,....,0) = (a_{11}, a_{12}, ...))$

$$= f(a_{11}v_1) + f(a_{12}v_2) + ...+ f(a_{1n}v_1)$$

$$= a_{11} \cdot \sum_{i=1}^{n} x_i \hat{v}_i(v_1) + a_{12} \cdot \sum_{i=1}^{n} x_i \hat{v}_i(v_2) + ...+ a_{1n} \cdot \sum_{i=1}^{n} x_i \hat{v}_i(v_n)$$

$$= a_{11}x_1 + a_{12}x_2 + ...+ a_{1n}x_n$$

$$....\hat{v}_i(v_j) = 0 \qquad i \neq j$$

$$= 1 \qquad i = j$$

$$= 0$$

$\Rightarrow$ This is true for the other vectors in U.

$\therefore f \in A(U)$

Every solution $(x_1, x_2, ..., x_n)$ of the system of homogenous equation it's an elements $x_1\hat{v}_1 + x_2\hat{v}_2 + ...+ x_n\hat{v}_n$ in $A(U)$.

Therefore, we see that the number of linearly independent solutions of the system of equation is the dimension of $A(U)$.

But we know,

$$\dim F^{(n)} = \dim U + \dim A(U)$$

$$\therefore \dim A(U) = n - r$$

Hence, the proof.

Corollary :

If n > m that is if no. of unknowns exceeds the number of equations then there is a solution $(x_1, x_2, ..., x_n)$ where not all of $(x_1, x_2, ..., x_n)$ are 0.

Proof : Since U is generated by m vectors and m < n also $r = \dim(U) < m$.

By above theorem the $\dim A(U) = n - r$ this number is nothing but no. of elements in the basis of $A(U)$ which are non-zero vectors.

Hence, the proof.

EXERCISE :

1. If $S, T \in Hom(V, W)$ and $S(v_i) = T(v_i)$ for all elements v_i of a basis of V, prove that S = T.

2. If V is finite dimensional and $v_1 \neq v_2$ are in V, prove that there is an $f \in \hat{V}$ such that $f(v_1) \neq f(v_2)$.

3. If F is the field of real numbers, find A (W), where W is spanned by (1, 2, 3) and (0, 4, –1).

4. If f and g are in $\hat{V}$ such that $f(v) = 0$ implies $g(v) = 0$, prove that $g = l\,f$ for some $l \in F$.

❑ ❑ ❑

INNER PRODUCT SPACES

Definition

The vector space V over F is said to be an inner product space, if there is defined for any two vectors $u, v \in V$ an element (u, v) in F such that,

1. $(u,v)=(\overline{v,u})$;

2. $(u,u)\geq 0$ and $(u,u)=0$ is and only if u = 0

3. $(a\,u+b\,v,w)=a\,(u,w)+b\,(u,w)$ for any $u,v,w\in V$ and $a\,,b\in F$.

Note : A function satisfying the properties 1, 2, 3 is called an inner product.

Example :

1) In $F^{(n)}$ define for $u=(u_1.....u_n)$ and $v=(v_1.....v_n)$. $(u,v)=u_1\overline{v}_1+......+u_n\overline{v}_n$, this defines an inner product of $F^{(n)}$.

2) $(u,v)=2u_1\overline{v}_1+u_1\overline{v}_2+u_2\overline{v}_1+u_2\overline{v}_2$ this defines an inner product on $F^{(2)}$.

3) Let V be the set of all continuous complex valued functions on the closed unit interval [0, 1].

If $f(t),\ g(t)\in V$ define $(f(t),g(t))=\int_0^1 f(t)\,g(\overline{t})\,dt$.

Definition :

If $v\in V$ then the length of v (norm of v) written as $\|v\|$ is defined by

$$\|u\|=\sqrt{(u,v)}$$

Leema :

If $u, v \in V$ and $a, b \in F$ then

$$(a u + b v, a u + b v) = a \bar{a} (u, u) + a \bar{b} (u, v)$$

Proof : by property 3,

$$(a u + b v, a u + b v) = a (u, a u + b v) + b (v, a u + b v)$$

but $(u, a u + b v) = \bar{a} (u, u) + \bar{b} (u + v)$ and $(v, a u + b v) = \bar{a} (v, u) + \bar{b} (v, v)$

$$\therefore (a u + b v, a u + b v) = a \bar{a} (u, u) + a \bar{b} (u, v) + \bar{a} b (v, u) + b \bar{b} (v, v)$$

Hence the result.

Corollary :

$$\|a u\| = |a| \|u\|$$

Proof : $\|a u\|^2 = (a u, a u) = a \bar{a} (u, u)$ since by above Lemma. $\left(\because a \bar{a} = |a|^2 \text{ and } (u, u) = \|u\|^2\right)$

$\therefore \|a u\|^2 = |a|^2 \|u\|^2$ taking positive square roots yields $\|a u\| = |a| \|u\|$.

Lemma :

If a, b, c are real numbers such that a > 0 and $a l^2 + 2 b l + c \geq 0$ for all real number l then $b^2 \leq ac$.

Proof : Completing the squares, $a l^2 + 2 b l + c = \frac{1}{a} (a l + b)^2 + \left(c - \frac{b^2}{a} \right)$.

Since it is greater than or equal to 0 for all l, in particular this must be true for $l = -\frac{b}{a}$. Thus $c - \left(\frac{b^2}{a} \right) \geq 0$ and since a > 0 we get $b^2 \leq ac$.

Theorem : If $u, v \in V$ then $|(u, v)| \leq \|u\| \|v\|$.

Proof : If u = 0 then both $(u, v) = 0$ and $\|u\| \|v\| = 0$ so that the result is true there.

Case - I :

Suppose (for the moment) that (u,v) is real and $u \neq 0$.

We know if $u, v \in V$ and $a, b \in F$ then

$$(a u + b v, a u + b v) = a\bar{a}\,(u,u) + a\bar{b}\,(u,v) + \bar{a}b\,(v,u) + b\bar{b}\,(v,v)$$

For any real number l,

$$0 \leq (l u + v, l u + v) = l^2 (u,u) + 2(u,v)\, l + (v,v)$$

Let $a = (u,u)$, $b = (u,v)$ and $c = (v,v)$ for these the hypothesis $\Rightarrow b^2 \leq ac$.

That is $(u,v)^2 \leq (u,u)(v,v)$; from this it is immediate that $(u,v) \leq \|u\|\|v\|$.

Case - II :

If $a = (u,v)$ is not real then it certainly is not 0 so that $\dfrac{u}{a}$ is meaningful.

Now $\left(\dfrac{u}{a}, v\right) = \dfrac{1}{a}\cdot(u,v) = \dfrac{1}{(u,v)}\cdot(u,v) = 1$ and it is certainly real.

Therefore by Case I, $1 = \left|\left(\dfrac{u}{a}, v\right)\right| \leq \left\|\dfrac{u}{a}\right\|\|v\|$.

$\therefore \left\|\dfrac{u}{a}\right\| = \dfrac{1}{|a|}\|u\|$ we get $1 \leq \dfrac{\|u\|\|v\|}{|a|}$.

Whence $|a| \leq \|u\|\|v\|$, putting $a = (u,v)$ we obtain $|(u,v)| \leq \|u\|\|v\|$ the desired result.

Example 1 : If $V = F^{(n)}$ with $(u,v) = u_1\bar{v}_1 + \ldots\ldots + u_n\bar{v}_n$ where $u = (u_1 \ldots\ldots u_n)$, $v = (v_1 \ldots\ldots v_n)$

then $$|u_1\bar{v}_1 + \ldots + u_n\bar{v}_n|^2 \leq \left(|u_1|^2 + \ldots + |u_n|^2\right)\left(|v_1|^2 + \ldots + |v_n|^2\right)$$

Example 2 : $$\left|\int_0^1 f(t)\overline{g(t)}\,dt\right|^2 \leq \int_0^1 |f(t)|^2 dt \int_0^1 |g(t)|^2 dt$$

Definition : If $u, v \in V$ then u is said to be orthogonal to v if $(u, v) = 0$.

Note : If u is orthogonal to v then v is orthogonal to u, for $(v,u) = \overline{(u,v)} = (\bar{0}) = 0$.

Definition : If *W* is subspace of V, the orthogonal complement of *W*, $W^{\perp}$ is defined by

$$W^{\perp} = \{x \in V \mid (x, w) = 0 \forall w \in W\}$$

Leema : $W^{\perp}$ is a subspace of V..

If $a, b \in W^{\perp}$ then for all $a, b \in w$ and

for $w \in W, (a\,a + b\,b, w) = a\,(aw) + b\,(bw)) = 0$

$\therefore aa + bb \in W^{\perp}$

Note : $W \cap W^{\perp} = (0)$ for if $w \in W \cap W^{\perp}$ it must be self orthogonal. if $(w, w) = 0$

$\therefore \|w\| = 0 \Rightarrow w = 0$

Example : W be subspace of V, $W^{\perp}$ is orthogonal complement of W which also subspace of V, then

$$V = W + W^{\perp}$$

(i) $V = W + W^{\perp}$ (ii) $W \cap W^{\perp} = \{0\}$

i) There exists an orthogonal basis $(w_1, w_2, ..., w_r)$ of *w* which is a part of an orthogonal basis $(w_1 ... w_r w_{r+1} ... w_n)$ of V so that

$$\langle w_i w_j \rangle = d_{ij} \begin{cases} 1 \text{ if } i = j \\ 0 \text{ if } i \neq j \end{cases}, \Rightarrow w_{r+1} ... w_n \in w^{\perp}$$

Let $v \in V$ be arbitrary and hence $\exists$ unique scalars $a_1 .... a_n$ such that

$$v = \sum_{i=1}^{n} a_i w_i = \sum_{i=1}^{r} a_i w_i + \sum_{i=r+1}^{n} a_i w_i \text{ taking } u = \sum_{i=1}^{r} a_i w_i, w = \sum_{i=r+1}^{n} a_i w_i$$

$\therefore v = u + w$ $u \in W, w \in W^{\perp}$

also this representation is unique for the scalars $a_1 ..... a_n$.

Thus $v = W + W^{\perp}$

ii) Let $u \in W \cap W^{\perp}$ be arbitrary than $u \in W$, $u \in W^{\perp}$.

$\Rightarrow (u,u) = 0 \Rightarrow \|u\| = 0 \Rightarrow u = 0$

$\therefore W \cap W^{\perp} = \{0\}$

Definition : The set of vectors $\{v_i\}$ in V is an orthonormal set if

(i) each v_i is of length 1, (if $(v_i, v_i) = 1$).

(ii) for if $i \neq j$, $(v_i, v_j) = 0$.

Lemma : If $B = \{v_1, v_2, ...v_n\}$ is an orthogonal set then the vectors in B are linearly independent.

If $w = \sum_{r=1}^{n} a_i v_i$ then $a_i = (w, v_i) v_i$.

Proof : Suppose that $a_1 v_1 + ... + a_n v_n = 0$.

$\therefore (a_1 v_1 + ... + a_n v_n, v_i) = 0 \Rightarrow a_1 (v_1, v_i) + ... + a_n (v_n, v_i) = 0$ since $(v_j, v_i) = 0$

For $j \neq i$ while $(v_i, v_i) = 1$ this equation reduces to $a_i = 0$.

Thus the v_j's arelinearly independent.

If $w = a_1 v_1 + ... + a_n v_n$ then

$(w, v_i) = (a_1 v_1 + ... + a_n v_n, v_i) = a_1 (v_1, v_i) + ... + a_n (v_1, v_n)$

$= a_i \qquad \because (v_j, v_i) = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$

$\Rightarrow a_i = (w, v_i)$.

Lemma : If $\{v_i ... v_n\}$ is an orthogonal set in V and if $w \in V$ then

$u = w - (w, v_1) v_1 - (w, v_2) .... - (w, v_i) v_i .... (w, v_n) v_n$ is orthogonal to each of $v_1, ..., v_n$.

Proof : Let $(u,v_i)=(w-(w,v_1)v_1\dots(w,v_n)v_n\ ,v_i)$

$$=(w,v_i)-(w,v_1)(v_1,v_i)\dots.(w,v_i)(v_i,v_i)\dots.(w,v_n)(v_n,v_i)$$

$$=(w,v_i)-(w,v_i) \qquad \because (v_j,v_i)=\begin{cases}0 & i\neq j\\ 1 & i=j\end{cases}$$

$= 0$ and v is arbitrary.

$\Rightarrow \therefore u$ is orthogonal to each $v_1\dots.v_n$.

Theorem :

Let V be a finite dimensional inner product space, then V has an orthogonal set as a basis.

Proof : Let V be of finite dimension n over F and let $v_1, v_2, \dots., v_n$ be a basis of V. Now from this basis we shall construct an orthogonal set of n vectors

Let $u_1 = v_1$

$$u_2 = v_2-(v_2,u_1)\frac{u_1}{\|u_1\|^2} \qquad \text{linear space of } u_2,u_1.$$

$$u_3 = v_3-(v_3,u_2)\frac{u_2}{\|u_2\|^2}-(v_3,u_1)\frac{u_1}{\|u_1\|^2} \qquad \text{linear space of } u_3,u_1,u_2.$$

$\vdots$

$$u_{i+1}=v_{i+1}-\sum_{j=1}^{i}(v_{i+j},u_j)\frac{u_j}{\|u_j\|^2} \qquad \text{linear space of } u_{i+1},u_1\dots u_i.$$

Now $$(u_1,u_2)=\left(v_1,v_2-(v_2,v_1)\frac{v_1}{\|v_1\|^2}\right)=(v_1,v_2)-\frac{(v_2,v_1)}{\|v_1\|^2}(v_1,v_1)$$

$$=(v_1,v_2)-(v_2,v_1)\frac{\|v_1\|^2}{\|v_1\|^2}=0$$

$$(u_2,u_3)=\left(v_2-(v_2,u_1)\frac{u_1}{\|u_1\|^2},v_3-(v_3,u_2)\frac{u_2}{\|u_2\|^2}-(v_3,u_1)\frac{u_1}{\|u_1\|^2}\right)$$

$$=(v_2,v_3)-\frac{(v_3,u_2)}{\|u_2\|^2}(u_2,u_2)-\frac{(v_3,u_1)}{\|u_1\|^2}(v_2,u_1)-\frac{(v_2,u_1)}{\|u_1\|^2}(u_1,u_3)$$

$$+\frac{(v_2,u_1)}{\|u_1\|^2}\frac{(v_3,u_2)}{\|u_2\|^2}(u_1,u_2)+\frac{(v_2,u_1)}{\|u_1\|^2}\frac{(v_3,u_1)}{\|u_1\|^2}(u_1,u_1)$$

$$=(v_2,v_3)-\frac{\left(v_3,v_2-(v_2,u_1)\frac{u_1}{\|u_1\|^2}\right)}{\|u_1\|^2}\left(u_2,v_2-(v_2,u_1)\frac{u_1}{\|u_1\|^2}\right)-\frac{(v_3,u_1)(v_2,u_1)}{\|u_1\|^2}$$

$$=(u_2,v_3)-\left(\frac{(u_3,v_2)}{\|u_2\|^2}-\frac{(v_3,u_1)(v_2,u_1)}{\|u_1\|^2\|u_2\|^2}\right)-\left((v_2,u_2)-\frac{(v_2,u_1)(v_2,u_1)}{\|u_1\|^2}\right)-\frac{(v_3,u_1)(v_1,u_1)}{\|u_1\|^2}$$

$$=(u_2,v_3)-(v_3,v_2)+\frac{(v_3,u_1)(v_2,u_1)}{\|u_1\|^2}-\frac{(v_3,u_1)(v_2,u_1)}{\|u_1\|^2}$$

$= 0$

$\therefore u_1,u_2,\ldots,u_n$ are orthogonal.

$$w_1=\frac{u_1}{\|u_1\|},\ w_2=\frac{u_2}{\|u_2\|},\ldots,\ w_n=\frac{u_n}{\|u_n\|}$$

$\therefore \{w_1,w_2,\ldots,w_n\}$ is orthogonal set which is linearly independent and dim (V) = no. of elements in this set. Therefore, it forms basis.

Linear Transformations

1) HOM (V, V) : the set of all vector space homomorphisms of V into itself.

2) HOM (V, V) forms a vector space over F under the operations addition and scalar multiplication defined as

$T_1,T_2\in HOM(V,V)$, then $(T_1+T_2)(v)=T_1(v)+T_2(v)$, $\forall u\in V$ and for $a\in F$,

$$(aT_1)(v)=a\,(T_1(v))$$

Example :

1) For $T_1,T_2\in HOM(V,V)$ and $T_1(v)\in V$ for any $v\in V$ then show that

$$T_1T_2\in HOM(V,V).$$

Solution : Let $T_1, T_2 \in HOM(V, V)$ we define $T_1T_2(v) = T_1(T_2(v))$ for any $v \in V$.

Let $a, b \in F$ and $u, v \in V$ to show that $(T_1T_2)(au + bv) = a(T_1T_2(u)) + b(T_1T_2(v))$.

Consider $T_1T_2(au + bv) = T_1[T_2(au + bv)] = T_1[aT_2(u) + bT_2(v)]$

$$= T_1(aT_2(u)) + T_1(bT_2(v)) = a(T_1(T_2(u))) + b(T_1(T_2(v)))$$

$$= a[(T_1T_2)(u)] + b[(T_1T_2)(v)]$$

Thus $T_1T_2 \in HOM(V, V)$.

2) $(T_1 + T_2)T_3 = T_1T_3 + T_2T_3$

Solution : Let $a, b \in F$ and $u, v \in V$.

$$(T_1 + T_2)T_3(au + bv) = (T_1 + T_2)[T_3(au + bv)] = (T_1 + T_2)[aT_3(u) + bT_3(v)]$$

$$= T_1[aT_3(u) + bT_3(v)] + T_2[aT_3(u) + bT_3(v)]$$

$$= aT_1T_3(u) + bT_1T_3(v) + aT_2T_3(u) + bT_2T_3(v)$$

$$= a[T_1T_3(u) + T_2T_3(u)] + b[T_1T_3(v) + T_2T_3(v)]$$

$$= a[(T_1T_3 + T_2T_3)(u)] + b[(T_1T_3 + T_2T_3)(v)]$$

$$= a[(T_1 + T_2)T_3](u) + b[(T_1 + T_2)T_3](v)$$

$\therefore (T_1 + T_2)T_3 = T_1T_3 + T_2T_3$

3) $T_3(T_1 + T_2) = T_3T_1 + T_3T_2$ same as above.

