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MEMBERS OF B.O.S. IN MATHEMATICS

Preface

Topology is the core course of Mathematics which acts as a foundation for many branches of mathematics like real analysis, functional analysis, algebraic topology, differentiable equations, dynamical systems, etc. The distance concept that appears in the analysis and metric spaces is attempted to be abstracted out by the subject topology. In this sense, the concept of open sets, closed sets, continuity of functions, convergence of sequences, and many other concepts that appears in the subjects analysis and metric spaces can be defined on any non-empty set without requiring the concept of distance.

The main objective of this self-instructional material is:

1. to provide the fundamental concepts in topological spaces.
2. to demonstrate the product spaces and continuous functions on topological spaces.
3. to analyze the compact and connected sets in topological spaces.
4. to study the theory and applications of separation and countability axioms, the Urysohn lemma, and the Urysohn Metrization Theorem.

This self-instructional material is written according to the syllabus of Centre for Distance Education, Shivaji University Kolhapur, and based on the following books.

1. J. R. Munkers, Topology, Second Edition, Pearson Education (Singapore), 2000.
2. W. J. Pervin, Foundations of General Topology, Academic Press, New York, 1964.

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M. Sc. (Mathematics)

General Topology

Contents

Each Unit begins with the section objectives -

Objectives are directive and indicative of :

1. what has been presented in the unit and
2. what is expected from you
3. what you are expected to know pertaining to the specific unit, once you have completed working on the unit.

The self check exercises with possible answers will help you understand the unit in the right perspective. Go through the possible answers only after you write your answers. These exercises are not to be submitted to us for evaluation. They have been provided to you as study tools to keep you in the right track as you study the unit.

Dear Students

The SIM is simply a supporting material for the study of this paper. It is also advised to see the new syllabus 2022-23 and study the reference books & other related material for the detailed study of the paper.

TOPOLOGICAL SPACES

1. Topological Spaces

Introduction

Various mathematicians like Frechet, Hausdorff, proposed different definitions for topology over a period of years during the first decades of the twentieth century, but it took quite a while to settle down to one definition for topology that seemed most suitable. In this unit, we learn the definition of a topological space and important examples of it.

Definition 1.1 : Topology

A topology of X is a collection of subsets X satisfying the following properties

1. $\phi, X \in \mathscr{T}$.
2. The union of the elements of any subcollection of $\mathscr{T}$ is in $\mathscr{T}$.

   (i.e. If $X_\alpha \in \mathscr{T}$ then $\bigcup_{\alpha \in \wedge} X_\alpha \in \mathscr{T}$).
3. Intersection of elements of any finite subcollection of $\mathscr{T}$ is in $\mathscr{T}$.

   (i.e. If $X_1, X, \ldots, X_n \in \mathscr{T}$ then $\bigcap_{i=1}^{n} X_i \in \mathscr{T}$)

The set X with topology $\mathscr{T}$ is called a topological space and is denoted by (X,$\mathscr{T}$) or simple X.

Example 1.2 : Let $X = \{a, b, c\}$

1. Then there are many topologies on X, for example, $\mathscr{T}_1 = \{\phi, X\}$ and $\mathscr{T}_2 = \{\phi, X, \{a\}, \{b,c\}\}$, are topologies on X. Infact, there are total 29 topologies on X.
2. But the set $\mathscr{T} = \{\phi, X, \{a\}, \{b\}\}$ is not a topology as $\{a\}, \{b\} \in \mathscr{T}$ but $\{a,b\} \notin \mathscr{T}$.

Definition 1.3 : Open Set

Let X be a topological space with topology $\mathscr{T}$. We say that a subset U of X is an open set if $U \in \mathscr{T}$.

Example 1.4 : Let $X = \{a,b,c\}$ and $\mathscr{T} = \{\phi, X, \{a\}, \{b,c\}\}$. Then $\{a\} \subset X$ is an open set where as $\{b\} \subset X$ is not an open set.

Definition 1.5 : Discrete Topology

Let X be any set. Then the collection $\mathscr{T}$ of all subsets of X is a topology and called the discrete topology. (i.e. $\mathscr{T} = P(X)$ is called the discrete topology). Equivalently if every singleton is open, then X is called a discrete topology.

We now give an equivalent definition of discrete topology in terms of singleton sets.

Theorem 1.6 : *A topology (X, $\mathscr{T}$) is discrete if and only if every singleton is open.*

Proof : If X is discrete, then every subset is open , so in particular, every singleton is open.

Conversely, suppose that for all $x \in X, \{x\} \in \mathscr{T}$.

Let Y be a subset of X. We have to show that $Y \in \mathscr{T}$.

We can write the set Y as $Y = \bigcup\{\{x\} \mid y \in Y\}$.

As $\{y\} \in \mathscr{T}$, and $\mathscr{T}$ is a topology, we get that $\bigcup\{\{y\}\} \in \mathscr{T}$.

That is $Y \in \mathscr{T}$ and thus $\mathscr{T}$ is discrete.

Definition 1.7 : Indiscrete Topology

Let X be any set. Then $\mathscr{T} = \{\phi, X\}$ is called the indiscrete topology.

Example 1.8 : Let X be a set and $\mathcal{T}_f = \{U \subset X \mid X - U$ is either finite or is all of $X\}$. Then $\mathcal{T}_f$ is a topology, called the finite complement topology.

Proof : Given $\mathcal{T}_f = \{U \subset X \mid X - U$ is finite or $U = \theta\}$

Since $X - X = \theta$ is Finite, $X \in \mathcal{T}_f$.

Also $X - \theta = X$ implies $\theta \in \mathcal{T}_f$.

Let $\{U_\alpha\}$ be the indexed collection of elements of $\mathcal{T}_f$.

If each U_α is empty, then their union is empty and hence belongs to $\mathcal{T}_f$.

So assume that there is at least one U_β which is non empty. Then $X - U_\beta$ is finite.

Now $X - \bigcup U_\alpha = \bigcap (X - U_\alpha) \subset X - U_\beta$

Since $X - U_\beta$ is finite, $X - \bigcup U_\alpha$ is also finite.

$\therefore \quad \bigcup U_\alpha \in \mathcal{T}_f$

Let $U_1, U_2, \ldots., U_n \in \mathcal{T}_f$. If one of U_i is empty, then their intersection is empty.

So assume that $U_i \neq \theta$ for all i. Then $X - U_1, X - U_2, \ldots., X - U_n$, are finite.

Then $X - \bigcap_{i=1}^{n} U_i = \bigcup_{i=1}^{n} (X - U_i)$ is finite.

$\therefore \quad \bigcap_{i=1}^{n} U_i \in \mathcal{T}_f$.

Hence $\mathcal{T}_f$ is a topology.

Example 1.9 : Let X be a set and $\mathcal{T}_c = \{U \subset X \mid X - U$ is countable or $U^c = X\}$.

Then $\mathcal{T}_c$ is a topology, called the countable complement topology.

Proof : Given $\mathcal{T}_c = \{U \subset X \mid X - U$ is countable or $U = \phi\}$.

Since X – θ = X and X – X = θ is countable, θ, $X \in \mathscr{T}_c$.

Let $U_\alpha \in \mathscr{T}_c, \alpha \in \Lambda$.

If each U_α is empty, then their union is empty and hence belongs to $\mathscr{T}_f$.

So assume that there is at least one U_β which is non empty. Then $X - U_\beta$ is countable.

$$X - \bigcup U_\alpha = \bigcap \left(X - U_\alpha \right) \subset X - U_\beta$$

Since $X - U_\beta$ is countable, $X - \bigcup U_\alpha$ is also countable.

$\therefore \quad \bigcup U_\alpha \in \mathscr{T}_c$

Let $U_1, U_2, ..., U_n \in \mathscr{T}_c$. If one of U_i is empty, then their intersection is empty.

So assume that $U_i \neq \varnothing$ for all *i*. Then $X - U_1, X - U_2, ..., X - U_n$, are countable.

Then $X - \bigcap_{i=1}^{n} U_i = \bigcup_{i=1}^{n} \left(X - U_i \right)$ is countable.

$\therefore \quad \bigcap_{i=1}^{n} U_i \in \mathscr{T}_c$.

Hence $\mathscr{T}_c$ is a topology.

Definition 1.10 : Suppose that $\mathscr{T}$ and $\mathscr{T}'$ are two topologies on a given set *X*. If $\mathscr{T} \subset \mathscr{T}'$ then $\mathscr{T}'$ is finer than $\mathscr{T}$. If $\mathscr{T}' \supsetneq \mathscr{T}$ then $\mathscr{T}'$ is strictly finer than $\mathscr{T}$. We also say that $\mathscr{T}$ is coarser (weaker) than $\mathscr{T}'$. We say that $\mathscr{T}$ and $\mathscr{T}'$ are comparable if either $\mathscr{T}' \supset \mathscr{T}$ or $\mathscr{T} \subset \mathscr{T}'$.

Remark 1.11 : We can understand the above definition better by thinking of a topological space as a truckload with full of pebble gravel and all unions of collections of pebbles being the open sets. Now by smashing the pebbles into smaller ones, the collection of open sets has been enlarged, and the topology, like the gravel, is said to have been made finer by the operation. We learn more about comparing topologies in the next unit.

EXERCISE-1

1. Which of the following is not a topology ?

   (A) The collection of all subsets U of X such that $X \setminus U$ either is finite or is all of X.

   (B) The collection of all subsets U of X such that $X \setminus U$ either is countable or is all of X.

   (C) The collection of all subsets U of X such that $X \setminus U$ either is infinite or is empty or is all of X.

   (D) None of the above

2. Let $X = \{a, b, c\}$. Then Which of the following is not a topology

   (A) $\{\varnothing, X\}$

   (B) $\{\varnothing, X, \{a\}\}$

   (C) $\{\varnothing, X, \{a\}, \{b, c\}\}$

   (D) $P(X)$, power set of X.

3. Let $X = \{a, b, c\}$ and $\mathscr{T} = \{\phi, X, \{a, b\}, \{b, c\}\}$. Is $\mathscr{T}$ a topology ? Justify.

4. Let $X = \{a, b, c, d, e, f\}$ and $\mathscr{T} = \{\phi, X, \{a\}, \{c, d\}, \{a, c, d\}, \{b, c, d, e, f\}\}$. Show that $\mathscr{T}$ is a topology.

5. Let $X = \mathbb{N}$ and $\mathscr{T} = \{\phi, X\} \cup \{\text{all finite subsets}\}$. Is $\mathscr{T}$ a topology ? Justify.

6. If $\{\mathscr{T}_\alpha\}$ is a family of topologies on X, show that $\bigcap \mathscr{T}_\alpha$ is a topology on X.

   Is $\bigcup \mathscr{T}_\alpha$ a topology on X ?

2. Basis and Sub basis for a topology

Introduction :

Specifying the topology by means of all its open sets is too difficult, in general. To over come this difficulty, we instead consider a smaller collection of subsets of X and defines the topology in terms of that. That particular collection satisfying some properties is called a basis, which we define explicitly in this unit.

Definition 2.1 : Base

Let X be any set and $\mathscr{B}$ be a collection of subsets of X. Then $\mathscr{B}$ is called a base for a topology on X if

1. For each $x \in X$, there exists $B \in \mathscr{B}$ such that $x \in B$.
2. For each $x \in B_1 \cap B_2$, there exists $B_3 \in \mathscr{B}$ such that $x \in B_3 \subset B_1 \cap B_2$.

Example 2.2 : For $X = \mathbb{R}$, the collection of open intervals $\mathscr{B} = \{(a,b) \mid a,b \in \mathbb{R}\}$ is a base.

Proof :

1. Let $x \in \mathbb{R}$. Then for any $\varepsilon > 0$, we have $x \in (x-\varepsilon, x+\varepsilon) \in \mathscr{B}$.
2. Let $x \in (a,b) \cap (c,d)$.

   $\Rightarrow x \in (a,b)$ and $x \in (c,d)$

   $\Rightarrow x < b$ and $c < x$

   $\Rightarrow c < x < b$

   $\therefore x \in (c,b)$.

   Let $a \leq c$. Then $(c,b) \subset (a,b) \cap (c,d)$.

   Therefore, $\mathscr{B}$ is a base for X.

Example 2.3 : Let $\mathscr{C}$ be the collection of all circular regions (interior of circles) in the plane. Then $\mathscr{B}$ is the base as given any $x \in X$, we can find a circular region around x and the second condition is explained in the following figure 1.1.

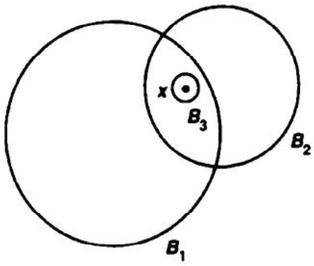

Figure 1:

Example 2.4 : Let $\mathscr{B}$ be the collection of all rectangular regions (interior of rectangles) in the plane. Then $\mathscr{B}$ is a base as shown in the following figure :

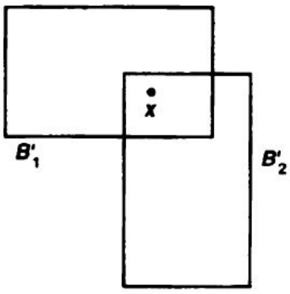

Figure 2:

Example 2.5 : If X is any set, then the collection $\mathscr{B}$ of all singletons of X is a base for the discrete topology on X.

Proof : Let $x \in X$. Then $\{x\} \in \mathscr{B}$ and $x \in \{x\}$.

Suppose $x \in B_1 \cap B_2$. Then $B_1 = B_2 = \{x\}$.

Therefore $x \in B_3 = B_1 \cap B_2$.

Theorem 2.6 : *Let $\mathscr{B}$ be a base for X. Then the collection* $\mathscr{T} = \{U \subset X \mid \forall x \in U,$ there exists $B_x \in \mathscr{B}$ such that $x \in B_x \subset U\}$ *is a topology and is called the topology generated by the base $\mathscr{B}$.*

Proof 1 : 1. Clearly $\phi \in \mathscr{T}$.

Let $x \in X$. Since $\mathscr{B}$ is a base there exists $B \in \mathscr{B}$ such that $x \in B \subset X$.

Hence $X \in \mathscr{T}$.

2. Let $U_\alpha \in \mathscr{T}$, $\alpha \in \Lambda$ and $x \in \bigcup_{\alpha \in \Lambda} U_\alpha$

$\Rightarrow x \in U_j$ for some $j \in \Lambda$.

Since $U_j \in \mathscr{T}$ there exists $B \in \mathscr{B}$ such that $x \in B \subset U_j$

Implies $x \in B \subset \bigcup_{\alpha \in \Lambda} U_\alpha$

$\Rightarrow \bigcup_{\alpha \in \Lambda} U_\alpha \in \mathscr{T}$.

3. Let U_1 and $U_2 \in \mathscr{T}$ and $x \in U_1 \bigcap U_2$

$\Rightarrow x \in U_1$ and $x \in U_2$

$\Rightarrow$ there exists $B_1, B_2 \in \mathscr{B}$ such that $x \in B_1 \subset U_1$ and $x \in B_2 \subset U_2$

$\therefore x \in B_1 \bigcap B_2$

Since $\mathscr{B}$ is a base there exists B_3 such that $x \in B_3 \subset B_1 \bigcap B_2$

$\Rightarrow x \in B_3 \subset B_1 \bigcap B_2 \subset U_1 \bigcap U_2$

$\Rightarrow U_1 \bigcap U_2 \in \mathscr{T}$

Hence the result is true for $n = 2$.

Now assume that the result is true for $n = k$.

That is if $U_1, U_2, ..., U_k \in \mathscr{T}$, then $\bigcap_{i=1}^{k} U_i \in \mathscr{T}$

Since $\bigcap_{i=1}^{k} U_i \in \mathscr{T}$ and $U_{k+1} \in \mathscr{T}$, we get that $\left(\bigcap_{i=1}^{k} U_i\right) \bigcap U_{k+1} \in \mathscr{T}$

Implies $\bigcap_{i=1}^{k+1} U_i = \bigcap_{i=1}^{k} U_i \bigcap U_{k+1} \in \mathscr{T}$

Thus $\mathscr{T}$ is a topology.

Lemma 2.7 : *Let X be a set and $\mathscr{B}$ be a base for a topology $\mathscr{T}$ on X. Then $\mathscr{T}$ equals the collection of all unions of elements of B.*

Proof : Given that $\mathscr{T} = \{U \subset X \mid \forall x \in U$ there exists $B_x \in \mathscr{B}$ such that $x \in B \subset U\}$

Let $U \in \mathscr{T}$. Then for each $x \in U$ there exists $B_x \in \mathscr{B}$ such that $x \in B_x \subset U$.

So we can write $U = \bigcup_{x \in U} B_x$.

Therefore, $\mathscr{T} \subset$ {all union of elements of $\mathscr{B}$}.

Since every $B_\alpha \in \mathscr{B}$ is in $\mathscr{T}$, and $\mathscr{T}$ is a topology, we get $\bigcup B_\alpha \in \mathscr{T}$.

$\therefore \mathscr{T}$ is equal to the collection of all union of elements of $\mathscr{B}$.

Remark 2.8 : If $\mathscr{B}$ is a base for a topology $\mathscr{T}$ on *X*, then

$\mathscr{T} = \{U \subset X \mid \forall x \in U,$ there exists $B_x \in \mathscr{B}$ such that $x \in B_x \subset U\}$

= {all union of elements of $\mathscr{B}$}

i.e. every element *U* of $\mathscr{T}$ (or every open sets *U* of *X*) can be expressed as a union of basis elements.

Lemma 2.9 : *Let X be a topological space. Suppose that $\mathscr{C}$ is a collection of open subsets of X such that for each open set U of X and each $x \in U$, there exists $C \in \mathscr{C}$ such that $x \in C \subset U$. Then $\mathscr{C}$ is a basis for the topology on X.*

Proof : Let $x \in X$. Since *X* is open, there exists $C \in \mathscr{C}$ such that $x \in C \subset X$.

Let $C_1, C_2 \in \mathscr{T}$ and $x \in C_1 \bigcap C_2$.

Since C_1 and C_2 are open, $C_1 \bigcap C_2$ is open.

Therefore, there exists $C_3 \in \mathscr{C}$ such that $x \in C_3 \subset C_1 \bigcap C_2$

$\Rightarrow \mathscr{T}$ is a basis for *X*.

Let $\mathscr{T}$ be the collection of open sets of *X* and $\mathscr{T}'$ is the topology generated by C.

We will show $\mathscr{T} = \mathscr{T}'$.

Let $U \in \mathscr{T}$ and $x \in U$.

Then by given hypothesis there exists $C \in \mathscr{T}$ such that $x \in C \subset U$

$\Rightarrow U \in \mathscr{T}'$

$\Rightarrow \mathscr{T} \subset \mathscr{T}'$.

Let $U \in \mathscr{T}'$. Then U is the union of elements of $\mathscr{C}$.

Since every element of $\mathscr{C}$ is open, union of these elements is also open.

$\therefore U = \bigcup_{\alpha} C_{\alpha} \in \mathscr{T}$

$\Rightarrow \mathscr{T}' \subset \mathscr{T}$

$\therefore \mathscr{T} = \mathscr{T}'$

Theorem 2.10 : *Let $\mathscr{B}$ and $\mathscr{B}'$ be bases for the topologies $\mathscr{T}$ and $\mathscr{T}'$ on X respectively. Then the following are equivalent.*

1. *$\mathscr{T}'$ is finer than $\mathscr{T}$ (i.e., $\mathscr{T}' \supset \mathscr{T}$).*
2. *For each $x \in X$ and for each $B \in \mathscr{B}$ containing x, there exists $B' \in \mathscr{B}$ such that $x \in B' \subset B$.*

Proof : $1 \Rightarrow 2$.

Suppose $\mathscr{T}' \supset \mathscr{T}$.

Let $x \in X$ and $B \in \mathscr{B}$ such that $x \in B$.

Since $B \in \mathscr{T} \Rightarrow B \in \mathscr{T}'$.

As $\mathscr{T}'$ is the topology generated by $\mathscr{B}'$, there exists $B' \in \mathscr{B}'$ such that $x \in B' \subset B$.

$2 \Rightarrow 1$.

Let $U \in \mathscr{T}$ and $x \in U$. Then there exists $B \in \mathscr{B}$ such that $x \in B \subset U$.

Then by assumption, there exists $B' \in \mathscr{B}$ such that $x \in B' \subset B$

$\therefore x \in B' \subset B \subset U$

$\Rightarrow x \in B' \subset U$

$\Rightarrow U \in \mathscr{T}'$

$\therefore \mathscr{T}' \supset \mathscr{T}$

Example 2.11 : Let $\mathscr{B}$ be the collection of circular regions and $\mathscr{B}'$ be the collection of rectangular regions in the plane. Let $\mathscr{T}$ and $\mathscr{T}'$ be the corresponding topologies. Then $\mathscr{T} = \mathscr{T}'$.

Proof : Clearly, given any circular region, we can find a rectangular region which is contained in the given circular region.

$\Rightarrow \mathscr{T}' \supset \mathscr{T}$.

Similarly, given any rectangular region, we can find a circular region which is contained in the given rectangular region.

$\Rightarrow \mathscr{T} \supset \mathscr{T}'$

$\therefore \mathscr{T} = \mathscr{T}'$.

Definition 2.12 : Let $\mathscr{B} = \{(a,b) \mid a,b \in \mathbb{R}\}$

$\mathscr{B}' = \{[a,b) \mid a,b \in \mathbb{R}\}$ and

$\mathscr{B}'' = \{(a,b) - K \mid a,b \in \mathbb{R}\} \cup \{(a,b) \mid a,b \in \mathbb{R}\}$

Then $\mathscr{B}$, $\mathscr{B}'$, $\mathscr{B}''$ are bases where $K = \left\{1, \frac{1}{2}, \frac{1}{3}, ...\right\}$.

The topology generated by $\mathscr{B}$ is called the standard topology on $\mathbb{R}$.

The topology generated by $\mathscr{B}'$ is called the lower limit topology on $\mathbb{R}$.

The topology generated by $\mathscr{B}''$ is called the $K-$topology on $\mathbb{R}$, denoted by $\mathbb{R}_K$.

Theorem 2.13 : *The topology of $\mathbb{R}_l$ and $\mathbb{R}_K$ are strictly finer than the standard topology on $\mathbb{R}$ but are not comparable with one another.*

Proof : Let $\mathscr{T}$, $\mathscr{T}'$ and $\mathscr{T}''$ be the topologies of $\mathbb{R}$, $\mathbb{R}_l$ and $\mathbb{R}_K$ respectively.

Let $x \in (a,b) \in \mathscr{B}$. Then $[x,b) \in \mathscr{B}'$ and $x \in [x,b) \subset (a,b)$.

$\Rightarrow \mathscr{T}' \supset \mathscr{T}$.

On the other hand $0 \in [0,1) \in \mathscr{B}'$ but there is no $(a,b) \in \mathscr{B}$ such that

$0 \in (a,b) \subset [0,1)$ (if there is, then $a < 0$ and as $(a,b) \subset [0,1)$, then $a \geq 0$)

$\therefore \mathscr{T}'$ is strictly finer than $\mathscr{T}$.

Clearly, $\mathscr{T}'' \supset \mathscr{T}$ as $\mathscr{B}'' \supset \mathscr{B}$.

We know that $0 \in (-1,1) - K \in \mathscr{B}''$ but there is no open interval (a, b) containing '0' such that $(a,b) \subset (-1,1) - K$.

Because if $0 \in (a,b)$ then $b > 0$. So by Archimedean property, there exists $n \in \mathbb{N}$ such that $nb > 1$.

$$\Rightarrow a < \frac{1}{n} < b$$

$$\therefore \frac{1}{n} \in (a,b) \text{ but } \frac{1}{n} \notin (-1,1) - K$$

$\therefore \mathscr{T}' \supseteq \mathscr{T}$

i.e. $\mathscr{T}''$ is strictly finer than $\mathscr{T}$.

Now we will show that $\mathscr{T}'$ and $\mathscr{T}''$ are not comparable.

Since $0 \in (-1,1) - K \in \mathscr{B}''$ and no interval $[a,b) \in \mathscr{B}'$ containing '0' such that $[a,b) \subset (-1,1) - K$

$\Rightarrow \mathscr{T}''$ is not contained in $\mathscr{T}'$.

Similarly $0 \in [0,1) \in \mathscr{B}'$ but no interval (a, b) or $(a, b) - K$ containing '0' will be contained in $[0,1)$.

$\therefore \mathscr{T}'$ is not contained in $\mathscr{T}''$.

Hence $\mathscr{T}'$ and $\mathscr{T}''$ are not comparable.

Definition 2.14 : Sub Basis

A sub basis $\mathscr{S}$ for a topology on X is a collection of subsets of X whose union is X i.e. $\forall x \in X$, there exists $S \in \mathscr{S}$ such that $x \in S$.

We will see that topology generated by sub basis is the collection of all the union of finite intersection of elements of sub basis.

Theorem 2.15 : *Let $\mathscr{S}$ be a sub basis for a topology on X, and $\mathscr{T}$ be the collection of all the union of finite intersection of elements of $\mathscr{S}$. Then $\mathscr{T}$ is a topology generated by $\mathscr{S}$.*

Proof : Let $\mathscr{S}^*$ be the collection of all finite intersection of elements of $\mathscr{T}$.

Now will show that $\mathscr{S}^*$ is a base.

Let $x \in X$. Since $\mathscr{S}$ is a subbase there exists $S \in \mathscr{S}$ such that $x \in S$

As $\mathscr{S} \subset \mathscr{S}^*$, $x \in S \in \mathscr{S}^*$.

Let $B_1,\ B_2 \in \mathscr{S}^*$. Then $B_1 = \bigcap_{i=1}^{n} S_i$ and $B_2 = \bigcap_{i=1}^{m} S_i$.

Then $B_1 \cap B_2 \in \mathscr{S}^*$ as $B_1 \cap B_2$ is the intersection of finite number of sets of $\mathscr{S}^*$.

$\therefore \mathscr{S}^*$ is a base and

$\mathscr{T}$ = {all unions of all finite intersection of elements of $\mathscr{S}$ }

= {all union of elements of $\mathscr{S}^*$}

Since $\mathscr{S}^*$ is a base, $\mathscr{T}$ is a topology.

Definition 2.16 : Order Topology

Let X be a set with order relation '<'. Let $a, b \in X$ with $a < b$. Then

$$(a,b) = \{x \mid a < x < b\}$$

$$(a,b] = \{x \mid a < x \le b\}$$

$$[a,b) = \{x \mid a \le x \le b\}$$

$$[a,b] = \{x \mid a \le x \le b\}$$

These four subsets of X are called intervals determine by a and b.

Example 2.17 : Let $X = \mathbb{N}$ with order '<' then (1;5) = {2, 3,4} and $[1,5) = \{1,2,3,4\}$.

Lemma 2.18 : *Let X be a set with simple order relation and assume that X has more than one element. Let $\mathscr{B}$ be the collection of all sets of the following :*

1. *All open intervals (a, b) in X.*
2. *All intervals of form $[a_0, b)$ where a_0 is the smallest element of X.*
3. *All intervals of the form $(a, b_0]$ where b_0 is the largest element of X.*

Then the collection $\mathscr{B}$ is a basis for the topology on X.

Proof : Let $x \in X$.

Suppose *x* is the smallest element of *X*.

Since *X* contains more than one element, there exists *b* such that $x < b$.

Then $x \in [x, b) \in \mathscr{B}$.

Similarly, if *x* is the largest element of *X*, there exists *a* such that $a < x$ implies $x \in (a, x] \in \mathscr{B}$.

If *x* is neither smallest nor largest, then there exists *a* and *b* such that $a < x < b$ and hence $x \in (a, b) \in \mathscr{B}$.

In any case, there exists $B \in \mathscr{B}$ such that $x \in B$.

Also if B_1 and $B_2 \in \mathscr{B}$ then $B_1 \cap B_2 \in \mathscr{B}$ because the intersection of B_1 and B_2 is any one of the form (a, b), $[a, b)$ or $(a, b]$.

Definition 2.19 : Order Topology

Let *X* be a set with simple order relation and assume that *X* has more than one element. Then the collection $\mathscr{B}$ consisting of all the sets of the form :

1. All open intervals (a, b) in *X*.
2. All intervals of form $[a_0, b)$ where a_0 is the smallest element of *X*.
3. All intervals of the form $(a, b_0]$ where b_0 is the largest element of *X*.

is a basis for the topology on *X*, called the order topology.

Example 2.20 : The standard topology on $X = \mathbb{R}$ is an order topology.

Proof : Since $\mathbb{R}$ has neither smallest element nor largest element,

We have $\mathscr{B} = \{(a,b) \mid a,b \in \mathbb{R}\}$.

This topology generated by $\mathscr{B}$ is same as the standard topology.

Example 2.21 : $X = \mathbb{Z}_+$, the set of positive integers. Then the order topology on X is the discrete topology.

Proof : Here $\mathscr{B} = \{(a,b) \mid a,b \in \mathbb{Z}_+\} \cup \{[1,c) \mid c \in \mathbb{Z}_+\}$

R or any $n > 1$; $\{n\} = (n-1, n+1) \in \mathscr{B}$

$\therefore$ Every singleton is open and hence the ordered topology on $\mathbb{Z}_+$ is the discrete topology.

Example 2.22 : $X = \mathbb{R} \times \mathbb{R}$ with dictionary order.

Here $\mathscr{B} = \{(a \times b, c \times d) \mid a < c$ and if $a = c$ then $b < d\}$

is a basis and the topology generated by this $\mathscr{B}$ is called the ordered topology on $\mathbb{R} \times \mathbb{R}$.

Definition 2.23 : If X is an ordered set and $a \in X$, then the rays determined by a are given by

$$(a,\infty) = \{x \mid x > a\}$$

$$[a,\infty) = \{x \mid x \geq a\}$$

$$(-\infty,a) = \{x \mid x < a\}$$

$$(-\infty,a] = \{x \mid x \leq a\}$$

Example 2.24 : The open ray (a,∞) is an open set, because if X has largest element b_0 then $(a,\infty)=(a,b_0]\in\mathscr{T}$.

