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(Mathematics)

For

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Preface

Large number of students appears for M.A./M. Sc. examinations externally every year. In view of this, Shivaji University has introduced the Distance and Online Education Mode for external students from the year 2008-09, and entrust the task to us to prepare the Self Instructional Material (SIM) for aspirants. An objective of the SIM is to provide students the material on the subject from which they can prepare for examination on their own without the help of a tutor. Today we are extremely happy to present the book on "Partial Differential Equations" for M.A./M.Sc. Semester-II students as a SIM prepared by well devoted expert Dr. L. N. Katkar. We hope that the exposition of the material in the book will meet the needs of all aspirants.

The mathematical formulation of the real world problems in science and engineering involves partial differential equations. In order to understand the physical behaviour of these real world problems, it is necessary to have some knowledge about the solutions of the governing partial differential equations. For example, transverse vibrations of an elastic string are governed by the wave equation; the temperature distribution in a homogeneous isotropic rod is governed by the heat equation etc. The wave equation; the heat equation and the Laplace equation have been derived by taking into account certain physical situations.

Partial differential equations of first order and various methods such as Charpit's method, Jacobi method of finding their complete integral; general integraol; singular integral and Cauchy integral surfaces are dealt in the first four units. The classification of second order partial differential equations and their canonical forms are given in the unit 5. Boundary value problems such as Dirichlet and Neumann boundary value problems are discussed in the subsequent units, besides maximum-minimum principle and families of equipotential surfaces. The well-known mathematical techniques namely, the most powerful method of separable of variables, Fourier transform techniques and Green's function approach are applied to solve various boundary value problems involving parabolic, elliptic and hyperbolic partial differential equations.

An attempt has been made to make the presentation of the various units comprehensive, rigorous and yet simple. One of the features of this book is that in all units numerous examples have been solved for the use of students working independently of a teacher. Although the book is aimed to M. Sc. Distance Education Students, even SET/NET aspirants and students of physics and engineering would find it useful.

We owe a deep sense of gratitude to the Ag. Vice-Chancellor who has given impetus to go ahead with ambitious projects like the present one. Dr. L. N. Katkar, of the Department of Mathematics, Shivaji University, has to be profusely thanked for the ovation he has poured to prepare the SIM on Partial Differential Equations. We also thank Prof. S. H. Thakar, Head, Department of Mathematics, Director of Distance Education Mode Prof. Cima Yeole, Shivaji University, for their help and keen interest in completion of the SIM. Thanks are also due to Mr. Sachin Kadam for computerizing the manuscript neatly and correctly.

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Partial Differential Equations

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M. Sc. (Mathematics)

Partial Differential Equations

Contents

Each Unit begins with the section objectives -

Objectives are directive and indicative of :

1. what has been presented in the unit and
2. what is expected from you
3. what you are expected to know pertaining to the specific unit, once you have completed working on the unit.

The exercises at the end of each unit are not to be submitted to us for evaluation. They have been provided to you as study tools to keep you in the right track as you study the unit.

Dear Students

The SIM is simply a supporting material for the study of this paper. It is also advised to see the new syllabus 2022-23 and study the reference books & other related material for the detailed study of the paper.

UNIT - I

FIRST ORDER PARTIAL DIFFERENTIAL EQUATIONS

Introduction :

The mathematical formulation of the real situations in science and engineering involves partial differential equations. In order to understand the physical behaviour of the real world situations, it is necessary to have some knowledge about the properties and the solutions of the governing partial differential equation. A partial differential equation is one involving more than one independent variable, a dependent variable and its partial derivatives with respect to the independent variables. In general, partial differential equations arise in physics in problems involving electric fields, fluid dynamics, wave motion etc. These equations are called Heat equations, Laplace equations wave equations. Each is profoundly significant in theoretical physics and their study is stimulated in the development of many mathematical ideas.

The basic concepts from solid geometry play important roles in the study of partial differential equations and it is essential that they should be understood thoroughly before the study of partial differential equations is begun. Hence we define some basic concepts from geometry.

Curves and Surfaces :

Curves in Space : Let I be an interval on the real line R and t a continuous variable which varies in I. If f_1, f_2, f_3 are continuous functions of t, then the equations.

$$x = f_1(t), y = f_2(t), z = f_3(t), \qquad \text{... (1.1)}$$

represent the parametric equations of a curve in three dimensional space.

Note : (i) t is called the parameter of the curve.

(ii) The standard parameter is the arc length of the curve measured from some fixed point on the curve to any current point, it is denoted by s instead of t.

(iii) The square of an infinitesimal arc length between two neighbouring points on the curve in 3-dim. space is given by

$$ds^2 = dx^2 + dy^2 + dz^2,$$

$$\Rightarrow \left(\frac{dx}{ds}\right)^2 + \left(\frac{dy}{ds}\right)^2 + \left(\frac{dz}{ds}\right)^2 = 1.$$

It follows from equation (1.1) that the condition that the parameter t be the arc length of the curve is that

$$f_1'^2 + f_2'^2 + f_3'^2 = 1.$$

Examples :

(i) The simplest example of a curve in space is a straight line with direction cosines (ℓ ,m,n) passing through a point (x_0, y_0, z_0) and is given by

$$x = x_0 + \ell s, y = y_0 + ms, z = z_0 + ns \text{ ,} \qquad \text{... (1.2)}$$

where s is the parameter.

(ii) A right circular helix lying on a circular cylinder is a space curve and is given by the parametric equations

$$x = a \cos \omega t, y = a \sin \omega t, z = kt \text{ ,} \qquad \text{... (1.3)}$$

where a, k, w, are constants.

Surface : Let (x, y, z) be the cartesian co-ordinates of a point in a 3-dimensional space. Then the functional relation between these variables x, y, z given by the equation

$$F(x, y, z) = 0 \qquad \text{... (1.4)}$$

is called a surface.

If F is linear, then equation (1.4) can be solved for one of the variables and it can be expressed in terms of the other two independent variables and we are left with only two degrees of freedom. Hence a surface is defined as the locus of a point moving in space with two degrees of freedom.

Parametric Equations of a Surface :

A set of those points of a 3-dimensional space which are expressed as a function of two parameters is called a surface. Thus a set of relations of the form.

$$x = F_1(u,v), y = F_2(u,v), z = F_3(u,v) \qquad \text{... (1.5)}$$

determines a surface.

Exaplanation : Solving the first pair of equations (1.5)

viz., $x = F_1(u,v), y = F_2(u,v)$,

for u and v as functions of x and y, we obtain

Say $u = \lambda(x,y), v = \mu(x,y)$.

This shows that once x and y are known, then u and v are determined. Then corresponding value of z is obtained by substituting these values of u and v in the third equation $z = F_3(u,v)$. In otherwards the value of z is determined once the values of x and y are known. Symbolically,

$$z = F_3\left(\lambda(x,y), \mu(x,y)\right). \qquad \text{... (1.6)}$$

Which is a functional relation between the coordinates x, y and z. Thus any point (x, y, z) determined from equations (1.5) always lies on a surface. The equations (1.5) therefore are called the parametric equations of the surface.

Note : Not every point in space corresponds to a pair of values of u and v.

For that $\frac{\partial(F_1, F_2)}{\partial(u, v)} \neq 0$

Note : Parametric equations of a curve and a surface are not unique.

Examples :

(1) The parametric equations of a surface of a sphere of radius 'a' are given by

$$x = \text{asinucosv},$$
$$y = \text{asinusinv}, \qquad a = \text{constant} \qquad \text{... (1.7)}$$
$$x = \text{acosu}.$$

The same surface is also represented by the set of equations

$$x = a\frac{(1-v^2)}{(1+v^2)}\cos u,$$

$$y = a\frac{(1-v^2)}{(1+v^2)}\sin u, \qquad \text{... (1.8)}$$

$$z = \frac{2av}{(1+v^2)}, \qquad a = \text{constant}$$

(2) The parametric equations of a cone $x^2 + y^2 = z^2 \tan^2\theta$ are given by

$$x = r\sin\theta\cos\phi,$$
$$y = r\sin\theta\sin\phi, \qquad \theta = \text{constant} \qquad \text{... (1.9)}$$
$$z = r\cos\theta \cdot$$

or $$x = r\cos\phi,\ y = r\sin\phi,\ z = r\cot\phi. \qquad \text{... (1.10)}$$

A Curve Through Surfaces :

Consider a surface f (x, y, z) = 0, ... (1.11)

and a plane z = k. ... (1.12)

A point whose co-ordinates satisfy equation (1.11) and which lies in the plane (1.12) has its co-ordinates satisfying the equations

$$z = k, f(x, y, k) = 0, \qquad \text{... (1.13)}$$

which represents a curve in the plane z = k.

For example : Let S be a sphere with equation

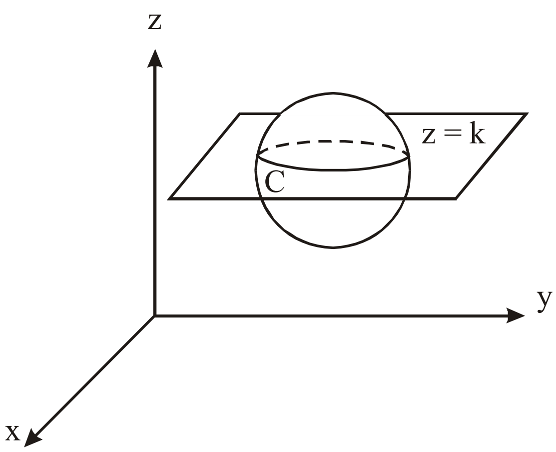
$$x^2 + y^2 + z^2 = a^2.$$

Then the points of S with z = k have

$$z = k \quad \text{and}$$

$$x^2 + y^2 = a^2 - k^2,$$

which is a curve C and the curve is a circle of radius $\sqrt{a^2 - k^2}$ for k < a.



Thus a curve can be thought of as the intersection of the surface (1.11) and the plane (1.12).

In general, the common points to the surfaces,

$$S_1 : f(x, y, z) = 0,$$

and $$S_2 : g(x, y, z) = 0,$$

lie on the curve C.

Thus, the locus of a point whose co-ordinates satisfies a pair of relations $S_1 = 0$ and $S_2 = 0$ is a curve in space.

Direction Cosines of a line passing through two points :

Consider a line through the points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$. The vector $\overline{PQ}$ is defined by

$$\overline{PQ} = (x_2, y_2, z_2) - (x_1, y_1, z_1),$$

$$\overline{PQ} = (x_2 - x_1, y_2 - y_1, z_2 - z_1),$$

$$\Rightarrow \overline{PQ} = (x_2 - x_1)i + (y_2 - y_1)j + (z_2 - z_1)k. \qquad \text{... (1.14)}$$

Direction cosines of a line $\overline{PQ}$ are the cosines of the angles made by the line $\overline{PQ}$ with co-ordinate axes. Let $\cos\alpha$, $\cos\beta$, $\cos\gamma$ be the direction cosines of the line $\overline{PQ}$, then we have from equations (1.14)

$$\overline{PQ} \cdot i = \left|\overline{PQ}\right| \cos\alpha = (x_2 - x_1)$$

$$\Rightarrow \cos\alpha = \frac{(x_2 - x_1)}{\left|\overline{PQ}\right|}.$$

Similarly, $$\cos\beta = \frac{(y_2 - y_1)}{\left|\overline{PQ}\right|}, \qquad \text{... (1.15)}$$

$$\cos\gamma = \frac{(z_2 - z_1)}{|\overline{PQ}|}.$$

Thus $\left(\frac{x_2 - x_1}{|\overline{PQ}|}, \frac{y_2 - y_1}{|\overline{PQ}|}, \frac{z_2 - z_1}{|\overline{PQ}|}\right)$ are the direction cosines of the line through the points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$.

Note : From equations (1.15), we see that $x_2 - x_1, y_2 - y_1, z_2 - z_1$ are proportional to the direction cosines of the line and hence they represent the direction ratios of the line $\overline{PQ}$.

The Direction Cosines of the tangent to the curve :

Example 1 : Show that the direction cosines of the tangent to the curve

$$x = x(s),\ y = y(s),\ z = z(s),$$

where s is the arc length of the curve measured from the fixed point p_0 on the curve to any point P on the curve are $\left(\frac{dx}{ds}, \frac{dy}{ds}, \frac{dz}{ds}\right)$.

Solution : Consider a curve in 3-dimensional space given by

$$x = x(s),\ y = y(s),\ z = z(s), \qquad \text{... (1.16)}$$

where s is the arc length measured from the fixed point p_0 to any point P on the curve.

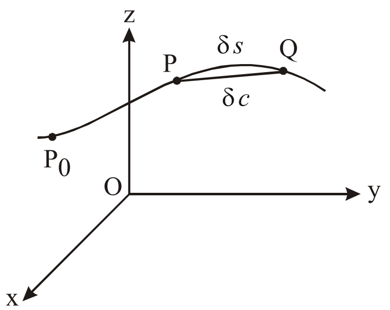
Thus $\qquad s = P_0P.$

Let Q be any other point at a distance δs from P,

$$\Rightarrow P_0Q = s + \delta s.$$

Consequently, the co-ordinates of the point Q are

$$Q = (x(s+\delta s), y(s+\delta y), z(s+\delta s)).$$



Since δs is measured along the curve from P to Q and is therefore greater than the length δc of the chord PQ.

In the limiting case as $\delta s \longrightarrow 0$, we have

$$\lim_{\delta s \to 0} \frac{\delta s}{\delta c} = 1. \qquad \text{... (1.17)}$$

The direction cosines of the chord PQ are given by

$$\left(\frac{x(s+\delta s) - x(s)}{\delta c}, \frac{y(s+\delta s) - y(s)}{\delta c}, \frac{z(s+\delta s) - z(s)}{\delta c}\right).$$

By Taylor's series expansion, we have

$$x(s+\delta s)-x(s)=\delta s\left(\frac{dx}{ds}\right)+O(\delta s)^2 .$$

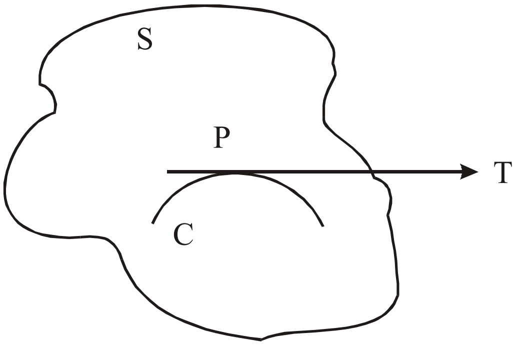
Hence direction cosines of the chord PQ are given by

$$\left(\frac{\delta s}{\delta c}\left(\frac{dx}{ds}+O(\delta s)\right),\frac{\delta s}{\delta c}\left(\frac{dy}{ds}+O(\delta s)\right),\frac{\delta s}{\delta c}\left(\frac{dz}{ds}+O(\delta s)\right)\right)$$

In the limiting case as $\delta s \longrightarrow 0$, the point Q tends towards the point P and the chord PQ takes up to the tangent to the curve at P. Thus as $\delta s \longrightarrow 0$, the direction cosines of the tangent to the curve (1.16) at apoint P are

$$\left(\frac{dx}{ds},\frac{dy}{ds},\frac{dz}{ds}\right).$$

Direction ratios of the normal to the surface :



Let us consider that the curve C (defined in equation (1.16)) lies on the surface S : F (x, y, z) = 0.

Any point (x(s), y(s), z(s)) on the curve lies on this surface satisfies the equation

$$F\left(x(s),y(s),z(s)\right)=0 . \qquad \text{... (1.18)}$$

If the curve lies entirely on the surface, then equation (1.18) is an identity for all values of s. Differentiating equation (1.18) with respect to s, we get

$$\frac{\partial F}{\partial x}\cdot\frac{dx}{ds}+\frac{\partial F}{\partial y}\cdot\frac{dy}{ds}+\frac{\partial F}{\partial z}\cdot\frac{dz}{ds}=0, \qquad \text{... (1.19)}$$

where $\left(\frac{dx}{ds},\frac{dy}{ds},\frac{dz}{ds}\right)$ are the direction cosines of the tangent to the curve C at the point P. Equation (1.19) shows that the tangent to the curve C at the point P is perpendicular to the line whose direction ratios are $\left(\frac{\partial F}{\partial x},\frac{\partial F}{\partial y},\frac{\partial F}{\partial z}\right)$.

$\Rightarrow \left(\frac{\partial F}{\partial x},\frac{\partial F}{\partial y},\frac{\partial F}{\partial z}\right)$ are the direction ratios of the normal to the surfaces S at the point P.

Example 2 : Find the direction cosines of the normal to the surface S of the form z = f (x, y).

Solution : Let the surface S : $z = f(x, y)$

$$\Rightarrow F(x,y,z)=f(x,y)-z \qquad \text{... (1.20)}$$

The direction ratios of the normal to the surface $S : F(x, y, z) = 0$ are $\left(\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z}\right)$,

where from equation (1.20), we have

$$\frac{\partial F}{\partial x} = \frac{\partial f}{\partial x} = \frac{\partial z}{\partial x},$$

$$\frac{\partial F}{\partial y} = \frac{\partial f}{\partial y} = \frac{\partial z}{\partial y}, \qquad \text{and}$$

$$\frac{\partial F}{\partial z} = -1.$$

Let us introduce the notations $\frac{\partial z}{\partial x} = p$ and $\frac{\partial z}{\partial y} = q$

Thus $$\left(\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z}\right) = (p, q, -1).$$

$\Rightarrow$ Direction ratios of the normal to the surface (1.20) are (p, q, -1). Hence the direction cosines of the normal to the surface S at the point P are

$$\frac{1}{\sqrt{p^2 + q^2 + 1}}(p, q, -1).$$

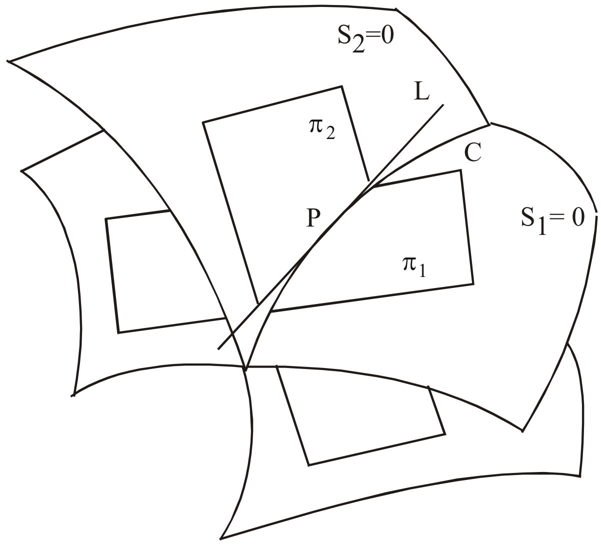
Equation of a line when two surfaces are given :

Let S_1 : F (x, y, z) = 0 and S_2 : G (x, y, z) = 0 be two surfaces. Then the equations of the tangent planes π_1 and π_2 at point P (x, y, z) to the surfaces $S_1 = 0$ and $S_2 = 0$ are given by

$$(X - x)\frac{\partial F}{\partial x} + (Y - y)\frac{\partial F}{\partial y} + (Z - z)\frac{\partial F}{\partial z} = 0, \qquad \text{... (1.21)}$$

and $$(X - x)\frac{\partial G}{\partial x} + (Y - y)\frac{\partial G}{\partial y} + (Z - z)\frac{\partial G}{\partial z} = 0, \qquad \text{... (1.22)}$$

where (x, y, z) are the co-ordinates of any other point on tangent plane. Let C be the locus of the intersection of two surfaces S_1 and S_2, and L the intersection of two planes π_1 and π_2. We see that the intersection L of the planes π_1 and π_2 is the tangent at P to the curve C, which is the intersection of the surfaces S_1 and S_2.



It follows from equations (1.21), (1.22) that the equation of the line L is

$$\frac{X-x}{\begin{vmatrix} F_y & F_z \\ G_y & G_z \end{vmatrix}} = \frac{Y-y}{\begin{vmatrix} F_z & F_x \\ G_z & G_x \end{vmatrix}} = \frac{Z-z}{\begin{vmatrix} F_x & F_y \\ G_x & G_y \end{vmatrix}}$$

or $$\frac{X-x}{\frac{\partial(F,G)}{\partial(y,z)}} = \frac{Y-y}{\frac{\partial(F,G)}{\partial(z,x)}} = \frac{Z-z}{\frac{\partial(F,G)}{\partial(x,y)}}. \qquad \text{... (1.23)}$$

This is the equation of line whose direction ratios are $\left(\frac{\partial(F,G)}{\partial(y,z)}, \frac{\partial(F,G)}{\partial(z,x)}, \frac{\partial(F,G)}{\partial(x,y)}\right)$, when two surfaces S_1 and S_2 are given.

2) Partial Differential Equations :

A partial differential equation is one involving more than one independent variables x, y, t, ..., one dependent variable $\theta \in C^n$ in some domain D and its partial derivatives $\theta_x, \theta_y, ..., \theta_{xx}, \theta_{xt}, ...$, such as,

$$f\left(x, y, t, ..., \theta, \theta_x, \theta_y, ..., \theta_{xt}, ...\right) = 0, \qquad \text{... (2.1)}$$

where C^n denotes a set of functions possessing continuous partial derivatives of order n.

Order of a partial differential equation :

The order of a partial differential equation is the order of the derivative of the highest order occuring in the equation.

In this unit and in the next three units, we shall consider partial differential equations of the first order with one dependent variable z and two independent variables x and y. Then the most general first order partial differential equation is given by

$$f(x, y, z, p, q) = 0, \qquad \dots (2.2)$$

where $$p = \frac{\partial z}{\partial x}, q = \frac{\partial z}{\partial y}.$$

Partial differential equations arise in a large variety of subjects in geometry, physics, mathematics etc.

Origin of first order Partial Differential Equations :

We shall examine in the following the intersecting question of how the first order partial differential equations arise.

Example 3 : Find the first order partial differential equation which represents the set of all spheres with centres on the z-axis and of radius a.

Solution : The set of all spheres with centres on the z-axis and of radius a is given by

$$x^2 + y^2 + (z-c)^2 = a^2, \qquad \dots (2.3)$$

where a and c are constants.

Differentiating equations (2.3) with respect to x and y we get,

$$x + p(z - c) = 0,$$

and $$y + q(z - c) = 0.$$

Eliminating the arbitrary constant c from the equations we obtain,

$$yp - xq = 0,$$

which is the required first order partial differential equation.

Example 4 : Find the partial differential equation which represents the set of all right circular cones with z-axis as the axis of symmetry.

Solution : The set of all right circular cones with z-axis as the axis of symmetry is given by the equation

$$x^2 + y^2 = (z-c)^2 \tan^2 \alpha, \qquad \dots (2.5)$$

where c is a constant and α is a constant semi-vertical angle of the cone.

Differentiating equation (2.5) with respect to x and y we get

$$x = p(z-c)\tan^2 \alpha,$$

$$y = q(z-c)\tan^2 \alpha.$$

Now eliminating c and α from the above equations we get

$$yp - xq = 0. \qquad \dots (2.6)$$

Thus the set of cones, vertex on the z-axis with semi-vertical angle α is characterized by the first order partial differential equation (2.6).

Example 5 : Find the partial differential equation which represents all surfaces of revolution with z-axis as the axis of revolution.

Solution : All surfaces of revolution with z-axis as the axis of revolution are of the form

$$z = F(r), \quad \text{... (2.7)}$$

where $r = \sqrt{x^2 + y^2}$ and F is an arbitrary function of class C^1 on some domain D.

On differentiating equation (2.7) first with respect to x and then with respect to y we get respectively

$$p = F'(r)\frac{\partial r}{\partial x}, \quad q = F'(r)\frac{\partial r}{\partial y}$$

where $$\frac{\partial r}{\partial y} = \frac{x}{r}, \quad \frac{\partial r}{\partial y} = \frac{y}{r}.$$

$$\Rightarrow p = \left(\frac{x}{r}\right)F'(r),$$

and $$q = \left(\frac{y}{r}\right)F'(r).$$

Eliminating the arbitrary function F from the above equation we get

$$yp - xq = 0, \quad \text{... (2.8)}$$

which is a partial differential equation of first order satisfied by all surfaces of revolution.

Note : We see from examples (3), (4) and (5) that the surfaces spheres, cones and in general all surfaces of revolution with z-axis as the axis revolution give rise to the same first order partial differential equation. What is common in all surfaces is that all surfaces of revolution have the z-axis as the axis of symmetry.

The obvious generlization of the surfaces of revolution with z-axis as the axis of symmetry is the relation between x, y and z of the form F (u, v) = 0, where u and v are functions of x, y, z. Hence we shall now generalize the above argument slightly in the following.

Example 6 : Find the partial differential equation satisfied by all surfaces of the form,

$$F(u, v) = 0,$$

where u = u (x, y, z) and v = v (x, y, z) are known functions of x, y and z and F is the arbitrary function of u and v.

Solution : The equations of all surfaces ingeneral is given by the equation.

$$F(u, v) = 0, \quad \text{... (2.9)}$$

where u = u (x, y, z) and v = v (x, y, z) are known functions of x, y and z.