4) $T_1(T_2, T_3) = (T_1, T_2)T_3$

Solution : $T_1(T_2, T_3)(au + bv) = T_1[(T_2, T_3)(au + bv)] = T_1[a(T_2, T_3)(u) + b(T_2, T_3)(v)]$

$$= a(T_1, T_2, T_3)(u) + b(T_1, T_2, T_3)(v)$$

$$= a(T_1, T_2)T_3(u) + b(T_1, T_2)T_3(v)$$

$$= (T_1, T_2)[a T_3(u)] + (T_1, T_2)[b T_3(v)]$$

$$= (T_1, T_2) T_3(a u) + (T_1, T_2) T_3(b v)$$

$$= (T_1, T_2) T_3(a u + b v)$$

5) $a\,(T_1, T_2) = (aT_1)T_2 = T_1(aT_2)$

Solution : $a\,(T_1, T_2)(u) = (aT_1)(T_2(u)) = (T_1)(aT_2(u)) = T_1(aT_2)(u)$

$\therefore$ From properties 1, gives clouser property w.r. to multplication 2, 3, 4 give HOM (V, V) an associative ring.

and $I \in HOM(V,V)$ defined as $Iv = v, \forall v \in V$ and

$TI = IT = T$ for every $T \in HOM(V,V)$.

$\therefore HOM(V,V)$ is ring with unity.

Definition : An associative ring A is called an algebra over F if A is a vector space over F such that $\forall a, b \in A$ and $a \in F$

$$a\,(ab) = (a\,a)b = a(a\,b)$$

Note : HOM (V, V) is an algebra over F. We denote if A (V) and whenever we want to emphasize the role of the field F. We shall denote it by A_F (V).

Definition : A linear transformation on V over F is an element of A_F (V). A (V) is the ring or algebra of linear transformations on V.

Lemma : If A is an algebra with unit element over F, then A is isomorphic to a subalgebra of A (V) for some vector sapce V over F.

Proof : Since A is an algebra over F it must be a vector space over F. We shall use V = A to prove the lemma.

If $a \in A$ let $T_a : A \to A$ be defined by $T_a(v) = va$ for every $v \in A$.

We assert that T_a is a linear transformation on V (= A).

By the right distribution law.

$$T_a\,(u_1+v_2)=(v_1+u_2)a=v_1a+u_2a$$

$$=T_a\,(v_1)+T_a\,(v_2)$$

Therefore, A is an algebra $T_a\,(a\,v)=(\,av)\,a=a\,(va)=a\,(T_a\,(v))$, for $v\in A$, $a\in F$.

Thus T_a is indeed a linear transformation on A. Consider the mapping $y\,:A\to A(V)$ defined as $y\,(a)=T_a$ for every $a\in A$.

Claim : y is an isomorphism of A into A (V).

a = b

$T_a=T_b\Rightarrow y\,(a)=y\,(b)$ y is well deined.

If $a,b\in A$ and $a\,,b\in F$ then $\forall v\in A$.

$T_{a\,a+b\,b}(v)=v\,(a\,a+b\,b)=a\,(va)+b\,(vb)$ $\because$ by left distribution law and AB algebra

$$=a\,(T_a\,(v))+b\,(T_b\,(v))=(a\,T_a+bT_b)(v)$$ T_a and T_b are L.T.

$\therefore T_{a\,a+b\,b}=a\,T_a+b\,T_b\Rightarrow y$ is a vector space homomorphous of A into A (V).

i.e. $y\,(a\,a+b\,b)=a\,y\,(a)+b\,y\,(b)$

Now $a,b\in A$.

$T_{ab}\,(v)=u\,(ab)=(va)b=T_b\,(T_a\,(v))=(T_aT_b\,)(v)$ $\because$ A is algebra associative law in A.

$\Rightarrow T_{ab}=T_aT_b\Rightarrow y\,(ab)=y\,(a)y\,(b)$

$\therefore y$ is also a ring homomorphism of A.

$\therefore y$ is a homomorphism of A as an algebra into A (V).

Now $Kev(y\,)=\{a\in A\,|\,y\,(a)=0\}$. i.e. $y\,(a)=0$ i.e. $T_a=0$ and $T_a\,(v)=0\,,\forall v\in V$.

Now V = A and A has a unit element e hence $T_a\,(e)=0$. However $0=T_a\,(e)=e_a=a$.

Providing that a = 0. The Kernel of y must consist of O.

y is one-one and $\dim(A)=\dim(A(V))$ gives y is onto

$\therefore y$ is an isomorphism of A into A (V). This completes the proof.

Lemma : Let A be an algebra with unit element over F, and suppose that A is of dimension *n* over F. Then every element in A satisfies some nontrial polynomial in F [x] of degree at most m.

Proof : Let e be the unit element of A, if $a \in A$ consider the elements $e, a, a^2, \ldots a^m$ in A. Since A is no dimensional over F, we know "If $v_1 \ldots . v_n$ is basis of V over F and if $w_1 \ldots . w_m$ in V are Linear independent over F then $m \leq n$".

Let $\dim A = m < m+1$

$\therefore$ $e, a, a^2, \ldots d^m$ be linearly dependent over F. In other words there are elements $a_0, a_1, \ldots, a_m \in F$ not all zero. Such that $a_0 e + a_1 a + \ldots . + a_m a^m = 0$. But then a satisfies the non-trivial polynomial $q(x) = a_0 + a_1 x + \ldots . + a_m x^m$ of degree at most m in F [x].

Theorem : If V is an n-dimensional vector space over F, then given any element T in A (V), there exists a non-trivial polynomial $q(x) \in f(x)$ of degree at most n^2 such that $q(T) = 0$.

Proof : As above.

Definition : V is finite dimensional, $T \in A(V)$ some polynomial $q(x)$ exist for which $q(T) = 0$ a non-trivial polynomial of lowest degree with this property $p(x)$ exists in $f(x)$ we call $p(x)$ a minimal polynomial for T over F. If T satisfies then $p(x) \mid h(x)$.

Definition : An element $T \in A(V)$ is called right invertible if there exists an $d \in A(V)$ such that *TS* = 1 (1 is unit element in A (V)). Similarly we can define left in verify if $U \in A(V)$ such that *UT* = 1.

If TS = UT = 1 then T is invertible.

Example : If TS = UT = I then S = U

Solution : S = IS = (UT) S = U (TS) = UI = U.

Definition : An element T in A (V) is invertible or regular if it is both right and left invertible i.e. if there is an element $S \in A(V)$ such that ST = TS = 1 we write S as T^{-1}.

Note : An element in A (V) which is not regular is called singular.

Example : Let F be the field of real numbers and let V be $f(x)$ the set of all polynomials on x over F.

Solution : Let S be defined by $S(q(x)) = \frac{d}{dx}(q(x))$ and T by $T(q(x)) = \int_1^x q(x)\,dx$

Then where as TS = 1.

Note : An element in A (V) is right invertible but is not invertible.

Theorem : If V is finite dimensional over F, then $T \in A(V)$ is invertible if and only if the constant term of the minimal polynomial for T is not 0.

Let $p(x) = a_0 + a_1x + ... + a_kx^k$, $a_k \neq 0$ be the minimal polynomial for T over F.

If $a_k \neq 0$ since $0 = p(T) = a_kT^k + a_{k-1}T^{k-1} + ... + a_1T + a_0$ we obtain

$$1 = T\left(-\frac{1}{a_0}\left(a_kT^{k-1} + a_{k-1}T^{k-2} + ... + a_1\right)\right)$$

$\therefore S = -\frac{1}{a_0}\left(a_kT^{k-1} + ... + a_1\right)$ acts as an inverse for T.

Whence T is invertible.

Suppose on the other hand that T is invertible, let $a_0 = 0$

Then $0 = a_1T + a_2T^2 + ... + a_kT^k = \left(a_1 + a_2 + ... + a_kT^{k-1}\right)T$

Multiplying this relation from the right by T^{-1} yields

$$a_1 + a_2T + ... + a_kT^{k-1} = 0$$

Whereby T satisfy the polynomial $q(x) = a_1 + a_2x + ... + a_kx^{k-1}$ in $f(x)$.

and deg $(q(x))$ is less than that of $f(x)$ this is impossible.

$\therefore p(x)$ is minimal polynomial consequently $a_0 \neq 0$.

Hence the Theorem.

Corollary : If V is finite dimensional over F and if $T \in A(V)$ is invertible then T^{-1} is a polynomial expression in T over T.

T is invertible

$\therefore a_0 + a_1 T + ... + a_k T^k = 0$ with $a_0 \neq 0$

Then $T^{-1} = -\frac{1}{a_0}\left(a_1 + a_2 T + ... + a_k T^{k-1}\right)$

Corollary : If V is finite-dimensional over F and if $T \in A(V)$ is singular then there exists an $S \neq 0$ in A (V) such that

$ST = TS = 0$

Proof : Because T is not regular the constant term of its minimal must be 0.

i.e. $p(x) = a_1 x + ... + a_k x^k$ where $0 = a_1 T + ... + a_k T^k$

If $S = a_1 + ... + a_k T^{k-1}$ then $S \neq 0$

$\therefore a_1 + ... + a_k x^{k-1}$ is of lower degree then $p(x)$.

$\therefore ST = TS = 0$

Corollary : If V is finite-dimensional over F and if $T \in A(V)$ is right invertible, then it is invertible.

Proof : Let TU = I, if T were singular there would be on $S \neq 0$ such that $ST = 0$.

However $0 = (ST)U = S(TU) = SI = S \neq 0$ a contradiction. Thus T is regular.

i.e. T is invertible.

Theorem : If V is finite-dimensional over F then $T \in A(V)$ is singular if and only if $\exists$ a $v \neq 0$ in V such that $T(v) = 0$.

Proof : We know T is singular if and only if there is an $S \neq 0$ in A (V) such that $ST = TS = 0$.

Since $S \neq 0$ there is an element $w \in V$ such that $S(w) \neq 0$.

Let $v = S(w)$ then $T(v) = T(S(w)) = (TS)(w)$

$$= 0(w) = 0$$

We produced anon-zero vector v in V which is annihilated by T.

Conversely, if $T(v)=0$ with $v \neq 0$.

Let $\exists$ of $S \in A(V)$, $v=S(w)$ for some $w \in V$.

$\therefore 0=T(S(w))=TS(w) \Rightarrow TS=0$

$\therefore T$ is singular, i.e. T is not invertible.

Definition : If $T \in A(V)$, then the range of T, T (V) is define by

$$T(V)=\{v \mid v \in V\}.$$

Theorem : If V is finite dimensional over F then $T \in A(V)$ is regular if and only if T maps V onto V..

V is finite dimensional vector space over F.

Proof : Let $T \in A(V)$ is regular. For $v \in V$ we have $T\left(T^{-1}(v)\right)=v$.

$\therefore T(V)=V$ and hence T is onto.

Conversly suppose T is onto.

Suppose that T is not regular.

$\therefore T$ is singular then there exists a vector $v_1 \neq 0$ in V such that $T(v_1)=0$, $\because v_1 \neq 0$.

We can extend to form a basis for V as $v_1, v_2, ..., v_n$. Then every element in T(V) is a linear combination of the elements $w_1=T(v_2)$ $w_1=T(v_2)....w_n=T(v_n)$.

Therefore $\dim T(V) \leq n-1 < n = \dim V$. But then T (V) must be different from V. i.e. T is not onto a contradiction hence T must be regular.

Definition : If V is finite dimensional over F, then the rank of T is the dimension of T (V), the range of T over F.

We denote rank of T by r (T).

Note :

1) If $r(T)=\dim V$, then T is regular.

2) If $r(T)=0$ then T = 0 and so T is singular.

Lemma : If V is finite dimensional over F then for $S, T \in A(V)$.

1) $r(ST) \leq r(T)$

2) $r(TS) \leq r(T)$ (so $r(ST) \leq \min\{r(T), r(S)\}$)

3) $r(ST) = r(TS) = r(T)$ for S regular in A (V).

Proof :

1) Since $S(V) < V$ $\therefore (TS)(V) = T(S(V)) \leq T(V)$

$\therefore \dim(TS(V)) \leq \dim T(V)$ i.e. $r(TS) < r(T)$

2) Suppose that $r(T) = m$, $\therefore T(V)$ has a basis of m elements $w_1 .....w_m$.

But the $S(T(V))$ is spanned by $S(w_1), S(w_2),...., S(w_n)$.

Hence has dimension at most m.

Since $r(ST) = \dim((ST)(V)) = \dim(S(T(V))) \leq m = \dim T(V) = r(T)$

3) If S is invertible then $S(V) = V$.

$\therefore TS(V) = T(S(V)) = T(V)$

$\therefore r(ST) = \dim((TS)(V)) = \dim(T(V)) = r(T)$

On the other hand if T (V) has $w_1 ....w_m$ as a basis the regularity of S implies that $S(w_1),....., S(w_m)$ are linearly independent.

Therefore for $a_1 ....a_m \in F$, $a_1 S(w_1) + ... + a_m S(w_m) = 0$

$S(a_1 w_1) + ... + S(a_m w_m) = 0$ $\because S$ is linear.

$\Rightarrow S(a_1 w_1 + ... + a_m w_m) = 0$ multiply by S' on left.

$a_1 w_1 + ... + a_m w_m = 0$ $\because S$ is regular.

$\Rightarrow a_1 = a_2 = ... = a_m = 0$ $\therefore w_1 .....w_m$ basis of T (V)

and it spans $(ST)(V)$ they form a basis of $ST(V)$.

But then $r(ST) = \dim(ST(V)) = \dim(T(V)) = r(T)$

Corollary : If $T \in A(V)$ and if $S \in A(V)$ is regular then

$$r(T) = r\left(STS^{-1}\right)$$

Proof : By 3 above $r\left(S^{-1}T\right) = r\left(TS^{-1}\right) = r(T)$

$$\therefore r\left(STS^{-1}\right) = r\left(S\left(TS^{-1}\right)\right) = r\left(\left(TS^{-1}\right)S\right) = r(T)$$

Example : Let V and w be vector space over the field F and let T be a linear transformation from V into w. If T is invertible then the inverse function T^{-1} is a linear transformation from w onto V.

Solution : When T is one-one and onto, there is a uniquely determined inverse function T^{-1} which maps w and V. such that $T^{-1}T$ identity on V and TT^{-1} identity on W.

Claim : T^{-1} is linear i.e. to show for $a, b \in F$, $w_1, w_2 \in W$.

$$T^{-1}\left(aw_1 + bw_2\right) = aT^{-1}\left(w_1\right) + bT^{-1}\left(w_2\right)$$

Now let $w_1, w_2 \in W$, $\therefore \exists v_1, v_2 \in V$ such that $T\left(v_1\right) = w_1$, $T\left(v_2\right) = w_2$.

i.e. $v_1 = T^{-1}\left(w_1\right)$ and $v_2 = T^{-1}\left(w_2\right)$

$$T\left(av_1 + bv_2\right) = aT\left(v_1\right) + bT\left(v_2\right) = aw_1 + bw_2$$

$$\therefore T^{-1}\left(aw_1 + bw_2\right) = T^{-1}\left[T\left(av_1 + bv_2\right)\right] = av_1 + bv_2 = aT^{-1}\left(w_1\right) + bT^{-1}\left(w_2\right)$$

Characteristics Roots

V will always denote a finite dimensional vector space over a field F.

Definition : If $T \in A(V)$ then $l \in F$ is called a characteristic root (or eigen value) of T if $lI - T$ is singular.

Theorem : The element $l \in F$ is a characteristic root of $T \in A(V)$ if and only if for some $v \neq 0$ in V, $T(v) = l(v)$.

Proof : If l is a characteristic root of T then $l - T$ is singular.

We know "If V–F.D.V.S. over F then $T \in A(V)$ is singular if and only if there exists a $v \neq 0$ in V such that $T(v) = 0$." ....

$\therefore$ There is a vector $v \neq 0$ in V such that $(l - T)(v) = 0$.

$\Rightarrow l(v) - T(v) = 0 \Rightarrow T(v) = lv$

Conversely, let $T(v) = lv$ for some $v \neq 0$ in **V.**

$\therefore lv - T(v) = 0$ i.e. $(l - T)(v) = 0$ by (*) must be singular and so l is a characteristics root of T.

Lemma : If $l \in F$ is a characteristic root of $T \in A(V)$, then for any polynomial $q(x) \in F[x]$, $q(l)$ is a characteristic root of $q(T)$.

Proof : Suppose that $l \in F$ is a characteristics root of T, by above theorem there is a non-zero vector v in V such that $Tv = lv$.

Now apply T on both side, we have

$$T^2(v) = T(l(v)) = T(lv) = lT(v) = l^2 v$$

Continuing this way, we obtain $T^k(v) = l^k v \ \forall$ positive integers k.

If $q(x) = a_0 x^m + ... + a_m$, $a_i \in F$ then $q(T) = a_0 T^m + ... + a_m$ apply on v.

$$q(T)(v) = \left(a_0 T^m + a_1 T^{m-1} + ... + a_m\right)(v) = a_0 T^m(v) + ... + a_m v$$

$$= \left(a_0 l^m + a_1 l^{m-1} + ... + a_m\right) v = q(l) v$$

Thus $\left(q(l) - q(T)\right)(v) = 0$ hence by above theorem $q(l)$ is a characteristic root of $q(T)$.

Theorem : If $l \in F$ is a characteristic root of $T \in A(V)$ then l is a root of the minimal polynomial of T. In particular T only has a finite number of characteristics roots in F.

Proof : Let $p(x)$ be the minimal polynomial over F or T, thus $p(T) = 0$.

If $l \in F$ is a characteristic root of T, there is a $v \neq 0$ in V with $T(v) = lv$.

As we know "If $l \in F$ is characteristic root of $T \in A(V)$ then for any polynomial $q(x) \in F[x]$, $q(l)$ is a characteristic root of $q(T)$."

Therefore, we have $p(T)(v) = p(l)v$ but $p(T) = 0$, which implies that $p(l)v = 0$, $\because v \neq 0$ by property of vector space we must have $p(l) = 0$. Therefore l is a root of $p(x)$. Since $p(x)$ has only a finite number of roots (in fact $\therefore \log p(x) \leq n^2$ where $\dim V = n^2$, $p(x)$ has at most n^2 roots) in F, there can only be a finite number of characteristic rots of T in F.