If X has no largest element then $(a,\infty)=\bigcup_{x>a}(a,x)$

$\therefore (a,\infty)$ is open.

Lemma 2.25 : *The open rays form a sub basis for the order topology on X. Also the topology generated by this sub basis is same as the order topology.*

Proof : Let $x\in X$.

If x is the smallest element then there exists a such that $x<a$ and

$x\in[x,a)=(-\infty,a)$.

Similarly, if x is the largest element then there exists a such that $a<x$ and

$x\in(a,x)=(-\infty,a)$.

Clearly, for any $x\in X$, $x\in(x-\varepsilon,\infty)$.

Hence the collection of open rays forms a sub basis.

Let $\mathscr{T}'$ be the topology generated by the subbasis and $\mathscr{T}$ be the order topology on X.

Since each open ray is an open set, we have $\mathscr{T}'\subset\mathscr{T}$.

Let $(a,b)\in\mathscr{B}$. Then

$$(a,b)=(-\infty,b)\bigcap(a,\infty)$$

$$[a_0,b)=(-\infty,b)$$

$$(a,b_0]=(a,\infty)$$

Implies $\mathscr{T}\subset\mathscr{T}'$ and hence $\mathscr{T}'=\mathscr{T}$.

EXERCISE - 2

1. Consider the following

   (I) The collection $\mathscr{B}_1 = \{(a,b) \mid a,b \in \mathbb{Q}\}$ is a base for $\mathbb{R}$.

   (II) The collection $\mathscr{B}_2 = \{(a,b) \mid a,b \in \mathbb{Z}\}$ is a base for $\mathbb{R}$.

   (A) Only (I) is true. (B) Only (II) is true.

   (C) Both (I) and (II) are true. (D) Both (I) and (II) are false.

2. Consider the following

   (I) Every basis element is an open set in *X*.

   (II) Every open set is a union of basis elements for *X*.

   (A) Only (I) is true. (B) Only (II) is true.

   (C) Both (I) and (II) are true. (D) Both (I) and (II) are false.

3. Consider the following

   (I) $\tau' \supset \tau$ if for each $x \in X$ and for each $B \in \mathscr{B}$ with $x \in B$, there exists $B' \in \mathscr{B}'$ such that $x \in B' \subset B$

   (II) $\tau \supset \tau'$ if for each $x \in X$ and for each $B \in \mathscr{B}$ with $x \in B$, there exists $B' \in \mathscr{B}'$ such that $x \in B' \subset B$

   (A) Only (I) is true. (B) Only (II) is true.

   (C) Both (I) and (II) are true. (D) Both (I) and (II) are false.

4. Which of the following is true

   (A) The topology of $\mathbb{R}_l$ is finer than the standard topology on $\mathbb{R}$

   (B) The standard topology on $\mathbb{R}$ is finer than the topology of $\mathbb{R}_l$

   (C) The topology of $\mathbb{R}_l$ is finer than the topology of $\mathbb{R}_K$

   (D) All of the above

5. Consider the following

(I) The order topology with usual order on $\mathbb{R}$ is the standard topology on $\mathbb{R}$.

(II) The order topology on the positive integers $\mathbb{Z}_+$ is the discrete topology.

(A) Only (I) is true. (B) Only (II) is true.

(C) Both (I) and (II) are true. (D) Both (I) and (II) are false.

6. Let $X = \mathbb{R}$ and $\mathscr{B} = \{(a,b) : a,b \in \mathbb{Q}\}$. Show that $\mathscr{B}$ is a base for *X*.

7. Let $X = \mathbb{R}$ and $\mathscr{B} = \{(n,n+1) : n \in \mathbb{Z}\}$. Show that $\mathscr{B}$ is not a base for *X*.

8. Let *X* be a topological space; let A be a subset of *X*. Suppose that for each $x \in A$, there is an open set *U* containing x such that $U \subset A$. Show that A is open in *X*.

9. Compare the finite complement and countable complement topologies.

3. Product Topology X × Y

Introduction :

The definition of the topological product of an infinite set of topological spaces was given by A.N. Tikhonov (1930). The construction of a topological product is one of the main tools in the formation of new topological objects from ones already exist. Using topological products, one can construct a number of fundamental standard objects of general topology. Another important topology is the subspace topology, which is also constructed from the existing one. In this unit, we focus on product topology and subspace topology and relate them using open sets.

Before defining the product topology, we prove the following lemma.

Lemma 3.1 : *The set* $\mathscr{B} = \{U \times V \mid U \text{ is open in } X \text{ and } V \text{ is open in } Y\}$ *is a basis for a topology on X × Y.*

Proof : Let $x \times y \in X \times Y$.

Since *X* is open in *X* and *Y* is open in *Y*, $X \times Y \in \mathscr{B}$ and $x \times y \in X \times Y \in \mathscr{B}$.

Let $U_1 \times V_1$, $U_2 \times V_2 \in \mathscr{B}$.

Then $x \times y \in (U_1 \times V_1) \cap (U_2 \times V_2) \Leftrightarrow x \in U_1 \cap U_2,\ y \in V_1 \cap V_2$

$$\Leftrightarrow x \times y \in (U_1 \cap U_2) \times (V_1 \cap V_2).$$

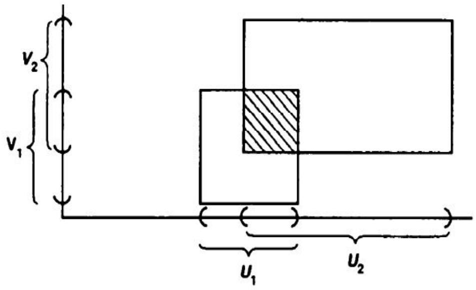

Figure 3:

Since $U_1 \cap U_2$ is open in *X* and $V_1 \cap V_2$ is open in *Y* (refer the Figure 3)

$\Rightarrow (U_1 \cap U_2) \times (V_1 \cap V_2) \in \mathscr{B}$

$\therefore (U_1 \times V_1) \cap (U_2 \times V_2) \in \mathscr{B}$

$\Rightarrow \mathscr{B}$ is a basis for $X \times Y$.

Definition 3.2 : [The product topology]

Let *X* and *Y* be topological spaces. The product topology on $X \times Y$ is the topology having basis as the collection of all sets of the form $U \times V$ where *U* is open in *X* and *V* is open in *Y*.

The next theorem characterize the base for the product topology $X \times Y$ using the bases for *X* and *Y*.

Theorem 3.3 : *If $\mathscr{B}$ is a basis for a topology on X and $\mathscr{C}$ is a basis for a topology on Y, then $\mathscr{D} = \{B \times C \mid B \in \mathscr{B}, C \in \mathscr{C}\}$ is a basis for a topology on $X \times Y$.*

Proof : Let *W* be an open set of $X \times Y$ such that $x \times y \in W$.

Then there exists $U \times V \in \mathscr{B}'$, such that $x \times y \in U \times V \subset W$, where $\mathscr{B}'$ is a basis for product topology X × Y.

$\Rightarrow x \in U$ and $y \in V$.

Since U is open in X and $x \in U$, there exists $B \in \mathscr{B}$ such that $x \in B \subset U$.

Similarly, there exists $C \in \mathscr{S}$ such that $y \in C \subset V$

$\Rightarrow x \times y \in B \times C \subset U \times V \subset W$

$\Rightarrow x \times y \in B \times C \subset W$ where $B \times C \in \mathscr{D}$.

$\therefore \mathscr{D}$ is a basis for the product topology of $X \times Y$.

Example 3.4 : The product of standard topology on $\mathbb{R}$ with itself is called the standard topology on $\mathbb{R}^2$. A basis for this product topology is

$$\mathscr{B} = \{(a,b) \times (c,d) \mid a,b,c,d \in \mathbb{R}\}.$$

Definition 3.5 : Let $\pi_1 : X \times Y \to X$ defined by $\pi_1(x,y) = x$ and $\pi_2 : X \times Y \to Y$ defined by $\pi_2(x,y) = y$.

Then π_1 is called a projection of $X \times Y$ onto X and π_2 is called a projection of $X \times Y$ onto Y.

Remark 3.6 : If $U \subset X$ is open, then

$$\begin{aligned} \pi_1^{-1}(U) &= \{(x,y) \mid \pi_1^{-1}(x,y) \in U\} \\ &= \{(x,y) \mid x \in U\} \\ &= U \times Y \end{aligned}$$

Since $U \times Y$ is open in $X \times Y$, $\pi_1^{-1}(U)$ is open in $X \times Y$.

Similarly, if $V \subset Y$ is open then $\pi_2^{-1}(V) = X \times Y$ is open in $X \times Y$.

Also $\pi_1^{-1}(U) \cap \pi_2^{-1}(V) = U \times V$

Since $U \times V$ open in $X \times Y$, $\pi_1^{-1}(U) \cap \pi_2^{-1}(V)$ is open in $X \times Y$.

Theorem 3.7 : *The collection*

$$\mathscr{S} = \{\pi_1^{-1}(U) \mid U \text{ is open in } X\} \cup \{\pi_1^{-1}(V) \mid V \text{ is open in } Y\}$$

is a subbasis for the product topology on $X \times Y$.

Proof : Let $x \times y \in X \times Y$. Since $X \times Y$ is open in the product topology, there exists $U \times V \in \mathscr{B}$ such that $x \times y \in U \times V \subset X \times Y$.

Since $x \in U$, $\{x\} \times Y \subset U \times Y = \pi_1^{-1}(U)$

$\Rightarrow x \times y \in \{x\} \times Y \subset \pi_1^{-1}(U)$

$\therefore x \times y \in \pi_1^{-1}(U)$.

Let $\mathscr{T}$ be the product topology on $X \times Y$ and $\mathscr{T}'$ be the topology generated by $\mathscr{S}$.

Since each element of $\mathscr{S}$ is open in the product topology, $\mathscr{T}' \subset \mathscr{T}$.

Now let $U \times V$ be a basis element for the product topology.

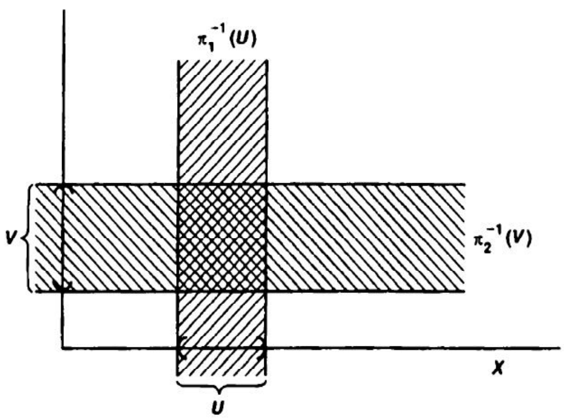

Figure 4 :

As explained in the remark ??, we have $U \times V = \pi_1^{-1}(U) \cap \pi_2^{-1}(V)$

$\Rightarrow U \times V \in \mathscr{T}'$

$\Rightarrow \mathscr{T} \subset \mathscr{T}'$

$\therefore \mathscr{T} = \mathscr{T}'$.

Lemma 3.8 : *Let $\mathscr{T}$ be a topology on X and* $Y \subset X$.

Then $\mathscr{T}_Y = \{Y \cap U \mid U \text{ is open in } X\}$ *is a topology on Y.*

Proof : Since $\varnothing,\ X \in \mathscr{T}$, we have $\varnothing = Y \cap \varnothing \in \mathscr{T}_Y$ and $Y = Y \cap X \in \mathscr{T}_Y$

Let $Y \cap U_\alpha \in \mathscr{T}_Y,\ \alpha \in \Lambda$.

Then $\bigcup_{\alpha\in\Lambda} (Y \cap U_\alpha) = Y \cap \left(\bigcup_{\alpha\in\Lambda} U_\alpha\right)$

Since $U_\alpha \in \mathscr{T} \Rightarrow \bigcup_{\alpha\in\Lambda} U_\alpha \in \mathscr{T}$

$\Rightarrow Y \cap \left(\bigcup_{\alpha\in\Lambda} U_\alpha\right) \in \mathscr{T}_Y$.

Let $Y \cap U_i \in \mathscr{T}_Y,\ i = 1, 2, ..., n.$

Then $\bigcap_{i=1}^{n} (Y \cap U_i) = Y \cap \left(\bigcap_{i=1}^{n} U_i\right) \in \mathscr{T}_Y$

$\therefore \mathscr{T}_Y$ is a topology on *Y*.

4. The Subspace Topology

Introduction :

Another important topology is the subspace topology, which is also constructed from the existing one. In this unit, we focus on product topology and subspace topology and relate them using open sets.

Definition 4.1 : Let $\mathscr{T}$ be a topology on X and Y be a subset of X. Then the topology $\mathscr{T}_Y = \{Y \cap U \mid U \text{ is open in } X\}$ on Y is called a subspace topology and with this $\mathscr{T}_Y$ we say that Y is a subspace of X.

We can construct the basis for the subspace topology Y using the base for the topology X as shown in the next lemma.

Lemma 4.2 : If $\mathscr{B}$ is a basis for a topology on X then the collection.

$\mathscr{B}_Y = \{Y \cap B \mid B \in \mathscr{B}\}$ is a basis for the topology $\mathscr{T}_Y$ on Y.

Proof : Let U be an open set in Y such that $x \in U$.

Since $U \in \mathscr{T}_Y$, $U = Y \cap V$ where V is open in X.

$\Rightarrow x \in V$.

Since $\mathscr{B}$ is a basis for a topology on X there exists $B \in \mathscr{B}$ such that $x \in B \subset V$.

$\Rightarrow x \in B \cap Y \subset V \cap Y$ where $B \cap Y \in \mathscr{B}_Y$.

$\therefore \mathscr{B}_Y$ is a basis for $\mathscr{T}_Y$.

Remark 4.3 : Every open set in a subspace topology need not be open in its parent topology, for example if $X = \mathbb{R}$ with usual topology, then $Y = [0,1)$ is open in the subspace topology Y, but not open in X. How ever thar is a special case, where every open set of Y is also open X, which we prove in the following lemma.

Lemma 4.4 : Let Y be a subspace of X. If U is open in Y and Y is open in X, then U is open in X.

Proof : Since U is open in Y, $U = Y \cap V$ for some V open in X.

As Y is open in X, we get that $U = Y \cap V$ is also open in X.

In the next theorem, we relate the subspace topology and the product topology.

Theorem 4.5 : If A is a subspace of X and B is subspace of Y, then the product topology on $A \times B$ is the same as the topology on $A \times B$ inherits as a subspace of $X \times Y$.

Proof : Let $\mathscr{T}$ be the product topology on $A \times B$.

Let $(A \times B) \cap (U \times V)$ be a basis element in the subspace topology on $A \times B$, where U is open in X and V is open in Y.

But $(A \times B) \cap (U \times V) = (A \cap U) \times (B \cap V)$.

Since $A \cap U$ is open in A and $B \cap V$ is open in B, we get that

$(A\cap U)\times(B\cap V)$ is open in the product topology on $A\times B$.

$\Rightarrow \mathscr{T}_{A\times B}\subset\mathscr{T}$

Let $U\times V$ be a basis element in the product topology $\mathscr{T}$ on $A\times B$.

$\Rightarrow U$ is open in A and V is open in B.

$\Rightarrow U = A\cap U'$ and $V = B\cap V'$ where U' is open in X and V' is open in Y.

$\therefore U\times V = (A\cap U')\times(B\cap V') = (A\times B)\cap(U'\times V')$

Since $U'\times V'$ is open in $X\times Y$, we get $U\times V$ is open in $\mathscr{T}_{A\times B}$

$\Rightarrow \mathscr{T}\subset\mathscr{T}_{A\times B}$

$\Rightarrow \mathscr{T}=\mathscr{T}_{A\times B}$.

Example 4.6 : Let Y = [0,1] be a subset of $X=\mathbb{R}$. Then the basis for subspace topology $\mathscr{T}_Y$ contains elements of the form $Y\cap(a,b)$ where (a,b) is a basis element for the topology on X then

$$Y\cap(a,b)=\begin{cases}(a,b), & \text{if } a,b\in Y;\\ [0,b), & \text{if } a\notin Y, b\in Y;\\ (a,1], & \text{if } a\in Y, b\notin Y;\\ \phi \text{ or } Y, & \text{if } a\notin Y, b\notin Y;\end{cases}$$

By definition of $\mathscr{T}_Y$, each of these sets are open in Y. (Note that the sets [0, b) and (a,1] are not open in X).

Since the collection of these sets form a basis for order topology in the case of Y = [0,1] its subspace topology and order topology are same.

However, next example shows that not every subspace topology is ordered.

Example 4.7 : Let X = $\mathbb{R}$ and $Y=[0,1)\cup\{2\}\subset\mathbb{R}$. Then Y is a subspace topology but not an order topology.

Proof : The set $\{2\}$ is open in the subspace topology on Y as

$$\{2\} = \left(\frac{3}{2}, \frac{5}{2}\right) \cap Y \text{ where } \left(\frac{3}{2}, \frac{5}{2}\right) \text{is open in X.}$$

But in the order topology on Y, $\{2\}$ is not open, because any basis element containing 2 is of the form $\{x \mid x \in Ya < x \leq 2\}$ for some $a \in Y$.

Clearly, $(a, 2]$ is not a subset of $\{2\}$.

$\therefore$ Sub space topology on Y is different from order topology on Y.

Definition 4.8 : Convex set

Let X be an order set and Y be a subset of X. Then Y is called a convex subset of X, if given $a, b \in Y$ with $a < b$ the entire interval (a, b) of points of X should contained in Y.

Example 4.9 : If $X = \mathbb{R}$, then all the intervals are convex.

Example 4.10 : If $X = \mathbb{R}$, then $\mathbb{N}$ is not convex as no interval is a subset of $\mathbb{N}$.

Example 4.11 : If $X = \mathbb{R}$, then $Y = [0,1) \cup \{2\}$ is not convex, because $0, 2 \in Y$, but $(0,2) \not\subseteq Y$.

The importance of convex sets is that, if subset is convex, then the order topology is same as the subspace topology (observer that in the example 4.7, Y is not convex). We prove this interesting result in the following theorem.

Theorem 4.12 : Let X be an ordered set in the order topology and Y be a convex subset of X. Then the order topology on Y is same as the topology Y inherits as a subspace of X.

Proof : Consider the ray (a,∞) in X.

If $a \in Y$, then $(a,\infty)\bigcap Y = \{x \mid x \in Y \text{ and } x > a\}$

which is an open ray in the order topology on Y.

If $a \notin Y$, then a is either lower bound for Y or upper bound for Y as Y is a convex subset of X.

($\because$ if there exists $x, y \in Y$ such that $x < a < y$, then as Y is convex, $a \in (x, y) \subset Y$)

If a is a lower bound then $(a,\infty)\bigcap Y = Y$.

If a is a upper bound then $(-\infty, a)\bigcap Y = \phi$.

$\therefore (a,\infty)\bigcap Y$ is open in the order topology on Y.

Similarly, $(-\infty, a)\bigcap Y$ is also open in the order topology on Y.

Since these sets $(a,\infty)\bigcap Y$ and $(-\infty, a)\bigcap Y$ form a subbasis for the subspace topology on Y, subspace topology is contained in the order topology on Y.

To prove the converse, since any open ray of Y is equal to the intersection of the open ray of X with Y, so it is open in subspace topology.

As open rays of Y form a subbasis for the order topology on Y, the order topology on Y is contained in the subspace topology.

EXERCISE - 3

1. Let *Y* be a subspace of *X*.

   (I) If *U* is open in *Y*, then *U* is open in *X*.

   (II) If *U* is open in *X*, then *U* is open in *Y*.

   (A) Only (I) is true. (B) Only (II) is true.

   (C) Both (I) and (II) are true. (D) Both (I) and (II) are false.

2. Consider the set $\mathscr{B} = \{U \times V \mid U$ is open in X and V is open in $Y\}$. Then

(I) $\mathscr{B}$ is a basis for $X \times Y$.

(II) $\mathscr{B}$ is a topology on $X \times Y$.

(A) Only (I) is true. (B) Only (II) is true.

(C) Both (I) and (II) are true. (D) Both (I) and (II) are false.

3. Show that if Y is a subspace of X, and A is a subset of Y, then the topology A inherits as a subspace of Y is the same as the topology it inherits as a subspace of X.

4. Let $X = \mathbb{R}$ be a usual topology and $Y = \mathbb{Z}$. Show that the subspace topology on Y is the discrete topology.

5. Consider the set $Y = [-1,1]$ as a subspace of $\mathbb{R}$. Which of the following sets are open in Y? Which are open in $\mathbb{R}$?

$$A = \left\{x \mid \frac{1}{2} < |x| < 1\right\}$$

$$B = \left\{x \mid \frac{1}{2} < |x| \le 1\right\}$$

$$C = \left\{x \mid \frac{1}{2} \le |x| < 1\right\}$$

$$D = \left\{x \mid \frac{1}{2} \le |x| \le 1\right\}$$

5. Closed Sets

Introduction

With the help of open sets, we can introduce some of the basic concepts of a topological space. In this unit we discuss the notion of closed set.

Definition 5.1 : Closed Set

A subset A of a topological space X is said to be closed if X – A (i.e., A^c) is open in X.

Example 5.2 : The subset [*a, b*] of $\mathbb{R}$ is closed as $[a,b]^c = (-\infty, a) \cup (b, \infty)$ is open in $\mathbb{R}$.

Example 5.3 : In the plane $\mathbb{R}^2$ the set $A = \{x \times y \mid x \geq 0 \text{ and } y \geq 0\}$ is closed.

Proof : The complement of A is given by

$$\mathbb{R}^2 - A = \{x \times y \mid x < 0 \text{ or } y < 0\}$$

$$= \{x \times y \mid x < 0 y \in \mathbb{R}\} \cup \{x \times y \mid x \in \mathbb{R}, y < 0\}$$

$$= ((-\infty, 0) \times \mathbb{R}) \cup (\mathbb{R} \times (-\infty, 0))$$

Since $(-\infty, 0) \times \mathbb{R}$ and $\mathbb{R} \times (-\infty, 0)$ are open in $\mathbb{R}^2$, we get

$\mathbb{R}^2 - A$ is open.

$\Rightarrow A$ is closed.

Example 5.4 : In the finite complement topology on X, the closed sets are finite subsets and X itself.

Proof : Let A be a closed set.

$\Rightarrow A^c$ is open in X.

$\Rightarrow X - A^c$ is finite or $A^c = \phi$.

$\Rightarrow A$ is finite or A = X.

Example 5.5 : In the discrete topology on X, each set A is closed because every subset of X is open implies X – A is open.

Example 5.6 : Let $X = \mathbb{R}$, $Y = [0,1] \cup [2,3]$. Then [0,1] is both open and closed in Y.

Proof : we can write [0,1] as $[0,1] = \left(-\frac{1}{2}, \frac{3}{2}\right) \cap Y$

$\left(-\frac{1}{2},\frac{3}{2}\right)$ is open X implies [0,1] is open in Y.

Also $Y-[0,1]=[2,3]$ and [2,3] is open implies that [0,1] is closed.

$\therefore$ [0,1] is both open and closed in Y.

Remark 5.7 : From the above examples, we can observe that sets are not doors as a door must be either open or closed, where as a set can be open, or closed, or both, or neither.

The collection of closed sets have the properties similar to open sets as we discuss in the next result.

Theorem 5.8 : Let X be a topological space. Then following holds :

1. ϕ, X are closed.
2. Arbitrary intersection of closed sets is closed.
3. Finite union of closed sets is closed.

Proof : ϕ and X are closed because they are compliments of open sets X and ϕ respectively.

Given a collection of closed sets $\{A_\alpha\}_{\alpha\in J}$, by Demorgan laws, we have

$$X-\bigcap_{\alpha\in J} A_\alpha=\bigcup_{\alpha\in J}\left(X-A_\alpha\right).$$

Since the sets $X-A_\alpha$ are open, their arbitrary union $\bigcup_{\alpha\in J}\left(X-A_\alpha\right)$ is open.

Implies $X-\bigcap_{\alpha\in J} A_\alpha$ is open.

Thus $\bigcap_{\alpha\in J} A_\alpha$ is closed.

If A_i is a closed for i = 1, 2, ..., n, then $X-\bigcup_{i=1}^{n} A_i=\bigcap_{i=1}^{n}\left(X-A_i\right)$

As the finite intersection of open sets is open, we have $\bigcap_{i=1}^{n}(X - A_i)$ is open.

Hence $\bigcap_{i=1}^{n} A_i$ is closed.

Definition 5.9 : Let Y be subspace of X. A subspace A of Y is said to be closed if Y – A is open in Y.

Theorem 5.10 : Let Y be a subspace of X then a set A is closed in Y if and only if it equals the intersection of a closed set of X with Y.

Proof : Suppose A is closed in Y.

$\Rightarrow Y - A$ is open in Y.

$\Rightarrow Y - A = U \cap Y$ where U is open in X.

$\Rightarrow A = U^c \cap Y$ (See the Figure 5)

$\therefore A = C \cap Y$ where $C = U^c$ is closed in X

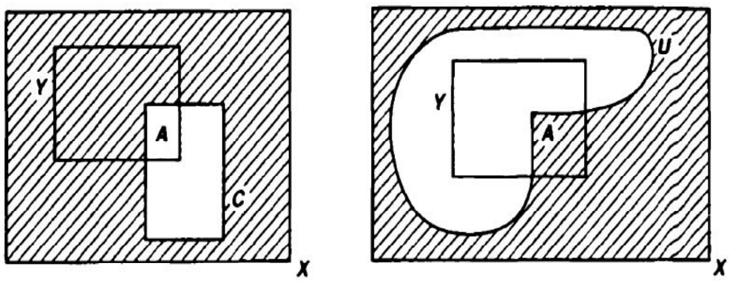

Figure 5

Conversely, suppose $A = C \cap Y$ for some closed set C in X.

$\Rightarrow X - C$ is open in X.

$\Rightarrow Y \cap (X - C)$ is open in Y.

But $Y - A = Y \bigcap (X - C)$ (refer the Figure 5)

$\therefore Y - A$ is open in Y.

$\Rightarrow A$ is closed in Y.

Remark 5.11 : A closed subset of Y need not be closed in X.

For example, consider $Y = \left[0, \frac{1}{2}\right)$ and $X = \mathbb{R}$ then $\left[0, \frac{1}{2}\right)$ is closed in Y but not closed in $\mathbb{R}$. However, we have the following.

Theorem 5.12 : Let Y be a subspace of X. If A is closed in Y and Y is closed in X then A is closed in X.

Proof : Given that A is closed in Y.

$\Rightarrow A = C \bigcap Y$, C is closed in X.

Since Y is closed in X we get that $C \bigcap Y$ is closed in X.

$\Rightarrow A$ is closed in X.

❖❖❖

UNIT - II

PRODUCT TOPOLOGY

1. Closure and Interior of a Set

Introduction :

With the help of open sets, we can introduce some of the basic concepts of a topological space. In this unit we discuss the notion of closure of a set and interior of a set.

Definition 1.1 : Interior of a set

Let X be a topological space and $A \subset X$. Then the interior of A is the union of all open sets contained in A and is denoted by Int A (or A°)

i.e. $IntA = \bigcup\{U \text{ is open in } X \mid U \subset A\}$.

Since each $U \subset A \Rightarrow IntA \subset A$.

Remark 1.2 : 1. Int A is the largest open set contained in A as Int A is the union of all such sets.

2. If A is open, then Int A = A as A is the largest set such that $A \subset A$.

Definition 6.3 : Closure of a Set

Let X be a topological space and $A \subset X$. Then closure of A is the intersection of all closed sets containing A and is denoted by $\overline{A}$.

i.e. $\overline{A} = \bigcap\{F \text{ is closed in } X \mid F \supset A\}$.

Since each $F \supset A \Rightarrow \overline{A} \supset A$.

Remark 1.4 :

1. $\overline{A}$ is the smallest closed set contained in A.
2. If A is closed, then $\overline{A} = A$.

Theorem 1.5 : Let Y be a subspace of X and A be a subset of Y . Let $\overline{A}$ denotes the closure of A in X. Then the closure of A in Y is $\overline{A} \cap Y$.

Proof : Let B be the closure of A in Y.

Since $\overline{A}$ is closed in X, we have $\overline{A} \cap Y$ is closed in Y.

Also $A \subset \overline{A} \cap Y$. Since B is the smallest closed set containing A, we get $B \subset \overline{A} \cap Y$.

As B is closed in Y, we have $B = C \cap Y$ for some closed set C in X.

Since $A \subset B$, we get $A \subset C$.

As $\overline{A}$ is the smallest closed set in X containing A, we get $\overline{A} \subset C$

$\Rightarrow \overline{A} \cap Y \subset C \cap Y = B$

$\Rightarrow \overline{A} \cap Y \subset B$

$\Rightarrow B = \overline{A} \cap Y$.

Theorem 1.6 : Let A be a subset of the topological space X. Then

(i) $x \in \overline{A}$ if and only if every open set U containing x intersects A

i.e. $x \in \overline{A}$ *iff* $U \cap A \neq \phi \forall x \in U$

(ii) Suppose the topology X is given by a basis, then $x \in \overline{A}$ if and only if every basis elements B containing x intersects A

i.e. $x \in \overline{A}$ *iff* $B \cap A \neq \phi \forall x \in B$.

Proof. (i) : We prove $x \notin \overline{A}$ iff there exists open set U containing x such that $U \cap A = \phi$.

Let $x \notin \overline{A}$

$\Rightarrow x \in X - \overline{A}$ and $X - \overline{A}$ is open.

By taking $U = X - \overline{A}$, we get $x \in U$ and $U \cap A = \phi$.

Conversely, suppose there exists open set $x \in U$ such that $U \cap A = \phi$

$\Rightarrow A \subset X - U$

Since $X - U$ is closed containing A and $\overline{A}$ is the smallest closed set containing A, we get $\overline{A} \subset X - U$

$\Rightarrow x \notin \overline{A}$.

(ii) Suppose $x \in \overline{A}$ and B is a basis element with $x \in B$.

As B is open, by (i), we get $B \cap A \neq \phi$.