Differentiating equation (2.9) with respect to x and y respectively, we get

$$F_u\left(u_x + u_z p\right) + F_v\left(v_x + v_z p\right) = 0, \quad \text{... (2.10)}$$

and $$F_u\left(u_y + u_z q\right) + F_v\left(v_y + v_z q\right) = 0. \qquad \text{... (2.11)}$$

Eliminating F_u and F_v between equations (2.10) and (2.11) we get

$$\frac{\left(v_x + p v_z\right)}{\left(u_x + p u_z\right)} = \frac{\left(v_y + q v_z\right)}{\left(u_y + q u_z\right)},$$

$$\Rightarrow p\left(u_y v_z - u_z v_y\right) + q\left(u_z v_x - u_x v_z\right) + \left(u_y v_x - u_x v_y\right) = 0,$$

$$\Rightarrow p\frac{\partial(u,v)}{\partial(y,z)} + q\frac{\partial(u,v)}{\partial(z,x)} = \frac{\partial(u,v)}{\partial(x,y)}. \qquad \text{... (2.12)}$$

This is the partial differential equation of first order satisfied by all surfaces of the form

F (u, v) = 0,

where $$\frac{\partial(u,v)}{\partial(x,y)} = u_x v_y - u_y v_x,$$

is the Jacobian of u, v with respect to x and y.

Theorem : A necessary and sufficient condition that there exists between two functions u (x, y) and v (x, y) a relation F (u, v) = 0 or u = H (v) not involving x or y explicitly is that

$$\frac{\partial(u,v)}{\partial(x,y)} = 0.$$

Proof : The necessary condition

Let there exist between two functions u (x, y) and v (x, y) a relation of the type

$$F(u, v) = 0 \qquad \text{... (2.13)}$$

not involving x and y explicitly.

Differentiating equation (2.13) with respect to x and then with respect to y we get

$$F_u u_x + F_v v_x = 0, \qquad \text{... (2.14)}$$

and $$F_u u_y + F_v v_y = 0. \qquad \text{... (2.15)}$$

Eliminating F_u and F_v between (2.14) and (2.15) we get

$$\begin{vmatrix} u_x & v_x \\ u_y & v_y \end{vmatrix} = 0.$$

$$\Rightarrow u_x v_y - u_y v_x = 0,$$

$$\Rightarrow \frac{\partial(u,v)}{\partial(x,y)} = 0 . \quad \text{... (2.16)}$$

The sufficient condition : Let u (x, y) and v (x, y) be two functions of x and y such that $\frac{\partial v}{\partial y} \neq 0$ and if $\frac{\partial(u,v)}{\partial(x,y)} = 0$ then we claim that there exists a relation *F (u, v) = 0* not involving x and y explicitly.

Eliminating y between the fuunctions u (x, y) and v (x, y) we get a relation

$$F(u, v, x) = 0 . \quad \text{... (2.17)}$$

Differentiating (2.17) with respect to x and y respectively we get

$$F_x + F_u u_x + F_v v_x = 0 , \quad \text{... (2.18)}$$

and

$$F_u u_y + F_v v_y = 0 . \quad \text{... (2.19)}$$

Eliminating F_v from these equations we get

$$F_x + F_u \left(u_x - \frac{u_y}{v_y} v_x \right) = 0 ,$$

$$\Rightarrow F_x v_y + F_u \left(u_x v_y - u_y v_x \right) = 0 ,$$

$$\Rightarrow F_x v_y + F_u \frac{\partial(u,v)}{\partial(x,y)} = 0 . \quad \text{... (2.20)}$$

Since

$$\frac{\partial(u,v)}{\partial(x,y)} = 0 ,$$

$$\Rightarrow F_x v_y = 0 ,$$

$$\Rightarrow F_x = 0 \text{ as } v_y \neq 0 .$$

$\Rightarrow$ the function F does not contain the variable x explicitly. Hence from the relation (2.17) we have

$$F(u, v) = 0.$$

Hence the condition is sufficient.

Remark : We have obtained partial differential equation of first order by eliminating arbitrary constants. (Refer examples (3), (4)).

Now consider two parameter family of surfaces given by the equation.

$$f(x, y, z, a, b) = 0 .$$

Solving for z, we get

$$z = F(x, y, a, b), \quad \text{... (2.21)}$$

where a and b are arbitrary constants. Differentiating equation (2.21) with respect to x and then with respect to y, we get

$$p = F_x \text{ and } q = F_y. \quad \text{... (2.22)}$$

The set of equations (2.21) and (2.22) constitute three equations involving two arbitrary constants 'a' and 'b'. Now eliminating 'a' and 'b' from these equations we obtain a relation of the type.

$$f(x, y, z, p, q) = 0 \quad \text{... (2.23)}$$

which is a partial differential equation of the first order. In general equation (2.23) need not be linear.

Example 7 : Obtain the partial differential equation of first order by eliminating arbitrary constants from the relation

$$(x-a)^2 + (y-b)^2 + z^2 = 1.$$

Solution : We are given two parameter family of surfaces

$$(x-a)^2 + (y-b)^2 + z^2 = 1. \quad \text{... (2.24)}$$

Equation (2.24) represents a set of all spheres of unit radius with centre in the xy plane. Differentiating equation (2.24) with respect to x and y we get respectively.

$$(x-a) + zp = 0,$$

$$\Rightarrow zp = -(x-a), \quad \text{... (2.25)}$$

and

$$(y-b) + zq = 0,$$

$$\Rightarrow zq = -(y-b). \quad \text{... (2.26)}$$

Eliminating the constants 'a' and 'b' from equation (2.24) we obtain

$$z^2\left(p^2 + q^2 + 1\right) = 1. \quad \text{... (2.27)}$$

This is the first order non-linear partial differential equation.

Example 8 : Obtain the partial differential equation of first order by eliminating arbitrary constants from the relation.

$$z^2\left(1 + a^3\right) = 8(x + ay + b)^3.$$

Solution : Two parameters family of surfaces are given by the equation

$$z^2\left(1 + a^3\right) = 8(x + ay + b)^3. \quad \text{... (2.28)}$$

Differentiating equation (2.28) with respect to x and y we get respectively.

$$z\left(a^3+1\right)p=12\left(x+ay+b\right)^2,$$

and $$z\left(a^3+1\right)q=12a\left(x+ay+b\right)^2.$$

$$\Rightarrow p=\frac{12}{z\left(a^3+1\right)}\left(x+ay+b\right)^2,$$

and $$q=\frac{12a}{z\left(a^3+1\right)}\left(x+ay+b\right)^2.$$

Consider

$$p^3+q^3=\frac{(12)^3}{z^3\left(a^3+1\right)^3}\left(x+ay+b\right)^6\left(1+a^3\right),$$

$$=\frac{(12)^3\left(x+ay+b\right)^6}{z^3\left(a^3+1\right)^2},$$

$$=\frac{(12)^3 z\left(x+ay+b\right)^6}{\left[z^2\left(a^3+1\right)\right]^2},$$

$$=\frac{(12)^3 z\left(x+ay+b\right)^6}{(8)^2\left(x+ay+b\right)^6},$$ by equation (2.28)

$$\Rightarrow p^3+q^3=27z. \quad \text{... (2.29)}$$

This is the required first partial differential equation.

Example 9 : Eliminate the arbitrary functions from the following equations and find the corresponding partial differential equations.

(i) $z=xy+F\left(x^2+y^2\right)$,

(ii) $F\left(x+y, x-\sqrt{z}\right)$,

(iii) $z=f\left(x+ct\right)+g\left(x-ct\right)$.

Solution :

(i) The equation of the surface is given by

$$z=xy+F\left(x^2+y^2\right), \quad \text{... (2.30)}$$

where F is arbitrary function. Differentiating equation (2.30) with respect to x and y we get respectively

$$p=y+2xF',$$

and $$q = x + 2yF'.$$

Eliminating F' between these equations we obtain

$$(p-y)y = (q-x)x,$$

$$\Rightarrow x^2 - y^2 - qx + py = 0.$$

This is the required partial differential equation.

(ii) The equation of the surface is given by the equation

$$F\left(x+y, x-\sqrt{z}\right) = 0, \qquad \text{... (2.31)}$$

where F is arbitray

Let $u = x + y$ and $v = x - \sqrt{z}$. ... (2.32)

Hence equation (2.31) becomes

$$F(u, v) = 0. \qquad \text{... (2.33)}$$

Differentiating equation (2.33) with respect to x and y respectively we get

$$F_u\left(u_x + u_z p\right) + F_v\left(v_x + v_z p\right) = 0 \text{ and}$$

$$F_u\left(u_y + u_z q\right) + F_v\left(v_y + v_z q\right) = 0,$$

where from equation (2.32), we find

$$u_x = 1,\ \ u_y = 1,\ \ v_x = 1,\ \ v_z = -\frac{1}{2\sqrt{z}},$$

$$\Rightarrow F_u + F_v\left(1 - \frac{1}{2\sqrt{z}}p\right) = 0, \qquad \text{... (2.34)}$$

and $$F_u - F_v\left(\frac{1}{2\sqrt{z}}q\right) = 0. \qquad \text{... (2.35)}$$

Eliminating F_u and F_v between equations (2.34) and (2.35) we get

$$p - q = 2\sqrt{z}. \qquad \text{... (2.36)}$$

This is the required partial differential equation

(iii) Here $$z = f(x+ct) + g(x-ct),$$

$$\Rightarrow z_x = f'(x+ct) + g'(x-ct),$$

$$z_{xx} = f''(x+ct) + g''(x-ct),$$

$$z_t = cf'(x+ct) - cg'(x-ct),$$

$$z_{tt} = c^2 f''(x+ct) + c^2 g''(x-ct),$$

$$\Rightarrow c^2 z_{xx} = z_{tt} \qquad \text{... (2.37)}$$

This is the required second order partial differential equation.

Exercise :

1. Obtain the partial differential equation of first order by eliminating arbitrary constants from the relations

(i) $z = x + ax^2y^2 + b,$ (iii) $z = (x+a)(y+b),$

(ii) $2z = (ax+y)^2 + b,$ (iv) $z = ax + by.$

2. Obtain the partial differential equation by eliminating arbitrary functions from the following relations.

(i) $z = x + y + F(xy),$ (ii) $F(x-z, y-z) = 0,$

(iii) $z = F\left(\frac{xy}{z}\right),$ (iv) $z = F\left(\frac{x}{y}\right),$

(v) $z = F(x-y),$ (vi) $f(x^2+y^2+z^2, z^2-2xy) = 0.$

3. Classification of First Order Partial Differential Equations :

1. Linear Equation : A first order partial differential equation is said to be a linear equation if it is linear in p, q and z. It is represented in the form

$$P(x,y)p + Q(x,y)q = R(x,y)z + S(x,y), \qquad \text{... (3.1)}$$

e.g. $yp - xq = xyz + x.$

2. Semi-linear Equation : A first order partial differential equation is said to be a semi-linear equation if it is linear in p and q and the coefficients of p and q are functions of x and y only. It is represented in the form.

$$P(x,y)p + Q(x,y)q = R(x,y,z), \qquad \text{... (3.2)}$$

e.g. $xp - yxq = xz^2.$

3. Quasi-linear Equation : A first order partial differential equation is said to be a quasi-linear equation if it is linear in p and q.

The equation of the type

$$P(x,y,z)p + Q(x,y,z)q = R(x,y,z) \qquad \text{... (3.3)}$$

is called quasi-linear equation.

e.g. $\left(x^2+z^2\right)p-xyq=z^2x+y^2$.

4. Non-linear Equation : The partial differential equations of the form *f (x, y, z, p, q) = 0* which do not come under the above three types are said to be non-linear equations.

eg. pq = z

This is a non-linear partial differential equation of first order.

Note : We observe that by eliminating arbitrary functions, we always produce quasi-linear partial differential equations only. However, we obtain both quasi-linear as well as non-linear partial differential equations when we eliminate arbitrary constants. If further, the number of constants to be eliminated from the given relation is just equal to the number of independent variables then the partial differential equation obtained by eliminating these constants is an equation of first order. However, if the number of constants to be eliminated is greater than the number of independent variables, the equation of second order will arise.

Classification of Integrals :

Consider a first order partial differential equation

$$f(x, y, z, p, q) = 0 \qquad \text{... (3.4)}$$

A solution of equation (3.4) in a region $D \subset \mathbb{R}\times\mathbb{R}$ is given by z = z (x, y) as a continuously differentiable function of x and y for $(x, y)\in D$ such that the value of p and q obtained from the relation z = z (x, y) must satisfy the equation (3.4). A solution z = z (x, y) of the first order partial differential equation represents a surface in 3-dimensional space. This surface in 3-dimensional space will be called an integral surface of the partial differential equation.

There are different types of solutions (integral surfaces) for the first order partial differential equation (3.4).

1. Complete Integral : A complete integral of partial differential equation (3.4) is a relation between the variables involving as many arbitrary constants as there are independent variables, provided the value of p and q obtained from it satisfies equation (3.4). Geometrically it represents doubly infinite system of surfaces.

Alternately, it is also defined as follows :

A two parameter family of solutions

$$z = F(x, y, a, b) \qquad \text{... (3.5)}$$

is called a complete integral of the first order partial differential equation (3.4) if in the region considered, the rank of the matrix

$$M=\begin{pmatrix} F_a & F_{xa} & F_{ya} \\ F_b & F_{xb} & F_{yb} \end{pmatrix}$$

is two.

2. **General Integral :** A solution of a partial differential equation (3.4) of the form

$$\phi(u,v)=0, \qquad \text{... (3.6)}$$

where u = u (x, y, z) and v = v (x, y, z) and ϕ is an arbitrary function, is called the general integral.

The complete integral (3.5) can be used in the derivation of general integral. Let the complete integral of the partial differential equation (3.4) by given by two parameter family of surfaces of the form.

$$z = F(x, y, a, b). \qquad \text{... (3.7)}$$

If we choose $b=\phi(a)$ we get one-parameter family of solution of equation (3.4) of the form

$$z=F(x,y,a,\phi(a)). \qquad \text{... (3.8)}$$

This is a sub-family of the two parameter family (3.7). The envelope of (3.8) if it exists and is obtained by eliminating 'a' between (3.8) and

$$F_a+F_b\phi'(a)=0. \qquad \text{... (3.9)}$$

Solving this equation for 'a' we get

$$a = a(x, y).$$

Substituting this in equation (3.8), we obtain an integral surface of (3.4) as

$$z=F(x,y,a(x,y),\phi(a(x,y))) \qquad \text{... (3.10)}$$

If the function a (x, y) is arbitrary, then such a solution is called a general integral (general solution) of (3.4). Geometrically it represents the envelope of one parameter family of surfaces.

Note : When $\phi(a)$ is a particular function, then we obtain a particular solution of the partial differential equation. Thus different choices of ϕ may give different particular solution of the partial differential equation.

Characteristic Curve :

Consider one-parameter family of surfaces

$$f(x, y, z, a) = 0. \qquad \text{... (3.11)}$$

For slightly different value of 'a' say $a+\delta a$, the system of surfaces becomes

$$f(x,y,z,a+\delta a)=0. \qquad \text{... (3.12)}$$

These two surfaces will intersect in a curve given by the equation

$$f(x,y,z,a)=0, \quad f(x,y,z,a+\delta a)=0. \qquad \text{... (3.13)}$$

Similarly, we can easily see that the curve may also be considered to be the intersection of the surface (3.11) with the surface whose equation is

$$\frac{1}{\delta a}\left[f(x,y,z,a+\delta a)-f(x,y,z,a)\right]=0.$$

As $\delta a \to 0$, we see that this curve of intersection is given by the equations

$$f(x,y,z,a)=0, \quad \frac{\partial}{\partial a} f(x,y,z,a)=0. \qquad \text{... (3.14)}$$

This limiting curve is called the 'characteristic curve' of equation (3.11). Geometrically, it is the curve on the surface (3.11) approached by the intersection of (3.11) and (3.12) as $\delta a \to 0$.

Envelope of the one-parameter family f (x, y, z, a) = 0 :

Consider a characteristic curve

$$f(x,t,a)=0, \frac{\partial}{\partial a} f(x,y,z,a)=0, \qquad \text{... (3.15)}$$

where 'a' is a parameter. As the parameter 'a' varies, the characteristic curve (3.15) will trace out a surface whose equation is obtained by eliminating 'a' between equations (3.15). Let this surface be given by

$$g(x,y,z)=0. \qquad \text{... (3.16)}$$

This surface is called the envelope of the one-parameter system *f (x, y, z, a) = 0.*

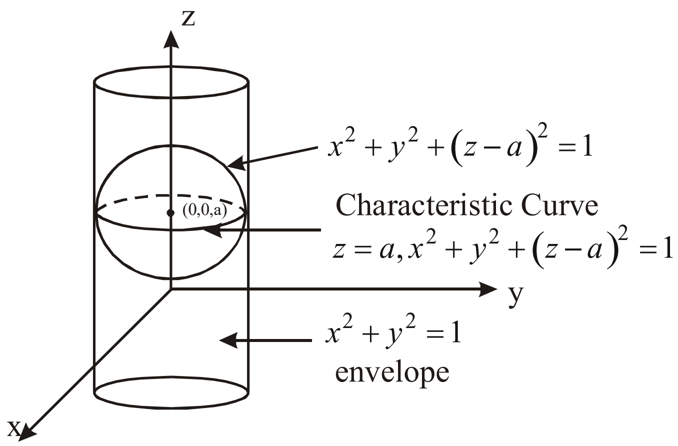
e.g. Consider one parameter family of surfaces

$$x^2+y^2+(z-a)^3=1. \qquad \text{... (3.17)}$$

It represents the family of spheres of unit radius with centres on the z-axis.

If $\quad f = x^2+y^2+(z-a)^2-1$,

then $\quad f_a = -2(3-a)$.



The characteristic curve is given by

$$x^2+y^2+(z+a)^2=1$$

and $\quad z=a \qquad \text{... (3.18)}$

Eliminating 'a' between equations (3.18) we get

$$x^2+y^2=1 \qquad \text{... (3.19)}$$

Which represents the envelope of the family and is the cylinder.

We shall show that the envelope of one-parameter family of surfaces if it exists is a solution of the given partial differential equation.

Result : Let $z = F(x, y, a)$ be a one-parameter family of solutions of the first order partial differential equation $f(x, y, z, p, q) = 0$. Then show that the envelope of this one parameter family, if it exists, is also a solution of the partial differential equation.

Proof : Consider the first order partial differential equation,

$$f(x, y, z, p, q) = 0. \qquad \text{... (3.20)}$$

The one-parameter family of solutions of (3.20) is given by

$$z = F(x, y, a). \qquad \text{... (3.21)}$$

Differentiating equation (3.21) with respect to a we get

$$F_a(x, y, a) = 0. \qquad \text{... (3.22)}$$

Now the envelope of the family of one-parameter is obtained by eliminating 'a' between the equations (3.21) and (3.22). Let the envelope be given by

$$z = G(x, y) = F(x, y, a(x, y)), \qquad \text{... (3.23)}$$

where $a(x, y)$ is obtained from equations (3.22) by solving for 'a' in terms of x and y.

Now we prove that the envelope (3.23) is a solution of equation (3.20). Hence differentiating equation (3.23) with respect to x and y we get

$$p = G_x = F_x + F_a a_x \text{ and } q = G_y = F_y + F_a a_y.$$

Using equation (3.22), we get

$$p = G_x = F_x, \quad q = G_y = F_y \qquad \text{... (3.24)}$$

This shows that the envelope will have the same partial derivatives as those of a member of the family. Since $p = F_x$ and $q = F_y$ satisfy the equation (3.20). This implies that $p = G_x$ and $q = G_y$ also satisfy the partial differential equation (3.20). This proves that the envelope of one parameter family of surfaces is also a solution of a partial differential equation.

Envelope of the two-parameter family of surfaces f (x, y, z, a, b) = 0 :

Consider the two-parameter system of surfaces defined by the equation

$$f(x, y, z, a, b) = 0, \qquad \text{... (3.25)}$$

where 'a' and 'b' are parameter.

Let $b = \phi(a)$. ... (3.26)

Differentiating equation (3.25) with respect to a we get

$$\frac{\partial f}{\partial a} + \frac{\partial f}{\partial b}\frac{\partial b}{\partial a} = 0 \qquad \text{... (3.27)}$$

The envelope is obtained by eliminating 'a' and 'b' from equations (3.25), (3.26) and (3.27).

3. The Singular Integral :

The envelope of the two-parameter family of surfaces $z = F(x, y, a, b)$, which is obtained by eliminating 'a' and 'b' from the equations

$$z = F(x, y, a, b),$$

$$F_a = 0, \quad F_b = 0,$$

is called the singular integral of the first order partial differential equation.

Note : This solution cannot be obtained by giving any values to the constants a and b and hence is not contained in the complete integral.

Result : Prove that singular integral is also a solution of the first order partial differential equation.

Proof : Let a two-parameter family of solutions

$$z = F(a, y, a, b) \qquad \text{... (3.28)}$$

be a complete integral of the first-order partial differential equation

$$f(x, y, z, p, q) = 0. \qquad \text{... (3.29)}$$

The singular solution of (3.29) is the envelope of (3.28). We will show that the envelope of this two parameter family (3.28), if it exists, is also a solution of (3.29). Hence differentiating equation (3.28) with respect to 'a' and 'b' we get respectively

$$F_a(x, y, a, b) = 0 \qquad \text{... (3.30)}$$

and $$F_b(x, y, a, b) = 0 \qquad \text{... (3.31)}$$

Now eliminating the parameters 'a' and 'b' between equations (3.28), (3.30) and (3.31) we obtain the envelope

$$z = G(x, y) = F(x, y, a(x,y), b(x,y)), \qquad \text{... (3.32)}$$

where a(x,y) and b(x,y) are obtained from equations (3.30), (3.31) by solving for 'a' and 'b' in terms of x and y.

Differentiating equation (3.32) with respect to x and then with respect to y we get respectively

$$p = G_x = F_x + F_a a_x + F_b b_x,$$

and $$q = G_y = F_y + F_a a_y + F_b b_y.$$

By virtue of (3.30) and (3.31) we have

$$p = G_x = F_x \text{ and } q = G_y = F_y.$$

This shows that the envelope will have the some partial derivatives as those of a member of the family. As the two-parameter family is the complete integral of the first order partial differential equation (3.29). Hence $p = G_x$ and $q = G_y$ also satisfy the equation (3.29). This proves that the envelope of the two parameter family (singular integral) is also a solution of the first order partial differential equation.

Note : The singular integral can also be found from the given partial differential equation without knowing the complete integral.

Result : Prove that singular solution is obtained by eliminating p and q from the equations.

$$f(x, y, z, p, q) = 0,\ f_p(x, y, z, p, q) = 0,\ f_q(x, y, z, p, q) = 0.$$

Proof : Consider the first order partial differential equation given by

$$f(x, y, z, p, q) = 0. \qquad \text{... (3.33)}$$

The complete integral of (3.33) is given by

$$z = F(x, y, a, b). \qquad \text{... (3.34)}$$

Differentiating (3.34) with respect to x and y we get respectively

$$p = F_x(x, y, a, b), \qquad \text{... (3.35)}$$

and
$$q = F_y(x, y, a, b). \qquad \text{... (3.36)}$$

Substituting equations (3.34), (3.35) and (3.36) in equation (3.33) we obtain

$$f\left(x, y, F(x, y, a, b), F_x(x, y, a, b), F_y(x, y, a, b)\right) = 0. \qquad \text{... (3.37)}$$

This holds identically for all 'a' and 'b'. Now we shall show that equation (3.37) satisfies

$$f_p = 0 \text{ and } f_q = 0.$$

Differentiating equation (3.37) with respect to 'a' and 'b' we get respectively.

$$f_z F_a + f_p F_{xa} + f_q F_{ya} = 0, \qquad \text{... (3.38)}$$

and
$$f_z F_b + f_p F_{xb} + f_q F_{yb} = 0. \qquad \text{... (3.39)}$$

However, on the singular solution, we have

$$F_a = 0 \text{ and } F_b = 0.$$

Hence equations (3.38) and (3.39) reduce to

$$f_p F_{xa} + f_q F_{ya} = 0, \qquad \text{... (3.40)}$$

$$f_p F_{xb} + f_q F_{yb} = 0. \qquad \text{... (3.41)}$$

Multiplying equation (3.40) by F_{yb} and equation (3.41) by F_{ya} and subtracting we get

$$f_p\left(F_{xa}F_{yb} - F_{xb}F_{ya}\right) = 0. \qquad \text{... (3.42)}$$

Since on the two-parameter family of surface (3.34)

$$F_{xa}F_{yb} - F_{xb}F_{ya} \neq 0.$$

If
$$F_{xa}F_{yb} - F_{xb}F_{ya} = 0$$

then
$$\begin{vmatrix} F_{xa} & F_{ya} \\ F_{xb} & F_{yb} \end{vmatrix} = 0,$$

and hence the matrix $\begin{pmatrix} F_a & F_{xa} & F_{ya} \\ F_b & F_{xa} & F_{yb} \end{pmatrix}$ will not have rank two (Since $F_a = 0, F_b = 0$), which contradicts the fact that $z = F(x, y, a, b)$ is a complete integral. Hence from equation (3.42) we have

$$f_p = 0.$$

Similarly, we prove $\quad f_q = 0.$

This proves the result.

4. **The Special Integral :** Usually (but not always) the three integrals viz., the complete integral, the general integral and the singular integral include all the integrals of the first order partial differential equation $f(x, y, z, p, q) = 0$. However, there are some solutions of certain first order partial differential equations which do not fall under any of the three classes. Such solutions are called "sepcial integrals".

Example 1 : Show that $z = ax + by + a^2 + b^2$ is a complete integral of $z = px + qy + p^2 + q^2$.

By taking (i) $b = \sqrt{1-a^2}$, (ii) b = a, find the envelope of the sub-family. Further find the singular integral.