Lemma : If $T, S \in A(V)$ and if S is regular, then T and STS^{-1} have the same minimal polynomial.

Let $T, S \in A(V)$ and S is reguar then we have

$$(STS^{-1})^2 = STS^{-1}STS^{-1} = STITS^{-1} = ST^2S^{-1}$$

$$(STS^{-1})^3 = STS^{-1}STS^{-1}STS^{-1} = STITITS^{-1} = ST^3S^{-1}....$$

$$(STS^{-1})^k = ST^kS^{-1}$$

Now for any $q(x) \in F[x]$, $q(STS^{-1}) = Sq(T)S^{-1}$

$\therefore$ if $q(x) = a_0 + a_1x + ... + a_mx^m$

$$q(STS^{-1}) = a_0 + a_1STS^{-1} + ... + a_m(STS^{-1})^m$$

$$= a_0 + a_1STS^{-1} + ... + a_mST^mS^{-1}$$

$$= Sa_0S^{-1} + Sa_1TS^{-1} + ... + Sa_mT^mS^{-1}$$

$$= S(a_0S^{-1} + a_1TS^{-1} + ... + a_mT^mS^{-1}) \qquad \because S \text{ is linear}$$

$$= S(a_0 + a_1T + ... + a_mT^m)S^{-1} \qquad \because S^{-1} \text{ is linear}$$

$$= Sq(T)S^{-1}$$

In particular if $q(T) = 0$ then $q(STS^{-1}) = 0$.

Thus if $p(x)$ is the minimal polynomial for T then it follows easily that $p(x)$ is also the minimal polynomial for STS^{-1}.

Hence the proof.

Definition : The element $0 \neq v \in V$ is called a characteristic vector of T. Belonging to the characteristic root $l \in F$ if $T(v) = lv$.

Theorem : If $l_1....l_k$ in F are distinct characteristic roots of $T \in A(V)$ and if $v_1....v_k$ are characteristic vectors of T belonging to $l_1....l_k$ respectively, then $v_1....v_k$ are linearly independent over F.

Proof : If k =1 the result trivialy true.

Therefore one assume that k > 1

Suppose $v_1....v_k$ are linearly dependent over F then there is a relation of the form $a_1v_1+...+a_kv_k=0$ where $a_1....a_k \in F$ and not all of them are o. In all such relations, there is one having as few non-zero coefficients as possible.

By suitable renumbering the vectors we can assume this shortest relation to be

$$b_1v_1+...+b_jv_j=0 \qquad b_1 \neq 0....b_j \neq 0 \qquad .... (1)$$

We know that $T(v_i)=l_iv_i$ so applying T to equation (1) we obtain

$$(l_2-l_1)b_2v_2+....+(l_j-l_1)b_jv_j=0$$

Now $l_i - l_1 \neq 0$ for i > 1 and $b_1 \neq 0$ whence $(l_i - l_1)b_i \neq 0$.

But then we have produced a shorter relation than that in (1) between $v_1....v_k$. This contradiction proves the theorem.

Corollary : If $T \in A(V)$ and if $\dim V = n$ then T can have at most n distinct characteristic roots in F.

Proof : Any set of linearly independent vectors in V can have at most n elements. Since any set of distinct characteristic roots of T by above theorem gives rise to a corresponding set of linearly independent characteristic vectors which is at most n.

Corollary : If $T \in A(V)$ and $\dim V = n$ if and if T has n distinct characteristic roots in F then there is a basis of V over F which consists of characteristic vectors of T.

Matrices

Let V be a n-dimensional vector space over F and let $v_1....v_n$ be basis of V over F.

If $T \in A(V)$ then T is determined on any vector as soon as we know its action on a basis of V.

$Tv_1, Tv_2, ..., Tv_n$ are the elements of V..

Each of these can be written as a linear combination of $v_1....v_n$ unique way.

Thus $T(v_1)=a_{11}v_1+a_{12}v_2+....+a_nv_n$, $T(v_2)=a_{21}v_1+a_{22}v_2+....+a_{2n}v_n$

$...T(v_n)=a_{n1}v_1+a_{n2}v_2+....+a_{nn}v_n$

This system can be written more compactly as

$$T(v_i) = \sum_{j=1}^{n} a_{ij} v_j \qquad \text{for i = 1,..., n}$$

The set of n² numbers $a_{ij} \in F$ completely discribes T.

Definition : Let V be an n-dimensional vector space over F and let $v_1 .... v_n$ be a basis of V over F. If $T \in A(V)$ then the matrix of T in the basis $v_1 .... v_n$ witten as m (T) is

$$m(T) = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & \vdots & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \qquad \text{where } T(v_i) = \sum_{j=1}^{n} a_{ij} v_j$$

Example : Let F be a field and V be the set of all polynomials in x of degree n–1 or less over F. On V let D be defined by

$$D\left(b_0 + b_1 x + ... + b_{n-1} x^{n-1}\right) = b_1 + 2b_2 x + ... + (n-1) b_{n-1} x^{n-2}$$

(it is called differentiation operator)

1) Show that D is L.T. on V.

2) Find m (D) w.r.t. basis $1, x, x^2, ..., x^{n-1}$

Solution :

1) $a, b \in F, \; p(x), q(x) \in V$

$a\, p(x) + b\, q(x) \in V$

$$D\left(a\, p(x) + b\, q(x)\right) = D\left(a\, p(x)\right) + D\left(b\, q(x)\right)$$

$$= D(a)\, p(x) + a\, D\left(p(x)\right) + D(b)\, q(x) + b\, D\left(q(x)\right)$$

$$= a\, D\left(p(x)\right) + b\, D\left(q(x)\right) \qquad \because D(a) = 0 = D(b)$$

$\therefore$ D is linear transformation.

2) The basis for V is $1, x, x^2, \ldots, x^{n-1}$

$\therefore D(v_1) = D(1) = 0$, $D(v_2) = D(x) = 1 = 0v_1 + 0v_2 + \ldots. + 0v_n$

$$= \sum_{i=1}^{n} ov_i$$

$D(v_3) = D(x^2) = 2x = 0v_1 + 2v_2 + 0v_3 + \ldots. + ov_n$

$\vdots$

$D(v_i) = D(x^{i-1}) = (i-1)x^{i-2} = 0v_1 + \ldots. + 2v_{i-2} + (i-1)v_{i-1} + 0v_i + \ldots + 0v_1$

$\vdots$

$D(v_n) = D(x^{(n-1)}) = (n-1)x^{n-2} = 0v_1 + \ldots. + 2v_{n-2} + (n-1)v_{n-1} + 0v_n$

gives basis

$$\therefore m(D) = \begin{bmatrix} 0 & 0 & 0 & --- & 0 & 0 \\ 1 & 0 & 0 & --- & 0 & 0 \\ 0 & 2 & 0 & --- & 0 & 0 \\ 0 & 0 & 3 & --- & 0 & 0 \\ \vdots & & & & & \\ 0 & 0 & 0 & --- & n-1 & 0 \end{bmatrix}$$

2) Find $m(D)$ for a basis $w_1 = x^{n-1}, w_2 = x^{n-2}, \ldots, w_n = 1$

Solution : Now $D(w_1) = D(x^{n-1}) = (n-1)x^{n-2} = 0w_1 + (n-2)w_2 + 0w_3 + \ldots + 0w_n$

$D(w_2) = D(x^{n-2}) = (n-2)x^{n-3} = 0w_1 + 0w_2 + (n-2)w_3 + \ldots + 0w_n$

$\vdots$

$D(w_i) = D(x^{n-i}) = (n-i)x^{n--i-1} = 0w_1 + \ldots. + 0w_i + (n-i)w_{i+1} + 0w_{i+2} + \ldots + 0w_n$

$\vdots$

$D(w_n) = D(x) = 0 = 0w_1 + \ldots. + 0w_n$

$$\therefore m(D)=\begin{pmatrix} 0 & (n-1) & 0 & 0 & 0 & 0 \\ 0 & 0 & (n-2) & 0 & 0 & 0 \\ \vdots & & & & & \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

3) $u_1=1, u_2=1+x, u_3=1+x^2, \ldots, u_n=1+x^{n-1}$

is it basis for V over F and what is matrix for D.

Solution : $a_1u_1+a_2u_2+\ldots+a_nu_n=0$

$a_1(1)+a_2(1+x)+\ldots+a_n(1+x^{n-1})=0$

$\Rightarrow (a_1+a_2+\ldots+a_n)+a_2x+\ldots+a_nx^{n-1}=0$

This is a linear combination of $1, x, x^2, \ldots, x^{n-1}$ and it is a basis for V..

Therefore all $a_i=0$.

$\therefore u_1 \ldots u_n$ are L.I. and it forms a basis of V..

$\therefore D(u_1)=D(1)=0=0u_1+0u_2+\ldots+0u_n$

$D(u_2)=D(1+x)=1=1u_1+0u_2+\ldots+0u_n$

$D(u_3)=D(1+x^2)=2x=2x+2-2=2(1+x-1)=2(u_2-u_1)$

$$=-2u_1+2u_2+0u_3+\ldots+0u_n$$

$\vdots$

$D(u_n)=D(1+x^{n-1})=(n-1)x^{n-2}=(n-1)(u_n-u_1)$

$$=(-n+1)u_1+0u_2+\ldots+0u_{n-2}+(n-1)u_{n-1}+0u_n$$

$$\therefore m(D)=\begin{pmatrix} 0 & 0 & 0 & --- & 0 & 0 \\ 1 & 0 & 0 & --- & 0 & 0 \\ -2 & 2 & 0 & --- & 0 & 0 \\ -3 & 0 & 3 & --- & 0 & 0 \\ \vdots & & & & & \\ -(n-1) & 0 & 0 & --- & (n-1) & 0 \end{pmatrix}$$

4) Let T is linear transformation of V of n-dimensional vector space V and if T has n distinct characteristic roots then what is the matrix for T.

Solution : Let T is linear transformation on V and $l_1 \dots l_n$ be n distinct characteristic roots of T.

We know "If $T \in A(V)$ and $\dim V = n$ and if T has n distinct and have ---- roots in F, then there is a basis of V over F which consists of characteristic vectors of T."

Therefore, we can find a basis $v_1, v_2, \dots. v_n$ of V over F such that $T(v_i) = l_i v_i$.

In this basis T has a matrix

$$m(T) = \begin{pmatrix} l_1 & 0 & 0 & --- & 0 \\ 0 & l_2 & 0 & --- & 0 \\ \vdots & & & & \\ 0 & 0 & 0 & & l_x \end{pmatrix}$$

Note :

If we have a basis $v_1 \dots. v_n$ of V over F a given matrix $\begin{pmatrix} b_{11} & \cdots & b_{1n} \\ \vdots & & \\ b_{n1} & \cdots & b_{nn} \end{pmatrix}$, $b_{i,j} \in F$ gives rise to a linear transformation T defined on V by $T(v_i) = \sum_{i=1}^{n} b_{ij} v_j$ on this basis.

Thus every possible square away serves as the matrix of some linear transformation in the basis $v_1 \dots. v_n$.

Let V is an n-dimensional vector space over F and $v_1 \dots. v_n$ be basis suppose that $S, T \in A(V)$, having matrices $m(S) = (a_{ij})$, $m(T) = (b_{ij})$ in the given basis.

Show that the collection of such matrices is an algebric structure.

S = T iff $S(v) = T(v)$ for any $v \in V$.

Hence iff $S(v_i) = T(v_i)$ for any $v_1 \dots. v_n$ forming a basis of W.

Equivalents S = T if and only if $a_{ij} = b_{ij}$ for each i and j.

If S = T if and only if $m(S) = m(T)$.

Now $m(S)=(a_{ij})$ and $S(v_i)=\sum_{j=1}^{n} a_{ij}v_j$ and $T(v_i)=\sum_{j=1}^{n} b_{ij}v_j$

$\therefore (S+T)(v_i)=S(v_i)+T(v_i)=\sum a_{ij}v_j+\sum b_{ij}v_j=\sum(a_{ij}+b_{ij})v_j$

$\therefore$ We can explicitly write down $m(S+T)$ for $m(S)=(a_{ij})$, $m(T)=(b_{ij})$

This is meant by the matrix of linear transformation in a given basis, $m(S+T)=(l_{ij})$ where $l_{ij}=a_{ij}+b_{ij}$ for every i and j.

Now for $g\in F$ show that $m(gS)=(m_{ij})$ when $m_{ij}=ga_{ij}$ for every i and j.

$$m(gS)=(ga_{ij})=(m_{ij})$$

$$gS(v_i)=g\sum_{j=1}^{n} a_{ij}v_j=\sum_{j=1}^{n}(ga_{ij})v_j=S(v_i)$$

For $m(ST)$ let $ST(v_i)=S(T(v_i))=S\left(\sum_{j=1}^{n} b_{ij}v_j\right)=\sum_{j=1}^{n} b_{ij}S(v_j)$

But $S(v_j)=\sum_{k=1}^{n} a_{jk}v_k$

$$\therefore ST(v_i)=\sum_{j=1}^{n} b_{jk}v_k=\sum_{j=1}^{n}\sum_{k=1}^{n}(b_{ij}a_{jk})v_k=\left(\sum_{k=1}^{n}\sum_{S=1}^{n}(b_{ij}a_{jk})v_k\right)$$

$$=\sum_{j=1}^{n} b_{ij}(a_{j1}v_1+a_{j2}v_2+\ldots+a_{jn}v_n)$$

$$=\sum_{j=1}^{n} b_{ij}a_{j1}v_1+\sum_{j=1}^{n} b_{ij}a_{j2}v_2+\ldots.+\sum_{j=1}^{n} b_{ij}a_{jn}v_n$$

$$=(b_{i1}a_{j1}+b_{i2}a_{21}+\ldots.+b_{in}a_{n1})v_1+\ldots.+(b_{i1}a_{1n}+\ldots.+b_{in}a_{nn})v_n$$

$$=\sum_{k=1}^{n}\sum_{j=1}^{n}(b_{ij}a_{jk})v_k$$

$\therefore m(ST)=(s_{ik})$ when for i and j

$$s_{ik}=\sum_{j=1}^{n} b_{ij}a_{jk}$$

F_n : set of all n x n matrices entry from F.

It is an algebra.

i) $(a_{ij}) = (b_{ij})$ two matrix in F_n iff $a_{ij} = b_{ij}$, $\forall$ i and j

ii) $(a_{ij}) + (b_{ij}) = (l_{ij})$ where $l_{ij} = a_{ij} + b_{ij}$, $\forall$ i and j

iii) $g \in F$, $g(a_{ij}) = (m_{ij})$ where $m_{ij} = ga_{ij}$, $\forall$ i and j

iv) $(a_{ij})(b_{ij}) = (s_{ij})$ where every i and j $s_{ij} = \sum_{K=1}^{n} a_{ik} b_{kj}$

Theorem : The set of all n x n matrices over F form an associative algebra, F_n over F. If V is an n-dimensional vector space over F, then A (V) and F_n are isomorphic as algebra over F.

Proof : Let $v_1, v_2, \ldots, v_n$ be a basis of V over F, $T \in A(V)$, $T : V \to V$, $m(T)$ is the matrix of T w.r.t. the basis $v_1, v_2, \ldots, v_n$.

We deine mapping $f : A(V) \to F_n$ as $T \to m(T)$.

$f(T) = m(T)$, claim f is well defined, 1 - 1, onto

Let $S, T \in A(V)$

if S = T then $S(v) = T(v)$ for every $v \in V$.

i.e. $S(v_i) = T(v_i)$, v_i in the basis of W..

iff $\sum_{j=1}^{n} a_{ij} v_j = \sum_{j=1}^{n} b_{ij} v_j$

iff, $a_{ij} = b_{ij}$, $\forall$ i and j

i.e. $(a_{ij}) = (b_{ij})$

one-one $\Rightarrow m(S) = m(T)$, f is well defined.

$A \in F_n$, $\exists T \in A(V)$, $f(T) = A_{m \times n}$.

f is onto.

$a, b \in F$, $T, S \in A(V)$, $aT + bS \in A(V)$.

$f(aT+bS)=m(aT+bS)=a\,m(T)+b\,m(S)$

$(aT+bS)(v_i)=(aT)(v_i)+(bS)(v_i)=aT(v_i)+b\left(S(v_i)\right)$

$$=a\sum_{j=1}^{n}a_{ij}v_j+b\sum_{j=1}^{n}b_{ij}v_j=\sum_{j=1}^{n}\left(aa_{ij}+bb_{ij}\right)v_j$$

$f(ST)=f(S)+f(T)=m(ST)=m(S)m(T)$

$\therefore ST(v_j)=\sum_{j=1}^{n}g_{ij}v_j$ where $g_{ij}=\sum_{j=1}^{n}a_{ik}b_{kj}$

$$g_{ij}=\sum_{j=1}^{n}\sum_{k=1}^{n}a_{ik}b_{kj}v_j$$

$\therefore f$ is homomorphic.

Hence f is isomorphic.

Theorem : If V is n-deimensional over F and if $T\in A(V)$ has the matrix $m_1(T)$ in the basis $v_1,...,v_n$ and the matrix $m_2(T)$ in the basis $w_1,...,w_n$ of V over F. Then there is an element $C\in F_n$ such that $m_2(T)=C^{-1}m_1(T)C$.

(In fact, if S is the linear transformation of V defined by $S(v_i)=w_i$, $\forall$ i = 1, ..., n then C can be closer to be $m_1(S)$)

Proof : Let $m_1(T)=(a_{ij})$ and $m_2(T)=(b_{ij})$

Thus $T(v_i)=\sum_{j=1}^{n}a_{ij}v_j$ and $T(w_i)=\sum_{j=1}^{n}b_{ij}w_j$

Let S be the linear transformation on V defined by $S(v_i)=w_i$, $\because v_1,...,v_n$ are basis of V over F. S maps V onto V.

We know " If V is finite dimensional over F then $T\in A(V)$ is regular iff T maps V onto V.."

$\therefore$ S is regular if S is invertible in A (V).