Conversely, suppose $B \cap A \neq \phi \forall x \in B$.

Let U be an open set such that $x \in U$.

Then there exists a basis element B such that $x \in B \subset U$

$\Rightarrow A \cap U \neq \phi$

$\Rightarrow x \in \overline{A}$.

Remark 1.7 : A open set U of X containing *x* is called a neighborhood of *x*. With this terminology, the first part of the above theorem can be stated as $x \in \overline{A}$ iff A intersects every neighbourhood of X.

Example 1.8 : If $X = \mathbb{R}$ and A = (0,1], then $\overline{A}$ = [0,1].

Proof : Since every neighborhood of 0 intersects A, we get $0 \in \overline{A}$

If *x* < 0, then $(-\infty, 0)$ is a neighborhood of *x* which doesn ft intersect A.

Similarly, *x* > 1, then $(1, \infty)$ is a neighborhood of *x* which doesn't intersect A.

Hence $\overline{A} = [0,1]$.

Example 1.9 : If $X=\mathbb{R}$ and $A=\left\{\frac{1}{n}, n\in\mathbb{Z}_+\right\}$ then $\overline{A}=A\cup\{0\}$.

Proof : Let B = (*a, b*) be a basis element with $0\in(a,b)$.

Then by Archimedean property, there exists *n* such that $a<\frac{1}{n}<b$

Implies B intersects A.

Also if *a* > 1, then $(1,\infty)$ is a neighborhood of *a* which doesn't intersect A.

And if 0 < *a* < 1, then $a\in(1/m,1/n)$ for some $m,n\in\mathbb{N}$.

Then $(a-\alpha,a+\alpha)$ is a neighborhood of *a* which doesn't intersect A,

where $\alpha=\frac{1}{2}\min\{1/m,1/n\}$.

Hence $\overline{A}=A\cup\{0\}$.

Example 1.10 : If $X=\mathbb{R}$ and $C=\{0\}\cup(1,2)$ then $\overline{C}=\{0\}\cup[1,2]$.

Example 1.11 : If $X=\mathbb{R}$, then $\overline{\mathbb{Q}}=\mathbb{R}$.

Proof : Let $x\in\mathbb{R}$ and B = (*a,b*) be a basis element with $x\in(a,b)$.

Since $a,b\in\mathbb{R}$, there exists $c\in\mathbb{Q}$ such that *a* < *c* < *b*.

Thus $B\cap\mathbb{Q}\neq\phi$.

Therefore $\overline{\mathbb{Q}}=\mathbb{R}$.

Example 1.12 : Consider the subspace Y = (0,1] of $\mathbb{R}$ and $A=\left(0,\frac{1}{2}\right)$.

Then $\overline{A}=\left(0,\frac{1}{2}\right]$ in Y.

2. Limit Points

Introduction :

With the help of open sets, we can introduce some of the basic concepts of a topological space. In this unit we discuss the notion of limit point of a set.

Definition 2.1 : Limit Point

Let A be a subset of a topological space X. Then a point $x \in X$ is called a limit point of A if $U \cap (A / \{x\}) \neq \emptyset$ for all open set U containing x.

(i.e. every neighborhood of x intersects some point of A other than x itself.)

The set of all limit points of A is denoted by A'.

Example 2.2 :

1. If A = (0, 1], then 0 is a limit point of A and also every element of A is a limit point.

   $\therefore A' = [0,1]$.

2. If $B = \left\{ \frac{1}{n} \middle| n \in \mathbb{Z}_+ \right\}$, then B' = {0} as zero is the only limit point of B.

3. $C = \{0\} \cup (1,2)$ then C'= [1, 2].

We give the relationship between the closure of a set and the limit points of that set in the following theorem.

Theorem 2.3 : *Let A be subset of the topological space X and A' be the set of all limit points of A. Then* $\overline{A} = A \cup A'$.

Proof : Let $x \in \overline{A}$.

If $x \in A$, then $x \in A \cup \overline{A}$.

Suppose $x \notin A$. Then $A \setminus \{x\} = A$.

As $x \in \overline{A}$, we get $U \cap A \neq \varnothing \quad \forall x \in U$

$\Rightarrow U \cap A \setminus \{x\} \neq \varnothing \quad \forall x \in U$

$\Rightarrow x$ is a limit point of A.

$\Rightarrow x \in A'$

$\Rightarrow x \in A \cup A'$

$\therefore \overline{A} \subset A \cup A'$.

Now conversely, let $x \in A'$.

Then $U \cap A \setminus \{x\} \neq \varnothing \quad \forall x \in U$

$\Rightarrow U \cap A \neq \varnothing \; \forall x \in U$

$\Rightarrow x \in \overline{A}$

$\therefore A' \subset \overline{A}$

$\Rightarrow A \cup A' \subset \overline{A}$

Hence $\overline{A} = A \cup A'$.

Corollary 2.4 : *A subset of a topological space is closed if and only it contains all its limit points.*

Proof : A is closed if and only $\overline{A} = A$

if and only $A \cup A' = \mathrm{A}$

if and only $A' \subset \mathrm{A}$

if and only A contains all its limit points.

3. Hausdorff Space

Introduction :

In this unit, we introduce the Hausdroff space and discuss the closed sets of the Hausdroff space.

Definition 3.1 : Hausdorff Space

A topological space X is called a Hausdorff space if for each pair x_1, x_2 of distinct points of X, there exist disjoint neighborhoods U_1 and U_2 of x_1 and x_2 respectively.

(i.e. $\forall x_1 \neq x_2$, there exists $x_1 \in U_1$ and $x_2 \in U_2$ such that $U_1 \cap U_2 = \varnothing$).

Example 3.2 : $\mathbb{R}$ with standard topology is Hausdorff.

Proof : Let $a, b \in \mathbb{R}$ with $a < b$.

Then there exists a rational number $r \in \mathbb{Q}$ such that $a < r < b$.

By taking $U_1 = (-\infty, r)$ and $U_2 = (r, \infty)$, we get that a $a \in U_1$ and $b \in U_2$ such that $U_1 \cap U_2 = \varnothing$.

Example 3.3 : Any non empty space X with indiscrete topology is not Hausdorff.

Proof : Let $\mathscr{T} = \{\varnothing, X\}$ be the topology.

Let $x, y \in X$ with $x \neq y$.

As there is only one non empty set X, we cannot separate these two points with two disjoint open sets. Hence X is not Hausdorff.

Example 3.4 : Let $X = \mathbb{R}$ with given topology $\mathscr{T} = \{(-n, n) \mid n \in \mathbb{Z}\}$. Then X is not Hausdorff.

Proof : Consider $0, \frac{1}{2} \in \mathbb{R}$.

As $0, \frac{1}{2} \in (-n, n)$ for each *n*, we can't have two disjoint open sets U and V such that $0 \in U$ and $\frac{1}{2} \in V$.

Hence X is not Hausdorff.

Example 3.5 : $\mathbb{R}$ with finite complement topology is not Hausdorff.

Proof : Let $x, y \in \mathbb{R}$ with $x \neq y$.

Suppose there exist open sets U and V such that $x \in U$; $y \in V$ such that $U \cap V = \varnothing$

Then $U \subset V^c$.

As V is open, V^c is finite and so U is finite.

Also as U is open, U^c is finite, then $U \cup U^c = \mathbb{R}$ is finite, which is a contradiction.

Therefore, $\mathbb{R}$ with finite complement topology is not Hausdorff.

Theorem 3.6 : Every finite point set in Hausdorff space is closed.

Proof : Since every finite point set is the finite union of single point set it is enough to prove that each singleton set is closed.

Let $A = \{x_0\}$ and $x \neq x_0$. Since X is Hausdorff, there exists $x \in U$ and $x_0 \in V$ such that $U \cap V = \varnothing$.

In particular, $U \cap \{x_0\} = \varnothing$

$\Rightarrow U \cap A = \varnothing$ and $x \in U$

$\Rightarrow x \notin \overline{A}$

$\therefore \overline{A} = \{x_0\} = A$

$\Rightarrow A$ is closed.

Theorem 3.7 : Let X be a space in which every finite set is closed and $A \subset X$. Then the point x is a limit point of A if and only if every neighbourhood of x contains infinitely many points of A.

Proof : Clearly, if every neighbourhood U of x contains infinitely many points of A, then $U \cap A \setminus \{x\} \neq \varnothing$

$\Rightarrow x$ is a limit point of A.

Suppose there exists a neighbourhood U of x which contains only finitely many points of A.

Then U also intersects A\{x} at finitely many points say at $x_1, x_2, ..., x_m$.

i.e, $U \cap (A \setminus \{x\}) = \{x_1, x_2, ..., x_m\}$.

As finite set is closed, we get $V = X \setminus \{x_1, x_2, ..., x_m\}$ is open.

Then $W = U \cap V$ is also open containing x.

But $W \cap (A \setminus \{x\}) = \varnothing$, which is a contradiction to x is a limit point.

Thus every neighbourhood of x contains infinitely many points of A.

Definition 3.8 : Let X be a topological space. Then a sequence (x_n) in X is said to be convergence to $x \in X$ if for every neighbourhood U of x there exists $N \in \mathbb{N}$ such that $x_n \in U \forall n \geq N$.

Remark 3.9 :

1. In a topological space, a sequence may converge to more than one point.
2. In ($\mathbb{N}$,finite complement topology), the sequence (1,2, ,3, ...) converges to every $n \in \mathbb{N}$ because for any open set $n \in U$, since X – U is finite, we get all but finitely many elements of the sequence lie in U.

Theorem 3.10 : If X is a Hausdorff space then a sequence (x_n) of points of X converges to at most one point of X.

Proof : Suppose x_n is a sequence which converges to $x \in X$ and $y \in X$ where $y \neq x$. Since X is Hausdorff and $x \neq y$ there exists U and V open sets such that $x \in U$ and $y \in V$ and $U \cap V = \varnothing$.

Since x_n converges to x and U is a neighborhood of $x, x_n \in U$ for all but finitely many.

$\Rightarrow$ only finitely many elements of (x_n) are outside U.

Also x_n converges to y implies V contains all x_n's but finitely many which is a contradiction.

$\therefore x_n$ converges to at most one point.

Remark 3.11 : If a sequence x_n converges in a Hausdorff space, then it converges to only one point say x and this x is called the limit of x_n and is denoted by $x_n \to x$.

Definition 3.12 : A topological space X satisfies T_1 axiom if every finite point set is closed in X.

Remark 3.13 : Every Hausdorff space satisfies T_1 axiom. The converse need not be true i.e. a topological space satisfying T_1 axiom need not be a Hausdorff space. e.g. Let X be a infinite set and consider the finite complement topology on X then this topological space satisfies T_1 axiom. But (X_1, Finite complement topology) is not Hausdroff. Infact we prove that any two open sets in X intersects i.e. $U \bigcap V \neq \varnothing$. Suppose not then $\exists U$ and V such that $U \bigcup V = \varnothing$

$\Rightarrow V \subset X - U$.

Since U is open, X – U is finite.

$\Rightarrow V$ is finite.

Since X is infinite, X – V is infinite.

$\Rightarrow V$ is not open.

Theorem 3.14 :

1. Every simple order set is a Hausdroff space in the order topology.
2. A subspace of a Hausdroff space is Hausdroff .
3. Product of two Hausdroff spaces is Hausdroff.

Proof :

1. Let A be a subset i.e. simply ordered subset of X and $x, y \in A$ with $x \neq y$.

   Consider, without loss of generality, $x < y, S = \{z \mid x < z < y\}$

If $S = \varnothing$ then $U(-\infty, y)$ and $V = (x, \infty)$ are neighbourhood of x and y such that $U \bigcap V = \varnothing$.

If $S \neq \varnothing$ let $z \in S$ then $U = (-\infty, z)$ and $V = (z, \infty)$ are neighbourhood of x and y such that $U \bigcap V = \varnothing$.

$\Rightarrow A$ is Hausdroff.

2. Let y be a subspace of a Hausdroff space X and $x, y \in Y$ with $x \neq y$

$\Rightarrow x, y \in X$.

Since X is Hausdroff $\exists U$ and V such that $x \in U$, $y \in V$, $U \bigcap V = \varnothing$

$\Rightarrow x \in U \bigcap Y$ and $y \in V \bigcap Y$ and $(U \bigcap Y) \bigcap (V \bigcap Y) = \varnothing$.

$\therefore Y$ is Hausdroff.

3. Suppose X and Y are Hausdroff space. Let $x_1 \times y_1$, $x_2 \times y_2 \in X \times Y$ such that $x_1 \times y_1 \neq x_2 \times y_2$

$\Rightarrow x_1 \neq x_2$ or $y_1 \neq y_2$.

If $x_1 \neq x_2$ then $\exists U, V$ such that $x_1 \in U$ then $x_2 \in V, U \bigcap V = \varnothing$.

$\Rightarrow x_1 \times y_1 \in U \times Y$ and $\Rightarrow x_2 \times y_2 \in V \times Y$ such that $(U \times Y) \bigcap (V \times Y) = \varnothing$.

$\therefore X \times Y$ is Hausdroff space.

EXERCISE - 4

1. Consider the following statements

(I) In the finite complement topology on a set X, every finite set is closed.

(II) In the discrete topology on a set X, every finite set is closed.

(A) Only (I) is true. (B) Only (II) is true.

(C) Both (I) and (II) are true. (D) Both (I) and (II) are false.

2. Let X be a topological space.

(I) Arbitrary intersection of open sets are open.

(II) Arbitrary union of closed sets are closed.

(A) Only (I) is true. (B) Only (II) is true.

(C) Both (I) and (II) are true. (D) Both (I) and (II) are false.

3. Consider the following

(I) If X is Hausdorff space, then every singleton set is closed.

(II) If every singleton set in X is closed, then X is Hausdorff.

(A) Only (I) is true. (B) Only (II) is true.

(C) Both (I) and (II) are true. (D) Both (I) and (II) are false.

4. Show that if A is closed in X, and B is closed in Y, then $A \times B$ is closed in $X \times Y$.

5. Show that if U is open in X, and A is closed in X, then $U \setminus A$ is open in X and $A \setminus U$ is closed in X.

6. For subsets A and B of X, show that

(a) If $A \subset B$, then $\overline{A} \subset \overline{B}$

(b) $\overline{A \bigcup B} = \overline{A} \bigcup \overline{B}$

4. Continuous Functions

INTRODUCTION

We have seen the concept of continuity on real line and in the plane. In this section, we define the continuity function which generalizes all these existing definitions. We also learn homeomorphism, which is analogous to the isomporphism between algebraic structures.

Definition 4.1 : Continuity

Let X and Y be topological spaces. Then $f : X \to Y$ is continuous if for each open set V in Y, $f^{-1}(V)$ is open in X.

Example 4.2 : If *X* is discrete topology then every function $f : X \to Y$ is continuous, because every subset of *X* is open and hence is $f^{-1}(V)$.

Example 4.3 : If *Y* is indiscrete topology, then any function $f : X \to Y$ is continuous.

Proof : Since *Y* is indiscrete, the only open sets are $\varnothing$ and *Y*.

Also $f^{-1}(\varnothing) = \varnothing$ and $f^{-1}(Y) = X$, which are open in *X*.

Hence $f : X \to Y$ is continuous.

In the definition of continuity, the condition on open sets can be reduced to basis elements, as we prove in the following lemma.

Lemma 4.4 : *A function $f : X \to Y$ is continuous if $f^{-1}(B)$ is open for every basis element $B \in \mathscr{B}$.*

Proof : Let *V* be open in *Y*.

Since $\mathscr{B}$ is a basis for *Y*, we can write $V = \bigcup_{\alpha \in I} B_\alpha$

$\Rightarrow f^{-1}(V) = f^{-1}\left(\bigcup_{\alpha \in I} B_\alpha\right) = \bigcup_{\alpha \in I} f^{-1}\left(B_\alpha\right).$

If $f^{-1}\left(B_\alpha\right)$ is open $\forall B_\alpha \in \mathscr{B}$ then $\bigcup_{\alpha \in I} f^{-1}\left(B_\alpha\right)$ is also open in *X*

$\Rightarrow f^{-1}(V)$ is open in *X*.

$\therefore f$ is continuous.

Theorem 4.5 : *Let X and Y be topological spaces and $f : X \to Y$ then the following are equivalent*

1. *f is continuous.*
2. *For every subset A of X, $f\left(\overline{A}\right) \subset \overline{f(A)}$.*
3. *For every closed set B in Y; $f^{-1}(B)$ is closed in X.*

4. *For every* $x \in X$ *and every neighbourhood V of f (x), there exists a neighbourhood U of x such that* $f(U) \subset V$.

Proof : (1) $\Rightarrow$ (2) Suppose f is continuous.

Let $A \subset X$ and $x \in \overline{A}$.

To show that $f(x) \in \overline{f(X)}$, let V be a neighborhood of $f(x)$.

Since f is continuous, $f^{-1}(V)$ is open in X.

Also $x \in f^{-1}(V)$ implies $f^{-1}(V) \bigcap A \neq \emptyset$.

Let $y \in f^{-1}(V) \bigcap A$

$\Rightarrow f(y) \in V$ and $f(y) \in f(A)$

$\Rightarrow f(y) \in V \bigcap f(A)$

$\therefore V \bigcap f(A) \neq \emptyset$

$\Rightarrow f(x) \in \overline{f(A)}$

$\Rightarrow f(\overline{A}) \subset \overline{f(A)}$

(2) $\Rightarrow$ (3) Let $A = f^{-1}(B)$

$\therefore f(A) = f\left(f^{-1}(B)\right) \subset B$

$\Rightarrow \overline{f(A)} \subset \overline{B} = B$.

Since $A \subset X$, we have $f(\overline{A}) \subset \overline{f(A)} \Rightarrow f(\overline{A}) \subset B$

$\Rightarrow \overline{A} \subset f^{-1} f(\overline{A}) \subset f^{-1}(B) = A$

$\Rightarrow \overline{A} \subset A$

$\Rightarrow A$ is closed.

$\Rightarrow f^{-1}(B)$ is closed.

(3) $\Rightarrow$ (4) Let $x \in X$ and *V* be an open set in *Y* such that $f(x) \in V$.

Then $Y - V$ is closed in *Y* and hence $f^{-1}(Y-V)$ is closed in *X*.

$\Rightarrow X - f^{-1}(V)$ is closed in *X*.

$\Rightarrow f^{-1}(V)$ is open in *X*.

So by letting $U = f^{-1}(V)$, we have $x \in f^{-1}(V) = U$ and

$f(U) = f \, o \, f^{-1}(V) \subset V$

$\Rightarrow f(U) \subset V$.

(4) $\Rightarrow$ (1) Let *V* be an open set in *Y* and $A = f^{-1}(V)$.

To show *A* is open in *X*, let $x \in A$. Then $f(x) \in V$.

Then there exists an open set *U* containing *x* such that $f(U) \subset V$

$U \subset f^{-1}(V) = A$

$\therefore x \in U \subset A$

$\Rightarrow A$ is open in *X*.

$\therefore f^{-1}(V)$ is open in *X*.

$\Rightarrow f$ is continuous.

As discussed in the introduction, we now show that continuity in real case is a special case of our definition.

Theorem 4.6 : *If* $f : \mathbb{R} \to \mathbb{R}$ *is continuous by means of topological spaces i.e.* $f^{-1}(U)$ *is open for all open set U then f is continuous by* $\varepsilon - \delta$ *definition i.e. given* $x_0 \in \mathbb{R}$ *and* $\varepsilon > 0$*, there exists* $\delta > 0$ *such that* $|x - x_0| < \delta$ *implies* $|f(x) - f(x_0)| < \varepsilon$.

Proof : Let $x_0 \in \mathbb{R}$ and $\varepsilon > 0$. Then $V = \left(f(x_0) - \varepsilon, f(x_0) + \varepsilon\right)$ is open in $\mathbb{R}$.

As f is continuous, $f^{-1}(V)$ is open.

Since $x_0 \in f^{-1}(V)$, for two real number *a* and *b*, we have $x_0 \in (a,b) \subset f^{-1}(V)$

Now take $\delta = \min\{x_0 - a, b - x_0\}$.

Let $x \in \mathbb{R}$ be such that $|x - x_0| \in \delta$.

Then $x \in (x_0 - \delta, x_0 + \delta) \subset (a,b)$

$\Rightarrow x \in (a,b) \subset f^{-1}(V)$

$\Rightarrow f(x) \in V = (f(x_0) - \varepsilon, f(x_0) + \varepsilon)$

$\Rightarrow |f(x) - f(x_0)| < \varepsilon$

$\therefore$ *f* continuous by $\varepsilon - \delta$ definition.

The next few results are about construction of continuous from one topological space to another.

Theorem 4.7 : If $f : X \to Y$ maps all of X into single point $y_0 \in Y$, then *f* is continuous.

Proof : Let U be an open set in Y. Then $f^{-1}(U) = \begin{cases} \varnothing, & \text{if } y_0 \notin U; \\ X, & \text{if } y_0 \in U. \end{cases}$

As $\varnothing$ and X are open in X, we get that $f^{-1}(U)$ is open in X.

Therefore, *f* is continuous.

Theorem 4.8 : If A is a subspace of X then the inclusion function $j : A \to X$ is continuous.

Proof : If U is open in X, then $j^{-1}(U) = A \cap U$.

Since $A \cap U$ is open in A $j^{-1}(U)$ is open in A.

$\Rightarrow j$ is continuous.

Theorem 4.9 : If $f : X \to Y$ and $g : Y \to Z$ are continuous then $g\, o f : X \to Z$ is continuous.

Proof : Let General Topology Page 50 and U is open in Z.

$\Rightarrow g^{-1}(U)$ is open in Y.

$\Rightarrow f^{-1}\left(g^{-1}(U)\right) = (g\, o f)^{-1}(U)$ is open in X.

$\therefore g\, o f$ is continuous.

Theorem 4.10 : If $f : X \to Y$ is continuous and A is a subspace of X then $f|_A : A \to Y$ is continuous.

Proof : We can observe that $f|_A = f\, o j$ where *j* is the inclusion map.

As *f* and *j* are continuous, there composition $f|_A = f\, o j$ is also continuous.

We now prove interesting result of continuity, called pasting lemma, which roughly states that under some conditions, two continuous functions pasted (glued) together gives a continuous function.

Lemma 4.11 : The pasting Lemma

Let $X = A \cup B$ where A and B are closed in X. Let $f : A \to Y$ and $g : B \to Y$ be continuous. If $f(x) = g(x)\, \forall x \in A \cap B$, then *f* and *g* combine to give a continuous function $h : X \to Y$ defined by $h(x) = \begin{cases} f(x) & x \in A \\ g(x) & x \in B \end{cases}$

Proof : Since $f(x) = g(x)\, \forall x \in A \cap B$, h is well defined.

To show $h : X \to Y$ is continuous, let C be closed in Y.

Then $h^{-1}(C) = f^{-1}(C) \cup g^{-1}(C)$ is closed as $f^{-1}(C)$ is closed in A and hence in X; similarly $g^{-1}(C)$ is closed in X.

Therefore, $h : X \to Y$ is continuous.

Remark 4.12 : This result is also true if A and B are open i.e. $X = A \bigcup B$ where A and B are open in X.

Example 4.13 : The function $h:\mathbb{R} \to \mathbb{R}$ defined by $h(x) = \begin{cases} x, & x \le 0; \\ \dfrac{x}{2}, & x \ge 0. \end{cases}$ is continuous.

Proof : Take $A = (-\infty, 0]$ and $B = [0, \infty)$.

Define $f : A \to \mathbb{R}$ by $f(x) = x$ and

$g : B \to \mathbb{R}$ by $g(x) = \dfrac{x}{2}$.

Then *f* and *g* are continuous.

Here $A \bigcap B = \{0\}$ and $f(0) = 0 = g(0)$.

As $h(x) = \begin{cases} f(x), & x \in A; \\ g(x), & x \in B. \end{cases}$.

by pasting lemma, *h* is continuous.

5. Homeomorphism

Definition 5.1 : Let *X* and *Y* be topological spaces and $f : X \to Y$ be a bijective map. Then *f* is called a homeomorphism if *f* and f^{-1} are continuous and in this case *X* and *Y* are said to be homeomorphic.

Lemma 5.2 : $f : X \to Y$ *is a homeomorphism* $\Leftrightarrow$ $f(U)$ *is open in Y if and only if U is open in X.*

Proof : To show $f : X \to Y$ is continuous, let *U* be open in *Y*.

$\Rightarrow U = f\left(f^{-1}(U)\right)$ is open in *Y*

$\Rightarrow f^{-1}(U)$ is open in *X*.

$\therefore$ *f* is continuous.

To show $f^{-1}: Y \to X$ is continuous, let *U* be open in *X*.

$\Rightarrow f(U)$ is open in *Y*.

$\Rightarrow (f^{-1})^{-1}(U) = f(U)$ is open in *Y*.

$\Rightarrow f^{-1}$ is continuous.

On the other hand assume that *f* is a homeomorphism. Then *f* and f^{-1} are continuous.

Suppose $f(U)$ is open in *Y*. As $f: X \to Y$ is continuous, we get $f^{-1}(f(U))$ is open in *X*.

$\Rightarrow U$ is open in *X*.

Now if *U* is open in *X*, as $f^{-1}: Y \to X$ is continuous, we have $(f^{-1})^{-1}(U)$ is open in *Y*.

$\Rightarrow f(U)$ is open in *Y*.

Example 5.3 : The map $f: \mathbb{R} \to \mathbb{R}$ defined by $f(x) = 3x+1$ is a homeomorphism.

Proof : Clearly *f* is bijective and continuous.

Also $f^{-1}(x) = \dfrac{y-1}{3}$ is continuous.

Hence *f* is a homeomorphism.

Example 5.4 : The function $f: (-1,1) \to \mathbb{R}$ defined by $f(x) = \dfrac{x}{1-x^2}$ is a homeomorphism.

Proof : Clearly f is continuous.

To show f is one one, let $f(x) = f(y)$

Then $x - y = xy(y - x)$

As $xy \neq -1$, we get $x = y$.

To show f onto, let $0 \neq y \in \mathbb{R}$.

Then $f\left(\dfrac{-1+\sqrt{1+4y^2}}{2y}\right) = y$

Also $f(0) = 0$.

Hence f is onto.

Also $f^{-1}(y) = \begin{cases} \dfrac{-1+\sqrt{1+4y^2}}{2y} & y \neq 0 \\ 0 & y = 0 \end{cases}$

Then $f^{-1}(y)$ is continuous and hence f is a homeomorphism.

Example 5.5 : The function $f : \mathbb{R}_l \to \mathbb{R}$ given by $f(x) = x$ is not a homeomorphism.

Proof : Here $f^{-1} : \mathbb{R} \to \mathbb{R}_l$ is not continuous, because the inverse image of the set [1,2) which is open in $\mathbb{R}_l$, is itself, which is not open in $\mathbb{R}$.

Hence f is not a homeomorphism.

EXERCISE - 5

1. Let X and Y be topological spaces and $f : X \to Y$.

(I) If f is continuous, then $f(\overline{A}) \subset \overline{f(A)}$, for every subset A of X.

(II) If for every subset A of X, $f(\overline{A}) \subset \overline{f(A)}$, then f is continuous.

(A) Only (I) is true. (B) Only (II) is true.

(C) Both (I) and (II) are true. (D) Both (I) and (II) are false.

2. Let X and Y be topological spaces and $f: X \to Y$.

(I) If X has discrete topology, then f is continuous.

(II) If Y has indiscrete topology, then f is continuous.

(A) Only (I) is true. (B) Only (II) is true.

(C) Both (I) and (II) are true. (D) Both (I) and (II) are false.

3. Let $\mathbb{R}$ denotes the set of all real numbers in its usual topology and $\mathbb{R}_l$ denotes same set in the topology generated by all intervals of the form [*a, b*). Let $f: \mathbb{R} \to \mathbb{R}_l$ be defined by $f(x) = x$ for every real number x. Then which of the following statements is true ?

A) f is not continuous B) f is continuous

C) f is a homeomorphism D) f^{-1} is not continuous

4. Prove that for functions $f: \mathbb{R} \to \mathbb{R}$, the $\varepsilon - \delta$ definition of continuity implies the open set definition.

5. Suppose that $f: X \to Y$ is continuous. If x is a limit point of the subset of A of X, is it necessary true that $f(x)$ is a limit point of $f(A)$?

6. Show that the subspace (a, b) of $\mathbb{R}$ is homeomorphic with (0, 1) and the subspace $[a, b]$ of $\mathbb{R}$ is homeomorphic with [0, 1]

7. Find a function $f: \mathbb{R} \to \mathbb{R}$ that is continuous at precisely one point.

6. Product Topology

Definition 6.1 : Let $\{A_\alpha\}_{\alpha \in J}$ be an indexed family of sets and $X = \bigcup_{\alpha \in J} A_\alpha$.

The cartesian product of this indexed family is denoted by $\prod_{\alpha \in J} A_\alpha$ is defined as $\prod_{\alpha \in J} A_\alpha = \left\{ x = (x_\alpha)_{\alpha \in J} \mid x_\alpha \in A_\alpha \text{ for each } \alpha \in J \right\}$.

Lemma 6.2 : *Let $\{X_\alpha\}_{\alpha \in J}$ be an indexed family of topological spaces let $\mathscr{B} = \{\prod_{\alpha \in J} U_\alpha \mid U_\alpha \text{ is open in } X_\alpha\}$ then $\mathscr{B}$ is a basis for the topology $\prod_{\alpha \in J} X_\alpha$.*

Proof : Let x $= x = (x_\alpha) \in \prod_{\alpha \in J} X_\alpha$ since each x_α is open in X_α,

$B = \prod_{\alpha \in J} X_\alpha \in \mathscr{B}$

$\Rightarrow x \in B \in \mathscr{B}$.

Let $B_1, B_2 \in \mathscr{B}$ then $B_1 = \prod_{\alpha \in J} U_\alpha, B_2 = \prod_{\alpha \in J} V_\alpha$

$$B_1 \cap B_2 = \left(\prod_{\alpha \in J} U_\alpha\right) \cap \left(\prod_{\alpha \in J} V_\alpha\right)$$

$$= \prod_{\alpha \in J} (U_\alpha \cap V_\alpha), \; U_\alpha \cap V_\alpha \text{ is open in } X$$

$\Rightarrow B_1 \cap B_2 \in \mathscr{B}$..