Solution : Let

$$z = F(x, y, a, b) = ax + by + a^2 + b^2 . \qquad \text{... (3.43)}$$

To prove $z = F(x, y, a, b)$ is a complete integral of equation

$$z = px + qy + p^2 + q^2 . \qquad \text{... (3.44)}$$

We prove the rank of the matrix $\begin{pmatrix} F_a & F_{xa} & F_{ya} \\ F_b & F_{xb} & F_{yb} \end{pmatrix}$ is two.

Thus from equation (3.43) we find

$$F_a = x + 2a, F_b = y + 2b, \quad F_{xa} = 1, \quad F_{ya} = 0, \quad F_{ya} = 1, \quad F_{xb} = 0$$

Hence the above matrix becomes

$$\begin{pmatrix} F_a & F_{xa} & F_{ya} \\ F_b & F_{xb} & F_{yb} \end{pmatrix} = \begin{pmatrix} x+2a & 1 & 0 \\ y+2b & 0 & 1 \end{pmatrix}.$$

Obviously, the rank of the matrix is 2. Hence equation (3.43) is a complete integral of (3.44).

Case 1 : Take $b = \sqrt{1-a^2}$

Then the one-parameter sub-family is given by

$$z = F\left(x, y, a\sqrt{1-a^2}\right) = ax + \sqrt{1-a^2}\,y + 1 . \qquad \text{... (3.45)}$$

Differentiating equation (3.45) with respect to 'a' we get

$$F_a = x - \frac{ay}{\sqrt{1-a^2}} = 0. \qquad \text{... (3.46)}$$

From (3.45) we find

$$z - 1 = a\left(x + \frac{\sqrt{1-a^2}}{a}y\right), \qquad \text{... (3.47)}$$

where from (3.46) we find

$$\frac{a}{\sqrt{1-a^2}} = \frac{x}{y} \Rightarrow \frac{a^2}{1-a^2} = \left(\frac{x}{y}\right)^2,$$

$$\Rightarrow a^2\left(y^2 + x^2\right) = x^2,$$

$$\Rightarrow a^2 = \frac{x^2}{x^2+y^2}.$$

Consequently, eliminating 'a' from equation (3.47) we obtain

$$(z-1)^2 = \left(\frac{x^2}{x^2+y^2}\right)\left(x + \frac{y^2}{x}\right)^2,$$

$$\Rightarrow (z-1)^2 = x^2 + y^2. \qquad \text{... (3.48)}$$

This is the envelope (particular solution) of the equation (3.44).

Case : If b = a, then the one parameter sub-family of surfaces is given by

$$z = a(x+y) + 2a^2. \qquad \text{... (3.49)}$$

Differentiating this with respect to a we get

$$F_a = 0 \Rightarrow x + y + 4a = 0,$$

$$\Rightarrow x + y = -4a.$$

Substituting this value in equation (3.49) we get

$$z = -\left(\frac{x+y}{4}\right) + 2\frac{(x+y)^2}{16},$$

$$\Rightarrow 8z = -(x+y)^2. \qquad \text{... (3.50)}$$

This is another envelope (particular solution) of the given partial differential equation (3.44).

Now to find singular integral of (3.44), we differentiate equation (3.43), with respect to 'a' and then with respect to 'b', we get respectively

$$F_a = x + 2a = 0, \quad \text{... (3.51)}$$

and $$F_b = y + 2b = 0. \quad \text{... (3.52)}$$

Eliminating 'a' and 'b' between equations (3.43), (3.51) and (3.52) we get

$$z = -\frac{x^2}{2} - \frac{y^2}{2} + \frac{x^2}{4} + \frac{y^2}{4}$$

$$\Rightarrow 4z = -\left(x^2 + y^2\right), \quad \text{... (3.53)}$$

which is a singular integral of (3.44).

Note : The singular integral of equation (3.44) can also be obtained directly by eliminating p and q between equations (3.44) and

$$f_p = x + 2p = 0 \Rightarrow p = -\frac{x}{2},$$

and $$f_q = y + 2q = 0 \Rightarrow q = -\frac{y}{2}.$$

Substituting these in equation (3.44) we get

$$4z = -\left(x^2 + y^2\right)$$

as the singular integral of equation (3.44)

Example 2 : Show that

$$(x-a)^2 + (y-b)^2 + z^2 = 1$$

is a complete integral of

$$z^2\left(1 + p^2 + q^2\right) = 1.$$

By taking (i) b = 2a, (ii) b = a, show that the envelopes of the subfamily are respectively.

$$(y-2x)^2 + 5z^2 = 5 \text{ and } (x-y)^2 + 2z^2 = 2,$$

which are particular integrals. Show further that $z = \pm 1$ are the singular integrals.

Solution : Let

$$f(x, y, z, p, q) = z^2\left(1 + p^2 + q^2\right) - 1 = 0 \quad \text{... (3.54)}$$

be the given partial differential equation.

Let $$F(x,y,z,a,b)=(x-a)^2+(y-b)^2+z^2-1 \quad \text{... (3.55)}$$

be the two-parameter family of surfaces.

Differentiating (3.55) with respect to a, b etc., we find

$$F_a=-2(x-a),\ F_b=-2(y-b),\ F_{xa}=-2,\ F_{xb}=0,\ F_{xb}=0,\ F_{yb}=-2$$

Hence the matrix

$$\begin{pmatrix} F_a & F_{xa} & F_{ya} \\ F_b & F_{xb} & F_{yb} \end{pmatrix}=\begin{pmatrix} -2(x-a) & -2 & 0 \\ -2(y-b) & 0 & -2 \end{pmatrix}$$

has rank 2.

$$\Rightarrow (x-a)^2+(y-b)^2+z^2=1.$$

is a complete integral of equation (3.54)

Case 1 : Take b = 2a.

Hence from (3.55) the one-parameter sub-family of surfaces becomes.

$$(x-a)^2+(y-2a)^2+z^2=1. \quad \text{... (3.56)}$$

Differentiating (3.56) with respect to 'a' we get

$$-2(x-a)-4(y-2a)=0,$$

$$\Rightarrow x+2y-5a=0, \quad \text{... (3.57)}$$

$$\Rightarrow a=\frac{x+2y}{5}.$$

Substituting this in equation (3.56) we get

$$\left(x-\frac{x+2y}{5}\right)^2+\left(y-\frac{2x+4y}{5}\right)^2+z^2=1,$$

$$\Rightarrow (y-2x)^2+5z^2=5. \quad \text{... (3.58)}$$

This is the envelope of one parameter sub-family.

Case 2 : If we take b = a, then the one parameter family of sub-system becomes

$$(x-a)^2+(y-a)^2+z^2=1. \quad \text{... (3.59)}$$

Differentiating this equation with respect to 'a' we get

$$x+y-2a=0,$$

$$\Rightarrow a=\frac{x+y}{2}. \quad \text{... (3.60)}$$

Substituting the value of 'a' in equation (3.59) we get

$$\left(x-\frac{x+y}{2}\right)^2+\left(y-\frac{x+y}{2}\right)^2+z^2=1,$$

$$\Rightarrow (x-y)^2+2z^2=2. \qquad \text{... (3.61)}$$

This is the envelope of one parameter family. Equations (3.58) and (3.61) are particular solutions of (3.54).

Now to find singular integral of equation (3.54), we differentiate 2-parameter family of surfaces (3.55) to get

$$F_a(x,y,z,a,b)=x-a=0, \qquad \text{... (3.62)}$$

$$F_b(x,y,z,a,b)=y-b=0. \qquad \text{... (3.63)}$$

The envelope is obtain by eliminating 'a' and 'b' between (3.55), (3.62) and (3.63). Thus

$$z=\pm 1\cdot$$

This shows that the envelope consists of the pair of planes $z=\pm 1$. These planes are integral surfaces of the equations (3.55). It is the singular integral of the equation.

Note : The characteristic curve of the two-parameter system (3.55) is the locus of points of intersection of (3.55) with the plane (3.57). Since this plane passes through the centre of the sphere (a, 2a, 0), hence the characteristic curve of the system is the great circle.

Example 3 : Show that $z=ax+\frac{y}{a}+b$ is a complete integral of pq = 1. This problem has no singular integral. Find the particular solution corresponding to the sub-family b = a.

Solution : Let the partial differential equation is given by

$$f(x,y,z,p,q)=pq-1=0. \qquad \text{... (3.64)}$$

Let also the two parameters family of surfaces be given by

$$z=F(x,y,a,b)=ax+\frac{y}{b}+b. \qquad \text{... (3.65)}$$

We find

$$F_a=x-\frac{y}{a^2},$$

$$F_b=1,\ \ F_x=a,\ \ F_y=\frac{1}{a}$$

$$\Rightarrow F_{xa}=1,\ \ F_{xb}=0,\ \ F_{ya}=-\frac{1}{a^2},\ \ F_{yb}=0.$$

Hence the matrix $\begin{pmatrix} F_a & F_{xa} & F_{ya} \\ F_b & F_{xb} & F_{yb} \end{pmatrix} = \begin{pmatrix} x-\frac{y}{a^2} & 1 & -\frac{1}{a^2} \\ 1 & 0 & 0 \end{pmatrix}$

has rank two.

This proves that $z = ax + \frac{y}{a} + b$ is a complete integral of equation (3.64).

Now if b = a, then from equation (3.65) we get one parameter sub-system as

$$z = ax + \frac{y}{a} + a. \qquad \text{... (3.66)}$$

Differentiating this with respect to 'a' we get

$$0 = x - \frac{y}{a^2} + 1,$$

$$\Rightarrow a^2 = \frac{y}{x+1}. \qquad \text{... (3.67)}$$

To eliminate 'a' from equation (3.66), we first write it as

$$za = a^2(x+1) + y,$$

$$z^2a^2 = a^4(x+1)^2 + y^2 + 2a^2(x+1)y.$$

Putting the value of a^2, we get

$$z^2 = \left(\frac{y}{x+1}\right)(x+1)^2 + y^2\frac{(x+1)}{y} + \frac{2y}{(x+1)}(x+1)y,$$

$$\Rightarrow z^2 = 2(x+1)y + 2y^2. \qquad \text{... (3.68)}$$

This is the envelope of one parameter family and is the required particular solution of equation (3.64).

Now differentiating equation (3.65) with respect to 'a' and then with respect to 'b' we get respectively

$$F_a = x - \frac{y}{a^2}, \quad F_b = 1$$

$$F_a = 0 \Rightarrow x - \frac{y}{a^2} = 0, \quad F_b = 0 \Rightarrow 1 = 0.$$

This is not true. $\Rightarrow$ the equation (3.64) has no singular integral.

Note : It is always possible to obtain different complete integrals which are not equivalent to each other. These are not obtained from one another merely by a change in the choice of arbitrary constants.

Exercise :

1. Show that $2z = (ax+y)^2 + b$ is a complete integral of $px + qy - q^2 = 0$.

LINEAR EQUATIONS OF THE FIRST ORDER

Introduction :

In this unit we study a method of finding a general integral of a quasi linear equation.

Theorem : The general solution of the Lagrange's equation (quasi-linear equation).

$$P(x,y,z)\,p+Q(x,y,z)\,q=R(x,y,z),$$

where P, Q and R are given continuously differentiable functions of x, y, and z (and not vanishing simultaneously) is F (u, v) = 0, where F is an arbitrary function of u and v and

$$u(x,y,z)=C_1, v(x,y,z)=C_2$$

are the solutions of the system

$$\frac{dx}{P(x,y,z)}=\frac{dy}{Q(x,y,z)}=\frac{dz}{R(x,y,z)}.$$

Proof : Given that

$$u(x,y,z)=C_1 \text{ and } v(x,y,z)=C_2 \qquad \text{... (1.1)}$$

are the solutions of the system of differential equations

$$\frac{dx}{P}=\frac{dy}{Q}=\frac{dz}{R}. \qquad \text{... (1.2)}$$

This implies that equation (1.1) satisfy equations (1.2),

$$\Rightarrow u_x dx+u_y dy+u_z dz=0 \text{ and} \qquad \text{... (1.3)}$$

$$\Rightarrow v_x dx+v_y dy+v_z dz=0. \qquad \text{... (1.4)}$$

This shows that the equations (1.3) and (1.4) must be consistent equations. Hence we have

$$u_x P+u_y Q+u_z R=0, \qquad \text{... (1.5)}$$

and

$$v_x P+v_y Q+v_z R=0. \qquad \text{... (1.6)}$$

Solving equations (1.5) and (1.6) for P, Q and R we obtain

$$\frac{P}{\begin{vmatrix} u_y & u_z \\ v_y & v_z \end{vmatrix}} = \frac{Q}{\begin{vmatrix} u_z & u_x \\ v_z & v_x \end{vmatrix}} = \frac{R}{\begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix}}$$

$$\Rightarrow \frac{P}{\left(u_y v_z - u_z v_y\right)} = \frac{Q}{\left(u_z v_x - u_x v_z\right)} = \frac{R}{\left(u_x v_y - u_y v_x\right)},$$

$$\Rightarrow \frac{P}{\frac{\partial(u,v)}{\partial(y,z)}} = \frac{Q}{\frac{\partial(u,v)}{\partial(z,x)}} = \frac{R}{\frac{\partial(u,v)}{\partial(x,y)}}. \qquad \text{... (1.7)}$$

Now we shall show that F (u, v) = 0 is a solution of $Pp + Qq = R$.

Consider the relation F (u, v) = 0. Differentiating this partially with respect to x and y we get

$$F_u\left(u_x + pu_z\right) + F_v\left(v_x + pv_z\right) = 0, \qquad \text{... (1.8)}$$

and

$$F_u\left(u_y + qu_z\right) + F_v\left(v_y + qv_z\right) = 0. \qquad \text{... (1.9)}$$

Eliminating F_u and F_v from equations (1.8) and (1.19) we get

$$\left(u_x + pu_z\right)\left(v_y + qv_z\right) = \left(u_y + qu_z\right)\left(v_x + pv_z\right),$$

$$\Rightarrow p\left(u_y v_z - u_z v_y\right) + q\left(u_z v_x - u_x v_z\right) = \left(u_x v_y - u_y v_x\right)$$

$$\Rightarrow p\frac{\partial(u,v)}{\partial(y,z)} + q\frac{\partial(u,v)}{\partial(z,x)} = \frac{\partial(u,v)}{\partial(x,y)}. \qquad \text{... (1.10)}$$

From equations (1.7) and (1.8) we find

$$Pp + Qq = R. \qquad \text{... (1.11)}$$

This shows that F (u, v) = 0 is a solution of equation (1.11), where $u(x, y, z) = C_1$ and $v(x, y, z) = C_2$ are solution of (1.2). This equivalently means that any surface F (u, v) = 0 generated by the integral curves (1.2) is a solution of (1.11).

General Case :

Theorem : If $u_i\left(x_1, x_2, ..., x_n, z\right) = C_i$, i = 1, 2, ..., n are independent solutions of the euqtions

$$\frac{dx_1}{P_1} = \frac{dx_2}{P_2} = ... = \frac{dx_n}{P_n} = \frac{dz}{R},$$

where $P_1, P_2, ..., P_n$ and R are continuous differentiable functions of $x_1, x_2, ..., x_n$ and z not simultaneously zero, then the relation

$$\phi(u_1, u_2, ..., u_n) = 0,$$

where ϕ is arbitrary, is a general solution of the quasi-linear partial differential equation

$$P_1 \frac{\partial z}{\partial x_1} + P_2 \frac{\partial z}{\partial x_2} + ... + P_n \frac{\partial z}{\partial x_n} = R .$$

Proof : We are given that

$$u_i(x_1, x_2, ..., x_n, z) = C_i, \quad i = 1, 2, ..., n \qquad ... (1.12)$$

are independent solutions of the equations

$$\frac{dx_1}{P_1} = \frac{dx_2}{P_2} = ... = \frac{dx_n}{P_n} = \frac{dz}{R} \qquad ... (1.13)$$

Differentiating equation (1.12) we get

$$\frac{\partial u_i}{\partial x_1} dx_1 + \frac{\partial u_i}{\partial x_2} dx_2 + ... + \frac{\partial u_i}{\partial x_n} dx_n + \frac{\partial u_i}{\partial z} dz = 0 . \qquad ... (1.14)$$

This shows that the equations (1.13) and (1.14) must be compatible (consistent).

$$\Rightarrow \sum_{j=1}^{n} \frac{\partial u_i}{\partial x_j} P_j + \frac{\partial u_i}{\partial z} R = 0 , \ i = 1, 2, ..., n. \qquad ... (1.15)$$

For each i, we have n-equations. Solving these n-equations for $P_1, P_2, ..., P_n$ and R we get

$$\frac{P_i}{\dfrac{\partial(u_1, u_2, ..., u_n)}{\partial(x_1, x_2, ..., x_{i-1}, z, x_{i+1}, ..., x_n)}} = \frac{R}{\dfrac{\partial(u_1, u_2, ..., u_n)}{\partial(x_1, x_2, ..., x_n)}}, \qquad ... (1.16)$$

where $$\frac{\partial(u_1, u_2, ..., u_n)}{\partial(x_1, x_2, ..., x_n)} = \begin{vmatrix} \frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial x_2} & \cdots & \frac{\partial u_1}{\partial x_n} \\ \frac{\partial u_2}{\partial x_i} & \frac{\partial u_2}{\partial x_1} & & \frac{\partial u_n}{\partial x_n} \\ \cdots & & & \\ \frac{\partial u_n}{\partial x_1} & \frac{\partial u_n}{\partial x_2} & & \frac{\partial u_n}{\partial x_n} \end{vmatrix}$$

is the Jacobian of the transformation. Now we shall show that the relation

$$F(u_1, u_2, ..., u_n) = 0 \qquad ... (1.17)$$

is a solution of quasi-linear partial differential equation

$$P_1 \frac{\partial z}{\partial x_1} + P_2 \frac{\partial z}{\partial x_2} + ... + P_n \frac{\partial z}{\partial x_n} = R$$

Differentiating equation (1.17) partially with respect to x_i we get

$$\sum_j \left(\frac{\partial F}{\partial u_j} \frac{\partial u_j}{\partial x_i} + \frac{\partial u_j}{\partial z} \frac{\partial z}{\partial x_i} \right) = 0 . \qquad ... (1.18)$$

Eliminating $\frac{\partial F}{\partial u_1}, \frac{\partial F}{\partial u_2}, ..., \frac{\partial F}{\partial u_n}$ from these n equations we get

$$\sum_j \frac{\partial z}{\partial x_j} \frac{\partial (u_1, u_2, ..., u_n)}{\partial (x_1, x_2, ..., x_{j-1}, z, x_{j+1}, ..., x_n)} = \frac{\partial (u_1, u_2, ..., u_n)}{\partial (x_1, x_2, ..., x_n)} . \qquad ... (1.19)$$

From equations (1.16) and (1.19) we obtain

$$\sum_{j=1}^{n} \frac{\partial z}{\partial x_j} P_j = R \qquad ... (1.20)$$

This proves that if $u_1, u_2, ..., u_n$ are independent solutions of (1.13) then F $(u_1, u_2, ..., u_n)$ is a solution of equations (1.20). This proves the theorem.

Example 1 : Find the general integral of

$$z(xp - yq) = y^2 - x^2 .$$

Solution : The given partial differential equations is

$$zxp - zyq = y^2 - x^2 . \qquad ... (1.21)$$

The integral surface of the equation (1.21) is generated by the integral curves of the auxilary euqtion

$$\frac{dx}{zx} = \frac{dy}{-yz} = \frac{dz}{y^2 - x^2} . \qquad ... (1.22)$$

Consider the first two ratios of the equation

$$\frac{dx}{xz} = \frac{dy}{-yz} \Rightarrow \frac{dx}{x} = \frac{dy}{-y} .$$

Integrating we get

$$\log x = -\log y + \log C_1 ,$$

$$\Rightarrow xy = C_1 . \qquad ... (1.23)$$

Now we consider each ratio of the equation (1.22)

$$= \frac{xdx + ydy + zdz}{x^2z - y^2z + zy^2 - zx^2},$$

$$\Rightarrow xdx + ydy + zdz = 0.$$

Integrating the equation we get

$$x^2 + y^2 + z^2 = C_2. \qquad \text{... (1.24)}$$

The curves given by equations (1.23) and (1.24) generate the integral surface

$$F\left(xy, x^2 + y^2 + z^2\right) = 0,$$

which is the general integral of equation (1.21).

Example 2 : Find the general integral of the partial differential equation

$$2x\left(y + z^2\right)p + y\left(2y + z^2\right)q = z^3.$$

Solution : The given partial differential equation is

$$2x\left(y + z^2\right)p + y\left(2y + z^2\right)q = z^3. \qquad \text{... (1.25)}$$

The integral surface of equation (1.25) is generated by the integral curves of the auxiliary equation

$$\frac{dx}{2x\left(y + z^2\right)} = \frac{dy}{y\left(2y + z^2\right)} = \frac{dz}{z^3}. \qquad \text{... (1.26)}$$

The first integral curve is obtained by considering each ratio of the equation (1.26) as

$$\frac{dx/x - dy/y - dz/z}{2y + 2z^2 - 2y - z^2 - z^2},$$

$$\Rightarrow \frac{dx}{x} - \frac{dy}{y} - \frac{dz}{z} = 0.$$

Integrating gives

$$\log x - \log y - \log z = \log C_1,$$

$$\Rightarrow \frac{x}{yz} = C_1. \qquad \text{... (1.27)}$$

Now to find the second integral curve, consider the ratios

$$\frac{dy}{y\left(2y + z^2\right)} = \frac{dz}{z^3},$$

$$\Rightarrow z^3 dy = 2y^2 dz + z^2 y dz\,,$$

$$\Rightarrow z^2\left(ydz - zdy\right) = -2y^2 dz\,,$$

$$\Rightarrow \frac{ydz - zdy}{y^2} = -\frac{2dz}{z^2}\,,$$

$$\Rightarrow d\left(\frac{z}{y}\right) = -2\frac{dz}{z^2}\,.$$

Integration gives

$$\frac{z}{y} = \frac{2}{z} + C_2\,. \qquad \text{... (1.28)}$$

Thus the curves given by equations (1.27) and (1.28) generate the integral surface

$$F\left(\frac{x}{yz}, \frac{z^2 - 2y}{yz}\right) = 0\,.$$

Example 3 : Find the general integral of the partial differential equation

$$x(x+y)p = y(x+y)q - (x-y)(2x+2y+z)\,.$$

Solution : The integral surface of the equation

$$x(x+y)p - y(x+y)q = -(x-y)(2x+2y+z) \qquad \text{... (1.29)}$$

is generated by the integral curves of the auxilary equation

$$\frac{dx}{x(x+y)} = \frac{dy}{-y(x+y)} = \frac{dz}{-(x-y)(2x+2y+z)}\,. \qquad \text{... (1.30)}$$

To find the first integral curve, consider the ratio

$$\frac{dx}{x(x+y)} = \frac{dy}{-y(x+y)} \Rightarrow \frac{dx}{x} = \frac{dy}{-y}\,.$$

Integration yields

$$\log x = -\log y + \log C_1\,,$$

$$\Rightarrow xy = C_1\,. \qquad \text{... (1.31)}$$

Similarly, to find the second integral curve, each ratio of equation (1.30) is

$$\frac{dx+dy}{(x-y)(x+y)} = \frac{dx+dy+dz}{-(x-y)(x+y+z)}\,,$$

$$\Rightarrow \frac{dx+dy}{x+y} = \frac{dx+dy+dz}{-(x+y+z)}.$$

Integration results in

$$\log(x+y) = -\log(x+y+z) + \log C_2.$$

$$\Rightarrow (x+y)(x+y+z) = C_2, \qquad \text{... (1.32)}$$

These curves (1.31) and (1.32) generate the integral surface

$$F\left(xy, (x+y)(x+y+z)\right) = 0.$$

Example 4 : Find the general integral of

$$\left(x^2+y^2\right)p + 2xyq = (x+y)z.$$

Solution : To find the integral surface of the equation

$$\left(x^2+y^2\right)p + 2xyq = (x+y)z, \qquad \text{... (1.33)}$$

we first find the integral curves of the auxiliary equation

$$\frac{dx}{x^2+y^2} = \frac{dy}{2xy} = \frac{dz}{(x+y)z}. \qquad \text{... (1.34)}$$

To get the integral curve, we consider the ratios

$$\frac{dx+dy}{(x+y)^2} = \frac{dz}{(x+y)z} \Rightarrow \frac{dx+y}{x+y} = \frac{dz}{z}.$$

Integration of which gives

$$\log(x+y) = \log z + \log C_1,$$

$$\Rightarrow \frac{x+y}{z} = C_1. \qquad \text{... (1.35)}$$

Similarly, the other integral curve is obtained by consider the ratios

$$\frac{dx}{x^2+y^2} = \frac{dx}{2xy} \Rightarrow \left(x^2+y^2\right)dy = 2xydx,$$

$$\Rightarrow y^2dy = -x^2dy + 2xydx,$$

$$dy = \frac{2xydx - x^2dy}{y^2},$$

$$\Rightarrow dy = d\left(\frac{x^2}{y}\right).$$

Integration results in

$$y - \frac{x^2}{y} = C_2 . \qquad \text{... (1.36)}$$

Hence the integral surface generated by the curves (1.35) and (1.36) is given by

$$F\left(\frac{x+y}{z}, \frac{y^2 - x^2}{y}\right) = 0 .$$

Example 5 : Find the general integral of the partial differential equation

$$\left(xy^3 - 2x^4\right)p + \left(2y^4 - x^3y^3\right)q = 9z\left(x^3 - y^3\right).$$