Now $T(w_i)=\sum_{j=1}^{n}b_{ij}w_j$, $\therefore w_i=S(v_i)$ on substituting this in the expression for $T(w_i)$ we obtain

$$T\left(S\left(v_i\right)\right)=\sum_{j=1}^{n} b_{ij}\left(S\left(v_i\right)\right)$$

$$\Rightarrow TS\left(v_i\right)=S\left(\sum_{j=1}^{n} b_{ij}v_j\right)$$

$\therefore$ S is invertible this further simplifies to

$$S^{-1}\left(TS\left(v_i\right)\right)=S^{-1}S\left(\sum b_{ij}v_j\right)$$

$$\Rightarrow\left(S^{-1}TS\right)\left(v_i\right)=\sum b_{ij}v_j$$

$\therefore$ by the definition of the matrix of linear transformation in the given basis,

$$m_1\left(S^{-1}TS\right)=\left(b_{ij}\right)=m_2\left(T\right)$$

However the mapping $T\rightarrow m_1\left(T\right)$ is an isomorphic of A (V) onto F_n.

$$\therefore m_1\left(S^{-1}TS\right)=m_1\left(S^{-1}\right)m_1\left(T\right)m_1\left(S\right)=\left(m_1\left(S^{-1}\right)\right)m_1\left(T\right)m_1\left(S\right)$$

$\therefore m_2\left(T\right)=m_1\left(S^{-1}\right)m_1\left(T\right)m_1\left(S\right)$ which is exactly what is claimed in the theorem.

Example 1 :

$$\underset{A}{\begin{bmatrix}1 & 1\\1 & -1\end{bmatrix}}\underset{B}{\begin{bmatrix}1 & 1\\0 & 1\end{bmatrix}}=\begin{bmatrix}1 & 2\\1 & 0\end{bmatrix} \qquad (AB-BA)=\begin{bmatrix}-1 & 2\\0 & 1\end{bmatrix}$$

$$\underset{B}{\begin{bmatrix}1 & 1\\0 & 1\end{bmatrix}}\underset{A}{\begin{bmatrix}1 & 1\\1 & -1\end{bmatrix}}=\begin{bmatrix}2 & 0\\1 & -1\end{bmatrix} \qquad (AB-BA)^2=\begin{bmatrix}-1 & 2\\0 & 1\end{bmatrix}\begin{bmatrix}-1 & 2\\0 & 1\end{bmatrix}=\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$$

$$\underset{A}{\begin{bmatrix}1 & 1\\1 & -1\end{bmatrix}}\underset{B}{\begin{bmatrix}2 & 3\\1 & 2\end{bmatrix}}=\begin{bmatrix}3 & 5\\1 & 1\end{bmatrix} \qquad (AB-BA)=\begin{bmatrix}-2 & 6\\-2 & 2\end{bmatrix}$$

$$\underset{B}{\begin{bmatrix}2 & 3\\1 & 2\end{bmatrix}}\underset{A}{\begin{bmatrix}1 & 1\\1 & -1\end{bmatrix}}=\begin{bmatrix}5 & -1\\3 & -1\end{bmatrix} \qquad (AB-BA)=\begin{bmatrix}-2 & 6\\-2 & 2\end{bmatrix}\begin{bmatrix}-2 & 6\\-2 & 2\end{bmatrix}=\begin{bmatrix}-8 & 0\\0 & -8\end{bmatrix}$$

$$=(-8)\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$$

PROBLEMS :

1. Prove that $S \in A(V)$ is regular if and only if whenever $v_1, \ldots, v_n \in V$ are linearly independent, then $S(v_1), S(v_2), \ldots, S(v_n)$ are also linearly independent.

2. Prove that $T \in A(V)$ is completely determined by its values on a basis of V..

3. Prove that the minimal polynomial of R over F divides all polynomials satisfied by T over F.

4. If V is two-dimensional over a field F prove that every element in A (V) satisfies a polynomial of degree 2 over F.

5. Prove that give the matrix $A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 6 & -11 & 6 \end{pmatrix} \in F_3$ (where the characteristic of F is not 2), then

   (a) $A^3 - 6A^2 + 11A - 6 = 0$

   (b) There exists a matrix $C \in F_3$ such that

$$CAC^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

6. If F is of characteristic 2, prove that if F_2 it is possible to find matrices A, B such that AB – BA = 1.

UNIT - III

CANONICAL FORMS

1) Triangular Form

Definition :

The linear transformations $S, T \in A(V)$ are said to be similar if there exists an invertible element $C \in A(V)$ such that $T = CSC^{-1}$.

Definition :

The relation on A (V) defined by similarity is an equivalence relation, the equivalence class of an element will be called its similarity class.

Note :

To check the two linear transformations are similar or not is difficult. Therefore, we can use similarity class which matrix in some basis. These matrices will be called the Canonical Forms.

Definition :

The subspace W of V is invarient under $T \in A(V)$ if $T(W) \subset W$.

Lemma :

If $W \subset V$ is invarient under T, then T induces a linear transformation $\overline{T}$ on $\frac{V}{W}$ defined by $\overline{T}(v+W) = T(v)+W$. If T satisfies the polynomial $q(x) \in F[x]$ then so does $\overline{T}$. If $p_1(x)$ is the minimal polynomial for $\overline{T}$ over F and if $p(x)$ is that for T then $p_1(x) | p(x)$.

Proof : Let $W \subset V$ is inveriant under T. T is linear transformation and $\overline{V} = \frac{V}{w}$ be vector space which contain the element as $\overline{v} = v + W$ for $v \in V$.

Define, $\overline{T}(\overline{v}) = T(v) + W$

Claim $\overline{T}$ is well defined and linear transform

$v_1 + W = \overline{v}_1, \overline{v}_2 = \overline{v}_2 + W$ for $v_1, v_2 \in V$ claim $\overline{T}(\overline{v}_1) = \overline{T}(\overline{v}_2)$.

$\therefore v_1 + W = v_2 + W \Rightarrow (v_1 - v_2) + W = 0 \Rightarrow v_1 - v_2 \in W$

$T(v_1 - v_2) \in W$ $\therefore$ W is invarient under T.

$\therefore T(v_1) - T(v_2) \in W = T(v_1) + W = T(v_2) + W$

$\Rightarrow \overline{T}(\overline{v}_1) = \overline{T}(\overline{v}_2)$, hence $\overline{T}$ is well defined.

$a, b, \in F$, $a\overline{v}_1 + b\overline{v}_2 \in \overline{V}$ for $\overline{v}_1, \overline{v}_2 \in \overline{V}$.

$$\therefore \overline{T}(a\overline{v}_1 + b\overline{v}_2) = \overline{T}\left(\overline{a\overline{v}_1 + b\overline{v}_2}\right) = T(av_1 + bv_2) + W$$

$$= (aT(v_1) + bT(v_2)) + W = aT(v_1) + W + bT(v_2) + W$$

$$= a[T(v_1) + W] + b[T(v_2) + W] = a\overline{T}(\overline{v}_1) + b\overline{T}(\overline{v}_2)$$

Hence, $\overline{T}$ is linear.

Now if $\overline{v} = v + W \in \overline{V}$ then $\overline{T^2}(\overline{v}) = T^2(v) + W = T(T(v)) + W$

$$\Rightarrow \overline{T^2}(\overline{v}) = \overline{T}(T(v) + W) = \overline{T}(\overline{T}(v + W)) = (\overline{T})^2(\overline{v})$$

Thus $\overline{T^2} = \overline{T}^2$. Similarly $(\overline{T^k}) = (\overline{T})^k$ for any $k \geq 0$ ... (1)

Consequently, for any polynomial $q(x) \in F[x]$, $\overline{q(T)} = q(\overline{T})$.

$\because$ for $q(x) = a_0 + a_1 x + ... + a_n x^n$

$\therefore q(T) = a_0 + a_1 T + ... + a_n T^m$

$$\Rightarrow \overline{q(T)} = \left(\overline{a_0 + a_1 T + ... + a_n T^m}\right) = \overline{a_0} + \overline{a_1 T} + ... + \overline{a_n T^m}$$

$$= a_0 + a_1\overline{T} + ... + a_n\overline{T}^n \quad \because a_i \in F$$

$$= a_0 + a_1\overline{T} + ... + a_n(\overline{T})^n = q(\overline{T}) \quad \because (1)$$

For any $q(x) \in F[x]$ with $q(T)=0$ $\because \overline{0}$ is the zero transformation on $\overline{V}$.

$$0=\overline{q(T)}=q(\overline{T}) \quad .... (2)$$

Let $p_1(x)$ be the minimal polynomial over F satisfied by $\overline{T}$.

If $q(\overline{T})=0$ for $q(x) \in F[x]$ then $p_1(x) \mid q(x)$ ... (3)

If $p(x)$ is the minimal polynomial for T over F then $p(T)=0$.

$\therefore p(\overline{T})=0$ by (2)

$\Rightarrow p_1(x) \mid p(x)$ by (3)

Hence proof.

Theorem :

If $T \in A(V)$ has all its charactristics roots in F, then there is a basis of V in which the matrix of T is triangular.

Proof :

We prove this theorem by using induction on the dimensions of V over F.

If $\dim V = 1$ then every element in A (V) is a scalar and also the theorem is true here.

Suppose that the theorem is true for all vectr spaces over F of dimension n–1 and let V be of dimension n over F.

The linear transformation T on V has all its characteristic roots in F let $l_1 \in F$ be a characteristic root of T. There exists a non-zero vector v_1 in V such that $T(v_1) = l_1 v_1$.

Let $W = \{ a v_1 \mid a \in F \}$; W is a one-dimensional subspace of V and is invarient under T.

Let $\overline{V} = V/W$ we know "If V is finite dimensional vector space and W be subspace of V then dim $\overline{V} = \dim V - \dim W$"

$$\therefore \dim \overline{V} = \dim V - \dim W = n-1$$

Also we know 'If $W \subset V$ is invariant under T then T induces a linear transformation $\overline{T}$ on V/W defined by

$$\overline{T}(v+W) = T(v)+W$$

Also, we know that, If T satisfies the polynomial $q(x) \in F[x]$ then so does $\overline{T}$. If $p_1(x)$ is the minimal polynomial for $\overline{T}$ over F and if $p(x)$ is that for T then $p_1(x) \mid p(x)$"

T induces a linear transformation $\overline{T}$ on $\overline{V}$ whose minimal polynomial over F divides the minimal polynomial of T over F.

Thus all the roots of the minimal polynomial of $\overline{T}$, being roots of the minimal polynomial of T, must lie in F.

The linear transformation $\overline{T}$ in its action on $\overline{V}$ satisfies the hypothesis of the theorem.

Since $\overline{V}$ is (n–1) dimensional over F, by our induction hypothesis there is a basis $\overline{v}_2,...,\overline{v}_n$ of over F such that

$$\overline{T}(\overline{v}_2) = a_{22}\overline{v}_2,\ \overline{T}(\overline{v}_3) = a_{32}\overline{v}_2 + a_{33}\overline{v}_3.......$$

$$\overline{T}(\overline{v}_i) = a_{i2}\overline{v}_2 + a_{i3}\overline{v}_3 + ... + a_{ii}\overline{v}_i.....\overline{T}(\overline{v}_n) = a_{n2}\overline{v}_2 + a_{n3}\overline{v}_3 + ... + a_{nn}\overline{v}_n$$

Let $v_2, v_3,..., v_n$ be elements of V mapping into $\overline{v}_2,...,\overline{v}_n$ respectively.

Then $v_1, v_2,...., v_n$ form a basis of V..

[$\because \overline{v}_2,...,\overline{v}_n$ be a basis if they are lineary independent and $v_2, v_3,...., v_n$ maps into these elementry $\therefore v_2, v_3,...., v_n$ linearly independent. Therefore we have if $v_1, v_2,...., v_n$ linearly independent then $Tv_1, Tv_2,...., Tv_n$ linearly independent Now let $v_1, v_2,...., v_n$ and $a_1,...., a_n \in F$ such that $a_1v_1 + ... + a_nv_n = 0$. i.e. $a_1 = 0$ the this linearly independent we are throw if not then $v_1 = -a_1^{-1}(a_2v_2 + ... + a_nv_n) \Rightarrow v_1$ is the linear combination of $T(v_1) = \overline{v}_1 = -\overline{a}_1^{-1}(a_2\overline{v}_2 + ... + a_n\overline{v}_n) \in W$ a contradiction to W is invariant under T and dim W = 1]

Since $\overline{T}(\overline{v}_2) = a_{22}\overline{v}_2 \Rightarrow \overline{T}(\overline{v}_2) - a_{22}\overline{v}_2 = 0 \Rightarrow T(v_2) - a_{22}v_2 \in W$

Thus $T(v_2) - a_{22}v_2$ is a multiple of v_1 say $a_{21}v_1$ yielding, after transforming

$$T(v_2) = a_{21}v_1 + a_{22}v_2$$

Similalry, $Tv_i - a_{i2}v_2 - a_{i3}v_3......a_{ii}v_i \in W$

$$\therefore T(v_i) - a_{i1}v_1 + a_{i2}v_2 + ... + a_{ii}v_i$$

The basis $v_1,...., v_n$ of V over F provides us with a basis where every $T(v_i)$ is a linear combination of v_i and its precedessors in the basis. Therefore the matrix of T in the basis is triangular.

Theorem :

If V is n-dimensional over F and if $T \in A(V)$ has all its characteristic roots in F, then T satisfies a polynomial of degree n over F.

Proof : By previous theorem we can find a basis $v_1, ..., v_n$ of V over F such that

$T(v_1) = l_1 v_1$, $T(v_2) = a_{21} v_1 + l_2 v_2$, ... $T(v_i) = a_{i1} v_1 + a_{i2} v_2 + ... + a_{ii} v_i$ for i = 1, 2, ..., n

Equivalenty $(T - l_1) v_1 = 0$, $(T - l_2) v_2 = a_{21} v_1$....

$$(T - l_i) v_i = a_{i1} v_1 + ... + a_{ii-1} v_{i-1} \qquad \text{for i = 1, 2, ..., n}$$

Now $(T - l_1)(T - l_2)(v_2) = (T - l_1)(a_{21} v_1) = a_{21}(T - l_1) v_1 = 0$

also $(T - l_1)(T - l_2) = (T - l_2)(T - l_1)$

Continuing this type of computation yields.

$$(T - l_1)(T - l_2)....(T - l_i) v_1 = 0, \; (T - l_1)(T - l_2)....(T - l_i) v_2 = 0$$

$$....(T - l_1)(T - l_2)....(T - l_i) v_i = 0 \qquad \text{for i = n}$$

The matrix $S = (T - l_1)(T - l_2) \in ....(T - l_n)$ satisfies $S(v_1) = S(v_2) = ... = S(v_n) = 0$.

Then, since S annihilates a basis of V, S must annihilate all of V. Therefore S = 0. Consequently T satisfies the polynomial $(x - l_1)(x - l_2)....(x - l_n)$ in F [x] of degree n proves the theorem.

2) Nilpotent Transformations

Lemma :

If $V = V_1 \oplus V_2 \oplus .... \oplus V_n$ where each subspace V_i is of dimension n_i and is invariant under T, an element of A (V), then a basis of V can be found so that the matrix of T in this basis is of the form

$$\begin{pmatrix} A_1 & 0 & \cdots & 0 \\ 0 & A_2 & \cdots & 0 \\ \vdots & & & \\ 0 & 0 & \cdots & A_n \end{pmatrix}$$

where each A_i is an $n_i \times n_i$ matrix and is the matrix of the linear transformation induced by T on V_i.

Choose a basis of V as follows : $v_1^{(1)}, v_2^{(1)}, ..., v_{n_1}^{(1)}$ is a basis of V_1, $v_1^{(2)}, v_2^{(2)}, ..., v_{n_2}^{(2)}$ is a basis of V_2 and so on .... $v_1^{(n)}, v_2^{(n)}, ..., v_{n_n}^{(n)}$ is a basis of V_n. Since each V_i is invariant under T.

$T\left(v_j^{(i)}\right) \in V_i$ so is a linear combination of $v_1^{(i)}, v_2^{(i)}, ..., v_{n_i}^{(i)}$ and only these. Thus the matrix of T in the basis so chosen is of the desired form. That each A_i is the matrix of T_i, the linear transformation induced on V_i by T is clear from the very definition of the matrix of a linear transformation.

Lemma :

If $T \in A(V)$ is nilpotent, then $a_0 + a_1 T + ... + a_m T^m$, when the $a_i \in F$ is invertible if $a_0 \neq 0$.

Proof :

If S is nilpotent and $a_0 \neq 0 \in F$ a simple computation shows that

$$(a_0 + S)\left(\frac{1}{a_0} - \frac{S}{a_0^{\,2}} + \frac{S^2}{a_0^{\,3}} + ... + (-1)^{r-1}\frac{S^{r-1}}{a_0^{\,r}}\right) = 1 \qquad \text{if } S^r = 0$$

Now if $T^r = 0$, $S = a_1 T + a_2 T^2 + ... + a_m T^m$ also must satisfy $S^r = 0$.

$$S^r = \left(a_1 T + a_2 T^2 + ... + a_m T^m\right)^r = 0 \qquad \because T^r = 0$$

Thus for $a_0 \neq 0$ in F, $a_0 + S$ is invertible.

Definition :

If $T \in A(V)$ is nilpotent then k is called index of nilpotent of T if $T^k = 0$ but $T^{k-1} \neq 0$.

Theorem :

If $T \in A(V)$ is nilpotent of index of nilpotence n_1 then a basis of V can be found such that the matrix of T in this basis has of the form

$$\begin{pmatrix} M_{n_1} & 0 & \cdots & 0 \\ 0 & M_{n_2} & & 0 \\ \vdots & & \ddots & \\ 0 & 0 & \cdots & M_{n_r} \end{pmatrix}$$

where $n_1 \geq n_2 \geq ..... \geq n_r$ and where $n_1 + n_2 + ... + n_r = \dim V$.

Proof : $T \in A(V)$ is nilpotent with index of nilpotence n_1.

$\therefore T^{n_1} = 0$ but $T^{n_1 - 1} \neq 0$

We can find a vector $v \in V$ such that $vT^{n_1 - 1} \neq 0$.

Let $v, T(v), T^2(v), ..., T^{n_1 - 1}(v)$ be n_1 vectors we claim that these are linearly independent over F.