$\therefore$ $\mathscr{B}$ is a basis for $\prod_{\alpha \in J} X_\alpha$.

Definition 6.3 : Box Topology

Let $\{X_\alpha\}_{\alpha \in J}$ be an indexed family of topological spaces then the collection $\mathscr{B} = \{\prod_{\alpha \in J} U_\alpha \mid U_\alpha \text{ is open in } X_\alpha \text{ for each } a \in J\}$ is a basis for $\prod_{\alpha \in J} X_\alpha$ and the topology generated by this basis is called the box topology.

Definition 6.4 : Product Topology on $\prod_{\alpha \in J} X_\alpha$

For each $\beta \in J$. Define $\prod_{\mathscr{B}} = \prod_{\alpha \in J} X_\alpha \to X_\beta$ by $\prod_\beta(x) = x_{\mathscr{B}}$ then $\prod_{\mathscr{B}}$'s are continuous.

Let $S_{\mathscr{B}} = \{\prod_{\mathscr{B}}^{-1}(U_{\mathscr{B}}) \mid U_{\mathscr{B}} \text{ is open in } X_{\mathscr{B}}\}$ and $S = \bigcup_{\mathscr{B} \in J} S_{\mathscr{B}}$ then S is a sub basis for $\prod X_\alpha$ and topology generated by S is called the product topology.

Theorem 6.5 : *The product topology on $\prod X_\alpha$ has a basis element in which elements are of the form $\prod U_\alpha$ where U_α is open in X_α for each $\alpha \in J$ such that U_α is open in X_α for each $\alpha \in J$ such that U_α equals X_α except for finitely many α's.*

Proof : Let $\mathscr{B}$ be the basis generated by the sub basis $\mathscr{S}$ then $\mathscr{B}$ consist of finite intersection of elements of $\mathscr{S}$. If we intersect element belonging to the same $\mathscr{S}_{\mathscr{B}}$ we don't get any thing new because

$\prod_{\mathscr{B}}^{-1}(U_{\mathscr{B}}) \cap \prod_{\mathscr{B}}^{-1}(V_{\mathscr{B}}) = \prod_{\mathscr{B}}^{-1}(U_{\mathscr{B}} \cap V_{\mathscr{B}}) \in \mathscr{S}_{\mathscr{B}}$ is again an element of $\mathscr{S}_{\mathscr{B}}$.

$\therefore$ Assume that basis element is the finite intersection of different $\mathscr{S}_{\mathscr{B}}$'s.

$B = \prod_{\mathscr{B}_1}^{-1}(U_{\mathscr{B}_1}) \cap \prod_{\mathscr{B}_2}^{-1}(U_{\mathscr{B}_1}) \cap .... \cap \prod_{\mathscr{B}_n}^{-1}(U_{\mathscr{B}_1})$ then

$x \in B$ if and only if x $x \in \prod_{\mathscr{B}_i}^{-1}(U_{\mathscr{B}_i}) \quad \forall i = 1, 2, ..., n$

if and only if $\prod_{\mathscr{B}_i}(x) \in U_{\mathscr{B}_i} \quad \forall i = 1, 2, ..., n$

if and only if $x_{\mathscr{B}_i} \in U_{\mathscr{B}_i} \quad \forall i = 1, 2, ..., n$

$\therefore$ There is no condition on x_α if $\alpha \neq \mathscr{B}_i$ $i = 1, 2, ..., n$

$\therefore$ $\mathscr{B} = \prod U_\alpha$ where $U_\alpha = X_\alpha$ if $\alpha \neq \mathscr{B}_i$

$\therefore B = \prod U_\alpha$ where $U_\alpha = X_\alpha$ except for finitely many α's.

Remark 6.6 :

1. In a finite product space $\prod_{i=1}^{n} X_i$ the box topology is same as the product topology.
2. Since every basis element in the product topology belongs to the basis for the box topology. We have box topology is finer than product topology.

Theorem 6.7 : *Let X_α be an indexed family of spaces and $A_\alpha \subset X_\alpha$ for each α if $\prod X_\alpha$ is given either the product or box topology then $\prod_{\alpha J} \overline{A_\alpha} = \overline{\prod_{\alpha \in J} A_\alpha}$.*

Proof : Let $x \in \prod_{\alpha J} \overline{A_\alpha}$

$\Rightarrow x \in \overline{A_\alpha} \quad \forall \alpha \in J$

To show $x \in \overline{\prod_{\alpha \in J} A_\alpha}$.

Let $x \in \prod_{\alpha \in J} U_\alpha$

$\Rightarrow x_\alpha \in U_\alpha \quad \forall \alpha \in J$.

Since $x_\alpha \in \overline{A}_\alpha$ and $x_\alpha \in U_\alpha$

$\Rightarrow A_\alpha \cap U_\alpha \neq \varnothing \quad \forall \alpha \in J$

$\therefore \left(\prod_{\alpha \in J} A_\alpha\right) \cap \left(\prod_{\alpha \in J} U_\alpha\right) = \prod_{\alpha \in J}\left(A_\alpha \cap U_\alpha\right) \neq \varnothing$

$\Rightarrow x \in \overline{\prod_{\alpha \in J} A_\alpha}$

$\Rightarrow \prod_{\alpha \in J} \overline{A}_\alpha \subset \overline{\prod_{\alpha \in J} A_\alpha}$

Conversely, let $x \in \overline{\prod_{\alpha \in J} A_\alpha}$.

To show $x \in \prod_{\alpha \in J} \overline{A}_\alpha$.

We have to show that $x_\alpha \in \overline{A}_\alpha \quad \forall \alpha \in J$.

Let $x_\alpha \in U_\alpha \quad \forall \alpha \in J$

$\Rightarrow x \in \prod_{\alpha \in J} U_\alpha$

$\Rightarrow \left(\prod_{\alpha \in J} A_\alpha\right) \cap \left(\prod U_\alpha\right) \neq \varnothing$

$\Rightarrow \prod_{\alpha \in J}\left(A_\alpha \cap U_\alpha\right) \neq \varnothing$

Say $y \in \prod_{\alpha \in J}\left(A_\alpha \cap U_\alpha\right)$

$\Rightarrow y_\alpha \in A_\alpha \cap U_\alpha \quad \forall \alpha$

$\therefore A_\alpha \cap U_\alpha \neq \varnothing, \ \forall \alpha$

$\Rightarrow x_\alpha \in \overline{A}_\alpha \quad \forall \alpha$

$\Rightarrow x_\alpha \in \prod_{\alpha J} \overline{A}_\alpha$

$\Rightarrow \overline{\prod_{\alpha \in J} A_\alpha} \subset \prod_{\alpha J} \overline{A}_\alpha$.

Theorem 6.8 : *Let* $f : A \to \prod_{\alpha \in J} X_\alpha$ *be given by* $f(a) = (f_\alpha(a))_{\alpha \in J}$ *where* $f_\alpha : A \to X_\alpha$ *for each* α.

Let $\prod X_\alpha$ *have the product topology then f is continuous if and only if* f_α *is continuous for each* α.

Proof : $f : A \to \prod_{\alpha \in J} X_\alpha$ is given by $f(a) = (f_\alpha(a))_{\alpha \in J}$ where $f_\alpha : A \to X_\alpha$.

Since $\prod_\alpha : \prod_{\alpha \in J} X_\alpha \to X_\alpha$, we get $f_\alpha = \prod_\alpha \circ f$.

Now suppose *f* is continuous. As $\prod_\alpha$ is continuous, $\prod_\alpha \circ f$ is continuous $\forall \alpha$.

$\Rightarrow f_\alpha$ is continuous $\forall \alpha$.

Conversely, suppose f_α is continuous for each α.

A typical basis element of the product topology is $\prod_{\mathscr{B}}^{-1}(U_{\mathscr{B}})$ where $U_{\mathscr{B}}$ is open in $X_{\mathscr{B}}$.

We have to show that $f^{-1}\left(\prod_{\mathscr{B}}^{-1}(U_{\mathscr{B}})\right)$ is open in *A*.

But $f^{-1}\left(\prod_{\mathscr{B}}^{-1}(U_{\mathscr{B}})\right) = \left(\prod_{\mathscr{B}} \circ f\right)^{-1}(U_{\mathscr{B}}) = f_{\mathscr{B}}^{-1}(U_{\mathscr{B}})$.

Since $f_{\mathscr{B}}$ is continuous. $f_{beta}^{-1}(U_{\mathscr{B}})$ is open.

$\Rightarrow f$ is continuous.

Remark 6.9 : The above result is not true for box topology.

For example, consider $\mathbb{R}^w = \prod_{n \in \mathbb{Z}^+} X_n$, infinite countable product of $X_n = \mathbb{R}, \forall n$.

Define $f : \mathbb{R} \to \mathbb{R}^w$ by $f(t) = (t, t, \ldots)$.

Suppose that $\mathbb{R}^w$ is given with box topology.

Here each $f_n \to \mathbb{R} \to \mathbb{R}$ given by $f_n(t) = t$ is continuous.

But we prove that *f* is not continuous. Consider,

$$B = (-1,1) \times \left(-\frac{1}{2}, \frac{1}{2}\right) \times \left(-\frac{1}{3}, \frac{1}{3}\right) \times \ldots$$

$$= \prod_{n \geq 1} \left(-\frac{1}{n}, \frac{1}{n}\right)$$

Then B is open in box topology.

Suppose $f^{-1}(B)$ is open in $\mathbb{R}$.

Since $0 \in f^{-1}(B)$, $f^{-1}(B)$ is open in $\mathbb{R}$ there exists δ >0 such that $0 \in (-\delta, \delta) \subset f^{-1}(B)$.

$$\Rightarrow f\left((-\delta, \delta)\right) \subset B$$

$$\Rightarrow \Pi_n\left((-\delta, \delta)\right) \subset \Pi_n(B) \forall$$

$$\Rightarrow f_n\left((-\delta, \delta)\right) \subset \left(-\frac{1}{n}, \frac{1}{n}\right) \forall$$

$$\Rightarrow (-\delta, \delta) \subset \left(-\frac{1}{n}, \frac{1}{n}\right) \forall$$

Which is a contradiction as there $\exists n_0$ such that $n_0 \delta > 1$

$\Rightarrow \delta > \dfrac{1}{n_0}$ and hence $(-\delta, \delta)$ is not subset of $\left(-\dfrac{1}{n_0}, \dfrac{1}{n_0}\right)$

Hence, $f^{-1}(B)$ is not open and hence *f* is not continuous.

7. The Metric Topology

Definition 7.1 : If *d* is a metric on *X*, then the collection $\mathscr{B} = \{B_d(x, \varepsilon) \mid x \in X, \varepsilon > 0\}$ is a basis for *X* and the topology generated by $\mathscr{B}$ is called the metric topology on *X* induced by *'d'*.

Definition 7.2 : Metrizable

A topological space X is said to be metrizable if there exists a metric d on X that induces the topology of X.

Example 7.3 : A metric space is metrizable with given metric on X.

Definition 7.4 : Bounded Metric

Let X be a metric space with metric d.

Define $\bar{d} : X \times X \to \mathbb{R}$ by $\bar{d}(x, y) \min\{d(x, y), 1\}$. Then $\bar{d}$ is a metric and called the standard bounded metric corresponding to d.

Theorem 7.5 : *The topology generated by d is same as the topology generated by* $\bar{d}$.

Proof : First we show that $\bar{d}(x, y) = \min\{d(x, y), 1\}$ is a metric on X.

1. $\bar{d}(x, y) \geq 0$
2. $\bar{d}(x, y) = 0$ if and only if $x = y$
3. $\bar{d}(x, y) = \bar{d}(y, z)$
4. Now we will show that $\bar{d}(x, z) \leq \bar{d}(x, y) + (y, z)$

   Suppose $\bar{d}(x, y) = 1$ or $\bar{d}(y, z) = 1$ then

   $\bar{d}(x, y) + \bar{d}(x, z) \leq 1 \leq \bar{d}(x, z)$

   $\Rightarrow \bar{d}(x, z) \leq \bar{d}(x, y) + \bar{d}(y, z)$

   Now suppose $\bar{d}(x, y) < 1$ and $\bar{d}(y, z) < 1$ then

   $\bar{d}(x, y) = d(x, y)$ and $\bar{d}(y, z) = d(y, z)$

$$\therefore \bar{d}(x, z) \leq \bar{d}(x, z) \leq d(x, y) + \bar{d}(y, z)$$

$$= \bar{d}(x, y) + \bar{d}(y, z)$$

$$\bar{d}(x,z) \le \bar{d}(x,y) + \bar{d}(y,z)$$

$\therefore \bar{d}$ is a metric on X.

Since the collection of ε-balls with ε < 1 forms a basis for the metric topology, it follows that d and $\bar{d}$ induces the same topology on X as the collection of ε balls with ε < 1 under these two metric are same.

Theorem 7.6 : *Let d and d' be two metrics on X ; $\mathscr{T}$ and $\mathscr{T}'$ be topologies induced by d and d'. Then $\mathscr{T}'$ is finer than $\mathscr{T}$ if and only if for each $x \in X$ and ε > 0 there exists δ > 0 such that $B_{d'}(x,\delta) \subset B_d(x,\varepsilon)$.*

Proof : Suppose $\mathscr{T}' \supset \mathscr{T}$. Let $x \in X$ and ε > 0.

Since $B_d(x,\varepsilon)$ is open in $\mathscr{T}$, there exists $B \in \mathscr{T}'$ such that $B \subset B_d(x,\varepsilon)$.

As B is open in $\mathscr{T}'$ there exists δ > 0 such that $B_d^{'}(x,\delta) \subset B$

Conversely, suppose assume that $\varepsilon - \delta$ criteria is true.

To show $\mathscr{T}' \supset \mathscr{T}$, let $B \in \mathscr{T}$.

As B is open in $\mathscr{T}$, there exists ε > 0 such that $B_d(x,\varepsilon) \subset B$.

Then by assumption there exists δ > 0 such that $B_d^{'}(x,\delta) \subset B_d(x,\varepsilon)$

$\Rightarrow B_d^{'}(x,\delta) \subset B$

$\Rightarrow \mathscr{T}'$ is finer than $\mathscr{T}$.

Theorem 7.7 : *The topologies on $\mathbb{R}^n$ induced by Euclidean metric d and square metric are the same as the product topology on $\mathbb{R}^n$.*

Proof : Here $d(x,y) = \left(\sum_{i=1}^{n}(x_i - y_i)^2\right)^{\frac{1}{2}}$ and

$\rho(x,y) = \max\{|x_i - y_i| \ldots |x_n - y_n|\}$. Since $|x_i - y_i| \le \left(\sum_{i=1}^{n}(x_i - y_i)^2\right)^{\frac{1}{2}}$

$$\Rightarrow \max\{1 \le i \le n\}\{|x_i - y_i|\} \le \left(\sum_{i=1}^{n}(x_i - y_i)^2\right)^{\frac{1}{2}}$$

$$\Rightarrow \rho(x, y) \le d(x, y).$$

Let $y \in B_d(x, \varepsilon)$

$$\Rightarrow d(x, y) < \varepsilon$$

$$\Rightarrow \rho(x, y) < \varepsilon$$

$$\Rightarrow y \in B_\rho(x, \varepsilon)$$

$$\therefore B_d(x, \varepsilon) \subset B_\rho(x, \varepsilon)$$

$$\Rightarrow \mathscr{T}_d \supset \mathscr{T}_\rho.$$

Also $(x_i - y_i)^2 \le \max_{1 \le i \le n}\left\{|x_i - y_i|^2\right\}$

$$\Rightarrow \sum_{i=1}^{n}(x_i - y_i)^2 \le n \max_{1 \le i \le n}\left\{|x_i - y_i|^2\right\}$$

$$\Rightarrow \left(\sum_{i=1}^{n}(x_i - y_i)^2\right)^{\frac{1}{2}} \le \sqrt{n} \max_{1 \le i \le n}\left\{|x_i - y_i|^2\right\}$$

$$\Rightarrow d(x, y) \le \sqrt{n}\rho(x, y)$$

Let $y \in B_\delta(x, \varepsilon)$

$$\Rightarrow \rho(x, y) < \varepsilon$$

$$\Rightarrow d(x, y) < \frac{\varepsilon}{\sqrt{n}}$$

$$y \in B_d\left(x, \frac{\varepsilon}{\sqrt{n}}\right)$$

$$B_\rho(x, \varepsilon) \subset B_d\left(x, \frac{\varepsilon}{\sqrt{n}}\right)$$

$\Rightarrow \mathscr{T}_\rho \supset \mathscr{T}_d$

$\therefore \mathscr{T}_\rho = \mathscr{T}_d$

Now we show that topology $\mathscr{T}_\rho$ generated by square metric is same as product topology $\mathscr{T}$.

Let $B = (a_1, b_1) \times (a_2, b_2) \times ... \times (a_n, b_n)$ be a basis element in $\mathbb{R}^n$ and $x = (x_1, x_2, ..., x_n) \in B$

$\Rightarrow x_i \in (a_i, b_i) \quad \forall i = 1, 2, ..., n$

Since (a_i, b_i) is open in $\mathbb{R}$ there exists $\varepsilon_i > 0$ such that

$(x_i - \varepsilon_i, x_i + \varepsilon_i) \subset (a_i, b_i) \forall i$

Take $\varepsilon = \min\{\varepsilon_1, \varepsilon_2, ..., \varepsilon_n\}$.

To show $B_\rho(x, \varepsilon) \subset B$, let $y \in B_\rho(x, \varepsilon)$

$\Rightarrow \rho(x, y) < \varepsilon$

$\Rightarrow |x_i - y_i| < \varepsilon \quad \forall i$

$\Rightarrow |x_i - y_i| < \varepsilon < \varepsilon_i \quad \forall i$

$\Rightarrow y_i \in (x_i - \varepsilon_i, x_i + \varepsilon_i) \subset (a_i, b_i)$

$\Rightarrow y \in B$

$\therefore B_\rho(x, \varepsilon) \subset B$

$\Rightarrow \mathscr{T}_\rho \subset \mathscr{T}$.

Let $B_\rho(x, \varepsilon)$ be a basis element in $\mathscr{T}_\rho$. Then

$B_\rho(x, \varepsilon) = (x_1 - \varepsilon, x_1 + \varepsilon) \times (x_2 - \varepsilon, x_2 + \varepsilon) \times ... \times (x_1 - \varepsilon, x_1 + \varepsilon)$ is open in $\mathbb{R}^n$

$\therefore B = (x_1 - \varepsilon, x_1 + \varepsilon) \times ... \times (x_n - \varepsilon, x_n + \varepsilon) \subset B_\rho(x, \varepsilon)$

$\mathscr{T} \supset \mathscr{T}_\rho$

$\therefore \mathscr{T} = \mathscr{T}_\rho = \mathscr{T}_d$

Hence the product topology on $\mathbb{R}^n$ is metrizable.

Definition 7.8 : Given an index set J and $x = (x_\alpha)$, $y = (y_\alpha)$, define $\bar{\rho}$ on $\mathbb{R}^J$ by $\bar{\rho}(x, y) = \sup\{\bar{d}(x_\alpha, y_\alpha) \mid \alpha \in J\}$ where $\bar{d}(x_\alpha, y_\alpha) = \min\{d(x_\alpha, y_\alpha), 1\}$. Then the topology induced by $\bar{\rho}$ is called the uniform metric.

Theorem 7.9 : *The uniform topology on $\mathbb{R}^J$ is finer than the product topology and coarser than the box topology.*

Proof : Let $B = \prod U_\alpha$ be a basis element in $\mathbb{R}^J$ and $x \in B$.

Since $U_\alpha = \mathbb{R}$ except for finitely many, let $U_{\alpha_i} = \mathbb{R}$ for $\alpha \neq 1, 2, \ldots n$.

As $x \in B$ then $x_\alpha \in U_\alpha \quad \forall \alpha$ as $U_{\alpha_1}, U_{\alpha_2}, \ldots U_{\alpha_n}$ are open in $\mathbb{R}$ there exists $\varepsilon_i > 0$ such that

$x_{\alpha_i} \in (x_{\alpha_i} - \varepsilon_i, x_{\alpha_i} + \varepsilon_i) \subset U_{\alpha_i}$

Take $\varepsilon = \min\{\varepsilon_1, \varepsilon_2, \ldots., \varepsilon_n\}$.

Let $y \in B_\rho(x, \varepsilon)$

$\Rightarrow \bar{\rho}(x, y) < \varepsilon$

$\Rightarrow \bar{d}(x_\alpha, y_\alpha) < \varepsilon \quad \forall \alpha$

In particular,

$\bar{d}(x_{\alpha_i}, y_{\alpha_i}) < \varepsilon < \varepsilon_i, \quad \forall i = 1, 2, \ldots, n$

$\Rightarrow y_{\alpha_i} \in U_{\alpha_i} \quad i = 1, 2, \ldots n$

$\therefore y \in \prod U_\alpha = B$

$\Rightarrow B_{\bar{f}}(x,\varepsilon) \subset B$.

Hence Uniform topology is finer than product topology.

Let B be a basis element in the uniform topology i.e. $B = B_{\rho}(x,\varepsilon)$.

Now take $U = \prod\left(x_{\alpha} - \frac{\varepsilon}{2}, x_{\alpha} + \frac{\varepsilon}{2}\right)$.

Then $U \subset B$ as for any y in U, $|y_{\alpha} - x_{\alpha}| < \varepsilon$

$\Rightarrow y \in B_{\bar{\rho}}(x,\varepsilon)$.

$\therefore$ Box topology is finer than uniform topology.

Theorem 7.10 : *Let $\bar{d}(a,b) = \min\{|a-b|,1\}$ be the standard bounded metric on $\mathbb{R}$. If x and y are two points of $\mathbb{R}^{w}$, define $D(x,y) = \sup\left\{\frac{\bar{d}(x_i,y_i)}{i}\right\}$. Then D is a metric that induces the product topology on $\mathbb{R}^{w}$.*

Proof: $\because \bar{d}(x_i,z_i) \le \bar{d}(x_i,y_i) + \bar{d}(y_i,z_i)$

$$\frac{\bar{d}(x_i,z_i)}{i} \le \frac{\bar{d}(x_i,y_i)}{i} + \frac{\bar{d}(y_i,z_i)}{i}$$

$$\le \sup\left\{\frac{\bar{d}(x_i,z_i)}{i}\right\} + \sup\left\{\frac{\bar{d}(x_i,z_i)}{i}\right\}$$

$$\frac{\bar{d}(x_i,z_i)}{i} \le D(x,y) + D(y,z)$$

$$\Rightarrow \sup\left\{\frac{\bar{d}(x_i,y_i)}{i}\right\} \le D(x,y) + D(y,z)$$

$$\Rightarrow D(x,z) \le D(x,y) + D(y,z)$$

$\therefore$ D is a metric on $\mathbb{R}^{w}$.

Let $\mathscr{T}_D$ be the topology generated by D and $\mathscr{T}$ be the product topology on $\mathbb{R}^w$.

To show $\mathscr{T} \subset \mathscr{T}_D$, let $B_D(x,\varepsilon)$ be a basis element of $\mathscr{T}_D$.

Since ε > 0 there exists N such that $\frac{1}{N} < \varepsilon$.

Let $V = (x_1 - \varepsilon, x_1 + \varepsilon) \times \ldots \times (x_N - \varepsilon, x_N + \varepsilon)$. Then V is open in the product topology.

To show that $V \subset B_D(x,\varepsilon)$, let $y \in V$.

Then $y_i \in (x_i - \varepsilon, x_i + \varepsilon)$ $\forall i = 1,2,\ldots,N$ and $y_i \in \mathbb{R}$ $\forall i > N$.

Clearly, $\frac{|x_i - y_i|}{i} \leq \frac{1}{N} < \varepsilon \quad \forall i > N$

$$\therefore D(x,y) \leq \max\left\{\frac{d(x_1,y_1)}{1}, \frac{d(x_2,y_2)}{2}, \ldots, \frac{d(x_N,y_N)}{N}, \varepsilon\right\}$$

$\Rightarrow y \in B_D(x,\varepsilon)$

$\therefore V \subset B_D(x,\varepsilon)$.

Conversely, consider a basis element $U = \prod_{i \in \mathbb{Z}_+} U_i$ in the product topology where U_i is open in $\mathbb{R}$ for $i = \alpha_1, \alpha_2, \ldots\ldots, \alpha_n$ and $U_j = \mathbb{R} \quad j \neq \alpha_i$.

Let $x \in U$. Then $x_i \in U_{\alpha i}$ $i = 1, 2, \ldots, n$.

Since U_i is open in $\mathbb{R}$ there exists $1 \geq \varepsilon_i > 0$ such that

$x_i \in (x_i - \varepsilon_i, x_i + \varepsilon_i) \subset U_{\alpha i}$ $i = 1, 2, \ldots, n$.

Now take $\varepsilon = \min\left\{\frac{\varepsilon_i}{i} \mid i = \alpha_1, \ldots \alpha_n\right\}$.

Claim : $B_D(x,\varepsilon) \subset U$.

Let $y \in B_D(x,\varepsilon)$

$\Rightarrow D(x, y) < \varepsilon$

$\Rightarrow \sup\left\{\dfrac{\bar{d}(x_i, y_i)}{1}\right\} < \varepsilon$

$\therefore \dfrac{\bar{d}(x_i, y_i)}{1} < \varepsilon \quad \forall i$

$\Rightarrow \bar{d}(x_i, y_i) < \varepsilon.i < \varepsilon_i \leq 1 \;\; \forall i = \alpha_1, \alpha_2, ... \alpha_n$

$\Rightarrow y_i \in U_{\alpha i}$ for $i = 1, 2, ..., n$

$\Rightarrow y \in U$

$\therefore y \in U$

$\therefore \mathscr{T}_D = \mathscr{T}$

$\Rightarrow \mathbb{R}^w$ is metrizable.

❖❖❖

UNIT - III

COMPACT SPACES AND COUNTABILITY AXIOMS

1. Connected Spaces

Introduction :

In this unit, we define connected topological space and construct new connected spaces from the existing ones. We also show that finite cartesian product of connected spaces is connected, but arbitrary product of connected spaces need not be connected.

Definition 1.1 : Let X be a topological space. A separation of X is a pair (U, V) of disjoint non-empty open sets of X whose union is X. If there is no separation of X, then X is called connected. If a separation exists for X, then X is called disconnected.

We give an equivalent definition of connectedness in terms of open and closed sets.

Lemma 1.2 : *A space X is connected if and if the only subsets of X that are both open and closed in X are the empty set and X itself.*

Proof : Suppose X is connected.

Let $A \subset X$ be closed and open in X such that $A \neq \varnothing$ and $A \neq X$.

Then $U = A$ and $V = A^c$ forms a separation of X, which is a contradiction to that X is connected.

Conversely, suppose X is not connected.

Then there exist disjoint nonempty open sets U and V such that $U \bigcup V = X$.

As $U = V^c$, U is both open and closed and $U \neq \varnothing$ and $U \neq X$, which is a contradiction.

Hence X is connected.

Theorem 1.3 : *If Y is a subspace of X, a separation of Y is a pair of disjoint nonempty sets A and B whose union is Y, neither of which contains a limit point of the other. The space Y is connected if there exists no separation of Y.*

Proof : Suppose A and B form a separation of Y *i.e.* $Y = A \bigcup B$, A and B are open, $A \bigcap B = \phi$.

$\Rightarrow A$ is both open and closed in Y.

Then the closure of A in $Y = \overline{A} \bigcap Y$.

Since A is closed in Y, closure of A is A.

i.e. $\overline{A} \bigcap Y = A$

$\Rightarrow \overline{A} \bigcap B = \phi$

Similarly, $A \bigcap \overline{B} = \phi$.

$\therefore$ No limit point of A is in B and vice-versa.

Conversely, suppose there exist A and B such that $A \bigcup B = Y$, $\overline{A} \bigcap B = \varnothing$, $A \bigcap \overline{B} = \varnothing$, $\overline{A} \bigcap Y = \overline{A} \bigcap (A \bigcup B) = (\overline{A} \bigcap A) \bigcup (\overline{A} \bigcap B) = A$.

$\Rightarrow A$ is closed in Y.

$\Rightarrow B$ is open in Y.

Similarly, A is open in Y.

Example 1.4 : Let X denote a two point space in the indiscrete topology. Then X is connected as there is no separation for X.

Example 1.5 : Let $X = \mathbb{R}$ and $Y = [-1, 0) \bigcup (0, 1]$. Then Y is disconnected as $A = [-1, 0)$ and $B = (0, 1]$ forms a separation of Y.

Lemma 1.6 : *If the sets C and D form a separation of X and if Y is connected subspace of X then Y lies entirely within either C or D.*

Proof : Since $X = C \bigcup D$; C and D are open in X, we have $C \bigcap Y$ and $D \bigcap Y$ are open in Y.

Also $(C \bigcap Y) \bigcup (D \bigcup Y) = Y$.

Since Y is connected, $C \bigcap Y = \varnothing$ or $D \bigcap Y = \varnothing$

$\Rightarrow Y \subset C^c = D$ or $Y \subset D^c = C$

i.e. $Y \subset D$ or $Y \subset C$.

As we have seen in the example ??, union of connected spaces need not be connected, but with some extra conditions we can prove that union of connected spaces is connected.

Theorem 1.7 : *The union of a collection of connected sub spaces of X that have a point in common is connected.*

Proof : Let $\{A_\alpha\}$ be a collection of connected subspaces and $p \in \bigcap A_\alpha$.

We prove that the space $Y = \bigcup A_\alpha$ is connected.

Suppose that $Y = C \bigcup D$ is a separation of Y.

Since $p \in Y$, we have $p \in C$ or $p \in D$; suppose $p \in C$.

As $\{A_\alpha\} \subset Y$ is connected and Y is not connected, we get either $A_\alpha \subset C$ or $A_\alpha \subset D$.