Solution : The general solution of the equation

$$\left(xy^3 - 2x^4\right)p + \left(2y^4 - x^3y^3\right)q = 9z\left(x^3 - y^3\right) \qquad \text{... (1.37)}$$

is the integral surface generated by the integral curves of the auxilliary equation

$$\frac{dx}{xy^3 - 2x^4} = \frac{dy}{2y^4 - x^3y} = \frac{dz}{9z\left(x^3 - y^3\right)} . \qquad \text{... (1.38)}$$

To find the integral curve, we first consider the ratios of the equation (1.38) as

$$\frac{\frac{dx}{x} + \frac{dy}{y}}{y^3 - 2x^3 + 2y^3 - x^3} = \frac{\frac{dz}{z}}{9\left(x^3 - y^3\right)},$$

$$\Rightarrow \frac{dx}{x} + \frac{dy}{y} = \frac{dz}{3z} .$$

Integration of which gives

$$\log x + \log y = -\frac{1}{3}\log z + \log C_1 ,$$

$$\Rightarrow x^3y^3z = C_1 \qquad \text{... (1.39)}$$

Now consider the ratios

$$\frac{dx}{xy^3 - 2x^4} = \frac{dy}{2y^4 - x^3y},$$

$$\Rightarrow \left(2y^4 - x^3y\right)dx - \left(xy^3 - 2x^4\right)dy = 0 .$$

Dividing by x^3y^3 we get

$$\frac{2y}{x^3}dx - \frac{dx}{y^2} - \frac{dy}{x^2} + \frac{2x}{y^3}dy = 0,$$

$$\Rightarrow x\frac{(xdy - 2ydx)}{x^4} + y\frac{(ydx - 2xdy)}{y^4} = 0,$$

$$\Rightarrow d\left(\frac{y}{x^2}\right) + d\left(\frac{x}{y^2}\right) = 0.$$

Integration yields

$$\frac{y}{x^2} + \frac{x}{y^2} = C_2. \qquad \text{... (1.40)}$$

Hence the general integral generated by the curves (1.39) and (1.40) is given by

$$F\left(x^3y^3z, \frac{y}{x^2} + \frac{x}{y^2}\right) = 0.$$

Example 6 : Find the general integral of

$$\left(x^3 + 3xy^2\right)p + \left(y^3 + 3x^2y\right)q = 2\left(x^2 + y^2\right)z.$$

Solution : The general solution of the equation

$$\left(x^3 + 3xy^2\right)p + \left(y^3 + 3x^2y\right)q = 2\left(x^2 + y^2\right)z \qquad \text{... (1.41)}$$

is the integral surface generated by the integral curves of the auxilliary equations

$$\frac{dx}{x^3 + 3xy^2} = \frac{dy}{y^3 + 3x^2y} = \frac{dz}{2\left(x^2 + y^2\right)z}. \qquad \text{... (1.42)}$$

The first integral curve of (1.42) is obtained by considering the ratio

$$\frac{dx/x + dy/y - 2\,dz/z}{x^2 + 3y^2 + y^2 + 3x^2 - 4\left(x^2 + y^2\right)},$$

$$\Rightarrow \frac{dx}{x} + \frac{dy}{y} - \frac{2dz}{z} = 0.$$

Integrating we obtain

$$\log x + \log y - 2\log z = \log C_1,$$

$$\Rightarrow \frac{xy}{z^2} = C_1. \qquad \text{... (1.43)}$$

The second integral curve of (1.42) is obtained by considering the ratio

$$\frac{xdx - ydy}{x^4 + 3x^2y^2 - y^4 3x^2y^2} = \frac{1}{2}\frac{dz/z}{x^2 + y^2} \Rightarrow \frac{2(xdx - ydy)}{x^2 - y^2} = \frac{dz}{z}.$$

Integrating we get

$$\log\left(x^2 - y^2\right) = \log z + \log C_2,$$

$$\Rightarrow \left(\frac{x^2 - y^2}{z}\right) = C_2. \qquad \text{.... (1.44)}$$

Hence the general integral is given by

$$F\left(\frac{xy}{z^2}, \frac{x^2 - y^2}{z}\right) = 0.$$

Exercise :

Find the general integral of the following partial differential equations.

1. $(y+1)p + (x+1)q = z$
2. $\left(z^2 - 2yz - y^2\right)p + x(y+z)q = x(y-z)$
3. $yzp + xzq = x + y$
4. $y^2 p - xyq = x(z - 2y)$
5. $xzp + yzq = xy$
6. $\left(x^2 - yz\right)p + \left(y^2 - zx\right)q = z^2 - xy$
7. $p - q = 2\sqrt{z}$

Answers :

1. $F\left(x^2 - y^2 + 2x - 2y, z(x-y)\right) = 0$
2. $F\left(x^2 + y^2 + z^2, y^2 - 2yz - z^2\right) = 0$
3. $F\left(x^2 - y^2, z^2 - 2(x+y)\right) = 0$

4. $F\left(x^2+y^2, y(y-z)\right)=0$

5. $F\left(\frac{x}{y}, xy-z^2\right)=0$

6. $F\left(\frac{x-y}{y-z}, \frac{y-z}{z-x}\right)=0$

7. $F\left(x+y, x=\sqrt{z}\right)=0$

2. Pfaffian Differential Equations :

Introduction : In this section, we introduce a Pfaffian difference equations. There is a fundamental difference between Pfaffian differential equations in two variables and those in higher number of variables. A Pfaffian differential equation in two variables always possesses an integrating factor. However, a Pfaffian differential equation in more than two variables may not be integrable in general. We shall derive in the following a necessary and sufficient condition for the integrability of a Pfaffian differential equation in three variables.

Definition : A Pfaffian differential equation is a differential equation of the form

$$\sum_{i=1}^{n} F_i\left(x_1, x_2, \ldots, x_n\right) dx_i = 0, \qquad \ldots (2.1)$$

where $F_i\left(i=1,2,\ldots,n\right)$ are continuous functions of some or all of the n-independent variables $x_1, x_2, \ldots, x_n$, is called a Pfaffian differential equation, and the expression $\sum_i F_i\left(x_1, x_2, \ldots, x_n\right) dx_i$ is called a Pfaffian differential form.

Definition : A Pfaffian differential form is said to be exact if we can find a continuously difefrentiable function $u\left(x_1, x_2, \ldots, x_n\right)$ such that

$$du = F_1\left(x_1, x_2, \ldots, x_n\right) dx_1 + F_2\left(x_1, x_2, \ldots, x_n\right) dx_2 + \ldots + F_n\left(x_1, x_2, \ldots, x_n\right) dx_n.$$

Definition : A Pfaffian differential equation is said to be integrable, if there exists a non-zero differentiable function $\mu\left(x_1, x_2, \ldots, x_n\right)$ such that the Pfaffian differential form

$$\mu\left[F_1\left(x_1, x_2, \ldots, x_n\right) dx_1 + \ldots + F_n\left(x_1, x_2, \ldots, x_n\right) dx_n\right]$$

is exact. In this case the function $\mu\left(x_1, x_2, \ldots, x_n\right)$ is called the integrating factor of the Pfaffian differential equation and $u\left(x_1, x_2, \ldots, x_n\right) = C$, where C is an arbitrary constant, is called the integral of the corresponding Pfaffian differential equation.

Note : Since the integral $u(x_1, x_2, ..., x_n) = C$ of the Pfaffian differential equation (2.1) represents a surface in $\mathbb{R}^n$, then it follows from Pfaffian differential equation that, at every point of the integral the normal has direction ratios ($F_1, F_2, ..., F_n$).

Result : A Pfaffian differential equation in two variables always possesses an integrating factor.

Proof : Consider a Pfaffian differential equation in two variables x and y in the form

$$P(x, y)dx + Q(x, y)dy = 0. \qquad ... (2.2)$$

If $Q(x, y) \neq 0$, then we write this as

$$\frac{dy}{dx} = -\frac{P(x, y)}{Q(x, y)} = f(x, y), \qquad ... (2.3)$$

where P (x, y) and Q (x, y) are known functions of x and y, so that f (x,y) is defined uniquely at each point of the xy plane, at which the functions P (x, y) and Q (x, y) are defined. From the existence theorem for a first order ordinary differential equation the equation (2.3) has a solution

$$F(x, y) = C_1 \qquad ... (2.4)$$

Result : If $\overline{X} = (P, Q, R)$ is a vector such that $\overline{X} \cdot curl\ \overline{X} = 0$ and μ is an arbitrary differentiable function of x, y and z then prove that

$$\mu\overline{X} \cdot curl(\mu\overline{X}) = 0$$

Proof : Let $\overline{X} = (P, Q, R) = Pi + Qj + Rk$, ... (2.5)

be a vector, where i, j, k are unit vectors in the positive x, y and z directions respectively, such that

$$\overline{X} \cdot curl\ \overline{X} = 0 \cdot \qquad ... (2.6)$$

By definition, we have

$$curl(\mu\overline{X}) = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \mu P & \mu Q & \mu R \end{vmatrix}$$

$$\Rightarrow curl(\mu\overline{X}) = i\left(\frac{\partial \mu R}{\partial y} - \frac{\partial \mu Q}{\partial z}\right) + j\left(\frac{\partial \mu P}{\partial z} - \frac{\partial \mu R}{\partial x}\right) + k\left(\frac{\partial \mu Q}{\partial x} - \frac{\partial \mu P}{\partial y}\right),$$

$$\Rightarrow (\mu\overline{X}) \cdot curl(\mu\overline{X}) = \mu P\left(\frac{\partial \mu R}{\partial y} - \frac{\partial \mu Q}{\partial z}\right) + \mu Q\left(\frac{\partial \mu P}{\partial z} - \frac{\partial \mu R}{\partial x}\right) + \mu R\left(\frac{\partial \mu Q}{\partial x} - \frac{\partial \mu P}{\partial y}\right),$$

$$\Rightarrow (\mu\overline{X})\cdot curl(\mu\overline{X}) = \mu^2\left[P\left(\frac{\partial R}{\partial y}-\frac{\partial Q}{\partial z}\right)+Q\left(\frac{\partial P}{\partial z}-\frac{\partial R}{\partial x}\right)+R\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)\right]+$$

$$+\mu\left[PR\frac{\partial \mu}{\partial y}-PQ\frac{\partial \mu}{\partial z}+PQ\frac{\partial \mu}{\partial z}-QR\frac{\partial \mu}{\partial x}+QR\frac{\partial \mu}{\partial x}-PR\frac{\partial \mu}{\partial y}\right].$$

$$\Rightarrow (\mu\overline{X})\cdot curl(\mu\overline{X}) = \mu^2\left[P\left(\frac{\partial R}{\partial y}-\frac{\partial Q}{\partial z}\right)+Q\left(\frac{\partial P}{\partial z}-\frac{\partial R}{\partial x}\right)+R\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)\right]. \quad \text{... (2.7)}$$

This can also be written as

$$(\mu\overline{X})\cdot curl(\mu\overline{X}) = \mu^2\sum_{x,y,z} P\left(\frac{\partial R}{\partial y}-\frac{\partial Q}{\partial z}\right). \quad \text{... (2.8)}$$

i.e. $$(\mu\overline{X})\cdot curl(\mu\overline{X}) = \mu^2\left(\overline{X}\cdot curl\ \overline{X}\right).$$

By virture of equation (2.6) we have

$$(\mu\overline{X})\cdot curl(\mu\overline{X}) = 0.$$

Conversely, let $(\mu\overline{X})\cdot curl(\mu\overline{X}) = 0$, for $\mu \neq 0$ then it follow from the definition

$$\overline{X}\cdot curl\overline{X} = 0\cdot$$

Note : The condition $\overline{X}\cdot curl\overline{X} = 0$ is equivalent to

$$P\left(\frac{\partial Q}{\partial z}-\frac{\partial R}{\partial y}\right)+Q\left(\frac{\partial R}{\partial x}-\frac{\partial P}{\partial z}\right)+R\left(\frac{\partial P}{\partial y}-\frac{\partial Q}{\partial x}\right)=0.$$

Criteria of Integrability of a Pfaffian Differential Equation :

Note that all Pfaffian differential equations do not possesses integral. If however, the equation is such that there exists a function $\mu(x, y, z)$ with the property that $\mu(Pdx + Qdy + Rdz)$ is an exact differential $d\phi$, then the equation is said to be integrable. The function ϕ is called the primitive of the differential equation. In the following theorem we find a necessary and sufficient condition that a Pfaffian differential equation is integrable.

Theorem : A necessary and sufficient condition that the Pfaffian differential equation $\overline{X}\cdot d\overline{r} = 0$ is integrable is that $\overline{X}\cdot curl\overline{X} = 0$, where $\overline{X} = (P, Q, R)$ is a vector.

Proof : Consider a Pfaffian differential equation in three variables x, y, z given by

$$P(x, y, z)dx + Q(x, y, z)dy + R(x, y, z)dz = 0. \quad \text{... (2.9)}$$

If $\overline{X}=(P,Q,R)$ is a vector and $\overline{r}=(x,y,z)$, $\Rightarrow$ $d\overline{r}=(dx,dy,dz)$, then equation (2.9) can also be written as

$$\overline{X}d\overline{r}=Pdx+Qdy+Rdz=0,$$

$$\Rightarrow \overline{X}d\overline{r}=0\cdot \qquad \text{... (2.10)}$$

Let us assume that the equation (2.10) is integrable. We claim that

$$\overline{X}\cdot curl\overline{X}=0\cdot$$

Since the equation (2.9) is integrable. This implies that there exist differentiable functions $\mu(x,y,z)$ and $u(x,y,z)$ such that.

$$du=\mu(x,y,z)\left[Pdx+Qdy++Rdz\right],\ \mu\neq 0 \qquad \text{... (2.11)}$$

where $\qquad u=u(x,y,z)$

$$\Rightarrow du=u_x dx+u_y dy+u_z dz\,. \qquad \text{... (2.12)}$$

From equations (2.11) and (2.12) we find

$$\mu P=u_x,\quad \mu Q=u_y,\quad \mu R=u_z$$

$$\Rightarrow u_x i+u_y j+u_z k=\mu\left(Pi+Qj+Rk\right)$$

$$\Rightarrow \nabla u=\mu\overline{X}\cdot$$

Taking the curl of the equation we get

$$curl\left(\nabla\phi\right)=curl\left(\mu\overline{X}\right).$$

Since the identity $curl\left(\nabla\phi\right)=0$

$$\Rightarrow curl\left(\mu\overline{X}\right)=0,$$

$$\Rightarrow \left(\mu\overline{X}\right)curl\left(\mu\overline{X}\right)=0,$$

$$\Rightarrow \overline{X}curl\overline{X}=0\cdot$$

$\Rightarrow$ the equation $\overline{X}d\overline{r}=0$ is integrable if $\overline{X}curl\overline{X}=0\cdot$

Conversely, assume that $\overline{X}curl\overline{X}=0\cdot$

We prove that the Pfaffian differential equation $Pdx+Qdy+Rdz=0$ is integrable.

Let us assume that one of the variables say z is a constant. Hence the Pfaffian differential equation becomes,

$$P(x,y,z)dx + Q(x,y,z)dy = 0 . \qquad \text{... (2.13)}$$

This is a Pfaffian differential equation in two variables, hence it is always integrable. This implies that there exists a function U and the integrating factor μ such that

$$dU = \mu\left(Pdx + Qdy\right),$$

$$\Rightarrow U_x dx + U_y dy = \mu\left(Pdx + Qdy\right),$$

$$\Rightarrow U_x = \mu P, \quad U_y = \mu Q . \qquad \text{... (2.14)}$$

Substituting the values of P and Q in equation (2.9) we get

$$U_x dx + U_y dy + U_z dz + \left(\mu R - U_z\right)dz = 0 .$$

This is equivalent to

$$dU + Kdz = 0, \qquad \text{... (2.15)}$$

where $K = \mu R - U_z .$ ... (2.16)

We are given that $\overline{X} \cdot curl \overline{X} = 0,$

$$\Rightarrow \mu\overline{X} \cdot curl \mu\overline{X} = 0, \qquad \text{... (2.17)}$$

where $\mu\overline{X} = \left(\mu P, \mu Q, \mu R\right),$

$$\mu\overline{X} = \left(\mu P, \mu Q, U_z + K\right), \qquad \text{due to (2.16)}$$

$$\Rightarrow \mu\overline{X} = \left(U_x, U_y, U_z\right) + \left(0,0,K\right),$$

$$\mu\overline{X} = \nabla U + \left(0,0,K\right).$$

Taking the curl of this equation and using the identity, curl of gradU = 0, we readily get

$$curl \mu\overline{X} = \frac{\partial K}{\partial y} i - \frac{\partial K}{\partial x} j ,$$

$$\Rightarrow curl \mu\overline{X} = \left(\frac{\partial K}{\partial y}, -\frac{\partial K}{\partial x}, 0\right). \qquad \text{... (2.18)}$$

Thus $\left(\mu\overline{X}\right)\left(curl\left(\mu\overline{X}\right)\right) = \left(U_x, U_y, U_z + K\right) \cdot \left(\frac{\partial K}{\partial y}, -\frac{\partial K}{\partial x}, 0\right),$

$$= \frac{\partial U}{\partial x}\frac{\partial K}{\partial y} - \frac{\partial U}{\partial y}\frac{\partial K}{\partial x},$$

$$\left(\mu\overline{X}\right)\left(curl\left(\mu\overline{X}\right)\right)=\frac{\partial(U,K)}{\partial(x,y)}.$$

Thus the condition (2.17) implies

$$\frac{\partial(U,K)}{\partial(x,y)}=0, \qquad \text{... (2.19)}$$

$\Rightarrow$ there exists a relation between U and K not involving x and y explicitly. Hence K can be expressed as a function of U and z alone. Therefore equation (2.15) becomes,

$$\frac{dU}{dz}+K(U,z)=0. \qquad \text{... (2.20)}$$

This is a first order ordinary differential equation, it possesses a solution. Let

$$\phi(U,z)=C,$$

where C is an arbitrary constant, be a solution of equation (2.20). On replacing U by its expression in terms of x, y and z we obtain the solution in the form

$$U(x,y,z)=C.$$

Hence the Pfaffian differential equation (2.9) is integrable.

Note : The Pfaffian differential equation (2.9) is in fact, exact if and only if $curl\overline{X}=0$.

Show that the following Pfaffian differential equations are integrable and hence find the corresponding integrals.

Example 1 : $$\left(y^2+yz\right)dx+\left(xz+z^2\right)dy+\left(y^2-xy\right)dz=0. \qquad \text{... (2.21)}$$

Solution : Here $P=y^2+yz, Q=xz+z^2, R=y^2-xy$.

Hence the vector $\overline{X}$ becomes

$$\overline{X}=\left(y^2+yz, xz+z^2, y^2-xy\right),$$

$$\Rightarrow curl\overline{X}=\begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2+yz & xz+z^2 & y^2-xy \end{vmatrix},$$

$$\Rightarrow curl\overline{X}=\left(2(y-x-z),2y,-2y\right),$$

$$\Rightarrow \overline{X}\cdot curl\overline{X}=\left(y^2+yz, xz+z^2, y^2-xy\right)\left(2(y-x-z),2y,-2y\right),$$

$$= 2\left[y(y+z)(y-x-z) + y(xz+z^2) - y^2(y-x) \right],$$

$$= 2\left(y^3 - xy^2 - y^2z + y^2z - xyz - yz^2 + xyz + z^2y - y^3 + xy^2 \right),$$

$$\Rightarrow \overline{X} \cdot curl\,\overline{X} = 0 \cdot$$

This proves that the Pfaffian differential equation (2.21) is integrable. Now to find the integral of (2.21) we assume z = constant $\Rightarrow$ dz = 0. Hence equation (2.21) becomes

$$\left(y^2 + yz\right)dx + \left(xz + z^2\right)dy = 0 . \qquad \text{... (2.22)}$$

We write this equation as

$$\frac{dx}{z(x+z)} + \frac{dy}{y(y+z)} = 0 ,$$

$$\Rightarrow \frac{dx}{x+z} + \frac{zdy}{y(y+z)} = 0 ,$$

$$\Rightarrow \frac{dx}{x+z} + \frac{dy}{y} - \frac{dy}{y+z} = 0 .$$

Since z is a constant. On integrating we get

$$\log(x+z) + \log y - \log(y+z) = \log C_1 ,$$

$$\Rightarrow \frac{y(x+z)}{y+z} = C_1,$$

where C_1 is a constant, may be function of z. Let the integral of (2.22) be denoted by U.

$$\Rightarrow U = \frac{y(x+z)}{y+z} . \qquad \text{... (2.23)}$$

Hence there exist a function μ such that

$$U_x = \mu P \Rightarrow \mu = \frac{1}{P}\left(\frac{y}{y+z} \right),$$

$$\Rightarrow \mu = \frac{1}{(y+z)^2} , \qquad \text{... (2.24)}$$

and U satisfies the equation

$$\frac{dU}{dz} + K = 0 , \qquad \text{... (2.25)}$$

where $$K = \mu R - U_z$$

$$K = \frac{y(y-x)}{(y+z)^2} - \frac{y}{y+z} + \frac{y(x+z)}{(y+z)^2},$$

K = 0.

Hence equation (2.25) becomes

$$\frac{dU}{dz} = 0,$$

$$\Rightarrow dU = 0,$$

$\Rightarrow U$ constant (independent of z),

$$\Rightarrow y(x+z) = C(y+z).$$

This is the required integral of (2.21).

Example 2 : $yzdx + xzdy + xydz = 0$.

Solution : The Pfaffian differential equation is

$$yzdx + xzdy + xydz = 0, \qquad \text{... (2.26)}$$

where the vector $\overline{X} = (P, Q, R) = (yz, xz, xy)$.

We see that

$$curl\overline{X} = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ yz & xz & xy \end{vmatrix} = i(x-x) - j(y-y) + k(z-z) = 0$$

$$\Rightarrow curl\overline{X} = 0,$$

$$\Rightarrow \overline{X} \cdot curl\overline{X} = 0. \qquad \text{... (2.27)}$$

This shows that the equation (2.26) is integrable.

Now to find the integral of (2.26) we treat z = constant

$$\Rightarrow dz = 0.$$

Hence equation (2.26) reduces to

$$yzdx + xzdy = 0$$

$$z(ydx + xdy) = 0 \quad \text{or} \quad d(xy) = 0$$

$$\Rightarrow xy = C_1, \qquad \text{... (2.28)}$$

where C_1 may be function of z.

Let $$U = xy. \qquad \text{... (2.29)}$$

There must exist a function μ such that $\frac{\partial U}{\partial x} = \mu P$

$$\Rightarrow \mu = \frac{1}{P}\frac{\partial U}{\partial x} = \frac{1}{yz}(y) \Rightarrow \mu = \frac{1}{z} \cdot \qquad \text{... (2.30)}$$

The function U in (2.29) therefore satisfies the equation

$$\frac{dU}{dz} + K = 0, \qquad \text{... (2.31)}$$

where $$K = \mu R - \frac{\partial U}{\partial z} \cdot$$

Thus $$K = \frac{1}{z}xy = 0 \Rightarrow K = \frac{xy}{z} \cdot$$

$$\Rightarrow K = \frac{U}{z} \qquad \text{As } U = xy \qquad \text{... (2.32)}$$

Hence equation (2.31) becomes

$$\frac{dU}{dz} + \frac{U}{z} = 0,$$

$$\Rightarrow \frac{dU}{U} + \frac{dz}{z} = 0.$$

Integrating we get

$$\log U + \log z = \log C \quad \text{or} \quad Uz = C,$$

$$\Rightarrow xyz = C, \qquad \text{... (2.33)}$$

which is the required integral.