Suppose that these are linearly dependent i.e. there are scalars $a_1, a_2, ..., a_{n_i} \in F$ not all zero such that

$$a_1 v + a_2 T(v) + ... + a_{n_i} T^{n_1 - 1}(v) = 0$$

Let a_s be the first non-zero scalar, hence

$$T^{s-1}\left(a_s + a_{s+1}T + ... + a_{n_1} T^{n_1 - s}\right)(v) = 0$$

We know "If $T \in A(V)$ is nilpotent then $a_0 + a_1 T + ... + a_m T^m$ when the $a_i \in F$ is invertible if $a_0 \neq 0$"

$\therefore a_S \neq 0$ we have $a_s + a_{s+1}T + ... + a_{n_1} T^{n_1 - s}$ is invertible, and therefore $T^{n_1 - 1}(v) \neq 0$.

Thus no such non-zero a_s exists and $v, T(v), T^2(v), ..., T^{n_1 - 1}(v)$ Linearly independent over F.

Let V_1 be the subspace of V spanned by $v_1 = v, v_2 = T(v), ..., v_{n_1} = T^{n_1 - 1}(v)$.

$\therefore V_1$ is invariant under T and have basis $v_1, v_2, ..., v_{n_1}$, T can be induces the linear transformation of V_1. The matrix representation of T_1 w.r.t. the above basis is M_{n_1}.

So far we have produced the upper left-hand corner of the matrix of the theorem.

Now $n_2 \leq n_1$, $\therefore T^{n_2} \neq 0$ for $n_1 \neq n_2$ (if $n_1 = n_2$ then do above process)

We can find $u \in V$ such that $T^{n_2}(u) \neq 0$.

$\therefore$ claim, $u, T(u), ..., T^{n_2 - 1}(u)$ linearly independent and spans V_2 subspace of V and V_2 is invariant under T. Therefore T induce a linear map on V_2 whose matrix representation is M_{n_2} and so on.

Similarly we can find other.

We can get basis for V and the matrix representation of the required form.

Lemma :

If $u_0 \in V_1$ is such that $T^{n_1-k}(u) = 0$ when $0 < K \le n_1$ then $u = T^k(u)$ for some $u_0 \in V_1$.

Proof : Since $u \in V$,

$$u = a_1 v + a_2 T(v) + ... + a_k T^{k-1}(v) + a_{k+1} T^k(v) + ... + a_{n_1} T^{n_1-1}(v)$$

Thus $\quad 0 = T^{n_1-k}(u) = a_1 T^{n_1-k}(v) + ... + a_k T^{n_1-1}(v)$

However $T^{n_1-k}(v), ..., T^{n_1-1}(v)$ are linearly independent over F

Whence $a_1 = a_2 = ... = a_k = 0$ and so $u = a_{k+1} T^k(v) + ... + a_{n_1} T^{n_1-1}(v) = T^k(u_0)$

When $u_0 = a_{k+1} v + ... + a_{n_1} T^{n_1-k-1}(v) \in V$. Hence the proof.

Lemma :

There exists a subspace W of V, invariant under T, such that $V = V_1 \oplus W$.

Proof :

Let W be a subspace of V of largest possible dimension such that

(1) $V_1 \cap W = (0)$ (2) W is invariant under T.

First we show that $V = V_1 + W$.

Suppose not, then there exist an element $z \in V$ such that $z \notin V_1 + W$.

Since $T^{n_1} = 0$, there exists an integer k, 0 < k < n, such that $T^k(z) \in V_1 + W$ and such that $T^i(z) \in V_1 + W$ for i < k.

Thus $T^k(z) = u + w$ when $u \in V_1$ and where $w \in W$. But then

$$0 = T^k(z) = T^{n_1-k}\left(T^k(z)\right) = T^{n_1-k}(u) + T^{n_1-k}(w)$$

However, since both V_1 and W are invariant under T, $T^{n_1-k}(u) \in V_1$ and $T^{n_i-k}(w) \in W$.

Now since $V_1 \cap W = (0)$ this leads to

$$T^{n_1-k}(u) = -T^{n_1-k}(w) V_1 \cap W = (0) \Rightarrow T^{n_i-k}(u) = 0$$

We know "If $u \in V_1$ is nsuch that $T^{n_1-k}u = 0$ where $0 < k \le n_1$ then $u = T^k u_0$ for some $u_0 \in V_1$."

We have $T^k(u_0) = u$ for some $u_0 \in V_1$.

Therefore $T^k(z) = u + w = T^k u_0 + w$

Let $z_1 = z - u_0$ then $T^k(z_1) = T^k(z) - T^k(u_0) = w \in W$ and since W is invariant under T this yields $T^m(z_1) \in W$ for all $m \geq k$.

On the other hand if i < k, $T^i(z_1) = T^i(z) - T^i(u_0) \notin V_1 + W$ for otherwise $T^i(z)$ must fall in $V_1 + W$ comtradicting the choice of k.

Let W_1 be the subspace of V spanned by W and $z_1, Tz_1, ..., T^{k-1}z_1$, $\because z_1 \notin W$ and since $W \subset W_1$, the dimension of W_1 must be larger than that of W, moreover, since $T^k(z_1) \in W$ and since W is invariant under T, W_1 must be invariant under T.

By the maximal nature of W, there must be an elements of the form

$$w_0 + a_1 z_1 + a_2 T(z_1) + ... + a_k T^{k-1}(z_1) \neq 0 \in W_1 \cap V_1$$

Where $w_0 \in W$. Not all of $a_1, ..., a_k$ can be 0, otherwise we would have $0 \neq w_0 \in W \cap V_1 = (0)$, a contradiction.

Let a_s be the first non-zero a_s, then $w_0 + T^{s-1}(a_s + a_{s+1}T + ... + a_k T^{k-s})(z_1) \in V_1$.

Since $a_s \neq 0 \Rightarrow a_s + a_{s+1}T + ... + a_k T^{k-s}$ is invertible and its inverse say R is a polynomial in T.

Thus W and V_1 are invariant under R.

However from the above $R(w_0) + T^{k-1}(z_1) \in R(V_1) \subset V_1$

Forcing $T^{s-1}(z_1) \in V_1 + R(W) \subset V_1 + W$ since $s - 1 < k$ this is impossible, therefore $V = V_1 + W$ and Because $V_1 \cap W = (0)$, $V = V_1 \oplus W$.

Hence the proof.

Problem : Let $V = V_1 \oplus W$ where W is invariant under T when we can find a basis of V, so that matrix representation of T in this basis is of the form $\begin{pmatrix} M_{n_1} & 0 \\ 0 & A_2 \end{pmatrix}$.

Definition : The integers $n_1, n_2, ..., n_r$ are called the invariants of T.

Definition : If $T \in A(V)$ is nilpotent the subspace M of V, if dimension m which is invariant under T, is called cyclic with respect to T if

1) $T^m(M) = (0)$, $T^{m-1}(M) \neq (0)$

2) There is an element $z \in M$ such that $z, T(z), ..., T^{m-1}(z)$ form a basis of M.

Lemma : If M of dimension m, is cyclic with respect to T, then the dimension of $T^k(M)$ is m–k for all $k \leq m$.

Proof : Let M of dimension m is cyclic w.r.t. T consider $z, T(z), T^2(z), ..., T^{m-1}(z)$ be basis of M.

$$\therefore T^k(z), T^{k+1}(z), T^{k+2}(z), ..., T^{m+k-1}(z) \in T^k(M) \qquad \text{(be basis of } \in \text{)}$$

But M is cyclic w.r.t. T i.e. $T^m(m) = 0$ i.e. $T^m(z) = 0$ and $T^{m-1}(z) \neq 0$

$$\therefore T^k(z), T^{k+1}(z), ..., T^{m-1}(z) \text{ be basis of } T^k(M).$$

Therefore m – k elements $\therefore \dim\left(T^k(M)\right)$ is m – k, $\forall k \leq m$.

Example : For a nilpotent T in A (V). Find integers $n_1 \geq n_2 \geq ..... \geq n_r$ and subspaces $V_1, V_2, ..., V_r$ of V cyclic with respect to T and of dimensions $n_1, n_2, ..., n_r$ respectively such that $V = V_1 \oplus ..... \oplus V_2$. Show that these are unique integers.

Solution : Let $T \in A(V)$ is nilpotent, we suppose index of nilpotent n_1.

$$\therefore T^{n_1} = 0 \text{ and } T^{n_1 - 1} \neq 0$$

Then $v \in V$, $v, T(v), T^2(v), ..., T^{n_1-1}(v)$ is Linearly independent set in V form a subspace of V generated by these element say V_1 and $\dim V_1 = n_1$.

Not let $u \in V$ and $u \notin V_1$ and $n_2 \leq n_1$ be integer $\therefore T^{n_2-1} \neq 0$.

And $u, T(u), T^2(u), ..., T^{n_2-1}(u)$ is linearly independent set form a subspace V_2 of V generated by these elements.

Continue this process until we cover V.

Suppose at the last we get $V_1, V_2, ..., V_r$ be subspace each invariant under T and

$$V_i \cap V_j = (0),\ \forall i \neq j \ \text{ and } \ V = V_1 + V_2 + ... + V_r \Rightarrow V = V_1 \oplus ..... \oplus V_r$$

Now we shows these integer are unique.

Suppose there are other integers $m_1 \geq m_2 \geq .... \geq m_s$ and subspace $U_1, ..., U_s$ of V cyclic w.r.t. to T and of dimensions $m_1, ..., m_s$ respectively such that $V = U_1 \oplus ..... \oplus U_s$.

Claim :

$s = r$ and $m_1 = n_1, ..., m_r = n_r$

Suppose that this were not the case then there is a first integer i such that $m_i \neq n_i$ we may assume that $m_i < n_i$.

Consider $T^m V$, therefore for $V = V_1 \oplus ..... \oplus V_r$ we have

$$T^{m_i} V = T^{m_i} V_1 \oplus .... \oplus T^{m_i} V_r,\ \therefore \dim T^{m_i} V_j = n_j .....m_i,\ \text{j = 1 ..... r}$$

Therefore above

$$\therefore \dim \left(T^{m_i} V\right) \geq \left(n_1 - m_i\right) + \left(n_2 - m_i\right) + ... + \left(n_i - m_i\right) \qquad .... (1)$$

and for $V = U_1 \oplus ..... \oplus U_s$ and since $T^{m_i}\left(U_j\right) = (0)$ for $j \geq i$

$$\therefore T^{m_i}(V) = T^{m_i} U_1 \oplus T^{m_i} U_2 \oplus ...... \oplus T^{m_i} U_{i-1}$$

Thus $\quad \dim T^{m_i}(V) = \left(m_1 - m_i\right) + \left(m_2 - m_i\right) + ... + \left(m_{i-1} - m_i\right)$

$$= \left(n_1 - m_i\right) + \left(n_2 - m_i\right) + ... + \left(n_{i-1} - m_i\right) \qquad \because n_j = m_j \text{ for j < i}$$

Contradict to the equation (1) $\because n_i - m_i > 0$ thus there is unique set of integers $n_1 \geq n_2 \geq ..... \geq n_r$ such that $V = V_1 \oplus ..... \oplus V_r$.

Equivalently we have shown that the invariants of T are unique.

Theorem : Two nilpotent linear transformation are similar if and only if they have the same invariants.

Proof : The above Example has proved that if the two nilpotent linear transformations have different invariants, then they cannot be similar for their respective matrices.

$$\begin{pmatrix} M_{n_1} & & & \\ & M_{n_2} & & \\ & & \ddots & \\ & & & M_{n_r} \end{pmatrix} \text{ and } \begin{pmatrix} M_{m_1} & & & \\ & M_{m_2} & & \\ & & \ddots & \\ & & & M_{m_r} \end{pmatrix} \quad \text{cannot be similar.}$$

In the other direction, if the two nilpotent linear transform S and T have the same invariants $n_1 \geq ..... \geq n_r$.

We know the result "If $T \in A(V)$ is nilpotent of index of nilpotent n_1 then basis of V can be found such that the matrix of T in this basis has of the form $\begin{pmatrix} M_{n_1} & & \\ & \ddots & \\ & & M_{n_r} \end{pmatrix}$ when $n_1 \geq ..... \geq n_r$ and $n_1 + n_2 + ... + n_r = \dim V$."

Therefore there are basis $v_1, v_2, ...v_n$ and $w_1, ...w_n$ of V such that the matrix of S in $v_1, ...v_n$ and that of T in $w_1, ..., w_n$ are each equal to $\begin{pmatrix} M_{n_1} & & \\ & \ddots & \\ & & M_{n_r} \end{pmatrix}$.

But of A is the linear transformation defined on V by $A(v_i) = w_i$ then $S = ATA^{-1}$ (Prove).

Hence S and T are similar.

Example : Let $m(T) = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \in F_3$. Find similar matrix. T act on $F^{(3)}$.

i.e. Find A such that $ATA^{-1} = S$ ·

Basis of $F^{(3)}$ is $u_1 = (1,0,0)$, $u_2 = (0,1,0)$, $u_3 = (0,0,1)$

Let $v_1 = u_1$, $v_2 = T(u_1) = u_2 + u_3$, $v_3 = u_3$

w.r.t. this basis

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad ATA^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

A Decomposition of V : Jordan Form

Example : Let V be finite dimensional vector space over F, $T \in A(V)$, V subspace of V invariant under T. T induces a linear transformation T_1 on V_1 defined by $T_1(u) = T(u)$ for every $u \in V_1$. Show that for any polynomial $q(x) \in F[x]$, the linear transformation induced by $q(T)$ on V_1 is precisely $q(T_1)$. In particular $q(T) = 0$ then $q(T_1) = 0$.

Let V be finite dimensional vector space over F. $T \in A(V)$, $V_1 \subseteq V$ invariant under T.

Therefore T induces a linear transformation T_1 on V_1 defined by $T_1(u) = T(u)$ for every $u \in V_1$.

Let $q(x) \in F[x]$ be any polynomial such that $q(T) = 0$.

$p(x)$ minimal polynomial for T and $p_1(x)$ is minimal for T.

$p(x) | q(x)$ we know that $p_1(x) | p(x) = p_1(x) | q(x) \Rightarrow q(T_1) = 0$

Lemma : Suppose that $V = V_1 \oplus V_2$ where $\mathbf{V}_1$ and V_2 are subspaces of V, invariant under T. Let T_1 and T_2 be the linear transformations induced by T on V_1 and V_2 respectively. If the minimal polynomial of T_1 over F is $p_1(x)$ while that of T_2 is $p_2(x)$, then the minimal polynomial for T over F is the least common multiple of $p_1(x)$ and $p_2(x)$.

Proof : If $p(x)$ is the minimal polynomial for T over F, as we know above example both $p(T_1)$ and $p(T_2)$ are zero, whence $p_1(x) | p(x)$ and $p_2(x) | p(x)$. But then the least common multiple of $p_1(x)$ and $p_2(x)$ must also divide $p(x)$.

On the other hand if $q(x)$ is the least common multiple of $p_1(x)$ and $p_2(x)$, consider $q(T)$. For $u_1 \in V_1$ since $p_1(x) | q(x)$

$q(T)(v_1) = q(T_1)(v_1) = 0$ similarly for $v_2 \in V_2$

$q(T)(v_2) = q(T_2)(v_2) = 0$ Given any $u \in V$, v can be written as $\cos q = (v_1 + v_2)$ when $v_1 \in V_1$, $v_2 \in V_2$ in consequence of which

$$q(T)(v) = q(T)(v_1 + v_2) = q(T)(v_1) + q(T)(v_2) = 0$$

Thus $q(T) = 0$ and T satisfies $q(x)$. $\therefore p(x) | q(x)$.

$$\Rightarrow p(x) = q(x)$$

Corollary :

If $V = V_1 \oplus .....\oplus V_k$ where each V_i is invariant under K and T if $p_i(x)$ is the minimal polynomial over F of T_i the linear transformation induced by T on V_i then minimal polynomial of T over F is least common multiple of $p_1(x), p_2(x),..., p_k(x)$.

Theorem :

For each i = 1, 2, ...K, $V_i \neq (0)$ and $V = V_1 \oplus V_2 \oplus ..... \oplus V_k$.

The minimal polynomial of T_i is $q_i(x)^{\ell i}$.

Proof : We prove this result using induction on k.

If k = 1 the $V = V_1$ and there is nothing that needs proving suppoose then that k > 1 and

$$p(x) = q_1^{\ell_1}(x) q_2^{\ell_2}(x).......q_k^{\ell_k}(x)$$

We first want to prove that each $V_1 \neq (0)$. We introduce the k polynomials

$$h_1(x) = q_2(x)^{\ell_2} ....q_k(x)^{\ell_k}$$

$$h_2(x) = q_1(x)^{\ell_1} q_3(x)^{\ell_3} ....q_k(x)^{\ell_k}$$

$$\vdots$$

$$h_i(x) = \prod_{i \neq j} q_j(x)^{\ell_j}$$

$$\vdots$$

$$h_k(x) = q_1(x)^{\ell_1} ....q_{k-1}(x)^{\ell_{k-1}}$$

Since k > 1, $h_i(x) = p(x)$ whence $h_i(T) \neq 0$, thus, given i, there is a $v \in V$ such that $w = h_i(T)v \neq 0$. But

$$q_i(T)^{\ell_i}(w) = q_i(T)^{\ell_i}\left(h_i(T)v\right) = p(T)(v) = 0$$

In consequence $w \neq 0$ is in V_i and so $V_i \neq (0)$. In fact, we have shown that $h_i(T)V \neq 0$ is in V_i and if $v_j \in V_j$ for $j \neq i$ since $q_j(x)^{\ell j} \mid h_i(x)$.

$$h_i(T)\left(v_j\right) = 0$$

The polynomials $h_1(x), h_2(x), ..., h_k(x)$ are relatively prime.

We know that "taken two polynomials $f(x)$, $g(x)$ in $F[x]$ they have a greatest common divisor $d(x)$ which can be realized as $d(x) = l(x) f(x) + m(x) g(x)$."

We can find polynomials $a_1(x), a_2(x), ..., a_k(x)$ in $F[x]$ such that $a_1(x) h_1(x) + ... + a_k(x) h_k(x) = 1$.