As $p \in A_\alpha$ for each α and $p \in C$ we get that $\{A_\alpha\} \subset C$ for every α

Hence $\bigcup \{A_\alpha\} \subset C$, contradicting the fact that D is nonempty.

Theorem 1.8 : *Let $\{A_\alpha\}$ be a sequence of connected subspaces of X, such that $A_n \bigcap A_{n+1} \neq \varnothing$ for all n. Then $\bigcup A_n$ is connected.*

Proof : Suppose $\bigcup A_n$ is disconnected.

Then there is a separation (U, V) of $\bigcup A_n$.

Since each A_n is connected , we get either $A_n \subset U$ or $A_n \subset V$.

Suppose that $A_n \subset U$. Since $A_n \bigcap A_{n+1} \neq \varnothing$, we get that $A_{n+1} \subset U$.

Then by induction each $A_n \subset U$.

Hence $\bigcup A_n \subset U$ and V is empty, which is a contradiction.

Therefore, $\bigcup A_n$ is connected.

Theorem 1.9 : *Let $\{A_\alpha\}$ be a collection of connected subspaces of X, let A be a connected subspace of X. If $A \bigcap A_\alpha \neq \varnothing$ for all α, then $A \bigcup \left(\bigcup A_\alpha\right)$ is connected.*

Proof : Suppose $A \bigcup \left(\bigcup A_\alpha\right)$ is disconnected.

Then there is a separation (*U, V*) of $A \bigcup \left(\bigcup A_\alpha\right)$.

Since *A* is connected , we get either $A \subset U$ or $A \subset V$.

Suppose that $A \subset U$. Since $A \bigcap A_\alpha \neq \varnothing$ for all α, $A_\alpha \subset U$.

Hence $\bigcup A_\alpha \subset U$ and V is empty, which is a contradiction.

Therefore, $A \bigcup \left(\bigcup A_\alpha\right)$ is connected.

Theorem 1.10 : *Let A be a connected subspace of X. If $A \subset B \subset \overline{A}$, then B is also connected.*

Proof : Suppose that *B* is disconnected.

Then there is a separation $B = C \bigcup D$ for *B*.

Since *A* is conencted and $A \subset B = C \bigcup D$, we get that either $A \subset C$ or $A \subset D$.

Suppose $A \subset C$. Then $\overline{A} \subset \overline{C}$.

Thus, $B \subset \overline{A} \subset \overline{C}$.

Since (*C, D*) is a separation for *B*, we get $\overline{C} \bigcap D = \varnothing$

Therefore, $B \subset \overline{C} \subset D^c$ and so $B \bigcap D = \varnothing$, which is a contradiction.

Hence B is connected.

Theorem 1.11 : *The image of a connected space under a continuous map is connected.*

Proof : Let $f : X \to Y$ be a continuous map and X be connected.

Since f is continuous, we know that $g : X \to f(X)$ is also continuous.

Now suppose that $f(X)$ is disconnected with separation $f(X) = A \bigcup B$.

Then $g^{-1}(A)$ and $g^{-1}(B)$ are disjoint open sets such that $X = g^{-1}(A) \bigcup g^{-1}(B)$

This is a contradiction to X is connected.

Theorem 1.12 : *A finite Cartesian product of connected spaces is connected.*

Proof : We prove that the product of two connected spaces X and Y is connected.

Let $a \times b \in X \times Y$ be a base point. Then the "horizontal slice" $X \times b$ is connected, being homeomorphic with X.

Also each "vertical slice" $x \times Y$ is connected, being homeomorphic with Y.

Since $x \times b \in (X \times b) \bigcap (x \times Y)$, each "T-shaped" space

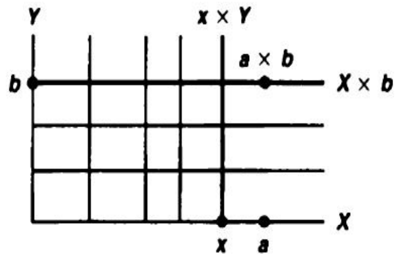

Figure 6:

$T_x = (X \times b) \bigcup (x \times Y)$ is connected.

Since $a \times b \in \bigcap_x T_x$ and each T_x is connected, therefore the union $\bigcap_x T_x$ is connected. As this union equals $X \times Y$, the space $X \times Y$ is connected.

Now suppose that the product space $X_1 \times ... \times X_{n-1}$ is connected.

Since the space $X_1 \times ... \times X_n$ is homeomorphic with $(X_1 \times ... \times X_{n-1}) \times X_n$, we get that $X_1 \times ... \times X_n$ is connected.

We now show that arbitrary product of connected spaces need not be connected.

Example 1.13 : The product space $\mathbb{R}^w$ is not connected in the box topology.

Proof : Consider the cartesian product $\mathbb{R}^w$ in the box topology.

We can write $\mathbb{R}^w = A \bigcup B$, where A is the set consisting of all bounded sequences of real numbers, and the set B of all unbounded sequences.

Then the sets A and B are disjoint and open in the box topology For if a is a point of $\mathbb{R}^w$, the open set

$$U = (a_1 - 1, a_1 + 1) \times (a_2 - 1, a_2 + 1) \times ...$$

consists entirely of bounded sequences if a is bounded, and of unbounded sequences if a if unbounded.

Thus, even though $\mathbb{R}$ is connected, the product space $\mathbb{R}^w$ is not connected in the box topology.

Theorem 1.14 : *The product space $\mathbb{R}^w$ is connected in the product topology.*

Proof : Now consider $\mathbb{R}^w$ in the product topology.

Let $\widetilde{\mathbb{R}}^n = \{x = (x_1, x_2, ...) \mid x_i = 0 \text{ for } i > n\}$.

The space $\widetilde{\mathbb{R}}^n$ is clearly homeomorphic to $\mathbb{R}^n$ so that it is connected.

Let $\mathbb{R}^\infty = \bigcup \widetilde{\mathbb{R}}^n$. Since each $\widetilde{\mathbb{R}}^n$ is connected and $0 = (0, 0, ...) \in \bigcap \widetilde{\mathbb{R}}^n$, it follows that the space $\mathbb{R}^\infty$ is connected.

To show that $\mathbb{R}^w$ is connected, it is enough to prove that the closure of $\mathbb{R}^\infty$ equals all of $\mathbb{R}^w$.

Let $a=(a_1,a_2,...)\in\mathbb{R}^w$ and $U=\prod U_i$ be a basis element for the product topology that contains a. We show that U intersects $\mathbb{R}^\infty$.

Since U is open in the product topology, there exists an integer N such that $U_i=\mathbb{R}$ for i > N.

Then the point x = (a1; :::an;0;0; :::) of R¥ belongs to U, since ai 2 Ui for all i, and 0 2Ui for $i > N$.

2. Connected Subspaces of the Real Line

Definition 2.1 : Linear Continuum

A simply ordered set L having more than one element is called a Linear Continuum if the following hold :

1. L ha the least upper bound property.
2. If $x < y$, there exists z such that $x < z < y$.

Theorem 2.2 : *If L is a linear Continuum in the order topology, L is connected, and so are intervals and rays in L.*

Proof : We know that a subspace Y of L is said to be convex, if for every pair of points *a, b* of Y with $a < b$, the entire interval $[a, b]$ of points of L lies in Y.

We first prove that if Y is a convex subspace of L, then Y is connected.

To contrary, assume that Y is the union of the disjoint nonempty sets A and B, each of which is open in Y.

Choose $a\in A$ and $b\in B$ such that $a < b$.

Then the interval $[a, b]$ of points of L is contained in Y.

Hence $[a, b]$ is the union of the disjoint sets $A_0=A\cap[a,b]$ and $B_0=B\cap[a,b]$, each of which is open in $[a, b]$ in the subspace topology, which is the same as the order topology.

The sets A_0 and B_0 are nonempty because $a\in A_0$ and $b\in B_0$. Thus, A_0 and B_0 forms a separation for $[a, b]$.

Let $c = \sup A_0$. We show that $c \notin A_0$ and $c \notin B_0$, which contradicts the fact that $[a, b]$ is the union of A_0 and B_0.

Case 1 : Suppose that $c \in B_0$. Then $c \neq a$, so either $c = b$ or $a < c < b$.

In either case, it follows from the fact that B_0 is open in $[a, b]$ that there is some interval of the form $(d, c]$ contained in B_0. If $c = b$, we have a contradiction at once, for d is a smaller upper bound on A_0 than c. If $c < b$, we note that $(c, b]$ does not intersect A_0 (because c is an upper bound on A_0).

Then $(d,b] = (d,c] \cup (c,b]$ does not intersect A_0.

Again, d is a smaller upper bound on A_0 than c, contrary to construction.

Case 2 : Suppose that $c \in A_0$. Then $c \neq b$, so either $c = a$ or $a < c < b$.

Because A_0 is open in $[a, b]$, there must be some interval of the form $[c, e)$ contained in A_0.

Because of order property (2) of the linear continuum L, we can choose a point z of L such that $c < z < e$.

Then $z \in A_0$, contrary to the fact that c is an upper bound for A_0.

We now prove that the intermediate value theorem of calculus is the special case of the following theorem that occurs when we take X to be a closed interval in $\mathbb{R}$ and Y to be $\mathbb{R}$.

Theorem 2.3 : Intermediate value theorem

Let $f : X \to Y$ be a continuous map, where X is a connected space and Y is an ordered set in the order topology. If a and b are two points of X and if r is a point of Y lying between $f(a)$ and $f(b)$, then there exists a point c of X such that $f(c) = r$.

Proof : The sets $A = f(X) \cap (-\infty, r)$ and $B = f(X) \cap (r, +\infty)$ are disjoint, and $f(a) \in A$ and $f(b) \in B$.

Also A and B are open in $f(X)$.

Suppose that there does not exist $c \in X$ such that $f(c) = r$.

Then $f(X)=A\bigcup B$ and hence A and B forms a separation of $f(X)$, which is a contradiction to the continuous image of a connected space is connected.

EXERCISE - 7

1. Consider the following statements

   (I) $\mathbb{R}_l$ is connected

   (II) $\mathbb{R}$ is connected

   (A) Only (I) is true. (B) Only (II) is true.

   (C) Both (I) and (II) are true. (D) Both (I) and (II) are false.

2. Consider the following

   (I) If f is continuous and E is connected, then $f^{-1}(E)$ is connected

   (II) If f is continuous and E is connected, then $f(E)$ is connected.

   (A) Only (I) is true. (B) Only (II) is true.

   (C) Both (I) and (II) are true. (D) Both (I) and (II) are false.

3. Let $\mathscr{T}$ and $\mathscr{T}'$ be two topologies on X. If $\mathscr{T}'\supset\mathscr{T}$, what does connectedness of X in one topology imply about connectedness in the other ?

4. Show that if X is an infinite set, it is connected in the finite complement topology.

5. Let $Y\subset X$; let X and Y be connected. Show that if A and B form a separation of $X\setminus Y$, then $Y\bigcup A$ and $Y\bigcup B$ are connected.

3. Local Connectedness

Introduction :

In this section, we discuss path connectedness, components, locally path connectedness and try to relate these concepts.

Definition 3.1 : Given points x and y of the space X, a path in X from x to y is a continuous map $f:[a,b]\to X$ such that $f(a)=x$ and $f(b)=y$. A space X is said to be **path connected** if every pair of points of X can be joined by a path in X.

Theorem 3.2 : *Every path connected space X is connected.*

Proof : Suppose $X = A \bigcup B$. Let $a \in A$ and $b \in B$. Then, since *X* is path connected, there exists a path $f:[c,d] \to X$ between *a* and *b*.

Since *f* is continuous and *A* and *B* are open, $f^{-1}(A)$ and $f^{-1}(B)$ are open and are disjoint. Therefore $[c,d] = f^{-1}(A) \bigcup f^{-1}(B)$ is disconnected, which is a contradiction.

Hence *X* is connected.

Remark 3.3 : The converse of the above theorem is not true. For example, let *S* denote the following subset of the plane.

$$S = \{x \times \sin(l/x) \mid 0 < x \leq l\}$$

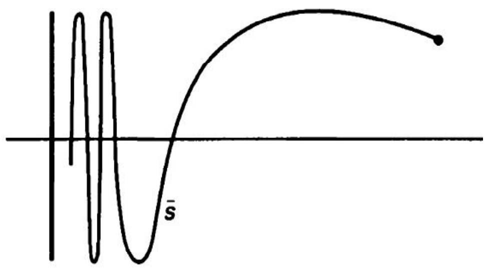

Figure 7:

Because *S* is the image of the connected set (0, 1] under a continuous map, *S* is connected. Therefore, its closure $\overline{S}$, called the topologist's sine curve, in $\mathbb{R}^2$ is also connected. But $\overline{S}$ is not path connected.

Definition 3.4 : Given *X*, define an equivalence relation on *X* by setting $x \sim y$ if there is a connected subspace of *X* containing both *x* and *y*. The equivalence classes are called the **components** (or the "connected components") of *X*.

Theorem 3.5 : *The components of X are connected disjoint subspaces of X whose union is X, such that each nonempty connected subspace of X intersects only one of them.*

Proof : Being equivalence classes, the components of X are disjoint and their union is X. Each connected subspace A of X intersects only one of them. Because, if A intersects the components C_1 and C_2 of X, say at points x_1 and x_2, respectively, then $x_1 \sim x_2$ by definition; this cannot happen unless $C_1 = C_2$.

To show the component C is connected, choose a point x_0 of C. For each point x of C, we know that $x_0 \sim x$, so there is a connected subspace A_x containing x_0 and x. By the result just proved, $A_x \subset C$. Therefore $C = \bigcup_{x \in C} A_x$.

Since the subspaces A_x are connected and have the point x_0 in common, their union is connected.

Definition 3.6 : A space X is said to be **locally connected** at x if for every neighborhood U of x, there is a connected neighborhood V of x contained in U. If X is locally connected at each of its points, it is said simply to be locally connected.

Theorem 3.7 : *A space X is locally connected if and only if for every open set U of X, each component of U is open in X.*

Proof : Suppose that X is locally connected; let U be an open set in X; let C be a component of U. To show C open, let $x \in C$. Since X is locally connected,there exists a connected neighborhood V of x such that $V \subset U$. Since V is connected and $V \bigcup C \neq \varnothing$, we get $x \in V \subset C$. Therefore, C is open in X.

Conversely, suppose that components of open sets in X are open.

Given a point $x \in X$ and a neighborhood U of x, let C be the component of U containing x. Since each component is conneceted, C is connected; since it is open in X by hypothesis, and $x \in C \subset U$. Therefore X is locally connected at x.

EXERCISE - 8

1. Consider the two statements

   (I) Every path connected space is connected.

   (II) Every connected space is path connected.

   (A) Only (I) is true. (B) Only (II) is true.

   (C) Both (I) and (II) are true. (D) Both (I) and (II) are false.

2. Let X be a locally path connected topological space. Then :

   (A) every connected open set in X is path connected

   (B) every connected set in X is path connected

   (C) every connected closed set in X is path connected

   (D) every open set in X is path connected

3. Consider the following statements :

   (a) A path connected set is connected

   (b) A connected set is path connected

   (c) Union of connected sets is connected

   Which of them are correct ?

   (A) 1 (B) 2 (C) 1, 3 (D) 2, 4

4. What are the components and path components of $\mathbb{R}_l$?

5. Show that the ordered square is locally connected but not locally path connected.

6. Let X be locally path connected. Show that every connected open set in X is path connected.

4. Compact Spaces

Introduction :

Frechet was the first to use the term "compact". Compactness was introduced into topology with the intention of generalizing the properties of the closed and bounded subsets of $\mathbb{R}^n$. In this unit, we discuss the properties of compact topological space and construct new from old ones. We also see under what conditions, compactness can be passed on to subspaces and products.

Definition 4.1 : A collection $\mathscr{A}$ of subsets of a space *X* is said to be a cover for *X*, if the union of the elements of $\mathscr{A}$ is equal to *X*. It is called an **open covering** of *X*, if its elements are open subsets of *X*.

Definition 4.2 : A topological space *X* is said to be **compact**, if every open covering $\mathscr{A}$ of *X* contains a finite sub collection that also covers *X*.

Example 4.3 : Any topological space *X* with finite number of elements is compact, as each open cover for *X* is itself a finite set.

Example 4.4 : The real line $\mathbb{R}$ is not compact.

Proof : Consider the set $\mathscr{A} = \{(n-1, n+1) + n \in \mathbb{R}\}$

Then for any $x \in \mathbb{R}$, $x \in ([x]-1, [x]+1)$, where $[x]$ is the greatest integer less than or equal to *x*. Implies $\mathscr{A}$ is an open cover for $\mathbb{R}$.

But no finite sub collection of $\mathscr{A}$ covers $\mathbb{R}$.

Hence $\mathbb{R}$ is not compact.

Example 4.5 : The subspace $X = \{0\} \cup \left\{\frac{1}{n} \mid n \in \mathbb{Z}_+\right\}$ of $\mathbb{R}$ is compact.

Proof : Let $\mathscr{A}$ be an open covering of *X*.

Then there is an element *U* of $\mathscr{A}$ containing 0.

Since *U* is open and $0 \in U$, there exists δ > 0 such that $(-\delta, \delta) \subset U$.

As δ > 0, by Archimedean property, there exists *N*, such that $\frac{1}{N} < \delta$.

Hence the set $\left\{\frac{1}{n} \mid n \geq N\right\} \subset U$.

So at most $1, \frac{1}{2}, \ldots, \frac{1}{N-1}$ are the elements of *X*, which are outside *U*, and these elements can be covered by finitely many open sets, say $U_1, \ldots, U_m$ of $\mathscr{A}$.

Then $\left(\bigcup_{i=1}^{m} U_i\right) \bigcup U$ is a finite sub collection of $\mathscr{A}$ which covers *X*.

Hence *X* is compact.

Lemma 4.6 : *Let Y be a subspace of X. Then Y is compact if and only if every covering of Y by sets open in X contains a finite sub collection covering Y.*

Proof : Suppose that *Y* is compact and $\mathscr{A} = \{A_\alpha\}_{\alpha \in J}$ is a covering of *Y* where A_α is open in *X*.

Then the collection $\{A_\alpha \bigcap Y \mid \alpha \in J\}$ is a covering of *Y* by sets open in *Y*.

Hence, a finite sub collection $\{A_{\alpha_1} \bigcap Y, ... A_{\alpha_n} \bigcap Y\}$ covers *Y*.

Implies $\{A_{\alpha_1}, ... A_{\alpha_n}\}$ is a sub collection of $\mathscr{A}$ that covers *Y*.

To prove converse, let $\mathscr{A}' = \{A_\alpha'\}$ be a covering of *Y* by sets open in *Y*.

For each α, choose a set A_α open in *X* such that $A_\alpha' = A_\alpha \bigcap Y$.

Then the collection $\mathscr{A} = \{A_\alpha\}$ is a covering of *Y* by sets open in *X*.

By hypothesis, some finite sub collection $\{A_{\alpha_1}, ... A_{\alpha_n}\}$ covers *Y*.

Then $\{A_{\alpha_1}', ... A_{\alpha_n}'\}$ is a sub collection of $\mathscr{A}'$ that covers *Y*.

Therefore, *Y* is compact.

Remark 4.7 : The subspace of a compact space need not be compact. For example, the interval [0, 1] is compact, which is known from analysis. But the subspace (0, 1) is not compact as $\mathscr{A} = \left\{\left(\frac{1}{n}, 1\right) \mid n \in Z_+\right\}$ is an open cover for (0, 1), which doesn't has a finite sub cover. Where as, if the given subspace is closed, then it is compact as we prove.

Theorem 4.8 : *Every closed subspace of a compact space is compact.*

Proof : Let Y be a closed subspace of the compact space X.

Let $\mathscr{A}$ be an open covering for Y by sets open in X.

Since Y is closed, $X - Y$ is open in X.

Therefore, $\mathscr{B} = \mathscr{A} \bigcup \{X - Y\}$ is an open covering for X.

Since X is compact, some finite sub collection of $\mathscr{B}$ covers X.

After discarding the set $X - Y$ from this finite sub collection, the resulting collection is a finite sub collection of $\mathscr{A}$ that covers Y.

Hence Y is compact.

Theorem 4.9 : *Every compact subspace of a Hausdorff space is closed.*

Proof : Let Y be a compact subspace of the Hausdorff space X.

To show Y is closed i.e. $X - Y$ is open, let x_0 be a point of $X - Y$.

Since X is Hausdorff, for each point $y \in Y$ and x_0, there exists disjoint open sets U_y and V_y containing x_0 and y, respectively.

Then the collection $\{V_y \mid y \in Y\}$ is a covering of Y by sets open in X.

As Y is compact, there exists a finite sub cover $V_{y_1}, ..., V_{y_n}$ for Y.

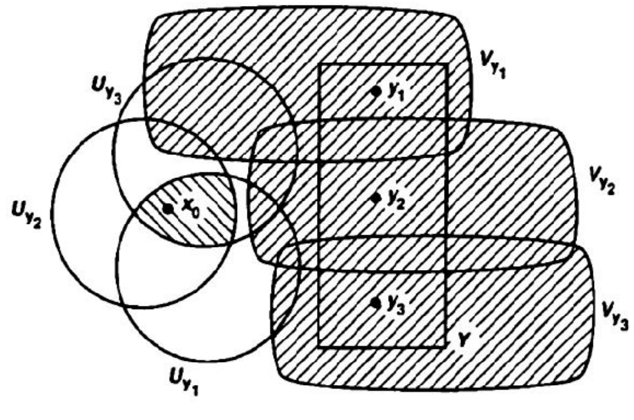

Figure 8:

Let $V = V_{y_1} \bigcup ... \bigcup V_{y_n}$ and $U = U_{y_1} \bigcap ... \bigcap U_{y_n}$.

Then $V \supset Y$, and $V \bigcap Y = \varnothing$, for if $z \in V$, then $z \in V_{y_1}$ for some *i*, hence $z \notin U_{y_i}$ and so $z \notin U$.

Therefore $x_0 \in U \subset X - V \subset X - Y$ and hence $X - Y$ is open.

Theorem 4.10 : *The image of a compact space under a continuous map is compact.*

Proof : Let $f : X \to Y$ be continuous and *X* be compact.

Let $\mathscr{A}$ be a covering for $f(X)$ by sets open in *Y*.

As *f* is continuous, the collection $\{f^{-1}(A) \mid A \in \mathscr{A}\}$ is an open cover for *X*.

Since *X* is compact, there exists a finite sub cover $f^{-1}(A_1),...., f^{-1}(A_n)$ for *X*.

Then $\{A_1,..., A_n\}$ is a finite sub cover for $f(X)$ and hence $f(X)$ is compact.

Theorem 4.11 : *Let $f : X \to Y$ be a bijective continuous function. If X is compact and Y is Hausdorff, then f is a homeomorphism.*

Proof : To prove $f^{-1} : Y \to X$ is continuous, let *A* be a closed subset of *X*.

Then *A* is compact. Since $f : X \to Y$ is continuous, $f(A)$ is compact.

Given that *Y* is Hausdorff, so $f(A)$ is closed in *Y*.

Therefore $\left(f^{-1}\right)^{-1}(A) = f(A)$ is closed in *Y*.

Remark 4.12 : According to above result, if a continuous bijective map $f : X \to Y$ is not a homeomorphism, then we can conclude that either *X* is not compact or *Y* is not Hausdorff, for example, The function $f : \mathbb{R}_l \to \mathbb{R}$ given by $f(x) = x$ is a bijective continuous function but not a homeomorphism. As $\mathbb{R}$ is Hausdorff, we can conclude that $\mathbb{R}_l$ is not compact.

We now prove tube lemma, which will be useful in proving that product of finitely many compact spaces is compact.

Lemma 1.13 : (The tube lemma) : *Consider the product space $X \times Y$, where Y is compact. If N is an open set of $X \times Y$ containing the slice $x_0 \times Y$ of $X \times Y$, then N contains some tube $W \times Y$ about $x_0 \times Y$, where W is a neighborhood of x_0 in X.*

Proof : Since *Y* is compact and $x_0 \times Y$ is homeomorphic to *Y*, we get that $x_0 \times Y$ is also compact. Let $x_0 \times y \in x_0 \times Y$. Since $x_0 \times Y \subset N$ and *N* is open subset of $X \times Y$, there exists open set $U_y \times V_y$ such that $x_0 \times y \in U_y \times V_y \subset N$.

Implies the collection $\{U_y \times V_y \mid y \in Y\}$ is an open cover for $x_0 \times Y$.

Therefore, there exists a finite sub cover $U_1 \times V_1, ..., U_n \times V_n$ for $x_0 \times Y$.

Without loss of generality, we can assume that $(U_i \times V_i) \cap (x_0 \times Y) \neq \emptyset$.

(as if some basis element is not intersecting $x_0 \times Y$, discard that from the collection.)

Let $W = U_1 \cap ... \cap U_n$. Then *W* is open and $x_0 \in W$. $(\because x_0 \in U_i \quad \forall i)$

We will prove that the sets $U_i \times V_i$ covers the tube $W \times Y$.

Let $x \times y \in W \times Y$. Then $x_0 \times y \in x_0 \times Y$.

$\Rightarrow x_0 \times y \in U_i \times V_i$ for some i,so that $y \in V_i$.

But $x \in U_j$ for every *j* (because $x \in W$).

Therefore, we have $x \times y \in U_i \times V_i$, as desired.

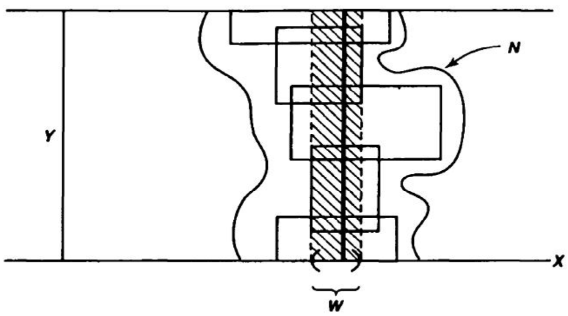

Figure 9 :

Since $U_i \times V_i \subset N$ for each i and $(W \times Y) \subset \bigcup (U_i \times V_i)$, we get that the tube $W \times Y \subset N$.

Theorem 4.14 : *The product of finitely many compact spaces is compact.*

Proof : We Prove the result by mathematical induction.

First we prove that the product $X \times Y$ of two compact spaces *X* and *Y* is compact.

Let $\mathscr{A}$ be an open covering of $X \times Y$.

Given $x_0 \in X$, the slice $x_0 \times Y$ is compact and may therefore be covered by finitely many elements $A_1, \ldots A_m$ of $\mathscr{A}$. ($\because$ $\mathscr{A}$ covers $x_0 \times Y$).

Their union $N = A_1 \cup \ldots \cup A_m$ is an open set containing $x_0 \times Y$.

Then by the tube lemma, the open set *N* contains a tube $W \times Y$ about $x_0 \times Y$, where *W* is open in *X*. Then $W \times Y$ is covered by finitely many elements $A_1, \ldots, A_m$ of $\mathscr{A}$.

Thus, for each *x* in *X*, we can choose a neighborhood W_x of *x* such that the tube $W_x \times Y$ can be covered by finitely many elements of $\mathscr{A}$.

Since *X* is compact and the collection of all the neighborhoods W_x is an open covering of *X*, there exists a finite sub collection $\{W_1, \ldots, W_k\}$ covering *X*.

Now as each $W_i \times Y$ is covered by finitely many elements of $\mathscr{A}$ and *X* is covered by these W_i, we get that The union of the tubes $W_1 \times Y, \ldots, W_k \times Y$ covers $X \times Y$.

Thus, $X \times Y$ is compact.

Now assume that $X_1 \times \ldots \times X_{n-1}$ is compact.

Then $X_1 \times \ldots \times X_n \equiv (X_1 \times \ldots \times X_{n-1}) \times X_n$ is compact as it is the product of two compact spaces.

We now give a equivalent definition of compact space interms of closed sets. We start with the following definition.

Definition 4.15 : A collection $\mathscr{C}$ of subsets of X is said to have the **finite intersection property** if $\bigcap_{i=1}^{n} C_i \neq \varnothing$, for every finite sub collection $\{C_1,...,C_n\}$ of $\mathscr{C}$.

Theorem 4.16 : *Let X be a topological space. Then X is compact if and only if for every collection $\mathscr{C}$ of closed sets in X having the finite intersection property, the intersection* $\bigcap_{C\in\mathscr{C}} C \neq \varnothing$.

Proof : Suppose X is compact and $\mathscr{C}$ be the collection of closed sets in X having finite intersection property.

Suppose $\bigcap_{C\in\mathscr{C}} C = \varnothing$.

$\Rightarrow \bigcup_{C\in\mathscr{C}} (X - C) = X$

Since X is compact, there exists a finite sub cover $X - C_1, ..., X - C_n$ such that $X \subset \bigcup_{i=1}^{n} (X - C_i)$.

$\Rightarrow \varnothing \supset \bigcap_{i=1}^{n} C_i$, which is a contradiction.

Therefore, $\bigcap_{C\in\mathscr{C}} C \neq \varnothing$.

To prove converse, let $\mathscr{A}$ be an open cover for X.

For contrary, suppose assume that there is no finite sub collection of $\mathscr{A}$ which covers X.

Then $\bigcup_{i=1}^{n} A_i \neq X$ for any $n \in \mathbb{N}$.

$\Rightarrow \bigcap_{i=1}^{n} (X - A_i) \neq \varnothing$.

So the collection $\mathscr{C} = \{X - A_\alpha \mid A_\alpha \in \mathscr{A}\}$ satisfies the finite intersection property.

Therefore, $\bigcap_{A\in\mathscr{A}} (X - A) \neq \varnothing$

$\Rightarrow \bigcup_{A\in\mathscr{A}} A \neq X$, which is a contradiction.

Therefore, there exists a finite subcover for X and hence X is compact.

5. Compact Subspaces of the Real Line

We end this unit by proving that $\mathbb{R}$ is uncountable without using algebraic properties. We start with a definition.

Definition 5.1 : For a topological space *X*, a point $x \in X$ is said to be an **isolated point**, if the one-point set $\{x\}$ is open in *X*.