Example 3 : $yzdx + \left(x^2 y - zx\right)dy + \left(x^2 z - xy\right)dz = 0$

Solution : The Pfaffian differential equation is given by

$$yzdx + \left(x^2 y - zx\right)dy + \left(x^2 z - xy\right)dz = 0 \qquad \text{... (2.34)}$$

where the vector

$$\overline{X} = (P, Q, R) = \left(yz, x^2 y - zx, x^2 z - xy\right).$$

Therefore, we find

$$curl\,\overline{X} = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ yz & x^2y - zx & x^2z - xy \end{vmatrix} = i(-x+x) - j(2xz - y - y) + k(2xy - z - z)$$

$$\Rightarrow curl\,\overline{X} = (0, -(2xz - 2y), 2xy - 2z).$$

Therefore, we see that

$$\overline{X}\cdot curl\,\overline{X} = \left(0 + \left(x^2y - zx\right)(2y - 2xz) + (2xy - 2z)\left(x^2z - xy\right)\right)$$

$$\Rightarrow \overline{X}\cdot curl\,\overline{X} = 2\left[x^2y^2 - xyz - x^3yz + x^2z^2 + x^3yz - x^2y^2 - x^2z^2 + xyz\right]$$

$$\overline{X}\cdot curl\,\overline{X} = 0. \qquad \text{... (2.35)}$$

This shows that the equation (2.34) is integrable. Now to find the integral of (2.34) we treat

$$z = \text{constant} \Rightarrow dz = 0.$$

Hence equation (2.34) reduces to

$$yzdx + \left(x^2y - zx\right)dy = 0,$$

$$\Rightarrow yzdx - zxdy + x^2ydy = 0,$$

or $$z(ydx - xdy) = -x^2ydy,$$

or $$z\left(\frac{ydx - xdy}{x^2}\right) = -ydy,$$

or $$z\left(\frac{xdy - ydx}{x^2}\right) = ydy \Rightarrow zd\left(\frac{y}{x}\right) = ydy.$$

Integrating we get

$$z\left(\frac{y}{x}\right) = \frac{y^2}{2} + C_1,$$

or $$\frac{zy}{x} - \frac{y^2}{2} = C_1,$$

$$\Rightarrow \frac{y(2z - xy)}{2x} = C_1,$$

Let $$U = \frac{y(2z - xy)}{2x}. \qquad \text{... (2.36)}$$

There must exist a function μ such that $\frac{\partial U}{\partial x} = \mu P$

or $$\mu = \frac{1}{P}\frac{\partial U}{\partial x} \Rightarrow \mu = \frac{1}{yz}\left(-\frac{yz}{x^2}\right),$$

$$\Rightarrow \mu = -\frac{1}{x^2}. \qquad \text{... (2.37)}$$

Also the function U in (2.36) satisfies the equation

$$\frac{dU}{dz} + K = 0, \qquad \text{... (2.38)}$$

where

$$K = \mu R - \frac{\partial U}{\partial z} \Rightarrow K = -\frac{1}{x^2}\left(x^2 z - xy\right) - \left(\frac{y}{x}\right),$$

$$\Rightarrow K = -z + \frac{y}{x} - \frac{y}{x},$$

$$\Rightarrow K = -z. \qquad \text{... (2.39)}$$

Hence equation (2.38) becomes

$$\frac{dU}{dz} - z = 0 \quad \text{or} \quad dU - zdz = 0.$$

Integrating we get

$$U - \frac{z^2}{2} = C.$$

i.e. $$\frac{yz}{x} - \frac{y^2}{2} - \frac{z^2}{2} = C,$$

or $$\frac{yz}{x} - \frac{1}{2}\left(y^2 + z^2\right) = C,$$

or $$2yz - x\left(y^2 + z^2\right) = 2xC, \qquad \text{... (2.40)}$$

which is the required integral.

Example 4 : $(6x+yz)dx+(xz-2y)dy+(xy+2z)dz=0$

Solution : The Pfaffian differential equation is given by

$$(6x+yz)dx+(xz-2y)dy+(xy+2z)dz=0,$$

... (2.41)

where the vector

$$\overline{X}=(6x+yz,xz-2y,xy+2z).$$

We find

$$Curl\,\overline{X}=\begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 6x+yz & xz-2y & xy+2z \end{vmatrix}=i(x-x)-j(y-y)+k(z-z)$$

$$Curl\,\overline{X}=0\Rightarrow\overline{X}\cdot curl\,\overline{X}=0 \quad \text{... (2.42)}$$

The equation (2.41) is integrable. To find the integral of (2.42) we treat

$$z=\text{constant}\Rightarrow dz=0\cdot$$

Hence equation (2.41) becomes

$$(6x+yz)dx+(xz-2y)dy=0$$

$$\Rightarrow 6xdx+yzdx+xzdy-2ydy=0$$

$$\Rightarrow 6xdx+z(ydx+xdy)-2ydy=0$$

or $$6xdx+zd(xy)-2ydy=0.$$

Integrating we get

$$3x^2+zxy-y^2=C_1, \quad \text{... (2.43)}$$

where C_1 may involved z.

Let $$U=3x^2+xyz-y^2.$$

There must exist a function μ such that

$$\frac{\partial U}{\partial x}=\mu P$$

$$\Rightarrow \mu=\frac{1}{P}\frac{\partial U}{\partial x}$$

$$=\frac{1}{6x+yz}(6x+yz)\Rightarrow\mu=1.$$

Also the function U in (2.44) must satisfy the equation

$$\frac{dU}{dz} + K = 0, \qquad \text{... (2.45)}$$

where

$$K = \mu R - \frac{\partial U}{\partial z}$$

$$= (xy + 2z) - xy$$

$$\Rightarrow K = 2z \cdot$$

Hence equation (2.45) becomes

$$\frac{dU}{dz} + 2z = 0 \quad \text{or} \quad dU + 2zdz = 0 \cdot$$

Integarting we get

$$U + z^2 = C_2,$$

ie. $$3x^2 + xyz - y^2 + z^2 = C_2, \qquad \text{... (2.46)}$$

which is the required integral of (2.41).

Example 5 : $\left(x^2 z - y^3\right)dx + 3xy^2 dy + x^3 dz = 0$.

Solution : To test the integrability of the equation (2.47) we note that

$$\overline{X} = \left(x^2 z - y^3, 3xy^2, x^3\right).$$

So that

$$curl\overline{X} = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x^2 z - y^3 & 3xy^2 & x^3 \end{vmatrix} = i(0-0) - j\left(3x^2 - x^2\right) + k\left(3y^2 + 3y^2\right)$$

$$Curl\overline{X} = \left(0, -2x^2, 6y^2\right)$$

Therefore, $$\overline{X} \cdot Curl\overline{X} = \left(0 - 6x^3 y^2 + 6x^3 y^2\right) = 0$$

$$\Rightarrow \overline{X} \cdot Curl\overline{X} = 0 \qquad \text{... (2.48)}$$

$\Rightarrow$ The equation (2.47) is integrable. Now to find the integral of (2.47) we treat

$$z = \text{constant} \Rightarrow dz = 0 \cdot$$

Hence equation (2.47) reduces to

$$\left(x^2 z - y^3\right)dx + 3xy^2 dy = 0 .$$

We write this equation as

$$zx^2 dx - y^3 dx + 3xy^2 dy = 0$$

$$\Rightarrow 3x^2 dx + 3xy^2 dy - y^3 dx = 0$$

$$\Rightarrow zdx + \frac{3xy^2 dy - y^3 dx}{x^2} = 0$$

$$\Rightarrow zdx + d\left(\frac{y^3}{x}\right) = 0 . \qquad z = \text{constant}$$

Integrating we get

$$zx + \frac{y^3}{x} = C_1 . \qquad \text{... (2.49)}$$

Let $$U = zx + \frac{y^3}{x} . \qquad \text{... (2.50)}$$

There must exist a function μ such that

$$\frac{\partial U}{\partial x} = \mu P \quad \text{or} \quad \mu = \frac{1}{P}\frac{\partial U}{\partial x}$$

$$\Rightarrow \mu = \frac{1}{\left(x^2 z - y^3\right)}\left(\frac{z - y^3}{x^2}\right) \Rightarrow \mu = \frac{1}{x^2} . \qquad \text{... (2.51)}$$

The function U in (2.50) also satisfies the equation

$$\frac{dU}{dz} + K = 0 , \qquad \text{... (2.52)}$$

where $$K = \mu R - \frac{\partial U}{\partial z} \Rightarrow K = \frac{1}{x^2}x^3 - x \Rightarrow K = 0$$

Therefore equation (2.52) becomes

$$\Rightarrow dU = 0 \Rightarrow U = C$$

i.e. $$x^2 z + y^3 = Cx , \qquad \text{... (2.53)}$$

which is required integral.

Example 6 : $(1+yz)dx + x(z-x)dy - (1+xy)dz = 0$.

Solution : The Pfaffian differential equation is given by

$$(1+yz)dx + x(z-x)dy - (1+xy)dz = 0, \qquad \text{... (2.54)}$$

where the vector

$$\overline{X} = (1+yz, x(z-x), -(1+xy)).$$

$$\Rightarrow curl\overline{X} = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 1+yz & x(z-x) & -1-xy \end{vmatrix} = i(-x-x) - j(-y-y) + k(z-2x-z)$$

$$\Rightarrow curl\overline{X} = (-2x, 2y+z, -2x).$$

We seet that

$$\overline{X}\cdot curl\overline{X} = -2x(1+yz) + 2y\left(xz - x^2\right) + 2x(1+xy)$$

$$= -2x - 2xyz + 2xyz - 2x^2y + 2x + 2x^2y$$

$$\overline{X}\cdot curl\overline{X} = 0 \qquad \text{... (2.55)}$$

$\Rightarrow$ the equation (2.54) is integrable. Now to find the integral of (2.54) we treat

$$z = \text{constant} \Rightarrow dz = 0\cdot$$

Therefore equation (2.54) reduces to

$$(1+yz)dx = x(z-x)dy = 0.$$

We write this as

$$\frac{dx}{x(z-x)} + \frac{dy}{1+yz} = 0,$$

or

$$\frac{zdx}{x(z-x)} + \frac{3dy}{1+yz} = 0,$$

$$\frac{dx}{x} + \frac{dx}{z-x} + \frac{dy}{y+1/z} = 0. \qquad \text{z = constant}$$

Integrating we get

$$\log x - \log(z-x) + \log\left(y + 1/z\right) = \log C_1,$$

$$\frac{x\left(y+\frac{1}{z}\right)}{(z-x)}=C_1$$

or $$\frac{x(yz+1)}{z(z-x)}=C_1.$$

Let $$U=\frac{x(yz+1)}{z(z-x)}. \qquad \dots (2.56)$$

Therefore, there must exist a function μ such that

$$\frac{\partial U}{\partial x}=\mu P$$

or $$\mu=\frac{1}{P}\frac{\partial U}{\partial x}=\frac{1}{z(1+yz)}\left[\frac{(yz+1)(z-x)+x(yz+1)}{(z-x)^2}\right]$$

$$=\frac{1}{z}\frac{(z-x+x)}{(z-x)^2}\Rightarrow \mu=\frac{1}{(z-x)^2}. \qquad \dots (2.57)$$

The function U in (2.56) therefore satisfies the equation

$$\frac{dU}{dz}+K=0, \qquad \dots (2.58)$$

where

$$K=\mu R-\frac{\partial U}{\partial z}$$

$$K=\frac{1}{(z-x)^2}\left[-(1+xy)\right]-\left[\frac{z(z-x)xy-x(yz+1)(2z-x)}{z^2(z-x)^2}\right]$$

$$K=\frac{1}{z^2(z-x)^2}\left[-z^2(1+xy)-\left(z^2xy-zx^2y-2xyz^2-2zx+x^2yz+x^2\right)\right]$$

$$=\frac{-\left(z^2-2xz+x^2\right)}{z^2(z-x)^2}=\frac{-(z-x)^2}{z^2(z-x)^2}$$

$$\Rightarrow K=-\frac{1}{z^2}. \qquad \dots (2.59)$$

Hence equation (2.58) becomes

$$\frac{dU}{dz} - \frac{1}{z^2} = 0 \text{ or } dU - \frac{dz}{z^2} = 0.$$

Integrating we get

$$U + \frac{1}{z} = C_2$$

or
$$\frac{x(yz+1)}{z(z-x)} + \frac{1}{z} = C_2$$

$$\Rightarrow (1+xy) = C_2(z-x).$$

Example 7 : $\left(2x + y^2 + 2xz\right)dx + 2xydy + x^2dz = 0$.

Solution : The Pfaffian differential equation is given by

$$\left(2x + y^2 + 2xz\right)dx + 2xydy + x^2dz = 0, \qquad \text{... (2.60)}$$

where the vector

$$\overline{X} = \left(2x + y^2 + 2xz, 2xy, x^2\right)$$

$$\Rightarrow curl\overline{X} = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 2x + y^2 + 2xz & 2xy & x^2 \end{vmatrix}$$

$$curl\overline{X} = i(0-0) - j(2x-2x) + k(2y-2y)$$

$$curl\overline{X} = 0 \Rightarrow \overline{X} \cdot curl\overline{X} = 0 \qquad \text{... (2.61)}$$

$\Rightarrow$ the equation (2.60) is integrable. To find the integral of (2.60) we treat

$$x = \text{constant} \Rightarrow dx = 0 \cdot$$

Thus equation (2.60) reduces to

$$2xydy + x^2dz = 0 \cdot$$

Integrating we get

$$2x\frac{y^2}{2} + x^2z = C_1$$

or
$$xy^2 + x^2z = C_1.$$

Let $$U = xy^2 + x^2 z.$$ ... (2.62)

There must exist a function μ such that

$$\mu Q = \frac{\partial U}{\partial y}, \quad \mu R = \frac{\partial U}{\partial z}$$

$$\Rightarrow \mu = \frac{1}{Q}\frac{\partial U}{\partial y} = \frac{1}{2xy} 2xy \Rightarrow \mu = 1.$$ ... (2.63)

Also the function U satisfies the equation

$$\frac{dU}{dx} + K = 0,$$ ... (2.64)

where $$K = \mu P - \frac{\partial U}{\partial x}$$

$$K = 2x + y^2 + 2xz - \left(y^2 + 2xz\right)$$

$$\Rightarrow K = 2x.$$ ... (2.65)

Hence equation (2.65) becomes

$$\frac{dU}{dx} + 2x = 0.$$

Integrating we get

$$U + x^2 = C_2$$

or $$xy^2 + x^2 z + x^2 = C_2$$ ... (2.66)

which is the solution of equation (2.60).

Example 8 : $$(1 + yz)dx + z(z - x)dy - (1 + xy)dz = 0$$ ... (2.67)

Solution : Here $$\overline{X} = \left(1 + yz, z(z - x), -(1 + xy)\right)$$

$$\Rightarrow curl\overline{X} = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 1 + yz & z(z - x) & -(1 + xy) \end{vmatrix}$$

$$curl\overline{X} = i\left(-\cancel{x} - 2z + \cancel{x}\right) - j\left(-y - y\right) + k\left(-z - z\right)$$

$$curl\overline{X} = -2zi + 2yj - 2zk$$

$$curl\,\overline{X} = (-2z, 2y, -2z)$$

$$\Rightarrow \overline{X}\cdot curl\,\overline{X} = \left(1+yz, z^2 - xz, -1-xy\right)(-2z, 2y, -2z)$$

$$\overline{X}\cdot curl\,\overline{X} = 0 \cdot \qquad \text{... (2.68)}$$

$\Rightarrow$ the given equation (2.67) is integrable. Consider,

$$x = \text{constant} \Rightarrow dx = 0 \cdot$$

Therefore, given equation becomes

$$z(z-x)dy - (1+xy)dz = 0,$$

$$\Rightarrow \frac{dy}{1+xy} - \frac{dz}{z(z-x)} = 0,$$

$$\Rightarrow \frac{xdy}{1+xy} - \frac{xdz}{z(z-x)} = 0,$$

$$\Rightarrow \frac{dy}{\left(y+\frac{1}{x}\right)} + \frac{dz}{z} - \frac{dz}{z-x} = 0. \qquad \text{x = constant}$$

Integrating we get

$$\log\left(y+\frac{1}{x}\right) + \log z - \log(z-x) = \log C_1,$$

$$\Rightarrow \frac{z\left(y+\frac{1}{x}\right)}{(z-x)} = C_1,$$

or $$\frac{z(yx+1)}{x(z-x)} = C_1.$$

Let $$U = \frac{z(yx+1)}{x(z-x)} \cdot \qquad \text{... (2.69)}$$

There must exist a function μ such that

$$\mu Q = \frac{\partial U}{\partial y} \quad \text{or} \quad \mu R = \frac{\partial U}{\partial z},$$

$$\Rightarrow \mu = \frac{1}{Q}\frac{\partial U}{\partial y} = \frac{1}{z(z-x)}\left[\frac{xz}{x(z-x)}\right],$$

$$\Rightarrow \mu = \frac{xz}{xz(z-x)^2} \Rightarrow \mu = \frac{1}{(z-x)^2}. \qquad \text{... (2.70)}$$

The function U satisfies the equation

$$\frac{dU}{dx} + K = 0, \qquad \text{... (2.71)}$$

where

$$K = \mu P - \frac{\partial U}{\partial x}$$

$$\Rightarrow K = \frac{1}{(z-x)^2}[1+yz] - \left[\frac{x(z-x)yz - z(xy+1)(z-2x)}{x^2(z-x)^2}\right]$$

On simplifying we get

$$K = \frac{1}{x^2(z-x)^2}(z-x)^2 \Rightarrow K = \frac{1}{x^2}. \qquad \text{... (2.72)}$$

$$\frac{dU}{dx} + \frac{1}{x^2} = 0 \text{ or } dU + \frac{dx}{x^2} = 0.$$

Integrating we get

$$U - \frac{1}{x} = C_2$$

$$\Rightarrow \frac{z(xy+1)}{x(z-x)} - \frac{1}{x} = C_2$$

$$\Rightarrow z(xy+1) - (z-x) = C_2 x(z-x)$$

$$\Rightarrow z(1+xy) = (z-x)(1+C_2 x). \qquad \text{... (2.73)}$$

Exercise :

Show that the following Pfaffian differential equations are integrable and hence find the corresponding integrals.

1. $z(z-y)dx + z(x+z)dy + x(x+y)dz = 0$

2. $yzdx + 2xzdy - 3xydz = 0$

3. $ydx + xdy + 2zdz = 0$

4. $(yz + xyz)dx + (zx + xyz)dy + (xy + xyz)dz = 0$

Answers :

1. $z(x+y) = C(x+z)$

2. $xy^2 = Cz^2$

3. $xy + z^2 = C$

UNIT - III

COMPATIBLE SYSTEMS OF FIRST ORDER PARTIAL DIFFERENTIAL EQUATIONS

Introduction :

In this unit we introduce a system of first order partial differential equations and find the conditions that the system has common solution. As discussed in the Unit 2, the method of finding the general integral of Lagrange's equation, in this unit we introduce methods due to Chanpits and Jacobi to find the complete integral of the first order partial differential equations.

Definition : Two first order partial differential equations

$$f(x, y, z, p, q) = 0 \qquad \text{... (1.1)}$$

and

$$g(x, y, z, p, q) = 0 \qquad \text{... (1.2)}$$

are said to be compatible (they have a common solution) on a domain D, if and only if

$$J = \frac{\partial(f, g)}{\partial(p, q)} \neq 0 \quad \text{on D} \qquad \text{... (1.3)}$$

and the equation

$$dz = \phi(x, y, z)\, da + \psi(x, y, z)\, dy \qquad \text{... (1.4)}$$

is integrable, where $p = \phi(x, y, z)$ and $q = \psi(x, y, z)$ are obtained by solving (1.1) and (1.2).

Theorem : A necessary and sufficient condition for the two partial differential equations

$$f(x, y, z, p, q) = 0 \text{ and } g(x, y, z, p, q) = 0$$

to be compatible is that

$$[f, g] = \frac{\partial(f, g)}{\partial(x, p)} + p\frac{\partial(f, g)}{\partial(z, p)} + \frac{\partial(f, g)}{\partial(y, q)} + q\frac{\partial(f, g)}{\partial(z, q)} = 0 .$$

Proof : Consider two first order partial differential equations

$$f(x, y, z, p, q) = 0 \qquad \text{... (1.5)}$$

and

$$g(x, y, z, p, q) = 0 . \qquad \text{... (1.6)}$$

By definition, equations (1.5) and (1.6) are said to compatible iff

$$J = \frac{\partial(f, g)}{\partial(p, q)} \neq 0 \qquad \text{... (1.7)}$$

and $\quad p=\phi(x,y,z)$ and $q=\psi(x,y,z)$

obtainable from (1.5) and (1.6) render the equation

$$dz=\phi(x,y,z)\,dx+\psi(x,y,z)\,dy \qquad \text{... (1.8)}$$

integrable. We write equation (1.8) as

$$\phi(x,y,z)\,dx+\psi(x,y,z)\,dy-dz=0 \qquad \text{... (1.9)}$$

We know the condition that the equation (1.9) is integrable iff

$$\overline{X}\cdot curl\overline{X}=0,$$

where

$$\overline{X}=(\phi,\psi,-1)$$

$$\Rightarrow curl\overline{X}=\begin{vmatrix} i & j & k \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ \phi & \psi & -1 \end{vmatrix}$$

$$\Rightarrow curl\overline{X}=i(-\psi_z)+\phi_z j+k(\psi_x-\phi_y)$$

$$\Rightarrow curl\overline{X}=(-\psi_z,\phi_z,\psi_x-\phi_y).$$

Thus the condition $\overline{X}\cdot curl\overline{X}=0$ becomes

$$(\phi,\psi,-1)(-\psi_z,\phi_z,\psi_x-\phi_y)=0$$

$$\Rightarrow \psi_x+\phi\psi_z=\phi_y+\psi\phi_z. \qquad \text{... (1.10)}$$

Substituting ϕ and ψ for p and q respectively in equation (1.5) we get

$$f(x,y,z,\phi,\psi)=0 \qquad \text{... (1.11)}$$

Differentiating equation (1.11) with respect to x and z we get

$$f_x+f_p\phi_x+f_q\psi_x=0 \qquad \text{... (1.12)}$$

and

$$f_z+f_p\phi_z+f_q\psi_z=0. \qquad \text{... (1.13)}$$

Multiplying equation (1.13) by ϕ and adding it to the equation (1.12) we get

$$f_x+\phi f_z+f_p(\phi_x+\phi\phi_z)+f_q(\psi_x+\phi\psi_z)=0. \qquad \text{... (1.14)}$$

Similarly, from equation (1.6) we obtain

$$g_x + \phi g_z + g_p(\phi_x + \phi\phi_z) + g_q(\psi_x + \phi\psi_z) = 0. \qquad \text{... (1.15)}$$

Multiplying equation (1.14) by g_p and (1.15) by f_p and subtracting we get

$$f_x g_p - f_p g_x + \phi(f_z g_p - f_p g_z) + (\psi_x + \phi\psi_z)(f_q g_p - f_p g_q) = 0$$

$$\Rightarrow (f_p g_q - f_q g_p)(\psi_x + \phi\psi_z) = (f_x g_p - f_p g_x) + \phi(f_z g_p - f_p g_z)$$

$$\Rightarrow \frac{\partial(f,g)}{\partial(p,q)}(\psi_x + \phi\psi_z) = \frac{\partial(f,g)}{\partial(x,p)} + \phi\frac{\partial(f,g)}{\partial(z,p)}$$

or $$\psi_x + \phi\psi_z = \frac{1}{J}\left(\frac{\partial(f,g)}{\partial(x,p)} + \phi\frac{\partial(f,g)}{\partial(z,p)}\right), \qquad \text{... (1.16)}$$

where $$J = \frac{\partial(f,g)}{\partial(p,q)} \neq 0.$$

Similarly, differentiating equation (1.11) with respect to y and z we obtain, after similar analysis, the equation

$$\phi_y + \psi\phi_z = \frac{-1}{J}\left(\frac{\partial(f,g)}{\partial(y,q)} + \psi\frac{\partial(f,g)}{\partial(z,q)}\right). \qquad \text{... (1.17)}$$

Now substituting equations (1.16) and (1.17) in the equation (1.10) we obtain

$$\frac{\partial(f,g)}{\partial(x,p)} + \phi\frac{\partial(f,g)}{\partial(z,p)} = -\frac{\partial(f,g)}{\partial(y,q)} - \psi\frac{\partial(f,g)}{\partial(z,q)}.$$

Replacing ϕ and ψ by p and q respectively, we get

$$\frac{\partial(f,g)}{\partial(x,p)} + p\frac{\partial(f,g)}{\partial(z,p)} + \frac{\partial(f,g)}{\partial(y,q)} + q\frac{\partial(f,g)}{\partial(z,q)} = 0. \qquad \text{... (1.18)}$$

This is the desired compatibility condition. This condition can also be written as

$$[f,g] = 0.$$

Example 1 : Show that the equations

$$f = p^2 + q^2 - 1 = 0 \text{ and } g = (p^2 + q^2)x - pz = 0$$

are compatible and find the one parameter family of common solutions.

Solution : Let the partial differential equations be given by

$$f(x,y,z,p,q) = p^2 + q^2 - 1 = 0 \qquad \text{... (1.19)}$$

$$g(x,y,z,p,q)=\left(p^2+q^2\right)x-pz=0. \qquad \text{... (1.20)}$$

We know the condition that the equations (1.19) and (1.20) are compatible iff

$$[f,g]=0$$

i.e.

$$\frac{\partial(f,g)}{\partial(x,p)}+p\frac{\partial(f,g)}{\partial(z,p)}+\frac{\partial(f,g)}{\partial(y,q)}+q\frac{\partial(f,g)}{\partial(z,q)}=0, \qquad \text{... (1.21)}$$

where from equations (1.19) and (1.20) we readily obtain

$$pz=x \Rightarrow p=\frac{x}{z}, \qquad \text{... (1.22)}$$

and

$$q^2=1-p^2 \Rightarrow q^2=1-\left(\frac{x}{z}\right)^2 \text{ or } q=\frac{1}{z}\sqrt{z^2-x^2}. \qquad \text{... (1.23)}$$

We find from equations (1.19) and (1.20) that

$$f_x=0,\ f_p=2p,\ g_x=p^2+q^2,\ g_p=2px-z$$

Therefore,

$$\frac{\partial(f,g)}{\partial(x,p)}=f_x g_p-f_p g_x=-2p\left(p^2+q^2\right)\Rightarrow\frac{\partial(f,g)}{\partial(x,p)}=-2\frac{x}{z}=-2p.$$

Similarly,

$$f_z=0,\ g_z=-p$$

Therefore,

$$\frac{\partial(f,g)}{\partial(z,p)}=f_z g_p-f_p g_z=+2p^2\Rightarrow\frac{\partial(f,g)}{\partial(z,p)}=2p^2=2\frac{x^2}{z^2}.$$

Similarly,

$$f_y=0,\ f_q=2q,\ g_y=0,\ g_q=2qx$$

Therefore,

$$\frac{\partial(f,g)}{\partial(y,q)}=f_y g_q-f_q g_y=0.$$

Next we find

$$\frac{\partial(f,g)}{\partial(z,q)}=f_z g_q-f_q g_z=-2q(-p)=2pq$$

$$\Rightarrow\frac{\partial(f,g)}{\partial(z,q)}=2pq=\frac{2x}{z^2}\sqrt{z^2-x^2}.$$

Substituting these in equation (1.21) we get

$$[f,g] = -2p + p\left(2p^2\right) + 0 + q(2pq)$$

$$= -2p + 2p\left(p^2 + q^2\right)$$

$$= -2p + 2p \qquad \text{As } p^2 + q^2 = 1 \text{ by ... (1.19)}$$

$$= 0$$

$$\Rightarrow [f,g] = 0. \qquad \text{... (1.24)}$$

This shows that the equations (1.19) and (1.20) are compatible.