From this we get $a_1(T) h_1(T) + ... + a_k(T) h_k(T) = 1$ whence, given $v \in V$,

$$v = v \cdot 1 = (a_1(T)) h_1(T) + ... + a_k(T) h_k(T)(v)$$

$$= a_1(T) h_1(T)(v) + ... + a_k(T) h_k(T)(v)$$

Now, each $a_i(T) h_i(v)$ is in $h_i(T)V$ and since we have shown above that $h_i(T)V \subset V_i$, we have now exhibited v as $v = v_1 + ... + v_k$ when each $v_i = a_i(T) h_1(T)(v)$ is in V_i.

Thus $V_i = V_1 + V_2 + ... + V_k$.

We must now verify that this sum is a direct sum. To show this, it is enough to prove that if $u_1 + u_2 + ... + u_k = 0$ with each $u_i = 0$. So suppose that $u_1 + u_2 + ... + u_k = 0$ and that some u_i say u_1 is not 0 apply $h_1(T)$ we obtain $h_1(T)(u_1) + ... + h_1(T)(u_k) = 0$.

However $h_1(T)(v_j) = 0$ for $j \neq i$ since $u_j \in V_j$, the equation reduced to $h_1(T)(u_1) = 0$. But $q_1(T)^{\ell_1}(u_1) = 0$ and since $h_1(x)$ and $q_1(x)$ are relatively prime we are led to $u_1 = 0$ which is of course in consistent with assumption that $u_1 \neq 0$.

$$\Rightarrow V = V_1 \oplus V_2 \oplus .... \oplus V_k$$

Now prove that the minimal polynomial of T_i on V_i is $q_i(T)^{\ell_i}$.

By definition of V_i, since $q_i(T)^{\ell_i} V_i = 0$, $q_i(T_i)^{\ell_i} = 0$ whence the minimal equation of T_i must be divisor of $q_i(x)^{\ell_i}$ thus of the form $q_i(x)^{f_i}$ with $f_i \leq \ell_i$. By "Corollary above".

The minimal polynomial of T over F is the least common multiple of $q_i(x)^{f_i}, ..., q_k(x)^{f_k}$ and so must be $q_i(x)^{f_i}, ..., q_k(x)^{f_k}$. Since this minimal polynomial is in fact $q_i(x)^{\ell_i}, ..., q_k(x)^{\ell_k}$ we must have that, $f_i \geq \ell_i, ..., f_k \geq \ell_k \Rightarrow \ell_i = f_i$ for i = 1, 2, ..., k and so $q_i(x)^{\ell_i}$ is minimal polynomial for T_i.

Hence the proof.

Note : If all the characteristic roots of T should happen to lie in F, then the minimal polynomial of T takes on the especially nice form $q(x)=(x-l_1)^{\ell_1}\ldots.(x-l_k)^{\ell_k}$ where $l_1,\ldots.,l_k$ are the distinct characteristic roots of T. The irreducible factors $q_i(x)$ are $x-l_i$. Noe that on V_i, T_i has only l_i as a characteristic root.

Corollary :

If all the distinct characteristic roots $l_i,\ldots.,l_k$ of **T** lie in F, then V can be written as $V=V_1\oplus\ldots..\oplus V_k$ where $V_i=\{u\in V \mid (T-l_i)^{\ell_i}v=0\}$ and where T_i has only one characteristic root l_i on V_i.

Note :

$\because V=V_1\oplus\ldots..\oplus V_k$ if $\dim V_i=n_i$ then we can find a basis of V such that in this basis the matrix of T is of the form $\begin{pmatrix} A_1 & & & \\ & A_2 & & \\ & & \ddots & \\ & & & A_k \end{pmatrix}$ where each A_i is an $n_i\times n_i$ matrix and is in fact the matrix of T_i.

Notation $\begin{pmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & & \vdots & \vdots \\ \vdots & & & & 0 & 1 \\ 0 & 0 & & & \cdots & 0 \end{pmatrix}_{n\times n} = M_n$

Definition : The matrix $\begin{pmatrix} l & 1 & 0 & \cdots & 0 \\ 0 & l & 1 & \cdots & 0 \\ \vdots & & & & \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & l \end{pmatrix}$ with l's on the diagonal, 1 is on the super diagonal and 0's elsewhere is a basic Jordan block belonging to l.

Theorem : Let $T \in A_p(V)$ have all its distinct roots, $l_i, \ldots, l_k$ in F. Then a basis of V can be fund in which the matrix T is of the form $\begin{pmatrix} J_1 & & & \\ & J_2 & & \\ & & \ddots & \\ & & & J_k \end{pmatrix}$ where each $J_i = \begin{pmatrix} B_{i1} & & & \\ & B_{i2} & & \\ & & \ddots & \\ & & & B_{ir} \end{pmatrix}$ and where $B_{i1}, \ldots B_i r_i$ are basic Jordan blocks belonging to l_i.

Proof : Note that an $m \times m$ basic Jordan block belonging to l is merely $l + M_m$, where,

$$M_m = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & & & & & \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}_{m \times m}$$

We know that "If $V = V_1 \oplus V_2 \oplus \ldots\ldots \oplus V_k$ where each subspace V_i is of dimension n_i and is invariant under T, an element of A (V), then a basis of V can be found so that the matrix of T in this basis is of the form $\begin{pmatrix} A_1 & 0 & \cdots & 0 \\ 0 & A_2 & \cdots & 0 \\ \vdots & & & \\ 0 & 0 & \cdots & A_k \end{pmatrix}$.

Where each A_i is an $n_i \times n_i$ matrix and is the matrix of the linear transformation induced by T on V_i."

Also we know that "If all the distinct characteristic roots $l_i, \ldots, l_k$ of T lie in F, then V can be written as $V = V_1 \oplus V_2 \oplus \ldots\ldots \oplus V_k$ where $V_i = \{v \in V \mid (T - l_i)^{\ell_i}(v) = 0\}$ and where T_i has only one charactristic root, l_i on V_i."

Therefore, we can reduce to the case when T has only one characteristic root l, that is $T - l$ is nilpotent.

Thus $T = l + (T - l)$ and since $T - l$ is nilpotent, there is a basis in which its matrix is of the form $\begin{pmatrix} M_{n_1} & & 0 \\ & \ddots & \\ 0 & & M_{n_r} \end{pmatrix}$

Therefore, "If $T \in A(V)$ is nilpotent of index of nilpotence n_1, then a basis of V can be found such that the matrix of T in this basis has the form $\begin{pmatrix} M_{n_1} & 0 & \cdots & 0 \\ 0 & M_{n_2} & \cdots & 0 \\ \vdots & & & \\ 0 & 0 & \cdots & M_{n_r} \end{pmatrix}$.

Where $n_1 \geq n_2 \geq ..... \geq n_r$ and $n_1 + n_2 + ... + n_r = \dim V$.

But then the matrix of T of the form

$$\begin{pmatrix} l & & & \\ & l & & \\ & & \ddots & \\ & & & l \end{pmatrix} + \begin{pmatrix} M_{n_1} & & \\ & \ddots & \\ & & M_{n_r} \end{pmatrix} = \begin{pmatrix} B_{n_1} & & \\ & \ddots & \\ & & B_{n_r} \end{pmatrix}$$

Using the first remark made in this proof about the relation of a basic Jordan block and the M_m's we have the required.

Example :

1) $$A = \begin{bmatrix} -2 & 5 & 1 & 0 \\ -2 & 4 & 1 & 0 \\ -1 & 2 & 1 & 0 \\ -1 & 2 & 0 & 1 \end{bmatrix}$$

Find Jordan form

Solution : The characteristic equation is $(x-1)^4$.

rank of $A - lI$ is 2.

Therefore, geometric multiplicity of equation is 2.

Hence there are two Jordan blocks.

of the form $dia\left(J_2(1), J_2(1)\right)$ or $diag\left(J_3(1), J_1(1)\right)$

The minimal polynomial is $(x-1)^3$

Therefore Jordan form $diag\left(J_3(1), J_1(1)\right)$

Example : Let $T \in A(V)$ and $F[x]$ ring of polynomials in x over F and define for any $f(x)$ in $F[x]$, $v \in V$, $f(x)v = f(T)v$. Prove that V is a module over $F[x]$.

[Let R ring $M \neq f$ is said to be an R-module if M is an abelian group under operation t such that $r \in R$ and $m \in M$ there exists an element $m \in M$ subject to (i) $r(a+b) = ra + rb$ (ii) $r(Sa) = (rS)a$ (iii) $(r+S)a = ra + Sa$, $\forall a, b \in M$ and $r_1, S \in R$]

Example : $F[x]$ is a Eucliaden ring. V is finitely generated module are $F[x]$.

V is the direct sum of a finite number of cyclic subv modules.

(On R-module M is said to be finitely generated if there exists element $a_1 .... a_n \in M$ such that every $m \in M$ is of the form $m = T_1 a_1 + ... + T_n a_n$.

M-cyclic if there is an element $m_0 \in M$ such that every $m \in M$ is of the form $m = a\, m_0$ for $a \in R$.

Problems :

1. Prove that the matrix $\begin{pmatrix} 1 & 1 & 1 \\ -1 & -1 & -1 \\ 1 & 1 & 0 \end{pmatrix}$ is nilpotent, and find its invariants and Jordan form.
2. Find all possible Jordan forms for all 8×8 matrices having $x^2(x-1)^3$ as minimal polynomial.
3. If the multiplicity of each characteristic root of T is 1, and if all the characteristic roots of T are in F, prove that T is diagonalizable over F.

❑ ❑ ❑

UNIT - IV

HERMITIAN, UNITARY AND NORMAL TRANSFORMATIONS

Fact - 1

A polynomial with coefficients which are complex numbers has all its roots in the complex field.

Fact - 2

The only irreducible, nonconstant, polynomials over the field of real numbers are either of degree 1 or of degree 2.

Lemma :

If $T \in A(V)$ is such that $(T(v), u) = 0$ for all $v \in V$, then T = 0.

Proof :

Since $(T(v), v) = 0$ for $v \in V$, given $u, w \in V$.

$$(T(u+w), u+w) = 0$$

Expansing this and use $(T(u), u) = 0$, $(T(w), w) = 0$, we obtain

$$(T(u)+T(w), u+w) = 0$$

$$(T(u), u) + (T(u), w) + (T(w), u) + (T(w), w) = 0$$

$$\therefore (T(u), w) + (T(w), u) = 0 \text{ for all } u, w \in V \quad \dots (1)$$

Since equation (1) holds for arbitrary w in V, it still must hold if we replace in it w by iw where $i^2 = -1$.

But $(T(u), iw) = -i(T(u), w)$ where as $(T(iw), u) = i(T(w), u)$.

Substituting these values in (1) and canceling i, we have

$$-(T(u), w) + (T(w), u) = 0 \quad \dots (2)$$

Adding (1) and (2) we get $(T(w), u) = 0$ for all $u, w \in V$.

Whence in particular $(T(w), T(w)) = 0$.

By the property of inner product space, we must have $T(w) = 0$ for all $w \in V$ hence T = 0.

Note : If V is an inner product space over the real field the lemma may be false.

For example, let $V = \{(a, b) \mid a, b \text{ real}\}$, where inner products is the dot product. Let T be linear transformation sending (a, b) into $(-b, a)$. This shows that $(T(v), v) = 0$ for all $v \in V$, **Yet** $T \neq 0$.

Definition : The linear transformation $T \in A(V)$ is said to be unitary if $(T(u), T(v)) = (u, v)$ for all $u, v \in V$.

Note : A unitary transformation is one which preserves all the structure of V, its addition, its multiplication by scalars and its inner product.

Note also that a unitary transformation preserves length for

$$\|v\| = \sqrt{(v, v)} = \sqrt{(T(v), T(v))} = \|T(v)\|$$

The converse is also true, which is proved in the next result.

Lemma :

If $(T(v), T(v)) = (v, v)$ for all $v \in V$ then T is unitary.

Proof :

Let $u, v \in V$ and $(T(u+v), T(u+v)) = (u+v, u+v)$

Expanding this we have

$$(T(u)+T(v), T(u)+T(w)) = (u+v, u+v)$$

$$(T(u), T(u)) + (T(u), T(v)) + (T(v), T(u)) + (T(v), T(v)) = (u, u) + (u, v) + (v, u) + (v, v)$$

Cancelling the same terms such as $(T(u), T(u)) = (u, u)$ we have

$$(T(u), T(v)) + (T(v), T(u)) = (u, v) + (v, u) \qquad \text{... (1)}$$

For $u, v \in V$. In equation (1) replace v by iv, we have

$$(T(u), T(iv)) + (T(iv), T(u)) = (u, iv) + (iv, u)$$

$$i(T(u), T(v)) + i(T(v), T(u)) = -i(u, v) + i(v, u)$$

Cancel i on both side we have,

$$-(T(u), T(v)) + (T(v), T(u)) = -(u, v) + (v, u) \qquad \text{... (2)}$$

Adding (1) and (2) results, we have

$(T(u), T(v)) = (u, v)$ for all $v \in V$.

Hence T is unitary.

Theorem :

The linear transformation T on V is unitary if and only if it takes an orthonormal basis of V into an orthonormal basis of V.

Proof :

Suppose that $\{v_1, ..., v_n\}$ is an orthonormal basis of V, thus $(v_i, v_j) = 0$ for $i \neq j$ while $(v_i, v_i) = 1$.

We will show that if T is unitary then $\{T(v_1), ..., T(v_n)\}$ is also an orthonormal basis of V. But

$$(T(v_i), T(v_j)) = (v_i, v_j) = 0 \quad \text{for} \quad i \neq j$$

and $\quad (T(v_i), T(v_i)) = (v_i, v_i) = 1$

Thus $\{T(v_1), ..., T(v_n)\}$ is an orthonormal basis of V..

On the other hand, if $T \in A(V)$ is such that both $\{v_1, ..., v_n\}$ and are orthonormal basis fo W,, if $u, w \in V$ then

$$u = \sum_{i=1}^{n} a_i v_i \qquad w = \sum_{i=1}^{n} b_i v_i$$

$$\therefore (u, w) = \left(\sum_{i=1}^{n} a_i v_i, \sum_{i=1}^{n} b_i v_i \right)$$

$$= (a_1 v_1 + ... + a_n v_n, b_1 v_1 + ... + b_n v_n)$$

$$=(a_1v_1, b_1v_1)+(a_1v_1, b_2v_2)+....+(a_1v_1, b_nv_n)$$

$$=(a_2v_2, b_1v_1)+(a_2v_2, b_2v_2)+....+(a_2v_2, b_nv_n)+....$$

$$+(a_nv_n, b_1v_1)+....+(a_nv_n, b_nv_n)$$

$$=a_1\overline{b}_1(v_1, v_1)+a_1\overline{b}_2(v_1, v_2)+......+a_1\overline{b}_n(v_1, v_n)+.....$$

$$+a_n\overline{b}_1(v_n, v_1)+......+a_n\overline{b}_n(v_n, v_n)$$

$$=a_1\overline{b}_1+a_2\overline{b}_2+....+a_n\overline{b}_n \qquad \because (v_i, v_j)=0 \quad i\neq j$$

$$=1 \quad i=j$$

However $T(u)=\sum_{i=1}^{n} a_iT(v_i)$ and $T(w)=\sum_{i=1}^{n} b_iT(v_i)$

$$(T(u), T(w))=\left(\sum_{i=1}^{n} a_iT(v_i), \sum_{i=1}^{n} b_iT(v_i)\right)$$

$$=\sum_{i=1}^{n} a_i\overline{b_i} \qquad \because (T(v_i), T(v_j))=0 \quad i\neq j$$

$$=1 \quad i=j$$

$$=(u, w)$$

Thus T is unitary.

Lemma :

If $T\in A(V)$ then given any $v\in V$ there exists an element $w\in V$, depending on *v* and T, such that $(T(u), v)=(u, w)$ for all $u\in V$.

This element *w* is uniquely determined by *v* and T.

Proof :

To prove the Lemma, it is sufficient to exhibit a $w\in V$ which works for all the elements of a basis of V. Let $\{u_1, ..., u_n\}$ be an orthonormal basis of V, we define

$$w=\sum_{i=1}^{n}\left(\overline{T(u_i), v}\right)u_i$$

Consider $$(u_i, w) = \left(u_i, \sum_{i=1}^{n} \left(\overline{T(u_i), v}\right) u_i \right)$$

$$= \left(T(u_1), v\right)(u_i, u_1) + \left(T(u_2), v\right)(u_i, u_2) + \ldots + \left(T(u_n), v\right)(u_i, u_n)$$

$$= \left(T(u_i), v\right) \qquad \because (u_i, u_j) = 0 \quad i \neq j$$

$$= 1 \quad i = j$$

Hence the element has the desired property.

For uniqueness, consider $\left(T(u), v\right) = (u, w_1)$ and $\left(T(u), v\right) = (u, w_2)$.

$$\therefore (u, w_1) = (u, w_2)$$

$$\therefore (u, w_1) - (u, w_2) = 0$$

$(u, w_1 - w_2) = 0$ for all $u \in V$.

Thus $u = w_1 - w_2$ and therefore

$$w_1 - w_2 = 0 \Rightarrow w_1 = w_2$$

Hence, the uniqueness of w.

Definition : If $T \in A(V)$ then the Hermitian adjoint of T written as T*, is defined by $\left(T(u), v\right) = \left(u, T^*(v)\right)$ for all $u, v \in V$.

Lemma : If $T \in A(V)$ then $T^* \in A(V)$ moreover,

1. $(T^*)^* = T_i$
2. $(S l T)^* = S^* + T^*$
3. $(l S)^* = \overline{l} S^*$
4. $(ST)^* = T^* S^*$ for all $S, T \in A(V)$ and all $l \in F$.

Proof : We must first prove that T* is a linear transformation on V. If u, v, w are in V, then

$$\left(u, T^*(v+w)\right) = \left(T(u), v+w\right) = \left(T(u), v\right) + \left(T(u), w\right)$$

$$= \left(u, T^*(v)\right) + \left(u, T^*(w)\right)$$

$$= \left(u, T^*(v) + T^*(w)\right)$$

$$\therefore T^*(v+w) = T^*(v) + T^*(w)$$

Similarly, for $l \in F$,

$$\left(u, T^*(lv)\right) = \left(T(u), lv\right)$$

$$= \overline{l}\left(T(u), v\right)$$

$$= \overline{l}\left(u, T^*(v)\right)$$

$$= \left(u, lT^*(v)\right)$$

Consequently $T^*(lv) = lT^*(v)$

Thus T* is linear transformation on V.