Theorem 5.2 : *Let X be a nonempty compact Hausdorff space. If X has no isolated points, then X is uncountable.*

Proof : **Step 1 :** We first show that given any nonempty open set *U* of *X* and $x \in X$, there exists a nonempty open set *V* contained in *U* such that $x \in \overline{V}$.

If $x \in U$, since *X* has no isolated points, $U \neq \{x\}$.

So there exists $y \in U$ such that $y \neq x$.

If $x \notin U$, since *U* is non empty, there exists $y \in U$.

So in any case, there exists $y \in U$ such that $y \neq x$.

As *X* is Hausdorff, there exists two disjoint open sets W_1 and W_2 containing *x* and *y*, respectively.

Let $V = W_2 \cap U$.

Since $y \in W_2 \cap U$, we get $V \neq \emptyset$ and $V \subset U$.

As $x \in W_1$ and $V \cap W_1 = \emptyset$, we get $x \notin \overline{V}$.

Step 2 : We show that given $f : \mathbb{Z}_+ \to X$, the function f is not surjective.

Let $x_n = f(n)$. Since *X* is non empty open set and $x_1 \notin \overline{V}_1$, by **Step 1**, there exists a nonempty open set $V_1 \subset X$ such that $x_1 \notin \overline{V}_1$.

As V_1 is non empty open subset of *X* and $x_2 \in X$, there exists a nonempty open set $V_2 \subset V_1$ such that $x_2 \notin \overline{V}_2$.

By induction, given a non empty open set V_{n-1}, there exists a nonempty open set $V_n \subset V_{n-1}$ such that $x_n \notin \overline{V}_n$.

Then the collection $\mathscr{C} = \{\overline{V}_i \mid i = 1, 2, 3, ...\}$ of nonempty closed sets of X satisfies finite intersection property.

Because X is compact, $\bigcap \overline{V}_n \neq \varnothing$, say $x \in \bigcap \overline{V}_n$.

Since $x \in \overline{V}_n$ but $x_n \notin \overline{V}_n$ for all n, we get that $x \neq x_n$ for all n.

Therefore, f is not surjective and hence X is uncountable.

Corollary 5.3 : *Every closed interval in* $\mathbb{R}$ *is uncountable.*

Corollary 5.4 : $\mathbb{R}$ *is uncountable.*

EXERCISE - 9

1. Which of the following is true ?

   (A) The real line $\mathbb{R}$ is compact

   (B) $\{0\} \cup \{1/n \mid n \in \mathbb{Z}_+\}$ is compact

   (C) The interval (0;1] is compact

   (D) All of the above are true.

2. Consider the following

   (I) If Y is a subspace of a compact space X, then Y is compact.

   (II) If Y is a compact subspace of X, then Y is closed.

   (A) Only (I) is true. (B) Only (II) is true.

   (C) Both (I) and (II) are true. (D) Both (I) and (II) are false.

3. Consider the following two statements

(I) If X is nonempty compact Hausdorff space with no isolated points, then X is uncountable.

(II) Every open interval in $\mathbb{R}$ is compact Hausdorff space and hence is uncountable.

(A) Only (I) is true.

(B) Only (II) is true.

(C) Both (I) and (II) are true.

(D) Both (I) and (II) are false.

4. Show that $\mathbb{R}$ with the finite complement topology is compact.

5. Prove that an infinite set X with the discrete topology is not compact.

6. Show that the union of a finitely many compact subsets of X is compact

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UNIT - IV

SEPARATION AXIOMS, NORMAL SPACES AND URYSOHN METRIZATION THEOREM

1. Forms of Compact Spaces

Introduction :

In early days of topology, a space is called compact if every infinite subspace of it has a limit point, where as the open covering formulation was called bicompactness. Later, the standard definition of compact is interms of open covering, the above compactness is renamed to limit point compactness. There is also another version of compactness called sequential compactness. In this unit we will compare these three versions of compactness and see when they all be same. We also study local compactness and one point compactification.

Definition 1.1 : A space X is said to be **limit point compact** if every infinite subset of X has a limit point.

Definition 1.2 : A space X is said to be **sequentially compact** if every sequence of points of X has a convergent subsequence.

The next few results emphasize the relation among these three versions of compactness.

Theorem 1.3 : *Every compact space is limit point compact.*

Proof : Let X be a compact space and $A \subset X$ be infinite.

Suppose A has no limit point.

Then $\overline{A} = A \bigcup \{\text{limit point of A}\} = A$, so that A is closed.

Since A has no limit point, for each $a \in A$, there exists a neighborhood U_a of a such that $U_a \cap A = \{a\}$.

Then X is covered by the open set $X - A$ and the open sets U_a.

As X is compact, it can be covered by finitely many of these sets

Say $X \subset \bigcup_{i=1}^{n} U_i \cup (X - A)$

Since $X - A$ does not intersect A, we have $A \subset \bigcup_{i=1}^{n} U_i$

As each set U_i contains only one point of A, the set A must be finite, which is a contradiction. Hence A has a limit point.

Example 1.4 : Limit point compactness need not implies compactness.

Proof : Let $Y = \{a, b\}$ be given with indiscrete topology, i.e. Y and $\varnothing$ are the only open sets in Y.

We show that the space $X = \mathbb{Z}_+ \times Y = \{(n, a), (m, b) \mid n, m \in \mathbb{Z}_+\}$ is limit point compact.

Let S be a non empty set of X, say $(n, a) \in S$.

Then (n, b) is a limit point of S as if $A \times Y$ is a neighborhood of (n, b), then $(n, a) \in (A \times Y) \cap S$.

We can observe that singleton $\{n\}$ is open in $\mathbb{Z}_+$ as $\{n\} = (n-1, n+1) \cap \mathbb{Z}_+$.

Thus the collection $U_n = \{n\} \times Y$ is an open cover for X but has no finite subcover for X. Therefore, X is not compact.

Theorem 1.5 : *Let X be a metrizable space. If X is limit point compact, then X is sequentially compact.*

Proof : Let X be a limit point compact space and (x_n) be a sequence in X.

Let $A = \{x_n \mid n \in Z_+\}$.

If the set A is finite, then there is a point x such that $x = x_n$ for infinitely many values of n. In this case, the sequence x_n has a constant subsequence and therefore converges.

Suppose A is infinite. Since X is limit point compact, A has a limit point x.

Then A intersects every neighborhood of x at infinitely many points.

Now We define a subsequence (x_n) converging to x as follows :

Since A intersects $B(x, 1)$, choose n_1 such that $x_{n_1} \in B(x,1)$.

Again as A intersects $B(x,1/2)$ at infinitely many points, choose $n_2 > n_1$ such that $x_{n_2} \in B(x,1/2)$.

In this way, we choose $n_k > n_{k-1}$ such that $x_{n_k} \in B(x,1/k)$.

Then the subsequence $x_{n_1}, x_{n_2}, ...$ converges to x.

Theorem 1.6 : *Let X be a metrizable space. If X is sequentially compact, then X is compact.*

Proof : We prove the result in 3 steps :

Step 1 : We show that if $\mathscr{A}$ is an open cover for X, there exists $\delta > 0$ (called Lebesgue number) such that if $A \subset X$ with *diam* $(A) < \delta$, then there exists $U \in \mathscr{A}$ such that $A \subset U$.

Let $\mathscr{A}$ be an open covering of X. Suppose that there is no $\delta > 0$.

Then for each positive integer n, there exists $C_n \subset X$ with $diam(C_n) < \frac{1}{n}$ but $C_n \not\subseteq U$ for all $U \in \mathscr{A}$.

Choose a point $x_n \in C_n$ for each n.

Since X is sequentially compact, there exists a subsequence (x_{n_k}) of the sequence (x_n) that converges, say to the point a.

Since $\mathscr{A}$ is an open cover for X, there exists $U \in \mathscr{A}$ such that $a \in U$.

Because X is metriazble and U is open, there exists ε > 0 such that $B(a,\varepsilon)\subset U$.

Since x_{n_k} converges to x, choose n_k large enough so that $x_{n_k}\in B\left(a,\frac{\varepsilon}{2}\right)$ and $\frac{1}{n_k}<\frac{\varepsilon}{2}$.

Then $C_{n_k}\subset B(a,\varepsilon)\subset U \Rightarrow C_{n_k}\subset U$, which is a contradiction.

So for every open cover for X, there exists a δ > 0 satisfying the condition mentioned in **Step 1**.

Step 2 : Given ε > 0, there exists a finite covering of X by open ε-balls.

Suppose assume that there exists an ε >0 such that X cannot be covered by finitely many ε-balls.

We Construct a sequence of points x_n of X as follows:

For any $x_1\in X$, $X\neq B(x_1,\varepsilon)$ (otherwise X could be covered by a single ε-ball).

Choose $x_2\in X-B(x_1,\varepsilon)$. Then $d(x_1,x_2)>\varepsilon$.

Again $X\neq B(x_1,\varepsilon)\cup B(x_2,\varepsilon)$.

Choose $x_3\in X-\left(B(x_1,\varepsilon)\cup B(x_2,\varepsilon)\right)$.

Then $d(x_1,x_3)>\varepsilon$ and $d(x_2,x_3)>\varepsilon$.

By continuing this way, we get $X\neq\bigcup_{i=1}^{n}B(x_i,\varepsilon)$ and $x_{n+1}\in\left[X-\bigcup_{i=1}^{n}B(x_i,\varepsilon)\right]$ such that $d(x_{n+1},x_i)>\varepsilon$ for all i = 1, 2, ... , n.

Therefore, the sequence x_n does not have any convergent subsequence as $d(x_n,x_m)>\varepsilon$ for all $n > m$.

Step 3 : Now we prove that X is compact.

Let $\mathscr{A}$ be an open covering of X.

By **Step 1**, the open cover $\mathscr{A}$ has a Lebesgue number δ.

Let $\varepsilon = \frac{\delta}{3}$. Then by **Step 2**, there exists a finite covering $\{B(x_j, \varepsilon)\}$ of *X* by open ε-balls.

Since $diam(B(x_j, \varepsilon)) = 2\varepsilon = \frac{2\delta}{3} < \delta$, there exists $U_j \in \mathscr{A}$ such that $B(x_j, \varepsilon) \subset U_j$ for all $j = 1, 2, \ldots, n$.

Then $X = \bigcup_{i=1}^{n} B(x_j, \varepsilon) \subset \bigcup_{i=1}^{n} U_j$.

Hence there is a finite subcollection of $\mathscr{A}$ that covers *X*.

Theorem 1.7 : *Let X be a metrizable space. Then the following are equivalent :*

1. *X is compact.*
2. *X is limit point compact.*
3. *X is sequentially compact.*

Proof : (1) $\Rightarrow$ (2) : Proof of theorem ??

(2) $\Rightarrow$ (3) : Proof of theorem ??

(3) $\Rightarrow$ (1) : Proof of theorem ??

Definition 1.8 : A space *X* is said to be **locally compact at** *x* if there is some compact subspace *C* of *X* that contains a neighborhood of *x*. If *X* is locally compact at each of its points, *X* is said simply to be **locally compact**.

Example 1.9 : The real line $\mathbb{R}$ is locally compact,

because $x \in (x - \varepsilon, x + \varepsilon) \subset [x - \varepsilon, x + \varepsilon]$.

Example 1.10 : The space $\mathbb{R}^n$ is locally compact,

because $x \in (a_1, b_1) \times \ldots \times (a_n, b_n) \subset [a_1, b_1] \times \ldots \times [a_n, b_n]$

Example 1.11 : The space $\mathbb{R}^{\omega}$ is not locally compact,

because if a basis element $B=(a_1,b_1)\times\ldots\times(a_n,b_n)\times\mathbb{R}\times\ldots\times\mathbb{R}\times\ldots$ contained in a compact subspace, then its closure $B=[a_1,b_1]\times\ldots\times[a_n,b_n]\times\mathbb{R}\times\ldots\times\mathbb{R}\times\ldots$ is compact, which is a contradiction.

Example 1.12 : Every simply ordered set *X* having the least upper bound property is locally compact: Given a basis element for *X*, it is contained in a closed interval in X, which is compact.

Definition 1.13 : If *Y* is a compact Hausdorff space and *X* is a proper subspace of *Y* whose closure equals *Y*, then *Y* is said to be a **compactification** of X. If $Y-X$ equals a single point, then *Y* is called the **one-point compactification** of *X*.

Theorem 1.14 : *Let X be a Hausdorff space Then X is locally compact if and only if given x in X, and given a neighborhood U of x, there is a neighborhood V of x such that* $\overline{V}$ *is compact and* $\overline{V}\subset U$ *(i.e.* $x\in V\subset\overline{V}\subset U$*).*

Proof : Suppose *X* is locally compact.

Let $x\in X$ and *U* be a neighborhood of *x*.

Since *X* is locally compact, there exists a one-point compactification *Y* of *X*.

Let *C* = *Y* – *U*. Then *C* is closed in *Y* implies *C* is a compact subspace of *Y*.

Since *X* is Hausdorff, there exist two open sets *V* and *W* such that $x\in V$ and $C\subset W$.

$$V\cap W=\varnothing\Rightarrow V\subset W^c\Rightarrow\overline{V}\cup W^c\subset C^c=U$$

Since W^c is closed, $\overline{V}$ is closed and hence compact.

Hence $\overline{V}\subset U$, and $\overline{V}$ is compact.

Suppose assume the converse part.

Let $x\in X$. Since *X* is open, by assumption, there is a neighborhood *V* of *x* such that $\overline{V}$ is compact.

Corollary 1.15 : *Let X be locally compact Hausdorff, let A be a subspace of X. If A is closed in X or open in X, then A is locally compact.*

Proof : Suppose that *A* is closed in *X*. Let $x \in A$. Then $x \in X$.

Since *X* is locally compact, there exists a compact subspace *C* of *X* containing the neighborhood *U* of *x* in *X*. Then $C \cap A$ is closed in *C* and thus compact, and $U \cap A \subset C \cap A$.

Suppose now that *A* is open in *X*. Given $x \in A$, by the preceding theorem there exists a neighborhood *V* of *x* in *X* such that $\overline{V}$ is compact and $\overline{V} \subset A$. Then $C = \overline{V}$ is a compact subspace of *A* containing the neighborhood *V* of x in *A*.

EXERCISE - 10

1. Consider the following.

   (I) Every compact space is limit point compact.

   (II) Every limit point compact space is compact.

   (A) Only (I) is true. (B) Only (II) is true.

   (C) Both (I) and (II) are true. (D) Both (I) and (II) are false.

2. Consider the two statements

   (I) Every sequentially compact space is compact.

   (II) Every limit point compact space is sequentially compact.

   (A) Only (I) is true. (B) Only (II) is true.

   (C) Both (I) and (II) are true. (D) Both (I) and (II) are false.

3. Show that [0, 1] is not limit point compact as a subspace of $\mathbb{R}_l$.

4. Let *X* be limit point compact. If $f : X \to Y$ is continuous, does it follow that $f(X)$ is limit point compact ?

5. Let X be limit point compact. If A is closed subset of X, does it follow that A is limit point compact ?

6. Show that the rationals $\mathbb{Q}$ are not locally compact.

2. Countability Axioms

Introduction :

The countable axioms do not arise naturally from the study of analysis. Problems like embedding a given space in a metric space or in a compact Hausdorff are purely from topology and these problems can be solved with the help of countable and separable axioms. In this section, we study the two countable axioms: first countable and second countable; relation among them.

Definition 2.1 : A space X is said to have a countable basis at x if there is a countable collection $\mathscr{B}$ of neighborhoods of x such that each neighborhood of x contains at least one of the elements of $\mathscr{B}$. A space that has a countable basis at each of its points is said to satisfy the **first countability axiom**, or to be **first-countable**.

Example 2.2 : ($\mathbb{R}$, usual topology) is first countable.

Proof : For each $x \in \mathbb{R}$, consider $\mathscr{B}_x = \left\{ \left(x - \frac{1}{n}, x + \frac{1}{n} \right) \mid n \in \mathbb{N} \right\}$.

Let U be a neighborhood of x.

Then there exists ε > 0 such that $x \in (x - \varepsilon, x + \varepsilon) \subset U$.

By Archimedean property, there exists $n \in \mathbb{N}$ such that $\frac{1}{n} < \varepsilon$.

Then, $x \in \left(x - \frac{1}{n}, x + \frac{1}{n} \right) \subset (x - \varepsilon, x + \varepsilon) \subset U$.

Therefore $\mathscr{B}_x$ is a countable base at x and hence ($\mathbb{R}$,; usual topology) is first countable.

Example 2.3 : Every metrizable space is first countable.

Proof : For each $x \in X$, let $\mathscr{B}_x = \left\{ B\left(x, \frac{1}{n} \right) | \, n \in \mathbb{N} \right\}$.

Let U be a neighborhood of x.

Then there is an $\varepsilon > 0$ such that $x \in B(x, \varepsilon) \subset U$.

By Archimedean property, there exists $n \in \mathbb{N}$ such that $\frac{1}{n} < \varepsilon$.

Then, $x \in B\left(x, \frac{1}{n} \right) \subset B(x, \varepsilon) \subset U$.

Therefore $\mathscr{B}_x$ is a countable base at x and hence X is first countable.

Example 2.4 : The real line $\mathbb{R}$ with countable complement topology (co-countable) is not first countable.

Proof : Let $x \in \mathbb{R}$.

Suppose $\mathscr{B} = \{ B_n \, | \, n \in \mathbb{N} \}$ is a countable base at x.

Here each B_n^c is countable and so $\bigcup_n B_n^c = \left(\bigcap_n B_n \right)^c$ is countable

Therefore $V = \bigcap_n B_n$ is open and $x \in V$.

Now take $y \in V \setminus \{x\}$ and $U = V \setminus \{y\}$.

Then $x \in U$ and U is open as $U^c = V^c \bigcup \{y\}$ is countable.

As $y \in B_n$ for each n and $y \notin U$, we get that $B_n \not\subset U$ for all n.

Which is a contradiction to the fact that $\mathscr{B}$ is countable base at x.

Therefore, $\mathbb{R}$ with countable complement topology is not first countable.

Theorem 2.5 : *Let X be a topological space.*

1. *Let A be a subset of X. If there is a sequence of points of A converging to x, then* $x \in \overline{A}$*; the converse holds if X is first-countable.*

2. *Let* $f : X \to Y$. *If f is continuous, then for every convergent sequence* $x_n \to x$ *in X, the sequence f (*x_n*) converges to f (x). The converse holds if X is first countable.*

Proof : (1) Let $(x_n) \subset A$ be a sequence such that $x_n \to x$, for some $x \in X$.

To show $x \in \overline{A}$, let U be an open set of X with $x \in U$.

Since $x_n \to x$, and $x \in U$, infinitely many $x_n^{'}s$ are in U.

Therefore, $U \cap A \neq \varnothing$ and hence $x \in \overline{A}$.

Conversely, suppose that X is first countable and $x \in \overline{A}$.

Let $\mathscr{B} = \{B_n \mid n \in \mathbb{N}\}$ be a countable basis at x.

We may assume that $x \in B_n$ for all n.

Let $U_n = B_1 \cap ... \cap B_n$.

Then U_n is open containing x and $U_n \subset B_n$.

If V is open set containing x, then there exist n such that $x \in B_n \subset V$

Since $U_n \subset B_n$, we get that $x \in U_n \subset V$.

Therefore, $\mathscr{U} = \{U_n \mid n \in \mathbb{N}\}$ is a countable basis for x.

Now, if $x \in A$, then $(x, x, x, ...)$ is the required sequence that converges to x.

If $x \notin A$, then x is a limit point of X and hence A intersects every neighbourhood of x.

As each U_n is open contains x, we get $U_n \cap A \neq \varnothing$ for all n.

Let $x_n \in U_n \cap A$. Then (x_n) is a sequence in A.

To show $x_n \to x$, let $x \in U$ be an open set of X.

Since $\mathscr{U}$ is a countable bais for *x*, there exists U_m such that $x \in U_m \subset U$.

Then $x_n \in U_n \subset U_m \subset U$ for all $n \geq m$. ($\because U_n \subset U_m \quad \forall n \geq m$)

Therefore $x_n \to x$.

(2) Suppose f is continuous and $x_n \to x$.

Let *V* be an open set of *Y* such that $f(x) \in V$.

Then $f^{-1}(V)$ is open in *X* and $x \in f^{-1}(V)$.

Therefore, $x_n \in f^{-1}(V)$ for infinitely many *n*'s.

$\Rightarrow f(x_n) \in V$ for infinitely many *n*'s and hence $f(x_n) \to f(x)$.

Conversely, assume that *X* is first countable and whenever $x_n \to x$, then $f(x_n) \to f(x)$.

To show f is continuous, we prove that $f(\overline{A}) \subset \overline{f(A)}$ for any subset *A* of *X*.

Let $y \in f(\overline{A})$. Then $y = f(x)$, for some $x \in \overline{A}$.

Since *X* is first countable and $x \in \overline{A}$, there exists a sequence $(x_n) \subset A$, such that $x_n \to x$.

Then by assumption, $f(x_n) \to f(x)$ and $f(x_n) \subset f(A)$.

Therefore, $f(x) \in \overline{f(A)}$. ($\because$ by first result)

$\Rightarrow y \in \overline{f(A)}$ and hence $f(\overline{A}) \subset \overline{f(A)}$.

Now we will go to the second countable spaces which have more impact than the first countable spaces.

Definition 2.6 : A topological space *X* is said to satisfy the **second countability axiom**, or to be **second-countable**, if it has a countable basis for its topology.

Theorem 2.7 : *Every second countable space is first countable, but not the converse.*

Proof : Let $\mathscr{B} = \{B_n \mid n \in \mathbb{N}\}$ be a countable basis for *X* and $x \in X$.

Then we prove that $\mathscr{B}_x = \{B_n \in \mathscr{B} \mid x \in B_n\}$ is a countable basis at *x*.

Let *U* be a neighborhood of *x*. Then there exists $B_n \in \mathscr{B}$ such that $x \in B_n \subset U$.

As $B_n \in \mathscr{B}_x$ we get that $\mathscr{B}_x$ is a basis at *x*.

Therefore *X* is first countable.

Converse is not true in general: Consider ($\mathbb{R}$, discrete topology).

As $\mathscr{B} = \{\{x\}\}$ is a countable basis at $x \in X$, we get that ($\mathbb{R}$, discrete topology) is first countable.

Now suppose $\mathscr{B}$ is a basis for ($\mathbb{R}$, discrete topology).

Since each $\{x\}$ is open, there exists $B_x \in \mathscr{B}$ such that $x \in B_x \subset \{x\} \Rightarrow B_x = \{x\}$.

As $\mathbb{R}$ is uncountable, $\{B_x \mid x \in \mathbb{R}\}$ is uncountable.

Therefore, $\mathscr{B}$ is uncountable and hence ($\mathbb{R}$, discrete topology) is not second countable.

We now show that the spaces satisfying countable axioms are nice in the sense that they can be passed onto subspaces and products.

Theorem 2.8 : *A subspace of a first-countable space is first-countable, and a countable product of first-countable spaces is first-countable. A subspace of a second countable space is second-countable, and a countable product of secondcountable spaces is second-countable.*

Proof : It is enough to prove for second countable spaces.

Let $\mathscr{B}$ be a countable basis for *X* and $A \subset X$.

Now consider $\mathscr{B}_A = \{B \cap A \mid B \in \mathscr{B}\}$.

Then clearly $\mathscr{B}_A$ is countable.

To show $\mathscr{B}_A$ is a basis for *A*, let *U* be a neighborhood of $a \in A$.

Then $U = A \cap V$ for some V open in X.

Since $\mathscr{B}$ is a basis for X, there exists $B \in \mathscr{B}$ such that $a \in B \subset V$.

As $A \cap B \subset A \cap V$, we get $a \in A \cap B \subset U$.

Therefore $\mathscr{B}_A$ is a countable basis for A.

If $\mathscr{B}_i$ is a countable basis for the space X_i, then the collection of all products $\prod U_i$, where $U_i \in \mathscr{B}_i$ for finitely many values of i and $U_i = X_i$ for all other values of i, is a countable basis for $\prod X_i$.

EXERCISE - 11

1. Consider the following

   (I) Every first countable space is second countable.

   (II) The discrete topology on $\mathbb{R}$ is second countable.

   (A) Only (I) is true. (B) Only (II) is true.

   (C) Both (I) and (II) are true. (D) Both (I) and (II) are false.

2. Consider the two statements

   (I) $\mathbb{R}$ with usual topology is second countable.

   (II) $\mathbb{R}$ with usual topology is first countable.

   (A) Only (I) is true. (B) Only (II) is true.

   (C) Both (I) and (II) are true. (D) Both (I) and (II) are false.

3. Show that $\mathbb{R}_l$ is first countable

4. Show that the real line $\mathbb{R}$ with finite complement topology(co-finite) is not first countable.

5. Is $\mathbb{R}_l$ is second countable? Justify.

3. Lindelof spaces

Introduction :

Other than the two countable axioms, there are other two alternative countable axioms, namely separable and Lindelof. Even though these two axioms are weaker than the second countable, they have their own importance.

Definition 3.1 : A subset A of a space X is said to be *dense* in X if $\overline{A} = X$.

Definition 3.2 : A space X is said to be **Lindelof space**, if every open covering of X contains a countable sub covering.

Definition 3.3 : A space X is said to be **separable**, if it has a countable dense subset.

We now prove that every second countable space is Lindelof as well as separable.

Theorem 3.4 : *Suppose that X has a countable basis. Then :*

(a) *Every open covering of X contains a countable sub collection covering X.*

(i.e. every second countable space is Lindelof)

(b) *There exists a countable subset of X that is dense in X.*

(i.e. every second countable space is separable)

Proof : Let $\mathscr{B} = \{B_n \mid n \in \mathbb{N}\}$ be a countable basis for X.

(a) Let $\mathscr{A}$ be an open covering of X.

Consider $\mathscr{B}' = \{B \in \mathscr{B} \mid \text{ there exists } U_\alpha \in \mathscr{A} \text{ such that } B \subset U_\alpha\}$.

We show that $\bigcup\limits_{B \in \mathscr{B}'} B = X$.

Let $x \in X$. Then there exists $U_\alpha \in \mathscr{A}$ such that $x \in U_\alpha$.

Since $\mathscr{B}$ is basis for X, there exists $B \in \mathscr{B}$ such that $x \in B \subset U_\alpha$.

Then $B \in \mathscr{B}'$ and $x \in B$.

Now for each $B \in \mathscr{B}'$, choose $U_B = U_\alpha$ such that $B \subset U_\alpha$.

Then $X = \bigcup_{B \in \mathscr{B}'} B \subset \bigcup_{B \in \mathscr{B}'} U_B$.

Therefore, $\{U_B \mid B \in \mathscr{B}'\}$ is a countable subcover for *X*.

(b) For each $n \in \mathbb{N}$, let $x_n \in B_n$.

Let $D = \{x_n \mid x_n \in B_n, n \in \mathbb{N}\}$.

To show *D* is dense in *X*, let $x \in X$ and $x \in U$ be an open set.

Since $\mathscr{B}$ is a basis for *X*, there exists $n \in \mathbb{N}$ such that $B_n \subset U$.

$x_n \in B_n$ implies $x_n \in U$ and hence $D \bigcap U \neq \varnothing$.

Example 3.5 : The space $\mathbb{R}_l$ (lower limit topology) is first countable, Lindelof, separable but not second countable.

Proof :

1. **First countable :** Given $x \in \mathbb{R}_l$, the collection

$$\mathscr{B}_x = \left\{ \left[x, x + \frac{1}{n} \right) \mid n \in \mathbb{N} \right\}$$ is a countable basis at *x*.

2. **Separable :** Clearly the set of rational numbers $\mathbb{Q}$ is dense in $\mathbb{R}_l$.

3. **Lindelof :** Let $\mathscr{A}$ be an open covering for $\mathbb{R}_l$.

Then for any $U \in \mathscr{A}$, there exists a basis element $[a_\alpha, b_\alpha)$ contained in *U*.

So if open cover of basis elements has a countable sub cover then $\mathscr{A}$ will have countable sub cover.

So without loss of generality, let $\mathscr{A} = \{[a_\alpha, b_\alpha) \mid \alpha \in J\}$ be an covering of $\mathbb{R}_l$.

Let *C* be the set $C = \bigcup_{\alpha \in J} (a_\alpha, b_\alpha)$ which is a subset of $\mathbb{R}$.

We show the set $\mathbb{R}-C$ is countable.

Let x be a point of $\mathbb{R}-C$.

Since $x \notin C$, x belongs to no open interval (a_α, b_α), therefore $x = a_\beta$ for some index β.

Choose such a β and then choose q_x to be a rational number belonging to the interval (a_β, b_β).

Define $f: \mathbb{R}-C \to \mathbb{Q}$ by $f(x) = q_x$.

To show f is injective, let $x, y \in \mathbb{R}-C$ with $x < y$.

Then $f(x) = q_x \in (a_\beta, b_\beta) \Rightarrow q_x < b_\beta$

Since $x < y$ and $y \notin (a_\beta, b_\beta)$, we get $y > b_\beta$.

Therefore, $f(x) = q_x < b_\beta < y < f(y)$, hence f is injective.

$(\because y = a_\gamma$ and $f(y) \in (a_\gamma, b_\gamma) \Rightarrow y = a_\gamma < f(y))$

Therefore $\mathbb{R}-C$ is countable.

Choose a countable sub collection $\mathscr{A}'$ of $\mathscr{A}$ that covers $\mathbb{R}-C$.

Since C is a subset of $\mathbb{R}$, C is a subspace of ($\mathbb{R}$, usual topology) and hence second countable. Now C is covered by the sets (a_α, b_α), which are open in $\mathbb{R}$ and hence open in C.

Then there exists a countable subcollection (a_α, b_α) for $\alpha = \alpha_1, \alpha_2, \ldots$ covering C.

Then the collection $\mathscr{A}'' = \{[a_\alpha, b_\alpha) \mid \alpha = \alpha_1, \alpha_2, \ldots\}$ is a countable subcollection of $\mathscr{A}$ that covers the set C.

Now $\mathscr{A}' \cup \mathscr{A}''$ is a countable sub collection of $\mathscr{A}$ that covers $\mathbb{R}_l$.