Now to solve these equations we have

$$dz = pdx + qdy$$

$$= \frac{x}{z}dx + \frac{1}{z}\sqrt{z^2 - x^2}\,dy$$

$$\Rightarrow zdz - xdx = \sqrt{z^2 - x^2}\,dy$$

$$\Rightarrow \frac{zdz - xdx}{\sqrt{z^2 - x^2}} = dy \Rightarrow d\left(\sqrt{z^2 - x^2}\right) = dy$$

Integrating we get

$$\sqrt{z^2 - x^2} = y + c$$

or $$z^2 - x^2 = (y+c)^2$$

or $$z^2 = x^2 + (y+c)^2, \qquad \text{... (1.25)}$$

which is a required one parameter family of common solution.

Example 2 : Show that the equations

$$xp = yq, \quad z(xp + yq) = 2xy$$

are compatible and find a one parameter family of common solution.

Solution : Let the partial differential equation be given by

$$f(x,y,z,p,q) = xp - qy = 0, \qquad \text{... (1.26)}$$

$$g(x,y,z,p,q) = z(xp + yq) - 2xy = 0. \qquad \text{... (1.27)}$$

We know the condition that the equations (1.26) and (1.27) are compatible iff

$$[f,g] = 0,$$

where

$$[f,g] = \frac{\partial(f,g)}{\partial(x,p)} + p\frac{\partial(f,g)}{\partial(z,p)} + \frac{\partial(f,g)}{\partial(y,q)} + q\frac{\partial(f,g)}{\partial(z,q)}. \qquad \text{... (1.28)}$$

From equations (1.26) and (1.27) we find

$$f_x = p, f_p = x, g_x = zp - 2y, g_p = zx$$

Therefore, $\dfrac{\partial(f,g)}{\partial(x,p)} = f_x g_p - f_p g_x = 2xy$.

Now we find

$$f_z = 0, g_z = xp + yq$$

Thus, $\dfrac{\partial(f,g)}{\partial(z,p)} = f_z g_p - f_p g_z = -x(xp+yq) \Rightarrow \dfrac{\partial(f,g)}{\partial(z,p)} = -x^2 p - xyq$.

Similarly,

$$f_y = -q, f_q = -y, g_y = zq - 2x, g_q = zy$$

Hence, $\dfrac{\partial(f,g)}{\partial(y,q)} = f_y g_q - f_q g_y = -2yx$,

and

$$\frac{\partial(f,g)}{\partial(z,q)} = f_z g_q - f_q g_z = y(xp+yq) \Rightarrow \frac{\partial(f,g)}{\partial(z,q)} = xyp + y^2 q.$$

Substituting these values in equation (1.28) we get

$$[f,g] = -\left(p^2x^2 - y^2q^2\right) = 0$$

$$\Rightarrow [f,g] = 0$$

$\Rightarrow$ Equations (1.26) and (1.27) are compatible. Now to solve these equations, we find from (1.26) and (1.27) that

$$p = \frac{y}{z}, \quad q = \frac{x}{z}.$$

Substituting this in $dz = pdx + qdy$ we get

$$dz = \frac{y}{z}dx + \frac{x}{z}dy$$

$$\Rightarrow zdz = ydx + xdy$$

$$zdz = d(xy).$$

Integrating we get

$$\frac{z^2}{2} = xy + C_1$$

or $$z^2 = 2xy + C.$$

Which is the required one parameter family of common solution.

Example 3 : Show that the equation z = px + qy is compatible with any equation $f(x, y, z, p, q) = 0$ that is homogeneous in x, y and z.

Solution : Since *f* is a homogeneous in *x, y, z,* therefore it can be written as

$$f = z^n \phi\left(\frac{x}{z}, \frac{y}{z}, p, q\right). \quad \text{... (1.29)}$$

Put $$u = \frac{x}{z}, v = \frac{y}{z}.$$

Therefore, the given equations reduce to

$$g(x, y, z, p, q) = px + qy - z = 0 \quad \text{... (1.30)}$$

and $$h(x, y, z, p, q) = \phi(u, v, p, q) = 0. \quad \text{... (1.31)}$$

We know equations (1.30) and (1.31) are compatible iff

$$[g, h] = 0,$$

where we know

$$[g, h] = \frac{\partial(g,h)}{\partial(x,p)} + p\frac{\partial(g,h)}{\partial(z,p)} + \frac{\partial(g,h)}{\partial(y,q)} + q\frac{\partial(g,h)}{\partial(z,q)}. \quad \text{... (1.32)}$$

Therefore, we have

$$\frac{\partial(g,h)}{\partial(x,p)} = g_x h_p - g_p h_x$$

$$= p\phi_p - x\phi_x$$

$$= p\phi_p - x\phi_u u_x$$

$$\frac{\partial(g,h)}{\partial(x,p)} = p\phi_p - \frac{x}{z}\phi_u.$$

Similarly,

$$\frac{\partial(g,h)}{\partial(z,p)} = g_z h_p - g_p q_z$$

$$= -\phi_p - x\phi_z$$

$$\Rightarrow \frac{\partial(g,h)}{\partial(z,p)} = -\phi_p - x\phi_z$$

$$\frac{\partial(g,h)}{\partial(y,q)} = g_y h_q - g_q h_y$$

$$= q\phi_q - y\phi_v v_y$$

$$\frac{\partial(g,h)}{\partial(y,q)} = q\phi_q - \frac{y}{z}\phi_v ,$$

and

$$\frac{\partial(g,h)}{\partial(z,q)} = g_z h_q - g_q h_z$$

$$\frac{\partial(g,h)}{\partial(z,q)} = -\phi_q - y\phi_z .$$

Substituting these in [*g, h*] we get

$$[g,h] = \cancel{p\phi_p} - \frac{x}{z}\phi_u - \cancel{p\phi_p} - px\phi_z + \cancel{q\phi_q} - \frac{y}{z}\phi_v - \cancel{q\phi_q} - q_y\phi_z$$

$$= -u\phi_u - v\phi_v - (px + qy)\phi_z$$

$$= -u\phi_u - v\phi_v - z(\phi_u \cdot u_z + \phi_v v_z) \qquad \text{by ... (1.30)}$$

$$= -u\phi_u - v\phi_v - z\left(-\phi_u \frac{x}{z^2} - \phi_v \frac{y}{z^2}\right)$$

$$= -u\phi_u - v\phi_v + \frac{x}{z}\phi_u + \frac{y}{z}\phi_v$$

$$= -u\phi_u - v\phi_v + u\phi_u + v\phi_v$$

$$[g,h] = 0$$

$\Rightarrow$ the given equations are compatible.

Exercise :

1. Show that the following first order partial differential equations are compatible and find a one-parameter family of common solution.

$$xp - yq = x,$$

$$x^2 p + q = xz.$$

2. Show that the equations $f(x, y, p, q) = 0$ and $g(x, y, p, q) = 0$ are compatible if

$$\frac{\partial(f,g)}{\partial(x,p)} + \frac{\partial(f,g)}{\partial(y,q)} = 0.$$

2. Charpit's Method :

In this section, we present a method of finding complete integral of a first order p.d.e. $f(x, y, z, p, q) = 0$ due to Charpit. Method is based on the concept of the last section viz. the concept of compatibility.

Definition : Let a first order p.d.e. be given by

$$f(x, y, z, p, q) = 0. \qquad \text{... (2.1)}$$

A one-parameter family of p.d. equations given by

$$g(x, y, z, p, q, a) = 0, \text{ a is a parameter}, \qquad \text{... (2.2)}$$

is said to be compatible with (2.1) if (2.2) is compatible with (2.1) for each value of a.

Result : Describe Charpit's Method of solving a first order partial differential equation

$$f(x, y, z, p, q) = 0.$$

Proof : Let the first order p.d.e. whose complete integral is to be determined be given by

$$f(x, y, z, p, q) = 0. \qquad \text{... (2.3)}$$

The fundamental idea in Charpit's method is the introduction of a second partial differential equation of the first order

$$g(x, y, z, p, q, a) = 0 \qquad \text{... (2.4)}$$

which contain an arbitrary constant 'a' and which is such that

(i) equations (2.3) and (2.4) can be solved for $p = p(x, y, z)$ and $q = q(x, y, z)$ and

(ii) the equation $dz = p(x, y, z)dx + q(x, y, z)dy$ is integrable.

i.e. we need only to seek an equation

$$g(x,y,z,p,q,a)=0$$

Compatible with the given equation

$$f(x,y,z,p,q)=0$$

We know equations (2.3) and (2.4) are compatible iff

$$[f,g]=0$$

i.e. $$\frac{\partial(f,g)}{\partial(x,p)}+p\frac{\partial(f,g)}{\partial(z,p)}+\frac{\partial(f,g)}{\partial(y,q)}+q\frac{\partial(f,g)}{\partial(z,q)}=0. \qquad \text{... (2.5)}$$

$$\Rightarrow (f_x g_p - f_p g_x)+p(f_z g_p - f_p g_z)+(f_y g_q - f_q g_y)+q(f_z g_q - f_q g_z)=0.$$

We write this as

$$-f_p\frac{\partial g}{\partial x}-f_q\frac{\partial g}{\partial y}-(pf_p+qf_q)\frac{\partial g}{\partial z}+(f_x+pf_z)\frac{\partial g}{\partial p}+(f_y+qf_z)\frac{\partial g}{\partial q}=0$$

or $$f_p\frac{\partial g}{\partial x}+f_q\frac{\partial g}{\partial y}+(pf_p+qf_q)\frac{\partial g}{\partial z}-(f_x+pf_z)\frac{\partial g}{\partial p}-(f_y+qf_z)\frac{\partial g}{\partial q}=0. \qquad \text{... (2.6)}$$

This is a quasi-linear first order partial differential equation for g with x, y, z, p and q as the independent variables.

Thus our problem of finding a one-parameter family of p.d. equations (2.4) which is compatible with the given p.d.e. (2.3) is equivalent to find a solution of equation (2.6) in as simple form as possible involving p or q or both and an arbitrary constant a.

This we do by finding an integral of the following subsidiary equations involving and arbitrary constant.

$$\frac{dx}{f_p}=\frac{d_y}{f_q}=\frac{dz}{pf_p+qf_q}=-\frac{dp}{f_x+pf_z}=-\frac{dq}{f_y+qf_z}. \qquad \text{... (2.7)}$$

Once an integral $g(x,y,z,p,q,a)$ of this kind has been found, solving the p.d.e. (2.3) and the integral thus obtained for p and q, we get

$$p=\phi(x,y,z,a),\quad q=\psi(x,y,z,a).$$

Then

$$dz=\phi dx+\psi dy \qquad \text{... (2.8)}$$

is integrable by virtue of the fact that the equations (2.1) and (2.2) are compatible.

Let the integral of (2.8) be of the form

$$F(x,y,z,a,b)=0\,. \qquad \text{... (2.9)}$$

This is a two-parameter family of solutions of (2.3), it is a complete integral of (2.3).

Example 1 : Find the complete integral of $z^2 - pqxy = 0$ by Charpit's method.

Solution : Let $f(x,y,z,p,q) = z^2 - pqxy = 0$ ... (2.10)

To find a one-parameter family of p.d.e. which is compatible with (2.10), we know the auxiliary equations are

$$\frac{dx}{f_p} = \frac{dy}{f_q} = \frac{dz}{pf_p + qf_q} = -\frac{dp}{f_x + pf_z} = -\frac{dq}{f_y + qf_z}\,, \qquad \text{... (2.11)}$$

where from equation (2.10) we have

$$f_p = -qxy,\; f_q = -pxy,\; f_x = -pqy,\; f_y = -pqx,\; f_z = 2z\,.$$

Hence the equations (2.11) become

$$\frac{dx}{-qxy} = \frac{dy}{-pxy} = \frac{dz}{-pqxy - pqxy} = -\frac{dp}{-pqy + 2pz} = -\frac{dq}{-pqx + 2qz}\,.$$

or
$$\frac{dx}{qxy} = \frac{dy}{pxy} = \frac{dz}{2pqxy} = \frac{dp}{2pz - pqy} = \frac{dq}{2qz - pqx} \qquad \text{... (2.12)}$$

Each ratio of (2.13) is also equal to

$$= \frac{pdx + qdy + xdp + ydq}{\cancel{pqxy} + \cancel{pqxy} + 2pxz - \cancel{pqxy} + 2qzy - \cancel{pqxy}}$$

each ratio
$$= \frac{pdx + qdy + xdp + ydq}{2z(px + qy)}\,.$$

Consider

$$\frac{dz}{2pqxy} = \frac{pdx + qdy + xdp + ydq}{2z(px + qy)}$$

Since from equation (2.10) we have

$$pq = \frac{z^2}{xy}$$

Hence,
$$\frac{dz}{2z^2} = \frac{pdx + qdy + xdp + ydq}{2z(px + qy)}$$

$$\Rightarrow \frac{dz}{z} = \frac{pdx + qdy + xdp + ydq}{px + qy}$$

$$\frac{dz}{z} = \frac{d(xp + yq)}{px + qy}.$$

Integrating we get

$$\log z = \log(px + qy) + \log a$$

$$\Rightarrow z = a(px + qy), \qquad \text{... (2.13)}$$

where a is an arbitrary constant.

Let $$g(x, y, z, p, q, a) = z - a(px + qy). \qquad \text{... (2.14)}$$

Thus equations (2.10) and (2.14) are compatible.

Solving equations (2.10) and (2.14) for p and q we obtain

$$p = \frac{z}{cx} \text{ and } q = \frac{cz}{y} \text{ with } a\left(c + c^{-1}\right) = 1.$$

Hence the equation $dz = pdx + qdy$ becomes

$$dz = \frac{z}{cx}dx + \frac{cz}{y}dy$$

$$\Rightarrow \frac{dz}{z} = \frac{1}{c}\frac{dx}{x} + c\frac{1}{y}dy.$$

Integrating we get

$$\log z = \frac{1}{c}\log x + c\log y + \log b$$

$$z = bx^{1/c}y^{c} \qquad \text{... (2.15)}$$

$$\Rightarrow F(x, y, z, b, c) = z - bx^{1/c}y^{c} \qquad \text{... (2.16)}$$

which is the complete integral of the first order p.d.e. (2.10). This is a two-parameter family of solutions of equation (2.10) and is the required complete integral.

Example 2 : Find the complete integral of

$$\left(p^2 + q^2\right)y - qz = 0.$$

Solution : Let the p.d.e. be given by

$$f(x,y,z,p,q)=(p^2+q^2)y-qz=0. \qquad \text{... (2.17)}$$

From equation (2.17) we have

$$f_p=2py,\ f_q=2qy-z,\ f_x=0,\ f_y=p^2+q^2,\ f_z=q.$$

Hence the auxiliary equations (2.7) becomes

i.e. $$\frac{dx}{2py}=\frac{dy}{2qy-z}=\frac{dz}{2p^2y+2q^2y-qz}=\frac{dp}{pq}=-\frac{dq}{p^2}. \qquad \text{... (2.18)}$$

Consider the ratio

$$\frac{dp}{q}=-\frac{dq}{p},$$

$$\Rightarrow pdp=-qdq.$$

Integrating we get

$$p^2+q^2=a^2, \qquad \text{... (2.19)}$$

where 'a' is a constant.

Let $$g(x,y,z,p,q,a)=p^2+q^2-a^2=0 \qquad \text{... (2.20)}$$

which is compatible with (2.17). Now to find the complete integral of (2.17) we solve equations (2.17) and (2.20) for p and q.

Hence we write from (2.17) and (2.20) that

$$(p^2+q^2)y=qz,$$

and $$(p^2+q^2)=a^2\Rightarrow qz=a^2y$$

or $$q=\frac{a^2y}{z}.$$

Hence $$p^2y=a^2y-ya^4\frac{y^2}{z^2}$$

or $$p^2=a^2-a^4\frac{y^2}{z^2}\Rightarrow p=\frac{a}{z}\sqrt{z^2-a^2y^2}.$$

Hence the equation

$$dz=pdx+qdy$$

becomes

$$dz = \frac{a}{z}\sqrt{z^2 - a^2y^2}\,dx + \frac{a^2y}{z}dy$$

or
$$\frac{zdz - a^2ydy}{\sqrt{z^2 - a^2y^2}} = adx$$

i.e.
$$d\left(\sqrt{z^2 - a^2y^2}\right) = adx\,.$$

Integrating we get

$$\sqrt{z^2 - a^2y^2} = ax + b$$

or
$$z^2 - a^2y^2 = (ax + b)^2\,.$$

Hence the equired complete integral is

$$z^2 = a^2y^2 + (ax + b)^2\,. \qquad \text{... (2.21)}$$

Example 3 : Find the complete integral of the p.d.e.

$$p^2q^2 + x^2y^2 = x^2q^2\left(x^2 + y^2\right)$$

Solution : Let the given p.d.e. be denoted by

$$f(x, y, z, p, q) = p^2q^2 + x^2y^2 = x^2q^2\left(x^2 + y^2\right) = 0\,. \qquad \text{... (2.22)}$$

From equation (2.22) we have

$$f_p = 2pq^2, f_q = 2qp^2 - 2x^2q\left(x^2 + y^2\right), f_z = 0,$$

$$f_x = 2xy^2 - q^2\left(4x^3 + 2xy^2\right), f_y = 2yx^2 - 2yx^2q^2\,.$$

Hence the auxiliary equations (2.7) become

$$\frac{dx}{2pq^2} = \frac{dy}{2qp^2 - 2x^2q\left(x^2 + y^2\right)} = \frac{dz}{2p^2q^2 + 2q^2p^2 - 2x^2q^2\left(x^2 + y^2\right)} =$$

$$= \frac{-dp}{2xy^2 - 2q^2\left(2x^3 + xy^2\right)} = \frac{-dq}{2yx^2 - 2x^2yq^2}\,.$$

Consider the ratios

$$\frac{qdy}{2p^2q^2 - 2x^2q^2\left(x^2 + y^2\right)} = \frac{-ydq}{2x^2y^2 - 2x^2y^2q^2}$$

$$\Rightarrow \frac{qdy}{-2x^2y^2} = \frac{-ydq}{-2x^2y^2\left(1 - q^2\right)} \qquad \text{by ... (2.22)}$$

$$\Rightarrow \frac{dy}{y} = \frac{dq}{q\left(1 - q^2\right)} \Rightarrow \frac{dy}{y} = \left(\frac{1}{q} - \frac{1}{2(1+q)} + \frac{1}{2(1-q)}\right)dq \cdot$$

Integrating we get

$$\log y = \log q - \frac{1}{2}\log(1+q) - \frac{1}{2}\log(1-q) + \log a$$

$$2\log y = 2\log q - \log(1+q) - \log(1-q) + 2\log a$$

$$\Rightarrow \log y^2 = \log\left(\frac{q^2}{1-q^2}\right)\cdot a^2$$

$$\Rightarrow y^2 = a^2\frac{q^2}{1-q^2}$$

or $$\left(1-q^2\right)y^2 = a^2y^2 \Rightarrow q^2\left(y^2 + a^2\right) = y^2$$

$$q = \frac{y}{\left(y^2 + a^2\right)^{1/2}} \cdot \qquad \text{... (2.23)}$$

Substituting this in equation (2.22) we get

$$p^2\left[\frac{y^2}{y^2 + a^2}\right] = -x^2y^2 + x^2\left(x^2 + y^2\right)\frac{y^2}{y^2 + a^2}$$

$$= \frac{x^2y^2}{\left(y^2 + a^2\right)}\left[-\left(y^2 + a^2\right) + x^2 + y^2\right]$$

$$\Rightarrow p^2 = x^2\left[x^2 - a^2\right]$$

$$p = x\left(x^2 - a^2\right)^{1/2}. \qquad \text{... (2.24)}$$

Substituting these values in the equation

$$dz = pdx + qdy$$

$$dz = x\left(x^2 - a^2\right)^{1/2} dx + \frac{y}{\left(y^2 + a^2\right)^{1/2}} dy$$

Integrating we get

$$z = \int x\sqrt{x^2 - a^2}\,dx + \int \frac{y}{\sqrt{y^2 + a^2}} dy + b$$

$$z = \frac{1}{3}\left(x^2 + a^2\right)^{3/2} + \left(y^2 + a^2\right)^{1/2} + b\,. \qquad \text{... (2.25)}$$

Example 4 : Find the complete integral of

$$px^5 - 4q^3x^2 + 6x^2z - 2 = 0$$

by Charpit's method.

Solution : Let

$$f\left(x, y, z, p, q\right) = px^5 - 4q^3x^2 + 6x^2z - 2 = 0\,. \qquad \text{... (2.26)}$$

From equation (2.26) we find

$$f_p = x^5, f_q = -12q^2x^2, f_x = 5px^4 - 8q^3x + 12xz, f_y = 0, f_z = 6x^2\,.$$

Hence the auxiliary equations (2.7) become

$$\frac{dx}{x^5} = \frac{dy}{-12q^2x^2} = \frac{dz}{px^5 - 12q^3x^2} = \frac{-dp}{5px^4 - 8q^3x + 12xz + 6px^2} = -\frac{dq}{6x^2q}\cdot \qquad \text{... (2.27)}$$

Consider the ratios

$$\frac{dx}{x^5} = -\frac{dq}{6x^2q}$$

$$\Rightarrow 6\frac{dx}{x^3} = -\frac{dq}{q}\cdot$$

Integrating we get

$$6\left(-\frac{1}{2x^2}\right) = -\log q + \log a$$

$$\Rightarrow \frac{3}{x^2} = \log\left(\frac{q}{C_1}\right) \Rightarrow q = a \cdot e^{\frac{3}{x^2}}$$

$$\Rightarrow q^3 = a^3 \cdot e^{\frac{9}{x^2}}\,.$$

Substituting in (2.26) we get

$$px^5 = 4a^3 e^{\frac{9}{x^2}} x^2 - 6x^2 z + 2 \Rightarrow p = \frac{4a^2 e^{\frac{9}{x^2}}}{x^3} - \frac{6z}{x^3} + \frac{2}{x^5}.$$

Hence the equation $dz = pdx + qdy$ becomes

$$dz = \left[\frac{4a^3 e^{\frac{9}{x^2}}}{x^3} - \frac{6z}{x^3} + \frac{2}{x^5}\right] dx + ae^{\frac{3}{x^2}} dy$$

$$dz + \frac{6z}{x^3} dx = \left[\frac{4a^3 e^{\frac{9}{x^2}}}{x^3} + \frac{2}{x^5}\right] dx + ae^{\frac{3}{x^2}} dy$$

or $$e^{-\frac{3}{x^2}} \left(dz + \frac{6z}{x^3} dx \right) = \left(\frac{4a^3 e^{\frac{6}{x^2}}}{x^3} + \frac{2e^{-\frac{3}{x^2}}}{x^5} \right) dx + ady$$

$$\Rightarrow d\left(ze^{-\frac{3}{x^2}} \right) = \left(\frac{4a^3 e^{\frac{6}{x^2}}}{x^3} + \frac{2e^{-\frac{3}{x^2}}}{x^5} \right) dx + ady\,.$$

Integrating we get

$$ze^{-\frac{3}{x^2}} = 4a^3 \int \frac{e^{\frac{6}{x^2}}}{x^3} dx + 2\int \frac{e^{-\frac{3}{x^2}}}{x^5} dx + ay + b \qquad \text{... (2.28)}$$

Consider $$I = \int \frac{e^{\frac{6}{x^2}}}{x^3} dx$$

Put $$\frac{6}{x^2} = t$$

$$\Rightarrow -\frac{12}{x^3} dx = dt \Rightarrow \frac{dx}{x^3} = \frac{-dt}{12}$$

Hence, $$I = -\frac{1}{12} \int e^t dt = -\frac{1}{2} e^{\frac{6}{x^2}}$$

$$\Rightarrow 4a^3 \int \frac{e^{\frac{6}{x^2}}}{x^3} dx = -\frac{a^3}{3} e^{\frac{6}{x^2}}$$

Now consider $2\int \frac{e^{-\frac{3}{x^2}}}{x^5}dx$

Put $-\frac{3}{x^2} = t \Rightarrow \frac{1}{x^2} = -\frac{t}{3} \Rightarrow \frac{6}{x^3}dx = dt$

$$\Rightarrow 2\int \frac{e^{-\frac{3}{x^2}}}{x^5}dx = 2\int e^t \frac{dt}{6}\left(-\frac{t}{3}\right) = -\frac{1}{9}\int \frac{te^t}{dt}$$

$$= -\frac{1}{9}\left[t \cdot e^t - e^t\right]$$

Thus $ze^{-\frac{3}{x^2}} = -\frac{a^3}{3}e^{\frac{6}{x^2}} + ay + b$

$$\Rightarrow z = (ay+b)e^{\frac{3}{x^2}} - \frac{a^3}{3}e^{\frac{9}{x^2}} - \frac{1}{9}e^{\frac{3}{x^2}} - \frac{3}{x^2}e^{\frac{3}{x^2}} - e^{-\frac{3}{x^2}}$$

$$z = (ay+b)e^{\frac{3}{x^2}} - \frac{a^3}{3}e^{\frac{9}{x^2}} - \frac{1}{3x^2} + \frac{1}{9}. \qquad \text{... (2.29)}$$

Example 5 : Find the complete integral of the p.d.e.

$$2z + p^2 + qy + 2y^2 = 0$$

by Charpit's method.