1. Consider $\left(u, (T^*)^*(v)\right) = \left(T^*(u), v\right) = \left(\overline{v, T^*(u)}\right)$

$$= \left(\overline{T(v), u}\right) = \left(u, T(v)\right)$$

for all $u, v \in V$, whence $(T^*)^*(v) = T(v)$

Which implies that $(T^*)^* = T$.

2. Consider $\left(u, (S+T)^*(v)\right) = \left((S+T)(u), v\right) = \left(S(u) + T(u), v\right)$

$\because$ by property of linear transformation

$$= \left(S(u), v\right) + \left(T(u), v\right)$$

$$= \left(u, S^*(v)\right) + \left(u, T^*(v)\right)$$

$$= \left(u, S^*(v) + T^*(v)\right)$$

$$= \left(u, (S^* + T^*)(v)\right)$$ $\because$ by property of Linear transformations

for all $u, v \in V$ whence

$$(S+T)^*(v) = (S^* + T^*)(v)$$

Which implies that

$$(S+T)^* = S^* + T^*$$

3. Consider $\left(u,(l\,S)^*(v)\right)=\left((l\,S)(u),v\right)=l\left(S(u),v\right)$

$$= l\left(u,S^*(v)\right)=\left(u,\bar{l}\,S^*(v)\right)$$

for all $u,v\in V$ whence

$$\Rightarrow (l\,S)^*(v)=\bar{l}\,S^*(v) \qquad \forall v\in V$$

implies that $(l\,S)^*=\bar{l}\,S^*$.

4. Consider $\left(u,(ST)^*(v)\right)=\left((ST)(u),v\right)=\left(T(u),S^*(v)\right)$

$$=\left(u,T^*(S^*(v))\right)=\left(u,T^*S^*(v)\right)$$

for all $u,v\in V$ this forces

$(ST)^*(v)=T^*S^*(v)$ for every $v\in V$

which implies that $(ST)^*=T^*S^*$

Hence the proof.

Lemma : $T\in A(V)$ is unitary if and only if T*T = 1.

Proof : If T is unitary, then for all $u,v\in V$,

$$\left(u,T^*T(v)\right)=\left(T(u),T(v)\right)=(u,v)$$

hence T*T = 1

On the other hand, if T*T = 1 then

$$(u,v)=\left(u,T^*T(v)\right)=\left(T(u),T(v)\right)$$

Which implies that T is unitary.

Note :

1. A unitary transformation is non-singular and its inverse is just its Herminrian adjoint.
2. From T*T = 1, we must have that TT* = 1.

Theorem : If $\{v_1,v_2,\ldots,v_n\}$ is an orthonormal basis of V and if the matrix of $T\in A(V)$ in this basis is (a_{ij}) then the matrix of T* in this basis is (b_{ij}).

Where $b_{ij}=\bar{a}_{ji}$.

Proof : Since the matrices of T and T* in this basis are, respectively (a_{ij}) and (b_{ij}) then

$$T(v_i) = \sum_{i=1}^{n} a_{ij} v_j \quad \text{and} \quad T^*(v_i) = \sum_{i=1}^{n} b_{ij} v_j$$

Now $$b_{ij} = (T^*(v_i), v_j) = (v_i, T(v_j)) = \left(v_i, \sum_{k=1}^{n} a_{jk} v_k\right)$$

$$= (v_i, a_{j1} v_1) + \ldots + (v_i, a_{j2} v_2) + \ldots + (v_i, a_{jn} v_n)$$

$$= \bar{a}_{j1}(v_i, v_1) + \ldots. + \bar{a}_{j2}(v_i, v_2) + \ldots. + \bar{a}_{jn}(v_i, v_n)$$

$$= \bar{a}_{ji} \qquad \because (v_i, v_j) = 0 \text{ if } i \neq 1$$

$$= 1 \quad i = 1$$

This proves the theorem.

Note : The Hermition adjoint for matrices on the Hermition adjoint for transformation are explicit same.

Using the matrix representation in an orthonormal basis use claim that $T \in A(V)$ is unitary if and only if, whenever (a_{ij}) is the matrix of T in this orthonormal basis, then

$$\sum_{i=1}^{n} a_{ij} \bar{a}_{ik} = 0 \text{ for } j \neq k$$

while $$\sum_{i=1}^{n} |a_{ij}|^2 = 1$$

Definition : $T \in A(V)$ is called self-adjoint or Hermintian if T* = T.

Definition : $T \in A(V)$ is called Skew-Hermitian if T* = – T.

Note : $S \in A(V)$ can be written in the form

$$S = \frac{S + S^*}{2} + i\left(\frac{S - S^*}{2i}\right)$$

Since $\frac{(S+S^*)}{2}$ and $\frac{(S-S^*)}{2i}$ are Hermitian

$\therefore S = A + iB$ when both A, B are Hermitrian.

Theorem : If $T \in A(V)$ is Hermitian, then all its characteristics roots are real.

Proof : Let l be a characteristic root of T, thus there is a $v \neq 0$ in V such that $T(v) = lv$.

We compute

$$l\,(v,v) = (lv,v) = (T(v),v) = (v,T^*(v))$$

$$= (v,T(v)) = (v,\,lv) = \bar{l}\,(v,v)$$

Since T is Hermition and $(v,v) \neq 0$.

We have $l = \bar{l}$.

Hence l is real.

Lemma : If $S \in A(V)$ and $S^*S\,(v) = 0$ then $S(v) = 0$.

Proof : Consider $(S^*S(v), v)$, since $S^*S\,(v) = 0$,

$$0 = (S^*S(v), v) = (S(v), S(v))$$

Impplies that $S(v) = 0$, therefore by definition of inner product space.

Corollary : If T is Hermitian and $T^k(v) = 0$ for $k \geq 1$ then $T(v) = 0$

Proof : We show that if $T^{2m}(v) = 0$ then $T(v) = 0$, for if $S = T^{2m-1}$, then $S^* = S$ and $S^*S = T^{2m}$.

Whence $(S^*S(v), v) = 0$ implies that $0 = S(v) = T^{2m-1}(v)$.

Continuing down in this way, we obtain $T(v) = 0$.

If $T^k(v) = 0$ then $T^{2m}(v) = 0$ for $2^m > k$ hence $T(v) = 0$.

Definition : $T \in A(V)$ is said to be normal if $T^*T = TT^*$.

Lemma : If N is a normal linear transformation and if $N(v)=0$ for $v \in V$, then $N^*(v)=0$.

Proof : Consider $(N^*(v), N^*(v))$.

Therefore by definition

$$(N^*(v), N^*(v)) = (NN^*(v), v)$$

$$= (N^*N(v), v) \qquad \because N^*N = NN^*$$

$$= (N(v), N(v))$$

However, $N(v)=0$, whence certainly $N^*N(v)=0$.

Thus, we obtain that $(N^*(v), N^*(v)) = 0$.

This forcing that $N^*(v)=0$.

Corollary : If l is a characteristic root of the normal transformation N and if $N(v)=lv$ then $N^*(v)=\bar{l}v$.

Proof : Since N is normal, $NN^* = N^*N$, therefore

$$(N-l)(N-l)^* = (N-l)(N^*-\bar{l}) = NN^* - \bar{l}N + l\bar{l}$$

$$= N^*N - lN^* - \bar{l}N + l\bar{l}$$

$$= (N^*-\bar{l})(N-l)$$

$$= (N-l)^*(N-l)$$

That is to say, $N-l$ is normal.

Since $(N-l)(v)=0$ by the normality of $N-l$; from the above lemma.

$(N-l)^*(v)=0$ hence $N^*(v)=\bar{l}v$

Hence the required.

Corollary : If T is unitary and if l is a characteristic root of T, then $|l|=1$.

Proof : Since T is unitary it is normal.

Let l be a characteristic root of T and suppose that $T(v)=lv$ with $v \neq 0$ in V. By previous corollary $T^*(v)=\bar{l}v$.

Thus $T^*T(v) T^*(lv) = lT^*(v) = l\bar{l}(v)$

Since $T*T = 1$ we have $v = l\,\bar{l}\,v$.

Thus we get $l\,\bar{l} = 1$ which of course says that .

Hence the required.

Lemma : If N is normal and if $N^k(v) = 0$, then $N(v) = 0$.

Proof : Let $S = N*N$; S is Hermitian, and by the normality of N,

$$S*(v) = (N*N)^k(v) = (N*)^k(N)^k(v) = 0 \qquad\qquad \because N^k(v) = 0$$

By "If T is Hermitian and $T^k(v) = 0$ for $k \geq 1$ then $T(v) = 0$."

We deduce that $S(v) = 0$ that is to say $N*N(v) = 0$.

Also we know "If $S \in A(V)$ and if $S*S\,(v) = 0$ then $S(v) = 0$."

Therefore $N(v) = 0$ as required.

Corollary : If N is normal and if for $l \in F$,

$$(N - l)^k(v) = 0, \text{ then } N(v) = lv .$$

Proof : From the normality of N it follows that $N - l$ is normal, whence by applyiong the lemma just proved to $N - l$ we obtain $(N - l)^k(v) = 0$ implies $(N - l)(v) = 0$.

Which implies that $N(v) = lv$.

Lemma : Let N be a normal transformation and suppose that l and m are two distinct characteristics roots of N. If v, w are in V and are such that $N(v) = lv$, $N(w) = mw$ then $(v, w) = 0$.

Proof : We compute $(N(v), w)$ in two different ways as a consequence of $N(v) = lv$.

$$(N(v), w) = (lv, w) = l\,(v, w) \qquad\qquad \text{... (1)}$$

From $N(w) = mw$

We know that "If l is a characteristic root of the normal tranformation N and if $N(v) = lv$ then $N*(v) = \bar{l}\,v$."

Therefore we have $N*(w) = \bar{m}w$.

Whence $\left(N(v), w\right) = \left(v, N^*(w)\right) = \left(v, \bar{m}w\right) = m\left(v, w\right)$ ... (2)

From equation (1) and (2) we have

$$l\left(v, w\right) = m\left(v, w\right)$$

and since $l \neq m$ this results in $\left(v, w\right) = 0$.

Hence the required.

Theorem : If N is a normal linear transformation ob V, then there exists orthoonormal basis consisting of characteristic vectors of N, in which the matrix of N is diagonal. Equivalently, if N is a normal matrix there exists a unitary matrix U such that is $UNU^{-1} (= UNU^*)$ diagonal.

Proof : Let N be normal and let $l_1, l_2, ..., l_k$ be the distinct characteristic roots of N. We know that "If all the distinct characteristic roots $l_1, ..., l_k$ of T lie in F, then V can be written as $V = V_1 \oplus ... \oplus V_k$ where $V_i = \left\{v \in V \,\middle|\, (T - l_i)^i (v) = 0\right\}$ and where T_i has only one characteristic root, l_i on V_i."

We can decompose V as $V = V_1 \oplus V_2 \oplus ... \oplus V_k$ where every $v_i \in V_i$ is annihilated by $\left(N - l_i\right)^{n_i}$.

We also know that "If N is normal and if for $l \in F$, $(N - l)^k (v) = 0$ then $N(v) = lv$."

Therefore V_i consists only of characteristic vectors of N belonging to the characteristic root l_i. The inner product of V induces an inner product on V_i.

We know that, "Let V be a finite dimentional inner product space, then V has an orthogonal set as a basis."

Therefore, we can find a basis of V_i orthonormal relative to this inner product.

By previous Lemma, let N be a normal transformation and suppose that l and m are two distinct characteristic roots of N. If v, w are in V and are such that $N(v) = lv$, $N(w) = mw$ then $\left(v, w\right) = 0$."

Elements lying in distinct V_i's are orthogonal.

Thus putting together the orthonormal basis of the V_i's provides us with an orthonormal basis of V. This basis consists of characteristic vector on N, hence in this basis the matrix of N is diagonal.

We know that, "The linear transformation T on V is unitary if and only if it takes an orthonormal basis of V into an orthonormal basis of V."

and

"If V is n-dimensional over F and if $T \in A(V)$ has the matrix $m_1(T)$ in the basis $v_1,...,v_n$ and the matrix $m_2(T)$ in the basis $w_1,...,w_n$ of V over F, then there is an element $C \in F_n$ such that $m_2(T) = C\left(m_1(T)\right)C^{-1}$."

These two results gives the matrix equivalence.

Corollary : If T is a unitary transformation, then there is an orthonormal basis in which the matrix of T is diagonal, equivalently, if T is a unitary matrix, then there is a unitary matrix U such that $UTU^{-1}(=UTU*)$ is diagonal.

Corollary : If T is a Hermitian linear transformation then there exists an orthonormal basis in which the matrix of T is diagonal, equivalently, if T is a Hermitian matrix, then there exists a unitary matrix U such that $UTU^{-1}(=UTU*)$ is diagonal.

Lemma : The normal transformation N is

1. Hermitian if and only if its characteristic roots are real.
2. Unitary if and only if its characteristic roots are all of absolute value 1.

Proof : We have this using matrices. If N is Hermitian, then it is normal and all its characteristic roots are real. If N is normal and has only real charactristic roots, then for some unitary matrix U, $UNU^{-1} = UNU* = D$ where, D is a diagonal matrix with real entries on the diagonal.

Thus, $D* = D$ since $D* = (UNU*)* = UN*U*$, the relation D* = D implies $UN*U* = UNU*$ and since U is invertible. We obtain N* = N. Thuis N is Hermition.

If N is unitary transformation, then its characteristic roots are all of aboslute value 1. Since "If T is unitary and if l is a characteristic root of T, then $|l| = 1$."

If N is normal and has its characteristic roots of absolute value 1.

Lemma : If N is normal and AN = NA, then AN* = N*A.

Proof : Let X = AN* – N*A, we claim that X = 0.

That is to show that for XX* = 0.

Since N commutes with A and with N*, it must commute with AM* – N*A thus

$$XX^* = (AN^* - N^*A)(NA^* - A^*N)$$

$$XX^* = (AN^* - N^*A)NA^* - (AN^* - N^*A)A^*N$$

$$= N\{(AN^* - N^*A)A^*\} - \{(AN^* - N^*A)A^*\}N$$

Being of the form NB – BN, the trace of XX* is 0. Thus X = 0 and AN* = N*A.

Lemma : The Hermition linear transformation T is nonnegative (positive) if and only if all of its characteristic roots are nonnegative (positive).

Proof : Suppose that $T \geq 0$, If l is a characteristic root of T, then $T(v) = l\,v$ for some $v \neq 0$.

Thus $0 \leq (T(v), v) = (l\,v, v) = l\,(v, v)$ since $(v, v) > 0$

We deduce that $l \geq 0$.

Conversly, if T is Hermitian with nonnegative characteristic roots, then we can find an orthonormal basis $\{v_1, ..., v_n\}$ consisting of chararteristic vectors of T. For each $v_i, T(v_i) = l_i v_i$ where $l_i \geq 0$.

Given $v \in V$, $v = \sum a_i v_i$

Hence $T(v) = \sum a_i T(v_i) = \sum l_i a_i v_i$

But $(T(v), v) = (\sum l_i a_i v_i, \sum a_i v_i) = \sum l_i a_i \bar{a}_i$

By the orthonormality of the V_i's, Since $l_i \geq 0$ and $a_i \bar{a}_i \geq 0$, we get that $(T(v), v) \geq 0$ hence $T \geq 0$.

Hence the required.

Lemma : $T \geq 0$ if and only if T = A*A for some A.

Proof : We first show that $A^*A \geq 0$. Given $v \in V$,

$$(A^*A(v), v) = (A(v), A(v)) \geq 0$$

Hence $A^*A \geq 0$.

On the other hand, if $T \geq 0$ we can find a unitary matrix U such that $UTU^* = \begin{pmatrix} l_1 & & \\ & \ddots & \\ & & l_n \end{pmatrix}$.

Where each l_i is a characteristic root of T, hence each $l_i \geq 0$.

Let $S = \begin{pmatrix} \sqrt{l_1} & & \\ & \ddots & \\ & & \sqrt{l_n} \end{pmatrix}$

Since each $l_i \geq 0$, each $\sqrt{l_i}$ is real, whence S is Hermitian. Therefore, U*SH is Hermitian,, but

$$(U^*SU)^2 = U^*S^2U = U^*\begin{pmatrix} l_1 & & \\ & \ddots & \\ & & l_n \end{pmatrix}U = T$$

We have represented T in the form AA*, where A = U*SU.

Note :

1. Unitary over the real field are called orthogonal and satisfy QQ' = 1.
2. Herimitian over the real field are just symmetric.

Problems :

1. If T is unitary just using the definition $(T(v), T(u)) = (v, u)$, Prove that T is nonsingular.
2. If T is skew-Herimitian, prove that all of its characteristic roots are pure imgainaries.
3. Prove that a normal transformation is unitary if and only if the characteristic roots are all of absolute value 1.
4. If N is normal, prove that $N^* = p(N)$ for some polynomial $p(x)$.
5. If $A \geq 0$ and $(A(v), v) = 0$, prove that $A(v) = 0$.

Bilinear Forms

Definition : Let V be a vector space over the field F. A bilinear form on V is a function *f*, which assigns to each ordered pair to vector *u, v* in V a scalar $f(u,v)$ in F, and which satisfies

$$f(cu_1 + u_2, v) = cf(u_1, v) + f(u_2, v)$$

$$f(u, cv_1 + v_2) = cf(u, v_1) + f(u, v_2)$$

[$f : V \times V \to F$, if *f* is linear as a function of either of its arguments when the other is fixed. The zero function from $V \times V$ into F is clearly bilinear form]

Note :

1. The set of all bilinear forms on V is a subspace of the space of all functions from $V \times V$ into F.
2. Any linear combination of bilinear forms on V is again a bilinear form.
3. The space of bilinear forms on V is denoted by $L(V, V, F)$.

Example : Let V be a vector space over the field F and let L_1 and L_2 be linear functions on W.

Define *f* by $f(u,v) = L_1(u)\ L_2(v)$.

If we fix *v* and regard *f* as a function of *u*, then we simply have a scalar multiple of the linear functional L_1, with *u* fixed, *f* is a scalar multiple of L_2. Thus it is clear that *f* is a bilinear form on W.

Definition : Let V be a finite dimensional vector space and let $B = (u_1, ..., u_n)$ be can ordered basis for V. If *f* is a bilinear form ob V. The matrix of *f* in the ordered basis B is the non matrix A with entries $A_{ij} = f(u_i, u_j)$. We shall denote this matrix by $[f]_B$.