Therefore $\mathbb{R}_l$ is Lindelof.

4. Not second countable : Suppose $\mathscr{B} = \{B_1, B_2, ...,\}$ is a countable basis for $\mathbb{R}_l$.

Let $b_n = \inf B_n$ and $J = \{b_n \mid n \in \mathbb{N}\}$.

Let $a \in \mathbb{R} \setminus J$. Then $a \neq b_n$ for all n.

Now consider $U = [a, \infty)$.

Suppose there exists $B_n \in \mathscr{B}$ such that $a \in B_n \subset U = [a, \infty)$, then $a = \inf B_n = b_n$, which is a contradiction.

Therefore, there doesn't exists a countable basis for $\mathbb{R}_l$ and hence is not second countable.

Linedlof spaces are not as nice as first and second countable spaces in the sense that they are not passed on to subspaces and products.

Example 3.6 : A subspace of a Lindelof space need not be Lindelof.

Proof : The ordered square $I_0^2 = [0,1] \times [0,1]$ is compact; therefore it is Lindelof.

Now consider the subspace $A = I \times (0,1)$.

Then A is the union of the disjoint sets $U_x = \{x\} \times (0,1)$, each of which is open in A. This collection of sets is uncountable, and no proper subcollection covers A. Therefore A is not Lindelof.

Example 3.7 : The product of two Lindelof spaces need not be Lindelof.

Proof : Even though the space $\mathbb{R}_l$ is Lindelof, we prove that the product space $\mathbb{R}_l \times \mathbb{R}_l = \mathbb{R}_l^2$ is not Lindelof.

Basis for $\mathbb{R}_l^2$ consists of the sets of the form $[a,b) \times [c,d)$.

To show it is not Lindelof, consider the subspace $L = \{x \times (-x) \mid x \in \mathbb{R}_l\}$.

Then L is closed in $\mathbb{R}_l^2$ and $\mathbb{R}_l^2 \setminus$ L is open.

Now we can cover $\mathbb{R}_l^2$ by the open set $\mathbb{R}_l^2 - L$ and by all basis elements of the form $[a,b)\times[-a,d)$.

Each of these open sets intersects L in at most one point.

Since L is uncountable, no countable subcollection covers $\mathbb{R}_l^2$.

Therefore $\mathbb{R}_l^2$ is not Lindelof.

Theorem 3.8 : *A closed subspace of a Lindelof space is Lindelof.*

Proof : Let Y be a closed subspace of a Lindelof space X.

To show Y is Lindelof, let $\mathscr{A} = \{U_\alpha \mid \alpha \in J\}$ be an open cover for Y.

Then $U_\alpha = U_\alpha' \cap Y$ where U_α' a is open in X.

Let $\mathscr{A}' = \{U_\alpha' \mid \alpha \in J\}$. Then $\mathscr{A}' \cup (X - Y)$ is an open cover for X.

Since X is Lindelof, this cover has a countbale sub cover, say $U_1', U_2', \ldots..$.

If some U_i' contains $(X - Y)$, drop that U_i'.

Then the collection U_1, U_2, ... is a countable subcover for Y.

EXERCISE - 12

1. Consider the following

(I) Every second countable space is Lindelof.

(II) Every second countable space is separable.

(A) Only (I) is true. (B) Only (II) is true.

(C) Both (I) and (II) are true. (D) Both (I) and (II) are false.

2. Consider the following

(I) Every separable space is first countable.

(II) Every first countable space is separable.

(A) Only (I) is true. (B) Only (II) is true.

(C) Both (I) and (II) are true. (D) Both (I) and (II) are false.

3. Which of the following is false ?

(A) The space $\mathbb{R}_l$ is first countable

(B) The space $\mathbb{R}_l$ is second countable

(C) The space $\mathbb{R}_l$ is Lindelof

(D) The space $\mathbb{R}_l$ is separable.

4. Show that every separable metric space is second countable.

5. Show that ($\mathbb{R}$, *cofinite*) is separable but not first countable.

6. Is ($\mathbb{R}$, *discrete*) separable ? Justify.

4. Separation Axioms

Introduction :

The separation axioms are about the use of topological means to distinguish disjoint sets and distinct points. Separation axioms depends on how rich is the topological space interms of open sets. More the open sets in a space, it separates more points and sets. The separation axioms are denoted with the letter “T”, as the word for separation in German is Trennung. In this section, we discuss three separation axioms: T_0, T_1 and T_2.

Definition 4.1 (T_0 axiom) : A topological space X is said to satisfy T_0 axiom, if given two distinct points x and y from X, there exists an open set U containing exactly one of these points, i.e. $x \in U$ and $y \notin U$ or $x \notin U$ and $y \in U$. A space is called T_0 if it satisfies T_0 axiom. T_0 space is also called Kolmogorov space.

Definition 4.2 (T_1 axiom) : A topological space X satisfies T_1 axiom, if for given two distinct points $x, y \in X$, there exists two open sets U and V such that $x \in U$ but $y \notin U$ and $y \in V$ but $x \notin U$.

Definition 4.3 (T_2 axiom) : A topological space X satisfies T_2 axiom, if for given two distinct points $x, y \in U$, there exists two open sets U and V such that $x \in U$ but $y \notin U$; $y \in V$ but $x \notin U$ and $U \cap V = \varnothing$. T_2 space is also called Hausdorff space, which we have seen already.

Remark 4.4 : The following observations justify why above axioms are called separation axioms:

1. In T_0 space, any two distinct points are separated (or distinguishable) by an open set.
2. In T_1 space, any two distinct points are separated (or distinguishable) by two open sets (need not be disjoint).
3. In T_2 space, any two distinct points are separated (or distinguishable) by two disjoint open sets.
4. We can understand these spaces through the following diagram :

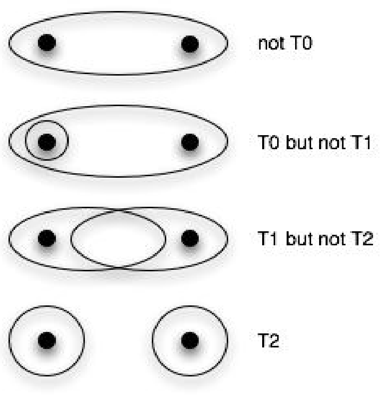

Figure 10:

5. We can also observe that $T_2 \Rightarrow T_1 \Rightarrow T_0$.

Example 4.5 : Let $X = \{a,b,c\}$ with topology $\mathscr{T} = \{\varnothing, X, \{a\}, \{b\}, \{a,b\}\}$. Then *X* is a T_0 space.

Proof : The open set $\{a\}$ separates *a* and *b*; *a* and *c*.

Similarly the open set $\{b\}$ separates *b* and c.

Therefore, *X* is a T_0 space.

Example 4.6 : The discrete topology with atleast two points is a T_0 space, as every singleton is open.

Example 4.7 : The indiscrete topology with atleast two points is not a T_0 space, as *X* is the only non empty open set.

Example 4.8 : Let $X = \{a,b,c\}$ with topology $\mathscr{T} = \{\varnothing, X, \{a\}, \{b\}, \{a,b\}\}$. Then *X* is a T_0 space but not T_1.

Proof : If we take *a* and *c*, then the only open set containing *c* is *X*, which also contains *a*. Thus we can not separate these two elements by two open sets.

Hence *X* is not T_1.

Theorem 4.9 : *A space X is T_1 space if and only if each singleton set is closed in X.*

Proof : Suppose *X* is T_1 space and $x \in X$.

To show $X \setminus \{x\}$ open, let $y \in X \setminus \{x\}$.

As $x \neq y$, there exists two open sets *U* and *V* such that $x \in U$ but $y \notin V$ and $y \in V$ but $x \notin V$.

Implies $y \in V \subset X \setminus \{x\}$.

Hence $X \setminus \{x\}$ open is open i.e. $\{x\}$ is closed.

Conversely, suppose each singleton is closed in *X*.

Let $x, y \in X$ with $x \neq y$.

Then $U = X \setminus \{y\}$ and $V = X \setminus \{x\}$ are open such that $x \in U$ but $y \notin V$ and $y \in V$ but $x \notin V$.

Therefore, X is T_1 space.

Theorem 4.10 : *A finite T_1 space is discrete.*

Proof : Let $X = \{x_1, ..., x_n\}$.

We have to show that each $\{x_i\}$ is open

But $\{x_i\}^c = \{x_1, ..., x_{i-1}, x_{i+1}, ..., x_n\}$ is finite and hence closed in *X*.

Thus *X* is discrete.

Example 4.11 : $\mathbb{R}$ together with finite complement topology is T_1 but not T_2.

Proof : Let $x, y \in \mathbb{R}$ with $x \neq y$.

Then $U = \mathbb{R} \setminus \{y\}$ and $V = \mathbb{R} \setminus \{x\}$ are open such that $x \in U$ but $y \notin V$ and $y \in V$ but $x \notin V$.

Therefore, $\mathbb{R}$ is T_1 space.

Suppose that $\mathbb{R}$ is T_2.

Then for $0, 1 \in \mathbb{R}$, there exists two open sets $0 \in U$; $1 \in V$ and $U \cap V = \emptyset$.

As *U* and *V* are open, U^c and V^c are finite.

Also $U \cap V = \emptyset$ implies $U \subset V^c$ and so *U* is finite.

Then $\mathbb{R} = U \cup U^c$ is finite, which is absurd.

Hence $\mathbb{R}$ together with finite complement topology is not T_2.

Remark 4.15 : The above theorem is not true if *Y* is not T_2.

To see this, consider *X* = ($\mathbb{R}$, *usual*), *Y* = ($\mathbb{R}$, *indiscrete*) and $D = \mathbb{Q}$.

Define $f : \mathbb{R} \rightarrow$ by $f(x) = \begin{cases} 1 & x \in D \\ 2 & x \notin D \end{cases}$

And $g:\mathbb{R}\to$ by $g(x)=\begin{cases}1 & x\in D\\ 3 & x\notin D\end{cases}$

Then f and g are continuous as Y is indiscrete.

Also $f(x) = g(x)$ for all $x\in D$.

But $f(x)\neq g(x)$ for $x\notin D$.

EXERCISE - 13

1. Consider the statements

   (I) Subspace of a T_1 space is T_1

   (II) Subspace of a T_2 space is T_2

   (A) Only (I) is true. (B) Only (II) is true.

   (C) Both (I) and (II) are true. (D) Both (I) and (II) are false.

2. Which of the following is true ?

   (I) A space is T_1 if and only every singleton is closed

   (II) A space is T_2 if and only every singleton is closed

   (A) Only (I) is true. (B) Only (II) is true.

   (C) Both (I) and (II) are true. (D) Both (I) and (II) are false.

3. Show that $\mathbb{R}$ with the topology $\mathscr{T}=\{A\subset\mathbb{R}\mid 0\in A\}\cup\{\varnothing\}$ is not T_1 space.

4. Show that any metric space is T_2 space.

5. Show that $\mathbb{R}$ with countable complement topology is not T_2, but any sequence has at most one limit.

6. Let X be a topological space and Y a Hausdorff space. Let f and g be a continuous function from X to Y. Show that the set $A=\{x\in X : f(x)=g(x)\}$ is a closed set.

5. Regular and Normal Spaces

Introduction :

In the previous section, we have seen three separation axioms. In this section we discuss two more important separation axioms: regular and normal. We also exhibit whether these can be passed on to subspaces and products.

Definition 5.1 : Let X be a topological space which is T_1. Then X is called regular if for every closed subset $B \subset X$ and for every $x \in X \setminus B$, there exist disjoint open sets U and V of X such that $x \in U$ and $A \subset V$.

Definition 5.2 : Let X be a topological space which is T_1. Then X is called normal if given two disjoint closed subsets A and B of X, there exist disjoint open sets U and V of X such that $A \subset U$ and $B \subset V$.

These two spaces are represented in the following diagram :

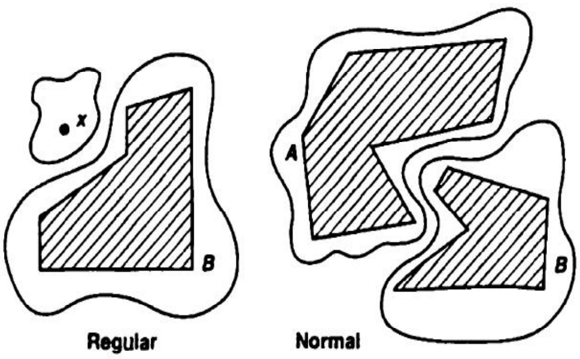

Figure 11:

In the following lemma, we give an equivalent definitions for regular and normal spaces.

Lemma 5.3 : *Let X be a topological space and one-point sets in X be closed.*

(a) *X is regular if and only if given a point x of X and a neighborhood U of x, there is a neighborhood V of x such that* $\overline{V} \subset U$.

(b) *X is normal if and only if given a closed set A and an open set U containing A, there is an open set V containing A such that* $\overline{V} \subset U$.

Proof :

(a) Suppose that X is regular.

Let $x \in X$ and U be a neighborhood of x.

Then $B = X - U$ is a closed set and $x \notin B$.

By hypothesis, there exist disjoint open sets V and W such that $x \in V$ and $B \subset W$

$\therefore B = X - U \subset W \Rightarrow U \supset X - W$.

Since $V \bigcap W = \varnothing$, $V \subset X - W$

Since $X - W$ is closed, we get $\overline{V} \subset X - W \subset U$. Therefore, $\overline{V} \subset U$.

To prove the converse, suppose the point x and the closed set B such that $x \notin B$ are given.

Then $U = X - B$ is open and $x \in U$.

By hypothesis, there is a neighborhood V of x such that $\overline{V} \subset U$.

Then the open sets V and $X - \overline{V}$ are disjoint such that $x \in V$ and $B \subset X - \overline{V}$.

Thus X is regular.

(b) Suppose that X is normal.

Let $A \subset X$ be closed and U be open such that $A \subset U$.

Then $B = X - U$ is a closed set and $A \bigcap B = \varnothing$.

By hypothesis, there exist disjoint open sets V and W such that

$A \subset V$ and $B \subset W$

$\therefore B = X - U \subset W \Rightarrow U \supset X - W$

Since $V \bigcap W = \varnothing$, $V \subset X - W$

Since $X - W$ is closed, we get $\overline{V} \subset X - W \subset U$. Therefore, $\overline{V} \subset U$.

To prove the converse, suppose that A and B are disjoint closed sets.

Then $U = X - B$ is open and $A \subset U$.

By hypothesis, there is an open set V containing A such that $\overline{V} \subset U$.

Then the open sets V and $X - \overline{V}$ are disjoint open sets containing A and B, respectively. Thus X is normal.

We prove that regularity can be passed onto subspaces and products.

Theorem 5.4 : *A subspace of a regular space is regular; a product of regular spaces is regular.*

Proof : Let Y be a subspace of the regular space X.

Then one-point sets are closed in Y.

($\because \{x\} = \{x\} \cap Y$ and $\{x\}$ is closed in X).

Let $x \in Y$ and B be a closed subset of Y such that $x \notin B$.

Now $\overline{B} \cap Y = B$, where $\overline{B}$ denotes the closure of B in X.

Therefore, $x \notin \overline{B}$. Since X is regular, there exist disjoint open sets U and V of X such that $x \in U$ and $\overline{B} \subset V$.

Then $U \cap Y$ and $V \cap Y$ are disjoint open sets in Y such that

$x \in U \cap Y$ and $\overline{B} \cap Y = B \subset V \cap Y$. Therefore, Y is regular.

Let $\{X_\alpha\}$ be a family of regular spaces and $X = \prod X_\alpha$.

Since each X_α is Hausdorff, $X = \prod X_\alpha$ is Hausdorff.

Hence one-point sets are closed in X.

To prove X regular, let $x = (X_\alpha) \in X$ and U be a neighborhood of x in X.

Since U is open, there exists a basis element $\prod U_\alpha$ about x such that $\prod U_\alpha \subset U$.

Since X_α is regular and $x_\alpha \in U_\alpha$, there exists a neighborhood V_α of x_α in X_α such that $\overline{V}_\alpha \subset U_\alpha$.

If $U_\alpha = X_\alpha$, then take $V_\alpha = X_\alpha$.

Then $V = \prod V_\alpha$ is a neighborhood of x in X.

Since $\overline{V} = \overline{\prod V_\alpha} = \prod \overline{V}_\alpha$, we get that $\overline{V} = \prod \overline{V}_\alpha \subset \prod U_\alpha \subset U$.

Therefore $\overline{V} \subset U$ and hence X is regular.

Example 5.5 : The space $\mathbb{R}_K$ is Hausdorff but not regular.

Proof : The basis for $\mathbb{R}_K$ is the union of all open intervals (a, b) and all sets of the form $(a,b)-K$, where $K = \{1/n \mid n \in \mathbb{Z}_+\}$.

This space is Hausdorff, because $\mathbb{R}$ with usual topology is Hausdorff, which is contained in $\mathbb{R}_K$.

We now show that $\mathbb{R}_K$ is not regular.

The set K is closed in $\mathbb{R}_K$, because $K^c = \mathbb{R} \setminus \left\{1, \frac{1}{2}, \ldots\right\}$ can be written as union of the basis elements: $K^c = (-\infty,-4) \cup \left[(-5,5) \setminus K\right] \cup (4,\infty)$ and thus K^c is open.

Also $0 \notin K$.

Suppose that there exist disjoint open sets U and V with $0 \in U$ and $K \subset V$.

Choose a basis element containing 0 and lying in U.

It must be a basis element of the form $(a, b) - K$, since each basis element of the form (a, b) containing 0 intersects K.

Choose n large enough that $1/n \in (a,b)$.

Then choose a basis element about $1/n$ contained in V; it must be a basis element of the form (c, d).

Finally, choose z so that $z < 1/n$ and $z > \max(c, 1/(n+1))$.

Then z belongs to both U and V, so they are not disjoint.

Example 5.6 : The space $\mathbb{R}_l$ is normal.

Proof : Since $\mathbb{R}_l$ is finer than $\mathbb{R}$ and one-point sets are closed in $\mathbb{R}$, we get that one-point sets are closed in $\mathbb{R}_l$.

To check normality, suppose that *A* and *B* are disjoint closed sets in $\mathbb{R}_l$.

Let $a \in A$. Since $\mathbb{R}_l - B$ is open and $a \in \mathbb{R}_l - B$, there exists a basis element $[a, x_a)$ such that $[a, x_a) \subset \mathbb{R}_l - B$. i.e. for each $a \in A$, $[a, x_a) \cap B = \varnothing$.

Similarly, for $b \in B$, choose a basis element $[b, x_b)$ such that $[b, x_b) \cap A = \varnothing$.

Now let $U = \bigcup_{a \in A} [a, x_a)$ and $V = \bigcup_{b \in B} [b, x_b)$.

Suppose $z \in U \cap V$, then there exists $a \in A$ and $b \in B$ such that $z \in [a, x_a)$ and $z \in [b, x_b)$. Suppose $a < b$. Then $a < b \leq z < z_a$ implies $b \in [a, x_a)$, which is a contradiction. $([a, x_a) \cap B = \varnothing)$

Therefore *U* and *V* are disjoint open sets containing *A* and *B* respectively.

Hence $\mathbb{R}_l$ is normal.

Remark 5.7 :

1. As $\mathbb{R}_l$ is normal, it is also regular, and hence $\mathbb{R}_l \times \mathbb{R}_l$ is regular.
2. The space $\mathbb{R}_l$ is normal, but $\mathbb{R}_l \times \mathbb{R}_l$ is not normal. Thus product of normal spaces need not be normal.
3. Also $\mathbb{R}_l \times \mathbb{R}_l$ is regular but not normal. So not every regular space is normal, but regular space with countable basis is normal as we prove in next result.

Theorem 5.8 : *Every regular space with a countable basis is normal.*

Proof : Let *X* be a regular space with a countable basis $\mathscr{B}$.

Let *A* and *B* be disjoint closed subsets of *X*.

As X is regular and $X - B$ is open, for each point $a \in A$, there exists a neighborhood $V_a \in \mathscr{B}$ such that $\overline{V}_a \subset X - B$.

Since $\mathscr{B}$ is countable, the collection $\{V_a \mid a \in A\}$ is countable and hence we can relabel them by V_i, $i \in \mathbb{N}$.

Therefore, $A \subset \bigcup_{i=1}^{\infty} V_i$ and $\overline{V}_i \cap B = \varnothing$ for all $i \in \mathbb{N}$.

Similarly there exists U_i in $\mathscr{B}$, $i \in \mathbb{N}$, such that $B \subset \bigcup_{i=1}^{\infty} U_i$ and $\overline{U}_i \cap A = \varnothing$ for all $i \in \mathbb{N}$.

Now let $U_1^{'} = U_1 - \overline{V}_1$ and $V_1^{'} = V_1 - \overline{U}_1$.

Then $U_1^{'}$ and $V_1^{'}$ are open such that $U_1^{'} \cap V_1^{'} = \varnothing$, $U_1^{'} \cap B = U_1 \cap B$ and $V_1^{'} \cap A = V_1 \cap A$.

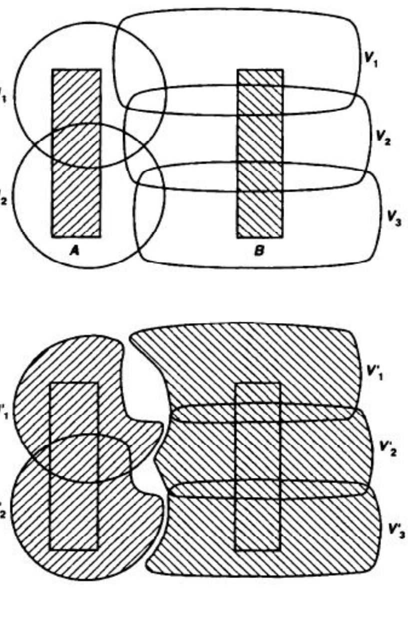

Figure 12:

By inductively we define,

$$U_n' = U_n \setminus \bigcup_{i=1}^{n} \overline{V}_i \text{ and } V_n' = V_n \setminus \bigcup_{i=1}^{n} \overline{U}_i$$

so that U_n' and V_n' are open such that $U_n' \cap V_n' = \varnothing$, $U_n' \cap B = U_n \cap B$ and $V_n' \cap A = V_n \cap A$.

Now let $U' = \bigcup_{i=1}^{\infty} U_n'$ and $V' = \bigcup_{i=1}^{\infty} V_n'$.

Then $A \subset V'$ and $B \subset U'$; U' and V' are open.

Suppose $U' \cap V' \neq \varnothing$, say $x \in U' \cap V'$.

Then $x \in U_i'$ and $x \in V_j'$. Supose $i \leq j$.

Since $x \in V_j'$ implies, $x \notin \overline{U}_k$ for all k = 1, 2, ..., j

In particular, $x \notin \overline{U}_i$, which is a contradiction. $\left(\because x \in U_i'\right)$

Therefore U' and V' are disjoint open sets containing A and B.

Hence X is normal.

Theorem 5.9 : *Every metrizable space is normal.*

Proof : Let X be a metrizable space with metric d.

Let A and B be disjoint closed subsets of X.

Let $a \in A$. Then $a \in X - B$ and $X - B$ is open.

So there exists $\varepsilon_a > 0$ such that $B(a, \varepsilon_a) \subset X - B$.

i.e., for each $a \in A$, there exists $\varepsilon_a > 0$ such that $B(a, \varepsilon_a) \cap B = \varnothing$.

Similarly for each $b \in B$, there exists $\varepsilon_b > 0$ such that $B(b, \varepsilon_b) \cap A = \varnothing$.

Now let $U = \bigcup_{a \in A} B\left(a, \frac{\varepsilon_a}{3}\right)$ and $V = \bigcup_{b \in B} B\left(b, \frac{\varepsilon_b}{3}\right)$

Then U and V are open sets containing A and B, respectively.

We prove that U and V are disjoint. For if $z \in U \cap V$, then $z \in B\left(a, \frac{\varepsilon_a}{3}\right) \cap B\left(b, \frac{\varepsilon_b}{3}\right)$ for some $a \in A$ and some $b \in B$.

By triangle inequality, we get $d(a,b) < \left(\frac{\varepsilon_a}{3} + \frac{\varepsilon_b}{3}\right)$.

If $\varepsilon_a \leq \varepsilon_b$, then $d(a,b) < \frac{2\varepsilon_b}{3}$ so that $a \in B(b, \varepsilon_b)$.

If $\varepsilon_b \leq \varepsilon_a$, then $d(a,b) < \frac{2\varepsilon_a}{3}$ so that $b \in B(a, \varepsilon_a)$, which is a contradiction in either case. Therefore U and V are disjoint containing A and B.

Hence X is normal.

Theorem 5.10 : *Every compact Hausdorff space is normal.*

Proof : Let X be a compact Hausdorif space.

Let A and B be disjoint closed subsets of X.

Then A and B are compact.

Let $a \in A$. Then since X is Hausdorff and B is compact, there exists disjoint open sets U_a and V_a containing a and B, respectively.

The collection $\{U_a\}$ covers A; because A is compact, A may be covered by finitely many sets $U_{a_1}, \ldots., U_{a_m}$.

Take $U = U_{a_1} \cup \ldots. \cup U_{a_m}$ and $V = V_{a_1} \cap \ldots. \cap V_{a_m}$

Suppose $x \in U \cap V$, then there exists j such that $x \in U_{a_j}$. As $x \in V$, $x \in V_{a_j}$, so $x \in U_{a_j} \cap V_{a_j}$, which is a contradiction.

Therefore, U and V are disjoint open sets containing A and B, respectively.

Hence every compact Hausdorff space is normal.

Theorem 5.11 : *Every well-ordered set X is normal in the order topology.*

Proof : Let X be a well-ordered set.

We prove that every interval of the form $(x, y]$ is open in X.

If y is the largest element of X, then $(x, y]$ is a basis element in the order topology.

If y is not the largest element of X, then $(x, y] = (x, y')$ where y' is the immediate successor of y.

Now let A and B be disjoint closed sets in X.

Suppose assume that neither A nor B contains the smallest element a_0 of X.

Let $a \in A$. Since $X - B$ is open and $a \in X - B$, there exists a basis element C such that $a \in C \subset X - B$.

Since a is not the smallest element of X, C contains some interval of the form $(x_a, a]$, i.e. for each $a \in A$, choose an interval $(x_a, a]$ such that $(x_a, a] \cap B = \varnothing$.

Similarly, for each $b \in B$, choose an interval $(y_b, b]$ such that $(y_b, b] \cap A = \varnothing$.

Then the sets $U = \bigcup_{a \in A} (x_a, a]$ and $V = \bigcup_{b \in B} (y_b, b]$ are open sets containing A and B, respectively;

We prove that they are disjoint. For suppose that $z \in U \cap V$.

Then $z \in (x_a, a] \cap (y_b, b]$ for some $a \in A$ and some $b \in B$. Assume that $a < b$.

Then if $a \le y_b$ the two intervals are disjoint, while if $a > y_b$, we have a $a \in (y_b, b]$, contrary to the fact that $(y_b, b] \cap A = \varnothing$.

A similar contradiction occurs if $b < a$.

Now suppose assume that A contains the smallest element a_0 of X.

The set $\{a_0\}$ is both open and closed in X.

Then $A - \{a_0\}$ and B are disjoint closed sets not containing the minimal element of X.

By the result of the preceding paragraph, there exist disjoint open sets U and V containing the closed sets $A - a_0$ and B, respectively.

Then $U \cup \{a_0\}$ and V are disjoint open sets containing A and B, respectively.

EXERCISE - 14

1. Which of the statements are true ?

   (I) Every regular space is Hausdorff.

   (II) Every normal space is regular.

   (A) Only (I) is true. (B) Only (II) is true.

   (C) Both (I) and (II) are true. (D) Both (I) and (II) are false.

2. Which of the following is false ?

   (A) The space $\mathbb{R}_l$ is regular

   (B) The space $\mathbb{R}_l$ is normal

   (C) The space $\mathbb{R}_l \times \mathbb{R}_l$ is regular

   (D) The space $\mathbb{R}_l \times \mathbb{R}_l$ is normal

3. Show that a closed subspace of a normal space is normal.

4. Prove that every regular Lindelof space is normal.

5. Show that every locally compact Hausdorff space is regular.

6. The Urysohn Lemma

Introduction :

In this section, we learn one of the deeper result, called Urysohn lemma, which guarantees the existence of continuous real valued function on a normal space. We also see one of the consequences of Urysohn lemma, namely, Tietze extension theorem which is an important result that asserts the extension of a continuous function defined on a subspace to the whole space.

Theorem 6.1 : (Urysohn's Lemma) : *Let X be a topological space; let A and B be disjoint closed subsets of X. Let [a, b] be a closed interval in the real line. Then there exists a continuous function* $f : X \to [a,b]$ *such that f (x) = a for all* $x \in A$*, and f (x) = b for all* $x \in B$.

Proof : It is enough to prove for $f : X \to [0,1]$ as $[a,b]$ is homeomorphic to [0, 1].

Let $D = \left\{ \frac{k}{2^n} : k = 1, \ldots, 2^n, n \in \mathbb{N} \right\}$ be the set of dyadic numbers in [0, 1].

We first prove that for each $p \in D$, there exists an open set U_p of X such that, whenever $p < q$, we have $A \subset U_p \subset \overline{U}_p \subset U_q \subset \overline{U}_q \subset X - B$.

Since A and B are disjoint closed sets, we get $A \subset X - B$ and $X - B$ is open.

As X is normal, there exists an open set $U_{\frac{1}{2}}$ such that $A \subset U_{\frac{1}{2}} \subset \overline{U}_{\frac{1}{2}} \subset X - B$.