Solution : Let

$$f(x, y, z, p, q) = 2z + p^2 + qy + 2y^2 = 0. \qquad \text{... (2.30)}$$

From equation (2.30) we have

$$f_p = 2p, f_q = y, f_x = 0, f_y = q + 4y, f_z = 2$$

Hence equations (2.7) become

$$\frac{dx}{2p} = \frac{dy}{y} = \frac{dz}{2p^2 + qy} = -\frac{dp}{2p} = -\frac{dq}{q + 4y + 2q}.$$

Consider the ratio

$$\frac{dx}{2p} = -\frac{dp}{2p} \Rightarrow dx = -dp.$$

Integrating we get

$$x = -p + a \qquad \text{... (2.31)}$$

or $$p = a - x.$$

Substituting this in (2.30) we get

$$2z + (a-x)^2 + qy + 2y^2 = 0.$$

$$\Rightarrow q = -\frac{1}{y}\left[2z + (a-x)^2 + 2y^2\right]. \qquad \text{... (2.32)}$$

Substituting these equation in $dz = pdx + qdy$ we get

$$dz = (a-x)dx - \frac{1}{y}\left[2z + (a-x)^2 + 2y^2\right]dy$$

$$\Rightarrow dz + \frac{2z}{y}dy = (a-x)dx - \frac{(a-x)^2}{y}dy - 2ydy$$

$$\frac{ydz + 2zdy}{y} = \frac{(a-x)ydx - (a-x)^2 dy}{y} - 2ydy$$

$$ydz + 2zdy = (a-x)ydx - (a-x)^2 dy - 2y^2 dy$$

Multiplying this equation by 2y we get

$$2y^2dz + 4zydy = 2y^2(a-x)dx - 2y(a-x)^2 dy - 4y^3dy$$

or $$d\left(2zy^2\right) = -\left[d\left(y^2(a-x)^2\right)\right] - d\left(y^4\right).$$

Integrating we get

$$2zy^2 + y^2(a-x)^2 + y^4 = b$$

or $$y^2\left[2z + (a-x)^2 + y^2\right] = b. \qquad \text{... (2.33)}$$

Which is the required complete integral.

Example 6 : Find the complete integral of

$$2(z + xp + yq) = yp^2$$

by Charpit's method

Solution : Let

$$f(x, y, z, p, q) = 2(z + xp + yq) - yp^2 = 0 \qquad \text{... (2.34)}$$

where from equation (2.34)

$$f_p = 2x - 2yp, f_q = 2y, f_x = 2p, f_y = 2q - p^2, f_z = 2.$$

Hence the auxiliary equations (2.7) become

$$\Rightarrow \frac{dx}{2(x - yp)} = \frac{dy}{2y} = \frac{dz}{2(xp + yq) - 2yp^2} = \frac{-dp}{4p} = \frac{-dq}{4q - p^2}.$$

Consider the ratios

$$\frac{dy}{y} = -\frac{dp}{2p} \quad \text{or} \quad \frac{2dy}{y} = -\frac{dp}{p}.$$

Integrating we get

$$2 \log y + \log p = \log a,$$

$$\Rightarrow y^2 p = a \quad \text{or} \quad p = \frac{a}{y^2}. \qquad \text{... (2.35)}$$

Substituting this in (2.34) we get

$$2\left(z + x\frac{a}{y^2} + yq \right) = y\frac{a^2}{y^4},$$

$$\Rightarrow 2yq = \frac{a^2}{y^3} - 2\left(z + x\frac{a}{y^2} \right),$$

or

$$q = \frac{a^2}{2y^4} - \left(\frac{z}{y} + \frac{xa}{y^3} \right). \qquad \text{... (2.36)}$$

Substituting in $dz = pdx + qdy$ we get

$$dz = \frac{a}{y^2}dx + \left[\frac{a^2}{2y^4} - \left(\frac{z}{y} + \frac{ax}{y^3} \right) \right] dy,$$

$$= \frac{a}{y^2}dx - \frac{ax}{y^3}dy + \frac{a^2}{2y^4}dy - \frac{z}{y}dy,$$

$$\Rightarrow dz + \frac{z}{y}dy = \frac{a}{y^2}dx - \frac{ax}{y^3}dy + \frac{a^2}{2y^4}dy,$$

$$\Rightarrow \frac{ydz + zdy}{y} = \frac{a}{y^2}dx - \frac{ax}{y^3}dy + \frac{a^2}{2y^4}dy,$$

$$d(yz) = \frac{a}{y}dx - \frac{ax}{y^2}dy + \frac{a^2}{2y^3}dy,$$

$$= a\left(\frac{ydx - xdy}{y^2}\right) + \frac{a^2}{2y^3}dy$$

$$d(yz) = a\left(\frac{x}{y}\right) + \frac{a^2}{2y^3}dy\,.$$

Integrating we get

$$yz = a\left(\frac{x}{y}\right) + \frac{a^2}{2}\left(\frac{y^{-2}}{-2}\right) + b$$

or

$$z = \frac{ax}{y^2} + \frac{b}{y} - \frac{a^2}{4y^3}$$

$$\Rightarrow z = \frac{ax}{y^2} + \frac{b}{y} - \frac{a^2}{4y^3}$$

which is the required complete integral.

Example 7 : Find the complete integral of the p.d.e.

$$z\left(p^2 + q^2\right) + px + qy = 0\,.$$

Solution : Let the given p.d.e. be denoted by

$$f(x, y, z, p, q) = z\left(p^2 + q^2\right) + px + qy = 0\,, \qquad \text{... (2.37)}$$

$$\Rightarrow f_p = 2pz + x,\ f_q = 2qz + y,\ f_z = \left(p^2 + q^2\right),\ f_x = p,\ f_y = q$$

Hence the auxiliary equations (2.7) reduce to

$$\frac{dx}{2pz + x} = \frac{dy}{2qz + y} = \frac{dz}{2p^2z + px + 2q^2z + qy} = \frac{-dp}{p + p\left(p^2 + q^2\right)} = \frac{-dq}{q + q\left(p^2 + q^2\right)}$$

Consider the ratios

$$\Rightarrow \frac{qdp}{pq + pq\left(p^2 + q^2\right)} = \frac{pdq}{pq + pq\left(p^2 + q^2\right)} \Rightarrow \frac{qdp - pdq}{0} \Rightarrow \frac{qdp - pdq}{q^2} = 0$$

$$\Rightarrow d\left(\frac{p}{q}\right) = 0$$

Integrating we get

$$\frac{p}{q} = a$$

or $$p = aq\,. \qquad \text{... (2.38)}$$

Substituting this in (2.37) we get

$$z\left(a^2q^2 + q^2\right) + aqx + qy = 0$$

$$q^2\left[z\left(a^2+1\right)\right] + q\left(ax+y\right) = 0$$

$$q\left[qz\left(a^2+1\right) + \left(ax+y\right)\right] = 0$$

for $$q \neq 0 \Rightarrow q = -\frac{(ax+y)}{z\left(a^2+1\right)} \qquad \text{... (2.39)}$$

Substituting these values in $dz = pdx + qdy$ we get

$$dz = aqdx + qdy$$

$$= -\frac{(ax+y)}{z\left(a^2+1\right)}\left[adx + dy\right]$$

or $$zdz = -\frac{1}{\left(a^2+1\right)}\left[(ax+y)(adx+y)\right]$$

$$\Rightarrow zdz = -\frac{1}{2\left(a^2+1\right)}2(ax+y)(adx+dy)$$

$$zdz = -\frac{1}{2\left(a^2+1\right)}d(ax+y)^2.$$

Integrating we get

$$\frac{z^2}{2} = -\frac{1}{2\left(a^2+1\right)}(ax+y)^2 + b$$

$$\Rightarrow z^2 = -\frac{(ax+y)^2}{\left(a^2+1\right)} + b$$

or $$z^2 + \frac{(ax+y)^2}{a^2+1} = b\,. \qquad \text{... (2.40)}$$

Example 8 : Find the complete integral of the equation

$$xp + 3yq = 2\left(z - x^2q^2\right)$$

by Charpit's method.

Solution : Let

$$f\left(x, y, z, p, q\right) = xp + 3yq = 2\left(z - x^2q^2\right) = 0\,. \qquad \text{... (2.41)}$$

From equation (2.41) we obtain

$$f_p = x, f_q = 3y + 4x^2q, f_x = p + 4xq^2, f_z = -2\,.$$

Hence the auxiliary equations (2.7) become

$$\frac{dx}{x} = \frac{dy}{3y + 4x^2q} = \frac{dz}{px + 3yq + 4q^2x^2} = \frac{-dp}{p + 4xq^2 - 2p} = \frac{-dq}{3q - 2q}\,.$$

Consider the ratio

$$\frac{dx}{x} = \frac{dq}{-q}\,.$$

Integrating we get

$$\log x + \log q = \log a \Rightarrow xq = a\,,$$

or

$$q = \frac{a}{x}\,. \qquad \text{... (2.42)}$$

Substituting (2.42) in (2.41) we get

$$xp + 3y\frac{a}{x} - 2\left(z - \frac{x^2a^2}{x^2}\right) = 0\,,$$

$$\Rightarrow px = 2z + 2a^2 - \frac{3ay}{x}\,,$$

or

$$p = \frac{2\left(z - q^2\right)}{x} - \frac{3qy}{x^2}\,. \qquad \text{... (2.43)}$$

Substituting these values in the equation $dz = pdx + qdy$ we get

$$dz = \left(\frac{2\left(z - q^2\right)}{x} - \frac{3ay}{x^2}\right)dx + \frac{a}{x}dy\,,$$

$$dz = \frac{1}{x^2}\left[2x\left(z - a^2\right) - 3ay\right]dx + \frac{a}{x}dy$$

or $$x^4\left[\frac{x^2dz-2x\left(x-a^2\right)dx}{x^2}\right]=-3aydx+axdy,$$

$$x^4d\left[\frac{\left(z-a^2\right)}{x^2}\right]=-3aydx+axdy,$$

or $$d\left[\frac{z-a^2}{x^2}\right]=\frac{a}{x^3}dy-\frac{3ay}{x^4}dx,$$

$$d\left(\frac{z-a^2}{x^2}\right)=d\left[\frac{ay}{x^3}\right].$$

Integrating we get

$$\frac{z-a^2}{x^2}=\frac{ay}{x^3}+b$$

or $$x\left(z-a^2\right)=ay+bx^3 \qquad \text{... (2.44)}$$

which is the required complete integral.

Example 9 : Find the complete integral of the p.d.e.

$$pxy+pq+qy=yz$$

by Charpit's method.

Solution : Let

$$f\left(x,y,z,p,q\right)=pxy+pq+qy-yz=0. \qquad \text{... (2.45)}$$

From equation (2.45) we have

$$f_p=xy+q, f_q=p+y, f_x=py, f_y=px+q=z, f_z=-y.$$

Hence the auxiliary equations (2.7) reduce to

$$\frac{dx}{xy+q}=\frac{dy}{p+y}=\frac{dz}{pxy+pq+pq+qy}=\frac{-dp}{py-py}=\frac{-dq}{px+q-z=qy}$$

we see that

$$dp=0,$$

$$\Rightarrow p=a. \qquad \text{... (2.46)}$$

Substituting in equation (2.45) we get

$$axy+aq+qy-yz=0,$$

$$q(a+y) = yz - axy$$

$$q = \frac{y(z-ax)}{(a+y)}. \qquad \text{... (2.47)}$$

Substituting these values in $dz = pdx + qdy$ we get

$$dz = adx + y\frac{(z-ax)}{(a+y)}dy.$$

We write this as

$$\Rightarrow dz - adx = y\frac{(z-ax)}{a+y}dy$$

$$\Rightarrow \frac{dz-adx}{z-ax} = \frac{ydy}{a+y} \Rightarrow \frac{dz-adx}{z-ax} = \left(1 - \frac{a}{a+y}dy\right).$$

Integrating we get

$$\log(z-ax) = y - a\log(y+a) + \log b$$

$$\Rightarrow \log\frac{(z-ax)\cdot(y+a)^a}{b} = y$$

$$\Rightarrow (z-ax)(y+a)^a = be^y$$

$$z - ax = be^y(y+a)^{-a}.$$

This is required complete integral.

Example 10 : Find the complete integral of the p.d.e.

$$x^2p^2 + y^2q^2 - 4 = 0$$

Solution : Let the p.d.e. be given by

$$f(x,y,z,p,q) = x^2p^2 + y^2q^2 - 4 = 0. \qquad \text{... (2.48)}$$

$$\Rightarrow f_p = 2px^2, f_q = 2qy^2, f_x = 2xp^2, f_y = 2yq^2, f_z = 0$$

Hence auxiliary equations (2.7) become

$$\frac{dx}{2px^2} = \frac{dy}{2qy^2} = \frac{dz}{2p^2x^2 + 2q^2y^2} = -\frac{dp}{2xp^2} = -\frac{dq}{2yq^2}. \qquad \text{... (2.49)}$$

Now consider the ratios

$$\frac{dx}{2px^2} = -\frac{dp}{2xp^2} \Rightarrow \frac{dx}{x} = -\frac{dp}{p} \cdot$$

Integrating we get

$$\log x = -\log p + \log a$$

$$\Rightarrow xp = a \ .$$

Let $$g(x, y, z, p, q, a) = xp - a = 0 \qquad \text{... (2.50)}$$

be the one-parameter family of p.d.c. which is compatible with (2.48).

Now to find the complete integral of (2.48) we solve

$x^2 p^2 + y^2 q^2 = 4$ and $xp = a$ for p and q to get

$$p = \frac{a}{x} \text{ and } q = \frac{\sqrt{4 - a^2}}{y}$$

Hence the equation

$$dz = pdx + qdy$$

becomes $$dz = \frac{a}{x} dx + \sqrt{4 - a^2} \frac{dy}{y} .$$

Integrating we get

$$z = a \log x + \sqrt{4 - a^2} \log y + b \cdot \qquad \text{... (2.51)}$$

This involves two arbitrary constants and hence it is called the complete integral of (2.48).

If however, we choose the ratio

$$\frac{dy}{2y^2 q} = \frac{dq}{-2yq^2} \Rightarrow \frac{dy}{y} = -\frac{dq}{q}$$

Integrating we get

$$\log y = -\log q + \log a$$

$$\Rightarrow yq = a$$

Let $$g(x, y, z, p, q) = yq - a = 0 \qquad \text{... (2.52)}$$

This is a one-parameter family of p.d.e. compatible with (2.48). Solving we get

$$q = \frac{a}{y} \cdot$$

Also from equation (2.48) we get

$$x^2p^2 + a^2 = 4 \Rightarrow p = \frac{\sqrt{4-a^2}}{x}.$$

Hence the equation $dz = pdx + qdy$ becomes

$$dz = \sqrt{4-a^2}\,\frac{dx}{x} + \frac{a}{y}dy.$$

Integrating we get

$$z = \sqrt{4-a^2}\log x + a\log y + b\cdot \qquad \text{... (2.53)}$$

This equation involves two auxiliary constants and hence is called a complete integral of (2.48).

Note : Equations (2.51) and (2.53) are not different.

Example 11 : Find the complete integral of

$$p^2x + q^2y = z$$

by Charpit's method.

Solution : Let

$$f(x, y, z, p, q) = p^2x + q^2y - z = 0 \qquad \text{... (2.54)}$$

From this equation we find

$$f_p = 2px, f_q = 2qy, f_x = p^2, f_q = q^2, f_z = -1.$$

Hence the auxiliary equations (2.7) become

$$\frac{dx}{2px} = \frac{dy}{2qy} = \frac{dz}{2p^2x + 2q^2y} = -\frac{dp}{p^2 - p} = -\frac{dq}{q^2 - q}.$$

Consider the ratios

$$\frac{p^2dx + 2pxdp}{\cancel{2p^3x} - \cancel{2p^3x} + 2p^2x} = \frac{q^2dy + 2qydq}{\cancel{2q^3y} - \cancel{2q^3y} + 2q^2y}$$

$$\Rightarrow \frac{p^2dx + 2pxdp}{p^2x} = \frac{q^2dy + 2qydq}{q^2y}.$$

Integrating we get

$$\log\left(p^2x\right) = \log\left(q^2y\right) + \log a$$

$$\Rightarrow p^2x = aq^2y \qquad \text{... (2.55)}$$

Using this in (2.54) we get

$$aq^2y + q^2y = z \Rightarrow q^2y = \frac{z}{a+1}$$

$$\Rightarrow q = \left(\frac{z}{(a+1)y}\right)^{1/2}. \qquad \text{... (2.56)}$$

Hence from equation (2.55) we get

$$p = \left(\frac{az}{x(z+1)}\right)^{\frac{1}{2}}$$

Substituting these values in

$$dz = pdx + qdy$$

We get $$dz = \left[\frac{az}{x(a+1)}\right]^{1/2} dx + \left[\frac{z}{(a+1)y}\right]^{1/2} dy$$

$$\Rightarrow \sqrt{1+a}\,\frac{dz}{\sqrt{z}} = \sqrt{a}\cdot\frac{dx}{\sqrt{x}} + \frac{dy}{\sqrt{y}}.$$

Integrating we get

$$\sqrt{(1+a)z} = \sqrt{ax} + \sqrt{y} + b$$

$$[(a+1)z]^{1/2} = (ax)^{1/2} + y^{1/2} + b.$$

This is the required complete Integral.

Example 12 : Solve the p.d.c. by Charpit's method

$$p = (z+qy)^2.$$

Solution : Let

$$f(x,y,z,p,q) = p - (z+qy)^2 = 0 \qquad \text{... (2.57)}$$

be the given non-linear p.d.e.

Where from equation (2.57) we find

$$f_p = 1, f_q = -2y(z+qy), f_x = 0, f_y = -2q(z+qy), f_z = -2(z+qy).$$

Hence the auxiliary equation (2.7) becomes

$$\frac{dx}{1} = \frac{dy}{-2y(z+qy)} = \frac{dz}{p-2qy(z+qy)} = \frac{-dp}{-2p(z+qy)} = \frac{-dq}{-2q(z+qy)} - 2q(z+qy) \qquad \text{... (2.58)}$$

Consider the ratios

$$\frac{dy}{-2y(z+qy)}=\frac{dp}{2p(z+qy)}\Rightarrow\frac{dy}{y}=-\frac{dp}{p}.$$

Integrating we get

$$\log y+\log p=\log a$$

$$\Rightarrow yp=a \qquad \text{... (2.59)}$$

or $$p=\frac{a}{y}.$$

Putting this value in (2.57) we get

$$\frac{a}{y}=(z+qy)^2 \text{ or } z+qy=\sqrt{\frac{a}{y}}$$

or $$q=\frac{1}{y}\left[\sqrt{\frac{a}{y}}-z\right]. \qquad \text{... (2.60)}$$

Substituting these values in

$$dz=pdx+qdy,$$

we get

$$dz=\frac{a}{y}dx+\frac{1}{y}\left[\sqrt{\frac{a}{y}}-z\right]dy,$$

$$dz=\frac{a}{y}dx-\frac{z}{y}dy+\frac{\sqrt{a}}{y^{3/2}}dy,$$

$$\Rightarrow ydz+zdy=adx+\sqrt{a}\cdot\frac{y}{y^{3/2}}dy,$$

$$d(yz)=adx+\sqrt{a}\frac{1}{\sqrt{y}}dy.$$

Integrating we get

$$yz=ax+2\sqrt{ay}+b. \qquad \text{... (2.61)}$$

Example 13 : Find the complete integral of the p.d.e.

$$z^2\left(p^2z^2+q^2\right)=1.$$

Solution : Let

$$f(x, y, z, p, q) = z^2\left(p^2z^2 + q^2\right) - 1 = 0\,, \qquad \text{... (2.62)}$$

where

$$f_p = 2pz^4, f_q = 2qz^2, f_x = 0, f_y = 0, f_z = 4z^3p^2 + 2zq^2\,.$$

Hence the auxiliary equation (2.7) become

$$\frac{dx}{2pz^4} = \frac{dy}{2z^2q} = \frac{dz}{2p^2z^4 + 2q^2z^2} = \frac{-dp}{4z^3p^3 + 2zpq^2} = \frac{-dq}{4z^3p^2q + 2zq^3}\,.$$

Consider the ratios

$$\frac{dp}{p\left(4z^3p^2 + 2zq^2\right)} = \frac{dq}{q\left(4z^3p^2 + 2zq^2\right)}\,,$$

$$\Rightarrow \frac{dp}{p} = \frac{dq}{q}\,.$$

On integrating we get

$$\log p = \log q + \log a \Rightarrow p = aq\,. \qquad \text{... (2.63)}$$

Substituting this value in equation (2.62) we get

$$z^2\left(a^2q^2z^2 + q^2\right) = 1 \Rightarrow q^2z^2\left(1 + a^2z^2\right) = 1\,,$$

$$q^2 = \frac{1}{z^2\left(1 + a^2z^2\right)}\,,$$

or

$$q = \frac{1}{z\sqrt{1 + a^2z^2}}\,. \qquad \text{... (2.64)}$$

Substituting in $dz = pdx + qdy$ we get

$$z\sqrt{1 + a^2z^2}\,dz = adx + dy\,\cdot$$

On integrating we get

$$\frac{\left(a^2z^2 + 1\right)^{3/2}}{3a^2} = ax + y + b\,. \qquad \text{... (2.65)}$$

Example 14 : Find the complete integral of the p.d.e.

$$2x\left(z^2q^2+1\right)=pz$$

by Charpit's method

Solution : Let

$$f(x,y,z,p,q)=2x\left(z^2q^2+1\right)-pz=0 \qquad \text{... (2.66)}$$

$$\Rightarrow f_p=-z, f_q=4xz^2q, f_x=2\left(z^2q^2+1\right), f_y=0, f_z=4xzq^2-p.$$

Hence the auxiliary equations (2.7) become

$$\frac{dx}{-z}=\frac{dy}{4xz^2q}=\frac{dz}{-zp+4q^2xz^2}=\frac{-dp}{2\left(z^2q^2+1\right)+4xzq^2p-p^2}=\frac{-dq}{4xzq^3-pq}$$

Consider the ratios

$$\frac{dz/z}{-p+4xq^2z}=\frac{-dq/q}{-p+4xzq^2}$$

$$\Rightarrow \frac{dz}{z}=-\frac{dq}{q}.$$

Integrating we get

$$\log z=-\log q+\log a$$

$$\Rightarrow zq=a$$

$$\Rightarrow q=\frac{a}{z}\cdot$$

Substituting this in (2.66) we get

$$2x\left(a^2+1\right)-pz=0$$

$$\Rightarrow p=2\left(a^2+1\right)\frac{x}{z}\cdot$$

Substituting this in $dz=pdx+qdy$ we get

$$dz=2\left(a^2+1\right)\frac{x}{z}dx+\frac{a}{z}dy$$

$$\Rightarrow zdz=2\left(a^2+1\right)xdx+ady$$

Integrating we get

$$\frac{z^2}{2} = (a^2+1)x^2 + ay + b,$$

or $$z^2 = 2(a^2+1)x^2 + 2ay + b.$$

This is the complete integral of (2.66).

Example 15 : Obtain the complete integral of the p.d.e.

$$z^2(1+p^2+q^2) = 1.$$

Solution : Let

$$f(x,y,z,p,q) = z^2(1+p^2+q^2) - 1 = 0 \quad \text{... (2.67)}$$

be the given p.d.e. Where from equation (2.67) we obtain

$$f_p = 2pz^2, f_q = 2qz^2, f_x = 0, f_y = 0, f_z = 2z(1+p^2+q^2).$$

Hence the auxiliary equations (2.7) become

$$\frac{dx}{2pz^2} = \frac{dy}{2qz^2} = \frac{dz}{2p^2z^2 + 2q^2z^2} = \frac{-dp}{2zp(1+p^2+q^2)} = \frac{-dq}{2zq(1+p^2+q^2)}. \quad \text{... (2.68)}$$

Consider the ratios

$$\frac{dp}{2zp(1+p^2+q^2)} = \frac{dq}{2zq(1+p^2+q^2)},$$

$$\Rightarrow \frac{dp}{p} = \frac{dq}{q}.$$

Integrating we get

$$\log p = \log q + \log a,$$

$$\Rightarrow p = aq.$$

Let $$g(x,y,z,p,q,a) = p - aq = 0 \quad \text{... (2.69)}$$

be the one-parameter family of p.d.e. compatible with (2.67). Solving equations (2.67) and (2.69) we get

$$z^2(1+a^2q^2+q^2) = 1$$

$$\Rightarrow z^2q^2(a^2+1) = 1 - z^2.$$

$$q^2 = \left(\frac{1}{a^2+1}\right)\left(\frac{1-z^2}{z^2}\right) \Rightarrow q = \frac{1}{\sqrt{a^2+1}}\left(\frac{1}{z^2} - 1\right)^{1/2}.$$

Therefore, the equation $dz = pdx + qdy$ becomes

$$dz = \frac{a}{\sqrt{a^2+1}}\left(\frac{1}{z^2}-1\right)^{1/2} dx + \frac{1}{\sqrt{a^2+1}}\left(\frac{1}{z^2}-1\right)^{1/2} dy$$

$$\Rightarrow \frac{dz}{\left(\frac{1}{z^2}-1\right)^{1/2}} = \frac{a}{\sqrt{a^2+1}} dx + \frac{1}{\sqrt{a^2+1}} dy$$

or $$\frac{zdz}{(1-z^2)^{1/2}} = \frac{a}{\sqrt{a^2+1}} dx + \frac{1}{\sqrt{a^2+1}} dy.$$

Integrating we get

$$\Rightarrow -\sqrt{1-z^2} = \frac{a}{\sqrt{a^2+1}} x + \frac{1}{\sqrt{a^2+1}} y + b$$

$$\Rightarrow (a^2+1)(1-z^2) = (y+ax+b)^2.$$

This is the required complete integral.