Theorem : Let V eb a finite dimensional vector space over the field F. For each ordered basis B of V, the function which associates with each bilinear form on V its matrix in the ordered bvasis B is an isomorphism of the space $L(V, V, F)$ onto the space of $n \times n$ matrices over the field F.

Proof : We observes that $f \to [f]$ is a one-one correspondence between the set of liniear forms on V and the set of all $n \times n$ matrices over F. This is a linear transformation is easy to see, becuase $(cf+g)(u_i,u_j)=(f(u_i,u_j)+g(u_i,u_j))$, for each i and j.

This simply says that $[cf+g]_B = c[f]_B + [g]_B$.

Corollary : If $B=\{u_1,....,u_n\}$ is an ordered basis for V, and $B^*=\{L_1,....,L_n\}$ is the dual basis for V* then the n^2 bilinear forms $f_{ij}(u,v)=L_i(u)\ L_j(v),\ 1 \le i \le n, 1 \le j \le n,$ form a basis for the space $L(V,V,F)$. In particular, the dimension of $L(V,V,F)$ is n^2.

Proof : The dual basis $\{L_1,...,L_n\}$ is essentially defined by the fact that $L_i(u)$ is the its coordinate of u in the ordered basis B.

Now the functions f_{ij} defined by,

$f_{ij}(u,v)=L_i(u)\ L_j(v)$ are bilinear forms of the type considered in the previous example.

If $u = x_1u_1+...+x_nu_n$ and $v=y_1u_1+...+y_nu_n$ then $f_{ij}(u,v)=x_iy_i$.

Let f be any bilinear form on V and let A be the matrix of f in the ordered basis B. Then

$$f(u,v)=\sum_{ij} A_{ij}x_iy_i$$

Which simly says that $f = \sum_{ij} A_{ij}\, f_{ij}$.

It is now clear that the n^2 forms f_{ij} comprise a basis for $L(V,V,F)$.

Example : Let V be the vector sapce R^2. Let f be the bilinear form defined on $u=(x_1,x_2)$ and $v=(y_1,y_2)$ by

$$f(u,v)=x_1y_1+x_1y_2+x_2y_1+x_2y_2$$

Now $$f(u,v)=[x_1,x_2]+\begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix}+[y_1,y_2]$$

and so the matrix of *f* in the standard ordered basis $B=\{e_1, e_2\}$ is

$$[f]_B = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

Let $B'=\{e_1', e_2'\}$ be the ordered basis defined by

$$e_1' = (1,-1), \quad e_2' = (1,1)$$

In this case the matrix p which changes coordinates from B' to B is

$$p = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$

Thus $[f]_{B'} = pt[f]_B\, p$.

$$= \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 2 \\ 0 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & 4 \end{bmatrix}$$

What this means is that if we express the vectors *u* and *v* by means of their coordinates in the basis B', say $u = x_1'e_1' + x_2'e_2'$, $v = y_1'e_1' + y_2'e_2'$ then $f(u,v) = 4x_2'y_2'$.

Theorem : Let *f* be a bilinear form on the finite dimensional vector space V. Let L_f and R_f be the linear transformations from V into V* defined by

$$(L_{fu})(v) = f(u,v) = (R_{fv})(u)$$

Then rank (L_f) = rank (R_f)

Proof : To prove rank (L_f) = rank (R_f), it will suffice to prove that L_f and R_f have the same nullity. Let B be an ordered basis for V, and let $A=[f]_B$. If *u* and *v* are vectors in V, with coordinate matrices X and Y in the ordered basis B, then $f(u,v) = X^tAY$. Now $R_f(v)=0$ means that

$f(u,v)=0$ for every u in V, i.e. that $X^t AY=0$ for every $n\times 1$ matrix X. The latter condition simply says that AY = 0. The nullity of R_f is therefore equal to the dimension of the space of solutions of A**Y** = 0.

Similarly, $L_f(u)=0$ if and only if $X^t AY=0$ for every $n\times 1$ matrix Y. Thus u is in the null space of L_f if and only if $X^t A=0$ if $A^t X=0$. The nullity of L_f is therefore equal to the dimension of the space of solutions of $A^t X=0$. Since the matrices A and $\mathbf{A}^t$ have the same column rank, we see that

$$\text{nullity}\,(L_f) = \text{nullity}\,(R_f)$$

Hence the required.

Definition : If f is a bilinear form on the finite dimensional space V, the rank of f is the integer

$$r = \text{rank}\,(L_f) = \text{rank}\,(R_f).$$

Corollary : The rank of a bilinear form is equal to the rank of the matrix of the form in any ordered basis.

Corollary : If f is a bilinear form on the n-dimensional vector space V, the following are equivalent.

(i) rank (f) = n

(ii) For each non-zero u in V, there is a v in V such that $f(u,v)\neq 0$.

(iii) For each non-zero v in V, there is an u in V such that $f(u,v)\neq 0$.

Proof : Statement (ii) simply says that the null space of L_f is the zero subspace. Statement (iii) says they the null space of R_f is the zero subspace. The linear transformations L_f and R_f have nullity 0 if and only if they have rank n i.e. if and only if rank (f) = n.

Definition : A bilinear form f on a vector space V is called non-degenerate (or non-singular) if it satisfies conditions (ii) and (iii) of above corollary.

EXERCISE :

1. Which of the following functions defined on vectors $u=(x_1,x_2)$ and $v=(y_1,y_2)$ in R^2, are bilinear forms ?

(a) $f(u,v)=1$

(b) $f(u,v)=(x_1+y_1)^2-(x_1-y_1)^2$

(c) $f(u,v)=x_1y_2-x_2y_1$

2. Describe the bilinear forms on R^3 which satisfy $f(u,v)=f(v,u)$ for all u,v.

Symmetric Bilinear Forms

Definition : Let f be a bilinear form on the vector space V. We say that f is symmetric if $f(u,v)=f(v,u)$ for all vector u, v in V.

Theorem : Let V be a finite dimensional vector space over a field of characteristic zero and let f be a symmetric bilinear form on V. Then there is an ordered basis for V in which f is represented by a diagonal matrix.

Proof : We must find an ordered basis $B=\{u_1,...,u_n\}$ such that $f(u_i,u_j)=0$ for $i\neq j$.

If f = 0 or n = 1 the theorem is onviously true. Thus we may suppose $f\neq 0$ and n > 1. If $f(u,u)=0$ for every u in V, the associated quadratic form q is identically 0, and the polarization identity

$$f(u,v)=\frac{1}{4}q(u+v)-\frac{1}{4}q(u-v)$$

Shows that f = 0. Thus there is a vector u in V such that $f(u,u)=q(u)\neq 0$. Let W be the one-demensional subspace of V which is spaned by u, and let $W^{\perp}$ be the set of all vectors v in V such that $f(u,v)=0$.

Now we claim that $V=W\oplus W^{\perp}$. Certainly the subspaces W and $W^{\perp}$ are independent. **A** typical vector in W is ca, where c is a scalar. If cu is also in $W^{\perp}$ then

$$f(cu,cu)=c^2f(u,u)=0$$

But $f(u,u) \neq 0$ thus c = 0. Also each vector in V is the sum of a vector in W and a vector in $W^{\perp}$ for, let V be any vector in V, and put

$$v = w - \frac{f(w,u)}{f(u,u)} u$$

Then $f(u,v) = f(u,w) - \frac{f(w,u)}{f(u,u)} f(u,u)$

$$= f(u,w) - f(w,u)$$

and since *f* is symmetric $f(u,v) = 0$.

Thus *v* is in the subspace $W^{\perp}$. The expression

$$w = \frac{f(w,u)}{f(u,u)} u + v$$

Shows that $V = W + W^{\perp}$

The restriction of *f* to $W^{\perp}$ is a symmetric bilinear form on $W^{\perp}$. Since $W^{\perp}$ has dimension (n – 1), we may assume by induction that $W^{\perp}$ has a basis $\{u_2, ..., u_n\}$ such that

$$f(u_i, u_j) = 0, \quad i \neq j, \quad (i \geq 2, j \geq 2)$$

Putting $u_1 = u$, we obtain a basis $\{u_1, u_2, ..., u_n\}$ for V such that $f(u_i, u_j) = 0$ for $i \neq j$.

Corollary : Let F be a subfield of the complex numbers, and let A be a symmetric $n \times n$ matrix over F. Then there is an invertible $n \times n$ matrix P over F such that $P^t AP$ is diagonal.

Theorem : Let V be a finite dimensional vector space over the field of complex numbers. Let *f* be a symmetric bilinear form on V, which has rank V. Then there is an ordered basis $B = \{v_1, ..., v_n\}$ for **V** such that

(i) the matrix of *f* in the ordered basis B is diagonal.

(ii) $f(v_j, v_j) = \begin{cases} 1, & j = 1, ..., r \\ 0, & j > r \end{cases}$

Proof : We know that "Let V be a finite dimensional vector space over a field of characteristic zero, and let *f* be a symmetric bilinear form on V. Then there is an ordered basis for V in which *f* is represented by a diagonal matrix."

Thus there is an ordered basis $\{u_1, ..., u_n\}$ for V such that $f(u_i, u_j) = 0$ for $i \neq j$.

Since *f* has rank *r*, so does its matrix in the ordered basis $\{u_1, ..., u_n\}$.

Thus we must have $f(u_j, u_j) \neq 0$ for precisely *r* values of *j*. By reordering the vectors u_j, we may assume that $f(u_j, u_j) \neq 0$, j = 1, ..., r.

Now we use the fact that the scalar field is the field of complex numbers. If $\sqrt{f(u_j, u_j)}$ denotes any complex square root of $f(u_j, u_j)$ and if we put

$$v_j = \begin{cases} \dfrac{1}{f(u_j, u_j)} u_j & j = 1.....r \\ u_j & j > r \end{cases}$$

Then the basis $\{v_1, ..., v_n\}$ satisfies conditions (i) and (ii).

Hence the required.

Theorem : Let V be an n-dimensional vector space over the field of real numbers, and let *f* be a symmetric bilinear form on V which has rank *r*. Then there is an ordered basis $\{v_1, v_2, ..., v_n\}$ for V in which the matrix of *f* is diagonal and such that $f(v_j, v_j) = \pm 1, \ j = 1.....r$.

Furthermore, the number of basis vectors v_j for which $f(v_j, v_j) = 1$ is independent of the choice of basis.

Proof : There is a basis $\{u_1, ..., u_n\}$ for V such that

$$f(u_i, u_j) = 0 \qquad i \neq j$$

$$f(u_j, u_j) \neq 0 \qquad 1 \leq j \leq r$$

$$f(u_j, u_j) = 0 \qquad j > r$$

Let $v_j = \left| f\left(u_j, u_j\right) \right|_{u_j}^{-1/2}$ $\qquad 1 \le j \le r$

$v_j = u_j \qquad j > r$

Then $\{v_1, ..., v_n\}$ is a basis with the stated properties.

Let p be the number of basis vectors v_j for which $f\left(v_j, v_j\right) = 1$, we must show that the number p is independent of the particular basis we have, satisfying the stated conditions. Let V$^+$ be the subspace of V spanned y the basis vectors v_j for which $f\left(v_j, v_j\right) = 1$ and V$^-$ be the subspace spanned by the basis vectors v_j for which $f\left(v_j, v_j\right) = -1$. Now $p = \dim V^+$, so it is the uniqueness of the dimension of V$^+$ which we must demonstrate. It is easy to see that if u is a non-zero vector in V$^+$, then $f(u,u) > 0$, in other words *f* is positive definite on the subspace **V**$^+$. Similarly, if *u* is a non-zero in V$^-$, then $f(u,u) < 0$ if *f* is negative definite on the subsace V$^-$. Nopw let $V^\perp$ be the subspace spaned by the basis vectors v_j for which $f\left(v_j, v_j\right) = 0$. If u is in $V^\perp$ then $f(u,v) = 0$ for all *v* in V.

Since $\{v_1, ..., v_n\}$ is a basis for V, we have $V = V^+ \oplus V^- \oplus V^\perp$.

Furthermore, we claim that if W is any subspace of V on which *f* is positive definite then the subsace W, V^- and $V^\perp$ are independent. For, suppose *u* is in W, *v* is in V^-, *w* is in $V^\perp$ and $u + v + w = 0$.

Then $\quad 0 = f(u, u + v + w) = f(u,u) + f(u,v) + f(u,w)$

$$0 = f(v, u + v + w) = f(v,u) + f(v,v) + f(v,w)$$

Since *w* is in $V^\perp$, $f(u,w) = f(v,w) = 0$ and since *f* is symmetric we obtain

$$0 = f(u,u) + f(u,v)$$

$$0 = f(v,v) + f(u,v)$$

Hence $f(u,u) = f(v,v)$. Since $f(u,u) \ge 0$ and $f(v,v) \le 0$ it follows that

$$f(u,u) = f(v,v) = 0$$

But *f* is positive definite on W and negative definite on V^-. We conclude that $u = v = 0$ and hence that $w = 0$ as well.

Since $V = V^{+} \oplus V^{-} \oplus V^{\perp}$

and W, V^{-}, $V^{\perp}$ are independent we see that $\dim W \le \dim V^{+}$. That is if W is any subspace of V on which *f* is positive definite, the dimension of W cannot exceed the dimension of V^{+}. If B_1 is another ordered basis for V which satisfies the conditions of the theorem, we shall have corresponding subspaced V_1^{+}, V_1^{-} and $V_1^{\perp}$ and the argument above shows that

$$\dim V_1^{+} \le \dim V^{+}$$

Reversing the argument, we obtain $\dim V^{+} \le \dim V_1^{+}$ and consequently $\dim V^{+} = \dim V_1^{+}$.

Hence the proof.

Note :

1. rank $f = \dim V^{+} + \dim V^{-}$
2. The number $\dim V^{+} - \dim V^{-}$ is often called signature of *f*.

Skew-Symmetric Bilinear Forms

Definition : A bilinear form *f* and V is called Skew-symmetic if $f(u,v) = -f(v,u)$ for all vectors u, v in V.

Theorem : Let V be an n-dimensional vector space over a subfield of the complex numbers, and let *f* be a Skew-symmetric bilinear form on V. Then the rank *r* of *f* is even and if *r* = 2k there is an ordered basis for V in which the matrix of *f* is the direct sum of the $(n-r)\times(n-r)$ zero matrix and *k* copies of the 2×2 matrix $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$.

Proof : Let $u_1, v_1, ..., u_k, v_k$ be vectors satisfying conditions

(a) $f(u_j, v_j) = 1$, j = 1,..., k.

(b) $f(u_i, u_j) = f(v_i, v_j) = f(u_i, v_j) = 0$, $i \ne j$

(c) If W_j is the two-dimensional subspace spanned by u_j and v_j then $V = W_1 \oplus W_2 \oplus ..... \oplus W_k \oplus W_0$.

Where every vector in W_0 is orthogonal to all u_j and v_j and the restriction of *f* to W_0 is the zero form.

Let $\{w_1,...,w_s\}$ be any ordered basis for the subspace W_0.

Then $B=\{u_1,v_1,u_2,v_2,....,u_k,v_k,w_1,...,w_s\}$ is an ordered basis for V..

From (a), (b) and (c) it is clear that the matrix of *f* in the ordered basis B is the direct sum of the $(n-2k)\times(n-2k)$ zero matrix and k copies of the 2×2 matrix $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$. Furthermore, it is clear that the rank of this matrix and hence the rank of *f* is 2k.

Hence the proof.

Groups Preserving Bilinear Forms

Let *f* be a bilinear form on the vector space V, and let T be a linear operator on V. We say that T preserves *f* if $f(T_u,T_v)=f(u,v)$ for all u, v in V.

Theorem : Let *f* be a non-degenerate bilinear form on a finite dimensional vector space V. The set of all linear operators on V, which preserve *f* is a group under the operation of composition.

Proof : Let G be the set of linear operators preserving *f* we observed that the identity operator is in G and theta whenever S and T are in G and the composition ST is also in G. From the fact that *f* is non-degenerate, we shallo prove that any operator T in G is invertible and T^{-1} is also in G. Suppose T preserves *f*. Let *u* be a vector in the null space of T. Then for any v in V we have

$$f(u,v)=f(T_u,T_v)=f(0,T_v)=0$$

Since *f* is non-degenerate, $u=0$. Thus T is invertible. Clearly T^{-1} also preserves *f*, for

$$f(T_u^{-1},T_v^{-1})=f(TT_u^{-1},TT_v^{-1})=f(u,v)$$

Hence the proof.

Let V be either the space R^n or the space C^n. Let *f* be the bilinear form

$$f(u,v)=\sum_{j=1}^{n} x_j y_j \text{ where } u=(x_1,....,x_n) \text{ and } v=(y_1,....,y_n)$$

The group preserving *f* is called the n-dimensional orthogonal group.

Let f be the symmetric bilinear form on R^n with quadratic form.

$$q(x_1,....,x_n) = \sum_{j=1}^{n} x_j^2 - \sum_{j=p+1}^{n} y_j^2$$

Then f is non-degenerate and has signature 2p – n. The group of matrices preserving a form of this type is called a pseudo-orthogonal group.

Theorem : Let V be an n-dimensional vector space over the field of complex numbers, and let f be a non-degenerate symmetric bilinear form on V. Then the group preserving f is isomorphic to the complex orthogonal group O (x, c).

Proof : Of course, by an isomorphism between two groups, we mean a one-one correspondence between their elements which preserves the group operation. Let G be the group of linear operators on V which preserve the bilinear form f. Since f is both symmetric and non-degenerate, the theorem "Let V be a finite-dimensional vector space over the field of complex numbers. Let f be a symmetric bilinear form on V which has rank w. Then there is an ordered basis $B = \{v_1,..., v_n\}$ for V such that

(i) the matrix of f in the orered basis B is diagonal.

(ii) $f(v_i, v_j) = \begin{cases} 1 & j = 1,2,...., r \\ 0 & j = r \end{cases}$

Tells us that there nis an ordered basis B for V in which f is represented by the $n \times n$ identity matrix. Therefore, a linear operator Y preserves f if and only if its matrix in the ordered basis B is a complex orthogonal matrix. Hence $T \to [T]_B$ is an isomorphism of G onto O (x, c).

Hence the proof.

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