Again as $A \subset U_{\frac{1}{2}}$ and $\overline{U}_{\frac{1}{2}} \subset X - B$, there exist open sets $U_{\frac{1}{4}}$ and $U_{\frac{3}{4}}$ such that

$A \subset U_{\frac{1}{4}} \subset \overline{U}_{\frac{1}{4}} \subset U_{\frac{1}{2}}$ and $\overline{U}_{\frac{1}{2}} \subset U_{\frac{3}{4}} \subset \overline{U}_{\frac{3}{4}} \subset X - B$. So we have,

$$A \subset U_{\frac{1}{4}} \subset \overline{U}_{\frac{1}{4}} \subset U_{\frac{1}{2}} \subset \overline{U}_{\frac{1}{2}} \subset U_{\frac{3}{4}} \subset \overline{U}_{\frac{3}{4}} \subset X - B.$$

Continuing by induction we obtain open sets of X such that

$$A \subseteq U_{2^{-n}} \subseteq \overline{U}_{2^{-n}} \subseteq U_{2.2^{-n}} \subseteq \overline{U}_{2.2^{-n}} \subseteq \ldots U_{(2^{-n}-1)2^{-n}} \subseteq \overline{U}_{(2^{-n}-1)2^{-n}} \subseteq X - B$$

Since for any $p \in D$, $p = k.2^{-n}$ for some $0 < k < 2^n$, there exists a open set U_p and if $p < q$, then $U_p \subset U_q$.

Now we define, $f : X \to [0,1]$ by $f(x) = \begin{cases} \inf\{d : x \in U_d\}, & \text{if } x \in \bigcup_{d \in D} U_d \\ 1, & \text{if } x \notin \bigcup_{d \in D} U_d \end{cases}$

Then $f(x) = 0$ for all $x \in A$ because, $A \subseteq U_d$, for all $d \in D$ and D is dense in [0, 1].

As $B \bigcap U_d = \varnothing$ for all $d \in D$, we get $f(b) = 1$ for all $b \in B$.

If $x \in \bar{U}_r$, then $x \in U_s$ for every $s > r$.

Therefore, $s \in \{p \mid x \in U_p\}$, so $f(x) = \inf\{p \mid x \in U_p\} \leq \inf\{s \mid r < s\} \leq r$.

If $x \notin U_r$, then $x \notin U_s$ for every $s < r$.

Therefore, $s \notin \{p \mid x \in U_p\}$ and hence $f(x) = \inf\{p \mid x \in U_p\} \geq r$.

Now we prove the continuity of f. Let $x_0 \in X$ and an open interval (c, d) in [0, 1] containing $f(x_0)$.

We will find a neighborhood U of x_0 such that $f(U) \subset (c,d)$.

Since D is dense in [0, 1], there exists $p, q \in D$ such that $c < p < f(x_0) < q < d$.

Let $U = U_q - \bar{U}_p$. Since $f(x_0) < q$, we have x0 2Uq. ($\because$ if $x_0 \notin U_q$, then $f(x_0) \geq q$)

Also as $f(x_0) > p$, we have $x_0 \notin \bar{U}_p$.

To show $f(U) \subset (c,d)$, let $x \in U$.

Then $x \in U_q \subset \bar{U}_q$ implies $f(x) \leq q$.

Since $x \notin \bar{U}_p$, we have $x \notin U_p$ and hence $f(x) \geq p$.

Thus $f(x) \in [p,q] \subset (c,d)$. Hence $f(U) \subset (c,d)$.

Theorem 6.2 : ***(Urysohn metrization theorem)****. Every regular space X with a countable basis is metrizable.*

Theorem 6.3 : ***(Tietze extension theorem).*** *Let X be a normal space; let A be a closed subspace of X.*

1. *Any continuous map of A into the closed interval [a,b] of $\mathbb{R}$ may be extended to a continuous map of all of X into [a, b].*

2. *Any continuous map of A into $\mathbb{R}$ may be extended to a continuous map of all of X into $\mathbb{R}$.*

Lemma 6.4 : *Let X be a set; let $\mathscr{A}$ be a collection of subsets of X having the finite intersection property. Then there is a collection $\mathscr{D}$ of subsets of X such that $\mathscr{D}$ contains $\mathscr{A}$, and $\mathscr{D}$ has the finite intersection property, and no collection of subsets of X that properly contains $\mathscr{D}$ has this property.*

We often say that a collection $\mathscr{D}$ satisfying the conclusion of this theorem is maximal with respect to the finite intersection property.

Proof : Let $\mathscr{A}$ be a collection of subsets of X that has the finite intersection property (in short, f.i.p).

Let $\mathbb{A} = \{\mathscr{B} \mid \mathscr{B} \supset \mathscr{A}$ and $\mathscr{B}$ has f.i.p$\}$

For $\mathscr{B}_1, \mathscr{B}_2, \in \mathbb{A}$, we define $\mathscr{B}_1 \leq \mathscr{B}_2$ if $\mathscr{B}_1 \subseteq \mathscr{B}_2$.

We show that $\mathbb{A}$ has a maximal element $\mathscr{D}$.

In order to apply Zorn's lemma, we must show that if $\mathbb{B}$ is a "subsuperset" of $\mathbb{A}$ that is simply ordered by proper inclusion, then $\mathbb{B}$ has an upper bound in $\mathbb{A}$.

We show that the collection $\mathscr{C} = \bigcup_{\mathscr{B} \in \mathbb{B}} \mathscr{B}$ is an element of $\mathbb{A}$, then it is the required upper bound on $\mathbb{B}$.

To show that $\mathscr{C}$ is an element of $\mathbb{A}$, we must show that $\mathscr{C} \supset \mathscr{A}$ and that $\mathscr{C}$ has the finite intersection property.

Clearly $\mathscr{C}$ contains $\mathscr{A}$, since each element of $\mathbb{B}$ contains $\mathscr{A}$.

To show that $\mathscr{C}$ has the finite intersection property, let $C_1, \ldots, C_n$ be elements $\mathscr{C}$.

Because $\mathscr{C}$ is the union of the elements of $\mathbb{B}$, there is, for each i, an element $\mathscr{B}_i$ of $\mathbb{B}$ such that $C_i \in \mathscr{B}$.

The superset $\{\mathscr{B}_1,...,\mathscr{B}_n\}$ is contained in $\mathbb{B}$, so it is simply ordered by the relation of proper inclusion.

Being finite, it has a largest element; that is, there is an index k such that $\mathscr{B}_i \subset \mathscr{B}_k$ for i = 1, ... , n. Then all the sets $C_1,....,C_n$ are elements of $\mathscr{B}_k$.

As $\mathscr{B}_k$ has the finite intersection property, the intersection of the sets $C_1,...,C_n$ is nonempty, as desired.

Theorem 6.5 : *Let X be a set; let $\mathscr{D}$ be a collection of subsets of X that is maximal with respect to the finite intersection property. Then:*

1. *Any finite intersection of elements of $\mathscr{D}$ is an element of $\mathscr{D}$.*
2. *If A is a subset of X that intersects every element of $\mathscr{D}$, then A is an element of $\mathscr{D}$.*

Proof :

1. Let B equal the intersection of finitely many elements of $\mathscr{D}$.

   Consider $\mathscr{E} = \mathscr{D} \bigcup \{B\}$. We show that $\mathscr{E}$ has the finite intersection property; then maximality of $\mathscr{D}$ implies that $\mathscr{E} = \mathscr{D}$, so that $B \in \mathscr{D}$ as desired.

   Take finitely many elements of $\mathscr{E}$. If none of them is the set B, then their intersection is nonempty because $\mathscr{D}$ has the finite intersection property.

   If one of them is the set B, then their intersection is of the form $D_1 \bigcap ... \bigcap D_m \bigcap B$. Since B equals a finite intersection of elements of $\mathscr{D}$, this set is nonempty.

2. Given A, define $\mathscr{E} = \mathscr{D} \bigcup \{A\}$.

   We show that $\mathscr{E}$ has the finite intersection property, from which we conclude that A belongs to $\mathscr{D}$.

   Take finitely many elements of $\mathscr{E}$. If none of them is the set A, their intersection is automatically nonempty.

   Otherwise, it is of the form $D_1 \bigcap ... \bigcap D_m \bigcap A$. Now $D_1 \bigcap ... \bigcap D_m$ belongs to $\mathscr{D}$, by (a); therefore, the intersection $D_1 \bigcap ... \bigcap D_m \bigcap A$ is nonempty, by hypothesis.

Theorem 6.6 : ***(Tychonoff theorem).*** *An arbitrary product of compact spaces is compact in the product topology*

Proof : Let $X = \prod_{\alpha \in J} X_\alpha$, where each space X_α is compact.

Let $\mathscr{A}$ be a collection of subsets of *X* having the finite intersection property.

We prove that the intersection $\bigcap_{A \in \mathscr{A}} \overline{A} \neq \varnothing$. Then *X* is compact.

Choose a collection $\mathscr{D}$ of subsets of *X* such that $\mathscr{D} \subset \mathscr{A}$ and $\mathscr{D}$ is maximal with respect to the finite intersection property. (Such $\mathscr{D}$ exists by previous lemma).

It will suffice to show that the intersection $\bigcap_{D \in \mathscr{D}} \overline{D} \neq \varnothing$ as $\mathscr{D} \subset \mathscr{A}$.

Given $\alpha \in J$, let $\pi_a : X \to X_\alpha$ be the projection map.

Consider the collection $\{\pi_a(D) \mid D \in \mathscr{D}\}$ of subsets of X_α. This collection has the finite intersection property because $\mathscr{D}$ does.

By compactness of X_α, for each α, we can choose a point x_α of X_α such that $x_\alpha \in \bigcap_{D \in \mathscr{D}} \overline{\pi_\alpha(D)}$.

Let *x* be the point $(x_\alpha)_{\alpha \in J}$ of *X*.

We shall show that $x \in \overline{D}$ for every $D \in \mathscr{D}$.

First we show that if $\pi_\beta^{-1}(U_\beta)$ is any subbasis element (for the product topologyon *X*) containing *x*, then $\pi_\beta^{-1}(U_\beta)$ intersects every element of $\mathscr{D}$.

The set U_β is a neighborhood of x_β in X_β.

Since $x_\beta \in \overline{\pi_\beta(D)}$ by definition, U_β intersects $\pi_\beta(D)$ in some point, say $\pi_\beta(y)$, where $y \in D$.

Then $y \in \pi_\beta^{-1}(U_\beta) \cap D$ and by previous results, every subbasis element containing *x* belongs to $\mathscr{D}$.

And then it follows that every basis element containing x belongs to $\mathscr{D}$, as every basis element is a finite intersection of subbasis elements.

i.e. if B is the basis element containing x, then $B \in \mathscr{D}$.

As $\mathscr{D}$ has the finite intersection property, $B \bigcap D \neq \varnothing$ for every $D \in \mathscr{D}$; hence $x \in \bar{D}$ for every $D \in \mathscr{D}$ as desired.

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UNIT - V

EXAMPLES, SEMINARS, GROUP DISCUSSIONS

Introduction :

In this unit, we go through several examples, results based on the topics discussed in the previous units. We also give some exercise problems that can be considered for seminar topics and for group discussions.

Theorem 1 : *(Urysohn metrization theorem).* *Every regular space X with a countable basis is metrizable.*

Proof :

Step 1 : We prove the following :

There exists a countable collection of continuous functions $f_n : X \to [0,1]$ having the property that given any point x_0 of X and any neighborhood U of x_0, there exists an index n such that f_n is positive at x_0 and vanishes outside U.

Let $\{B_n\}$ be a countable basis for X. Let $x \in X$ and U be a neighborhood of x.

Then there exists B_m such that $x \in B_m \subset U$.

Since X is regular, there exists V containing x such that $x \in \overline{V} \subset B_m$.

Again we can find a basis element B_n such that $x \in B_n \subset V$.

Therefore, $x \in \overline{B_n} \subset \overline{V} \subset B_m$.

For each pair n, m of indices for which $\overline{B}_n \subset B_m$, apply the Urysohn lemma to choose a continuous function $g_{n,m} : X \to [0,1]$ such that $g_{n,m}(\overline{B}_n) = \{1\}$ and $g_{n,m}(X - B_m) = \{0\}$.

Then the collection $\{g_{n,m}\}$ satisfies our requirement.

Because the collection $\{g_{n,m}\}$ is indexed with a subset of $\mathbb{Z}_+ \times \mathbb{Z}_+$, it is countable; therefore it can be re indexed with the positive integers, giving us the desired collection $\{f_n\}$.

Step 2 : Given the functions f_n as in **Step 1**, take $\mathbb{R}^w$ in the product topology and define a map $F : X \to \mathbb{R}^w$ by $F(x) = (f_1(x), f_2(x), ...)$

We show that F is an imbedding.

First, F is continuous because $\mathbb{R}^w$ has the product topology and each f_n is continuous. Second, F is injective because given $x \neq y$, there exists an open set U such that $x \in U$ and $y \notin U$.

Then by step 1, there is an index n such that $f_n(x) > 0$ and $f_n(y) = 0$; therefore, $F(x) \neq F(y)$.

Finally, we prove that F is a homeomorphism of X onto its image, the subspace $Z = F(X)$ of $\mathbb{R}^w$.

We know that F defines a continuous bijection of X with Z.

Let z_0 be a point of $F(U)$. We Shall find an open set W of Z such that $z_0 \in W \subset F(U)$.

Let x_0 be the point of U such that $F(x_0) = z_0$.

Choose an index N for which $f_N(x_0) > 0$ and $f_N(X | U) = \{0\}$.

Take the open ray $(0, +\infty)$ in $\mathbb{R}$, and let V be the open set $V = \pi_N^{-1}((0, +\infty))$ of $\mathbb{R}^w$.

Let $W = V \bigcap Z$. Then W is open in Z, by definition of the subspace topology.

$z_0 \in W$ because $\pi_N(z_0) = \pi_N(F(X_0)) = f_N(x_0) > 0$.

And $W \subset F(U)$. For if $z \in W$, then $z = F(x)$ for some $x \in X$, and $\pi_N(z) \in (0, +\infty)$.

Since $\pi_N(Z)=\pi_N(F(x))=f_N(x)$, and f_N vanishes outside *U*, the point *x* must be in *U*.

Then $z=F(x)$ is in *F*(*U*), as desired.

Thus *F* is an imbedding of *X* in $\mathbb{R}^w$.

Since $\mathbb{R}^w$ is metrizable, *X*, as a subspace of $\mathbb{R}^w$, is metrizable.

Definition 2 : If *A* and *B* are two subsets of the topological space *X*, and if there is a continuous function $f:X\to[0,1]$ such that $f(A)=\{0\}$ and $f(B)=\{1\}$, we say that *A* and *B* can be **separated by a continuous function**.

Definition 3 : A space *X* is **completely regular** if one-point sets are closed in *X* and if for each point x0 and each closed set *A* not containing x0, there is a continuous function $f:X\to[0,1]$ such that $f(x_0)=1$ and $f(A)=\{0\}$.

Example 4 : Every normal space *X* is completely regular.

Proof : Let *A* be a closed set and $x_0\in X$ such that $x_0\notin A$.

Since $\{x_0\}$ is closed, and *X* is normal, by Urysohn lemma, there exists a continuous function $f:X\to[0,1]$ such that $f(x)=0$ for all $x\in A$ and $f(x_0)=1$.

Hence *X* is completely regular.

Example 5 : Every completely regular space *X* is regular.

Proof : Let *A* be a closed subset of *X* and $x_0\in X\setminus A$.

Then by definition of completely regular, there exists a continuous function $f:X\to[0,1]$ such that $f(x_0)=1$ and $f(A)=\{0\}$.

Observe that the sets $\left[0,\frac{1}{2}\right)$ and $\left(\frac{1}{2},1\right]$ are open in [0, 1].

As f is continuous, we get that the sets $f^{-1}\left(\left[0,\frac{1}{2}\right)\right)$ and $f^{-1}\left(\left(\frac{1}{2},1\right]\right)$ are open and they are disjoint.

Also $x_0 \in f^{-1}\left(\left(\frac{1}{2},1\right]\right)$ and $A \subset f^{-1}\left(\left[0,\frac{1}{2}\right)\right)$.

Hence X is regular.

We now show that completely regular spaces can be passed onto subspaces and products.

Theorem 6 : *A subspace of a completely regular space is completely regular. Product of completely regular spaces is completely regular.*

Proof : Let X be completely regular and Y be a subspace of X.

Let x_0 be a point of Y, and let A be a closed set of Y not containing x_0.

Let $\overline{A}$ denotes the closure of A in X. As A is closed in Y, we get $A = \overline{A} \cap Y$.

As $x_0 \notin A$, we have $x_0 \notin \overline{A}$.

Since X is completely regular, there exists a continuous function $f : X \to [0,1]$ such that $f(x_0) = 1$ and $f(\overline{A}) = \{0\}$.

Then $f|_Y : Y \to [0,1]$ satisfies $f|_Y(x_0) = 1$ and $f|_Y(A) = \{0\}$.

Let $X = \prod X_\alpha$ be a product of completely regular spaces.

Let $b = (b_\alpha) \in X$ and A be a closed set of X not containing b.

Since $X \setminus A$ is open and $b \in X \setminus A$, there exists a basis element $\prod U_\alpha$ such that $b \in \prod U_\alpha \subset X \setminus A$

As we know $U_\alpha = X_\alpha$ except for finitely many a, say $\alpha = \alpha_1, ..., \alpha_n$.

Given i = 1, ...,n choose a continuous function $f_{\alpha_i} : X_{\alpha_i} \to [0,1]$ such that $f_{\alpha_i}\left(b_{\alpha_i}\right)=1$ and $f_{\alpha_i}\left(X-U_{\alpha_i}\right)=0$.

Now define $f : X \to [0,1]$ by $f(y)=\prod_{i=1}^{n} f_{\alpha_i}\left(y_\alpha\right)$.

Then $f(b)=1$. To show $f(A)=0$, let $a \in A$.

Then there exists α_j, for some j =1,....,n such that $a_\alpha \notin U_{\alpha_j}$ and hence $f_{\alpha_i}\left(a_\alpha\right)=0$ implies $f(A)=0$.

Definition 7 : A **compactification** of a space X is a compact Hausdorff space Y containing X as a subspace such that $\overline{X}=Y$. Two compactifications Y_1 and Y_2 of X are said to be **equivalent** if there is a homeomorphism $h: Y_1 \to Y_2$ such that $h(x)=x$ for every $x \in X$.

Lemma 8 : *If Y is a compactification of X, then X is completely regular.*

Proof : Suppose X has a compactification Y.

Since Y is compact and Hausdorff, Y is normal.

As every normal space is completely regular, Y is completely regular

Thus A is completely regular being a subspace of Y.

In the next lemma, we prove the converse of the above statement i.e. if X is completely regular, then X has a compactification.

Lemma 9 : *Let X be a space; suppose that* $h: X \to Z$ *is an imbedding of X in the compact Hausdorff space Z. Then there exists a corresponding compactification Y of X; it has the property that there is an imbedding* $H: Y \to Z$ *that equals h on X. The compactification Y is uniquely determined up to equivalence. We call Y the compactification* ***induced*** *by the imbedding h.*

Proof : Let X_0 denote the subspace $h(x)$ of Z, and Y_0 denote its closure in Z.

Then Y_0 is a compact Hausdorff space and $\overline{X}_0 = Y_0$

Hence Y_0 is a compactification of X_0.

We now construct a space Y containing X such that (X, Y) is homeomorphic (X_0, Y_0).

We choose a set A disjoint from X that is in bijective correspondence with the set $Y_0 - X_0$ under some map $k : A \to Y_0 - X_0$.

Define $Y = X \cup A$, and define a bijective correspondance $H : Y \to Y_0 - X_0$ by

$$H(x) = h(x) \qquad \text{for } x \in X,$$

$$H(a) = k(a) \qquad \text{for } a \in A.$$

Then Y is a topology with open set U if and only if $H(U)$ is open in Y_0.

Now the map H is a homeomorphism; and the space X is a subspace of Y because $h = H|_X$.

By expanding the range of H, we obtain the required imbedding of Y into Z.

Now suppose Y_i is a compactification of X and that $H_i : Y_i \to Z$ is an imbedding that is an extension of h, for $i = 1, 2$.

Now each H_i maps X onto $h(X) = X_0$.

As H_i is continuous, it maps Y_i into $\overline{X}_0$.

Hence $H_i(Y_i) = \overline{X}_0$, and $(H_2^{-1})_o H_1$ defines a homeomorphism of Y_1 with Y_2 that equals the identity on X.

Example 10 : Let Y be the space [0, 1]. Then Y is a compactification of $X = (0,1)$; obtained by adding one point at each end of (0, 1).

Lemma 11 : *Let $A \subset X$; let $f : A \to Z$ be a continuous map of A into the Hausdorff space Z. There is atmost one extension of f to a continuous function $g : \overline{A} \to Z$.*

Proof : Suppose that $g, g' : \overline{A} \to X$ are two different extensions of f.

Choose x so that $g(x) \neq g'(x)$.

Let U and U' be disjoint neighborhoods of $g(x)$ and $g'(x)$, respectively.

Choose a neighborhood V of x so that $g(V) \subset U$ and $g'(V) \subset U'$.

Now V intersects A ,say at point y.

Then $g(y) \in U$ and $g'(y) \in U'$.

But since $y \in A$, we have $g(y) = f(y)$ and $g'(y) = f(y)$.

This contradicts the fact that U and U' are disjoint.

Theorem 12 : (The Stone-Cech compactification) : *Let X be a completely regular space. There exists a compactification Y of X having the property that every bounded continuous map $f : X \to \mathbb{R}$ extends uniquely to a continuous map of Y into $\mathbb{R}$.*

Proof : Let $\{f_\alpha\}_{\alpha \in J}$ be the collection of all bounded continuous real-valued functions on X, indexed by some index set J.

For each $\alpha \in J$, let $I_\alpha = [\inf f_\alpha(X), \sup f_\alpha(X)]$. Then define $h : X \to \prod_{\alpha \in J}$ by $h(x) = (f_\alpha(x))_{\alpha \in J}$.

Since each I_α is compact, $\prod I_\alpha$ is compact.

Because X is completely regular, the collection $\{f_\alpha\}$ separates points from closed sets in X.

Therefore the map h is an imbedding.

Let Y be the compactification of X induced by the imbedding h.

Then there is an imbedding $H : Y \to \prod I_\alpha$ that equals h when restricted to the subspace X of Y.

We now show that a bounded continuous real-valued function f on X extends to Y.

Since f is bounded and continuous, $f = f_\beta$ for some index $\beta \in J$.

Let $\pi_\beta : \prod I_\alpha \to I_\beta$ be the projection mapping.

Then the continuous map $\pi_\beta \circ H : Y \to I_\beta$ is the required extension of f as if $x \in X$ we have $\pi_\beta(H(x)) = \pi_\beta(h(x)) = \pi_\beta\left(\left(f_\alpha(x)\right)_{\alpha \in J}\right) = f_\beta(x)$.

Uniqueness of the extension is a consequence of the lemma ?? .

Theorem 13 : *Let X be a completely regular space. If Y_1 and Y_2 are two compactification of X satisfying the extension property of Theorem ??, then Y_1 and Y_2 are equivalent.*

Proof : Consider the inclusion mapping $j_2 : X \to Y_2$.

It is a continuous map of X into the compact Hausdorff space Y_2.

Because Y_1 has the extension property, we can extend the j_2 to a continuous map $f_2 : Y_1 \to Y_2$.

Similarly, we can extend the inclusion map $j_1 : X \to Y_1$ to a continuous map $f_1 : Y_2 \to Y_1$.

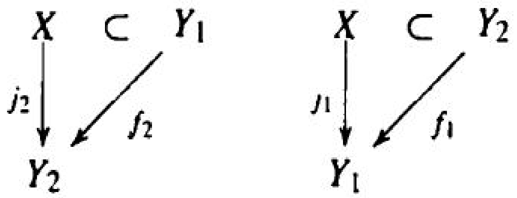

Figure 13:

Then the composite map $f_1 \circ f_2 : Y_1 \to Y_1$ satisfies $f_1(f_2(x)) = x$ for all $x \in X$.

Therefore $f_1 \circ f_2$ is a continuous extension of the identity map $i_X : X \to X$.

But the identity map of Y_1 is also continuous extension of i_X.

Then by lemma ??, $f_1 \circ f_2$ is equal to the identity map of Y_1.

Similarly, $f_2 \circ f_1$ is equal to the identity map of Y_2.

Thus f_1 and f_2 are homeomorphisms.

Remark 14 : For each completely regular space X, there exists a unique compactification of X satisfying the extension condition of Theorem ??. We will denote this compactification of X by $\beta(X)$ and call it the **Stone-Cech compactification** of X. It is characterised by the fact that any continuous map $f : X \to C$ of X into a compact Hausdorff space C extends uniquely to a continuous map $g : \beta(X) \to C$.

SELF - TEST 4.4

1. The one point compactification of $\mathbb{R}^2$ is homeomorphic with:

   A) $\mathbb{R}^2$

   B) $\mathbb{R}$

   C) S^2

   D) S^1

2. Which of the following statements is not equivalent to any two of the remaining statements for a topological space X, where one point sets are closed ?

   A) X is completely regular

   B) X is metrizable

   C) X is homeomorphic to a subspace of a compact Hausdorff space

   D) X is homeomorphic to a subspace of a normal space

SHOT ANSWER QUESTIONS 4.4

1. Give an example of a regular space which is not completely regular.
2. Give an example of a completely regular space which is not normal.

TOPICS FOR SEMINARS AND GROUP DISCUSSIONS

1. If $\{\mathscr{T}_\alpha\}$ is a family of topologies on X, show that $\bigcap \mathscr{T}_\alpha$ is a topology on X. Is $\bigcup \mathscr{T}_\alpha$ is a topology on X?

2. Let $\{\mathscr{T}_\alpha\}$ be a family of topologies on X. Show that there is a unique smallest topology on X containing all the collections $\mathscr{T}_\alpha$ and a unique largest topology ontained in all $\mathscr{T}_\alpha$.

3. If $X = \{a,b,c\}$ let $\mathscr{T}_1 = \{\varnothing, X, \{a\}, \{a,b\}\}$ and $\mathscr{T}_2 = \{\varnothing, X, \{a\}, \{b,c\}\}$. Find the smallest topology containing $\mathscr{T}_1$ and $\mathscr{T}_2$, and the largest topology contained in $\mathscr{T}_1$ and $\mathscr{T}_2$.

4. Consider the following topologies on $\mathbb{R}$

   (a) $\mathscr{T}_1$ = the standard topology

   (b) $\mathscr{T}_2$ = the topology of $\mathbb{R}_K$

   (c) $\mathscr{T}_3$ = the finite complement topology

   (d) $\mathscr{T}_4$ = the upper limit topology, having all sets $(a, b]$ as basis

   (e) $\mathscr{T}_5$ = the topology having all sets $(-\infty, a)$ as basis.

   Determine, for each of these topologies, which of the others it contains.

5. A map $f : X \to Y$ is said to be an open map if for every open set U of X, the set $f(U)$ is open in Y. Show that $\pi_1 : X \times Y \to X$ and $\pi_2 : X \times Y \to Y$ are open maps.

6. Show that every order topology is Hausdorif.

7. In the finite complement topology on $\mathbb{R}$, to what point or points does the sequence $x_n = 1/n$ converge ?

8. Show the T_1 axiom is equivalent to the condition that for each pair of points of X, each has a neighborhood not containing the other.

9. Let Y be an ordered set in the order topology. Let $f, g : X \to Y$ be continuous.

(a) Show that the set $\{x \mid f(x) \le g(x)\}$ is closed in X.

(b) Let $h : X \to Y$ be the function $h(x) = \min\{f(x), g(x)\}$. Use the pasting lemma to show that h is continuous.

10. Justify the following statements

(a) Is a product of path connected spaces necessarily path connected ?

(b) If $A \subset X$ and A is path connected, is $\overline{A}$ necessarily path connected ?

(c) If $f : X \to Y$ is continuous and X is path connected, is $f(X)$ necessarily path connected ?

(d) If $\{A_\alpha\}$ is collection of path connected subspaces of X and if $\bigcap A_\alpha \neq \varnothing$, is $\bigcup A_\alpha$ necessarily path connected ?

11. If $A \subset X$, a retraction of X onto A is a continuous map $r : X \to A$ such that $r(a) = a$ for each $a \in A$. Show that a retraction is a quotient map.

12. Show that no two of the spaces (0, 1), (0, 1], and [0, 1] are homeomorphic.

13. Show $\mathbb{R}^n$ and $\mathbb{R}$ are not homeomorphic if $n > 1$.

14. Let $p : X \to Y$ be a quotient map. Show that if X is locally connected, then Y is locally connected.

15. Show that if Y is compact, then the projection $\pi_1 : X \times Y \to X$ is a closed map.

16. Show that in $\mathbb{R}$ with countable complement topology, finite sets are compact.

17. Show that $\mathbb{R}$ with discrete topology is locally compact.

18. Continuous image of a locally compact space is locally compact. True or False ? Justify.

19. Continuous image of a first countable space is first countable. True or False ? Justify.

20. Show that continuous image of a separable space is separable.

21. Is subspace of a separable space, separable? Justify.

22. Show that $\mathbb{R}$ with finite complement topology is Lindelof.

23. Give an example (other than already discussed) of a space which is T_0 but not T_1.

24. Give an example (other than already discussed) of a space which is T_1 but not T_2.

25. Give an example (other than already discussed) of a space which is T_2 but not T_3.

26. Give an example (other than already discussed) of a space which is T_3 (*regular*) but not $T_{3\frac{1}{2}}$.

27. Give an example (other than already discussed) of a space which is regular but not normal.

28. Give an example (other than already discussed) of a space which is completely regular but not normal.

29. Show that every locally compact Hausdorff space is completely regular.

30. Is [0, 1] a compactification of (0, 1) ? Is it a one point compactification ?

REFERENCES

The content writers are here by thank James R. Munkres, as we have taken the most of the material from his book: Topology. Definitions, statements and some diagrams have been taken from this book as it is so that students can understand better. We duly acknowledge the content taken from this book. Most of the proofs have been altered according to our understanding. These are the reference books:

1. Topology : First course, J R Munkres, Prentice Hall Inc., New Jersey .
2. Theory and Problems of Set Theory and Related Topics, Lipshutz Seymour, Schaum Publishing Co. New York
3. Foundations of General Topology, Pervin William J, Academic Press.

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