Note : A first order p.d.e. can have several complete integrals. Note however that the two complete integrals are equivalent, in the sense that one can be obtained from another merely by changing the arbitrary constants.

Remark : However, when one complete integral has been obtained, every other solution, including every other complete integral can be obtained. We shall explain the procedure in the next section.

3. Some Standard Types of p.d.e.

Type (I) : This type of equation is of the form

$$f(p,q) = 0 \qquad \text{... (3.1)}$$

i.e. The given partial differential equation does not involve x, y and z.

Hence $f_x = 0, f_y = 0, f_z = 0$.

From auxiliary equations (2.7) we have

$$\frac{dx}{f_p} = \frac{dy}{f_q} = \frac{dz}{pf_p + qf_q} = \frac{dp}{0} = \frac{dq}{0}.$$

Solving the last equation we get either p = a or q = a.

Putting this in (3.1) we get

$$f(a,q)=0 \quad \text{or } f(p,a)=0$$

$$\Rightarrow q=Q(a) \text{ or } p=P(a).$$

Therefore, putting this in $dz = pdx + qdy$ we get

$$dz = adx + Q(a)dy.$$

Integrating we get

$$z = ax + Q(a)y + b$$

or $$z = P(a)x + ay + b.$$

Type (II) : This type of equation is of the form

$$f(z,p,q)=0 \qquad \text{... (3.2)}$$

i.e. the given p.d.e. does not involve x and y explicitly.

The auxiliary equations become

$$\frac{dx}{f_p}=\frac{dy}{f_q}=\frac{dz}{pf_q+qf_q}=\frac{dp}{-pf_z}=\frac{dq}{-qf_z} \qquad \text{... (3.3)}$$

Consider the last two ratios

$$\frac{dp}{p}=\frac{dq}{q}\Rightarrow p=aq \qquad \text{... (3.4)}$$

Substituting in (3.2) we get

$$f(z,aq,q)=0 \text{ or } q=Q(a,z) \qquad \text{... (3.5)}$$

Therefore, the equation $dz = pdx + qdy$ becomes

$$dz = aQ(a,z)dx + Q(a,z)dy$$

or $$\frac{1}{Q(a,z)}dz = adx + dy$$

Integrating we get

$$\int \frac{dz}{Q(a,z)} = ax + y + b. \qquad \text{... (3.6)}$$

Type (III) : This type of equations is of the form

$$g(x,p) = h(y,q) \qquad \text{(separable type)} \qquad \text{... (3.7)}$$

and not z is involved.

The auxiliary equations are

$$\frac{dx}{g_p} = \frac{dy}{-hq} = \frac{dz}{pg_p - qh_q} = \frac{dp}{-g_x} = \frac{dq}{h_y}$$

Consider the ratios

$$\frac{dx}{g_p} = \frac{-dx}{g_x}$$

$$\Rightarrow g_x dx = -g_p dp$$

$$\Rightarrow g_x dx + g_p dp = 0.$$

Integrating we get

$$g(x, p) = a \qquad \text{... (3.8)}$$

$\Rightarrow$ from (1) that

$$h(y, q) = a. \qquad \text{... (3.9)}$$

Solving these equations (3.8) and (3.9) for p and q we get

$$p = G(a, x), \quad q = H(a, y).$$

Therefore, the equation $dz = pdx + qdy$ becomes

$$dz = G(a, x)dx + H(a, y)dy.$$

Integrating we get

$$z = \int G(a, x)\,dx + \int H(a, y)\,dy + b,$$

which is the complete integral.

Type (IV) : This type of equation is of the form

$$z = px + qy + f(p, q). \qquad \text{... (3.10)}$$

This is called Clairaut form of partial differential equation.

The auxiliary equations are

$$\frac{dx}{-x - g_p} = \frac{dy}{-y - g_q} = \frac{dz}{-xp - pg_p - yq - qg_q} = \frac{-dp}{-p + p} = \frac{-dq}{-q + q}$$

$$\Rightarrow dp = o \Rightarrow p = a \text{ and } q = b$$

Substituting this in equation (3.10), we obtain its complete integral in the form

$$z = ax + by + f(a, b). \qquad \text{... (3.11)}$$

e.g. Find the complete integral of the p.d.e.

$$pqz = p^2\left(xq + p^2\right) + q^2\left(yp + q^2\right).$$

Solution : The given partial differential equation is in Clairaut form, hence its complete integral is given by

$$z = ax + by + \frac{a^4 + b^4}{ab}.$$

4. Jacobi's Method

Introduction : Let a partial differential equation be

$$F\left(x, y, z, p, q\right) = 0 \qquad \text{... (4.1)}$$

and $$u\left(x, y, z\right) = 0 \qquad \text{... (4.2)}$$

be the solution of (4.1)

Differentiating (4.2) w.r.t. y we get

$$u_y + u_z q = 0 \Rightarrow q = -\frac{u_y}{u_z}.$$

Substituting the values for p and q in equation (4.1), let the equation (4.1) reduce to

$$f\left(x, y, z, u_x, u_y, u_z\right) = 0. \qquad \text{... (4.3)}$$

This is a p.d.e., in which x, y and z are the independent variables and the dependent variable u does not appear explicitly in the equation.

Complete integral of $f\left(x, y, z, u_x, u_y, u_z\right) = 0$:

A function $u\left(x, y, z, a, b, c\right)$ is said to be a complete integral of (4.3) if it satisfies the p.d.e. and the associated matrix

$$\begin{pmatrix} F_a & F_{ax} & F_{ay} & F_{az} \\ F_b & F_{bx} & F_{by} & F_{bz} \\ F_c & F_{cx} & F_{cy} & F_{cz} \end{pmatrix}$$

is of rank three.

Theorem : Let $$f\left(x, y, z, u_x, u_y, u_z\right) = 0 \qquad \text{... (4.4)}$$

be a p.d.e. Show that any function h given by

$$h\left(x, y, z, u_x, u_y, u_z\right) = 0$$

is compatible with (4.4) is

$$\frac{\partial(f,h)}{\partial(x,u_x)}+\frac{\partial(f,h)}{\partial(y,u_y)}+\frac{\partial(f,h)}{\partial(z,u_z)}=0.$$

Proof : Let $f\left(x,y,z,u_x,u_y,u_z\right)=0$ ... (4.5)

be a given p.d.e. in which x, y, z are independent variables. Differentiating (4.5) w.r.t. x, y and z we get

$$f_x+\frac{\partial f}{\partial u_x}u_{xx}+\frac{\partial f}{\partial u_y}u_{yx}+\frac{\partial f}{\partial u_z}u_{zx}=0, \quad \text{... (4.6)}$$

$$f_y+\frac{\partial f}{\partial u_x}u_{xy}+\frac{\partial f}{\partial u_y}u_{yy}+\frac{\partial f}{\partial u_z}u_{zy}=0, \quad \text{... (4.7)}$$

$$f_z+\frac{\partial f}{\partial u_x}u_{xz}+\frac{\partial f}{\partial u_y}u_{yz}+\frac{\partial f}{\partial u_z}u_{zz}=0. \quad \text{... (4.8)}$$

Consider

$$h\left(x,y,z,u_x,u_y,u_z\right)=0, \quad \text{... (4.9)}$$

where $h=h_i$, i = 1, 2, ...

On differentiating equation (4.9) w.r.t. x, y and z we obtain

$$h_x+\frac{\partial h}{\partial u_x}u_{xx}+\frac{\partial h}{\partial u_y}u_{yx}+\frac{\partial h}{\partial u_z}u_{zx}=0, \quad \text{... (4.10)}$$

$$h_y+\frac{\partial h}{\partial u_x}u_{xy}+\frac{\partial h}{\partial u_y}u_{yy}+\frac{\partial h}{\partial u_z}u_{zy}=0, \quad \text{... (4.11)}$$

and

$$h_z+\frac{\partial h}{\partial u_x}u_{xz}+\frac{\partial h}{\partial u_y}u_{yz}+\frac{\partial h}{\partial u_z}u_{zz}=0. \quad \text{... (4.12)}$$

Multiply equation (4.6) by $\frac{\partial h}{\partial u_x}$ and (4.12) by $\frac{\partial f}{\partial u_x}$ and subtracting we get

$$f_x\frac{\partial h}{\partial u_x}-\frac{\partial f}{\partial u_x}h_x+u_{xy}\left(\frac{\partial f}{\partial u_y}\cdot\frac{\partial h}{\partial u_x}-\frac{\partial f}{\partial u_x}\cdot\frac{\partial h}{\partial u_y}\right)+u_{xz}\left(\frac{\partial f}{\partial u_z}\cdot\frac{\partial h}{\partial u_x}-\frac{\partial f}{\partial u_x}\cdot\frac{\partial h}{\partial u_z}\right)=0. \quad \text{... (4.13)}$$

Now multiplying equation (4.7) by $\frac{\partial h}{\partial u_y}$ and (4.11) by $\frac{\partial f}{\partial u_y}$ and subtracting we get

$$f_y \frac{\partial h}{\partial u_y} - \frac{\partial f}{\partial u_y} h_y + u_{xy}\left(\frac{\partial f}{\partial u_x}\cdot\frac{\partial h}{\partial u_y} - \frac{\partial f}{\partial u_y}\cdot\frac{\partial h}{\partial u_x}\right) + u_{yz}\left(\frac{\partial f}{\partial u_z}\cdot\frac{\partial h}{\partial u_y} - \frac{\partial f}{\partial u_y}\cdot\frac{\partial h}{\partial u_z}\right) = 0 \cdot \quad \text{... (4.14)}$$

Similarly, on multiplying equation (4.8) by $\frac{\partial h}{\partial u_z}$ and (4.12) by $\frac{\partial f}{\partial u_z}$ and subtracting we obtain

$$f_z \frac{\partial h}{\partial u_z} - \frac{\partial f}{\partial u_z} h_z + u_{xz}\left(\frac{\partial f}{\partial u_x}\cdot\frac{\partial h}{\partial u_z} - \frac{\partial f}{\partial u_z}\cdot\frac{\partial h}{\partial u_x}\right) + u_{yz}\left(\frac{\partial f}{\partial u_y}\cdot\frac{\partial h}{\partial u_z} - \frac{\partial f}{\partial u_z}\cdot\frac{\partial h}{\partial u_y}\right) = 0 \cdot \quad \text{... (4.15)}$$

Adding equations (4.13), (4.14) and (4.15) we get

$$\left(f_x \frac{\partial h}{\partial u_x} - \frac{\partial f}{\partial u_x} h_x\right) + \left(f_y \frac{\partial h}{\partial u_y} - \frac{\partial f}{\partial u_y} h_y\right) + \left(f_z \frac{\partial h}{\partial u_z} - \frac{\partial f}{\partial u_z} h_z\right) = 0 . \quad \text{... (4.16)}$$

or

$$\frac{\partial(f,h)}{\partial(x,u_x)} + \frac{\partial(f,h)}{\partial(y,u_y)} + \frac{\partial(f,h)}{\partial(z,u_z)} = 0 \cdot \quad \text{... (4.17)}$$

Equation (4.16) can also be written as

$$f_{u_x} \frac{\partial h}{\partial x} + f_{u_y} \frac{\partial h}{\partial y} + f_{u_z} \frac{\partial h}{\partial z} - f_x \frac{\partial h}{\partial u_x} - f_y \frac{\partial h}{\partial u_y} - f_z \frac{\partial h}{\partial u_z} = 0 \quad \text{... (4.18)}$$

which is the required result.

Jacobi's Method :

Result : Describe Jacobi's Method of solving the first order partial differential equation of the form

$$f\left(x, y, z, u_x, u_y, u_z\right) = 0 .$$

Proof : Let the first order partial differential equation whose complete integral is to be determined be given by the equation.

$$f\left(x, y, z, u_x, u_y, u_z\right) = 0 , \quad \text{... (4.19)}$$

where x, y, z are independent variables.

The fundamental idea of Jacobi's method is the introduction of two partial differential equations of the first order

$$h_1\left(x, y, z, u_x, u_y, u_z, a\right) = 0 , \quad \text{... (4.20)}$$

$$h_2\left(x, y, z, u_x, u_y, u_z, b\right) = 0 , \quad \text{... (4.21)}$$

each involving one arbitrary constant 'a' and 'b' such that

(i) $\dfrac{\partial(f,h_1,h_2)}{\partial(u_x,u_y,u_z)} \neq 0$ on D and

(ii) the Pfaffian equation

$$du = u_x(x,y,z)dx + u_y(x,y,z)dy + u_z(x,y,z)dz$$

is integrable, where u_x, u_y, u_z are obtained by solving equations (4.19), (4.20) and (4.21)

i.e. we seek functions h_1, and h_2 such that the equations (4.20) and (4.21) are compatible with (4.19). We know that any h ($= h_i$, i = 1,2) compatible with equation (4.19) is given by

$$\frac{\partial(f,h)}{\partial(x,u_x)} + \frac{\partial(f,h)}{\partial(y,u_y)} + \frac{\partial(f,h)}{\partial(z,u_z)} = 0 \qquad \text{... (4.22)}$$

$$\Rightarrow f_x\frac{\partial h}{\partial u_x} - f_{u_x}\frac{\partial h}{\partial x} + f_y\frac{\partial h}{\partial u_y} - f_{u_y}\frac{\partial h}{\partial y} + f_z\frac{\partial h}{\partial u_z} - f_{u_z}\frac{\partial h}{\partial z} = 0.$$

We write this as

$$f_{u_x}\frac{\partial h}{\partial x} + f_{u_y}\frac{\partial h}{\partial y} + f_{u_z}\frac{\partial h}{\partial z} - f_x\frac{\partial h}{\partial u_x} - f_y\frac{\partial h}{\partial u_y} - f_z\frac{\partial h}{\partial u_z} = 0. \qquad \text{... (4.23)}$$

This is the first order partial differential equation for h with x, y, z, u_x, u_y and u_z as the independent variables.

Hence the subsidiary equations of (4.23) are

$$\frac{dx}{f_{u_x}} = \frac{dy}{f_{u_y}} = \frac{dz}{f_{u_z}} = \frac{du_x}{-f_x} = \frac{du_y}{-f_y} = \frac{du_z}{-f_z}. \qquad \text{... (4.24)}$$

From equation (4.24) we find two integrals involving arbitrary constants 'a' and 'b' of the form

$$h_1(x,y,z,u_x,u_y,u_z,a) = 0, \qquad \text{... (4.25)}$$

and

$$h_2(x,y,z,u_x,u_y,u_z,b) = 0. \qquad \text{... (4.26)}$$

These integrals are such that, equations (4.19), (4.25) and (4.26) can be solved for u_x, u_y, u_z. These values of u_x, u_y and u_z are then substituted in

$$du = u_x dx + u_y dy + u_z dz \qquad \text{... (4.27)}$$

which is integrable. The integral satisfying (4.19) is of the form

$$\phi(x,y,z,a,b,c) = 0. \qquad \text{... (4.28)}$$

This is the required complete integral of equation (4.19).

Remark : The conditions for the equation

$$du = u_x dx + u_y dy + u_z dz$$

to be exact are

$$\frac{\partial u_x}{\partial y} = \frac{\partial u_y}{\partial x}, \quad \frac{\partial u_y}{\partial z} = \frac{\partial u_z}{\partial y}, \quad \frac{\partial u_z}{\partial x} = \frac{\partial u_x}{\partial z}$$

These conditions are obviously true. Hence the equation is either exact or not integrable at all.

Example 1 : Solve the equation

$$z^2 + zu_z - u_x{}^2 - u_y{}^2 = 0$$

by Jacobi's method.

Solution : Let

$$f\left(x, y, z, u_x, u_y, u_z\right) = z^2 + zu_z - u_x{}^2 - u_y{}^2 = 0 \qquad \text{... (4.29)}$$

be the first order partial differential equation.

Where from equation (4.29) we find

$$f_{u_x} = -2u_x, f_{u_y} = -2u_y, f_{u_z} = z, f_x = 0, f_y = 0, f_z = 2z + u_z$$

Hence the auxiliary equations (4.24) become

$$\frac{dx}{-2u_x} = \frac{dy}{-2u_y} = \frac{dz}{z} = \frac{du_x}{0} = \frac{du_y}{0} = \frac{du_z}{-2z - u_z}. \qquad \text{... (4.30)}$$

From this we obtain two independent solutions from the ratios

$$\frac{du_x}{0} = \frac{du_y}{0}$$

$$\Rightarrow du_x = 0 \Rightarrow u_x = a,$$

$$\Rightarrow du_y = 0 \Rightarrow u_y = b,$$

which are the two integral of (4.29). Substituting these in the equation (4.29) we obtain

$$z^2 + zu_z - a^2 - b^2 = 0,$$

or $$u_z = \frac{a^2 + b^2 - z^2}{z}.$$

Substituting these values in the equation

$$du = u_x dx + u_y dy + u_y dy$$

we get

$$du = adx + bdy + \left(\frac{a^2 + b^2 - z^2}{z}\right)dz,$$

$$du = adx + bdy + \left(a^2 + b^2\right)\frac{dz}{z} - zdz.$$

Integrating we get

$$u = ax + by + \left(a^2 + b^2\right)\log z - \frac{z^2}{2} + C.$$

This is the required complete integral of (4.29).

Example 2 : Solve the equation

$$z + 2u_z - \left(u_x + u_z\right)^2 = 0$$

by Jacobi's method.

Solution : Let

$$f\left(x, y, z, u_x, u_y, u_z\right) z + 2y_z - \left(u_x + u_z\right)^2 = 0. \qquad \text{... (4.31)}$$

We first find two one-parameter family of p.d.e. which are compatible with (4.31)

From equation (4.31) we find

$$f_{u_x} = -2\left(u_x + u_y\right), f_{u_y} = -2\left(u_x + u_y\right), f_{u_z} = 2, f_x = 0, f_y = 0, f_z = 1.$$

Hence the auxiliary equations (4.24) reduce to

$$\frac{dx}{-2\left(u_x + u_y\right)} = \frac{dy}{-2\left(u_x + u_y\right)} = \frac{dz}{0} = \frac{du_x}{0} = \frac{du_y}{0} = \frac{du_z}{-1}.$$

From which we obtain two independent integrals by considering the ratios

$$\frac{du_x}{0} = \frac{du_y}{0},$$

$$\Rightarrow du_x = 0 \Rightarrow u_x = a,$$

$$du_y = 0 \Rightarrow u_y = b.$$

Let $\quad h_1\left(x, y, z, u_x, u_y, u_z, a\right) = u_x - a = 0, \qquad \text{... (4.33)}$

and $\quad h_2\left(x, y, z, u_x, u_y, u_z, b\right) = u_y - b = 0, \qquad \text{... (4.34)}$

which are compatible with (4.31). Substituting this in (4.31) we get

$$z = 2u_z - (a+b)^2 = 0,$$

or $$u_z = \frac{1}{2}\left((a+b)^2 - z\right).$$

Substituting these values in the equation

$$du = u_x d_x + u_y d_y + u_z d_z,$$

we get

$$du = adx + bdy + \frac{1}{2}\left((a+b)^2 - z\right)dz.$$

Integrating we get

$$u = ax + by + \frac{1}{2}(a+b)^2 z - \frac{z^2}{4} + C$$

which is the required complete integral of (4.31).

Example 3 : Solve the equation

$$u_x x^2 - u_y{}^2 - au_z{}^2 = 0$$

by Jacobi's method.

Solution : Let

$$f\left(x, y, z, u_x, u_y, u_z\right) = u_x x^2 - u_y{}^2 - au_z{}^2 = 0 \qquad \text{... (4.35)}$$

be a given partial differential equation. From equation (4.35) we find

$$f_{u_x} = x^2, f_{u_y} = -2u_y, f_{u_z} = -2au_z, f_x = 2xu_x, f_y = 0, f_z = 0.$$

Hence, the auxiliary equations (4.24) become

$$\frac{dx}{x^2} = \frac{dy}{-2u_y} = \frac{dz}{-2au_z} = -\frac{du_x}{2xu_x} = -\frac{du_y}{0} = -\frac{du_z}{0} \qquad \text{... (4.36)}$$

From which we obtain two integrals by considering the ratios

$$\frac{du_y}{0} = \frac{du_z}{0}$$

$$\Rightarrow u_y = 0 \Rightarrow u_y = b,$$

and $$du_z = 0 \Rightarrow u_z = c .$$

Let these be denoted by

$$h_1\left(x, y, z, u_x, u_y, u_z, a\right) = u_y - b = 0 , \quad \text{... (4.37)}$$

and $$h_2\left(x, y, z, u_x, u_y, u_z, b\right) = u_z - c = 0 . \quad \text{... (4.38)}$$

Substituting these in (4.35) we obtain

$$u_x x^2 - b^2 - ac^2 \Rightarrow u_x = \frac{b^2 + ac^2}{x^2} . \quad \text{... (4.39)}$$

Substituting these values in the equation

$$du = u_x dx + u_y dy + u_z dz$$

we get

$$du = \frac{\left(b^2 + ac^2\right)}{x^2} dx + bdy + cdz .$$

Integrating

$$u = -\frac{\left(b^2 + ac^2\right)}{x} + by + cz + d \quad \text{... (4.40)}$$

Example 4 : Solve the equation by Jacobi's method

$$u_x{}^2 + u_y{}^2 + u_z = 1$$

Solution : Let

$$f\left(x, y, z, u_x, u_y, u_z\right) = u_x{}^2 + u_y{}^2 + u_z - 1 = 0 \quad \text{... (4.41)}$$

be a given p.d.e. Where from equation (4.41) we find

$$f_{u_x} = 2u_x, f_{u_y} = 2u_y, f_{u_z} = 1, f_x = 0, f_y = 0, f_z = 0 .$$

Substituting these in auxiliary equations (4.24) we find

$$\frac{dx}{2u_x} = \frac{dy}{2u_y} = \frac{dz}{1} = -\frac{du_x}{0} = -\frac{du_y}{0} = -\frac{du_z}{0} . \quad \text{... (4.42)}$$

Consider the ratios

$$\frac{du_x}{0} = \frac{du_y}{0} \Rightarrow du_x = 0$$

$$u_x = a ,$$

and $\qquad du_y = 0 \Rightarrow u_y = b$.

Let these be denoted by

$$h_1(x, y, z, u_x, u_y, u_z, a) = u_x - a = 0, \qquad \text{... (4.43)}$$

$$h_2(x, y, z, u_x, u_y, u_z, b) = u_y - b = 0. \qquad \text{... (4.44)}$$

Substituting these in (4.41) we get

$$a^2 + b^2 + u_z - 1 = 0$$

$$\Rightarrow u_z = 1 - (a^2 + b^2). \qquad \text{... (4.45)}$$

Substituting these values in

$$du = u_x dx + u_y dy + u_z dz,$$

we get

$$du = adx + bdy + \left[1 - (a^2 + b^2)\right] dz.$$

Integrating we get

$$u = ax + by + (1 - a^2 - b^2) z + c.$$

Example 5 : Solve the equation by Jacobi's method

$$xu_x + yu_y = u_z^{\,2}.$$

Solution : Let

$$f(x, y, z, u_x, u_y, u_z) = xu_x + yu_y - u_z^{\,2} = 0. \qquad \text{... (4.46)}$$

We find two one-parameter family of p.d.e. which are compatible with (4.46).

From equation (4.46) we have

$$f_{u_x} = x, f_{u_y} = y, f_{u_z} = -2u_z, f_x = u_x, f_y = u_y, f_z = 0.$$

Hence the auxiliary equations (4.24) become

$$\frac{dx}{x} = \frac{dy}{y} = \frac{dz}{-2u_z} = -\frac{du_x}{u_x} = -\frac{du_y}{u_y} = -\frac{du_z}{0}.$$

From which we obtain two independent integrals by considering

$$\Rightarrow du_z = 0 \Rightarrow u_z = a, \qquad \text{... (4.47)}$$

and $\qquad \dfrac{dx}{x} = -\dfrac{du_x}{u_x} \Rightarrow \log x = -\log u_a + \log b$

$$xu_x = b. \qquad \text{... (4.48)}$$

Equations (4.47) and (4.48) are compatible with (4.46). Therefore, substituting (4.47) and (4.48) in (4.46) we get

$$b + yu_y = a^2 \Rightarrow u_y = \frac{1}{y}\left(a^2 - b\right). \qquad \text{... (4.49)}$$

Substituting the values from equations (4.47), (4.48) and (4.49) in equation

$$du = u_x dx + u_y dy + u_z dz,$$

we get

$$du = b\frac{dx}{x} + \left(a^2 - b\right)\frac{1}{y}dy + adz.$$

Integrating we get

$$u = b\log x + \left(a^2 - b\right)\log y + ax + C.$$

This is the required complete integral of (4.46)

Example 6 : Solve the equation by Jacobi's method

$$xu_x + yu_y = u_z^{\,2}.$$

Solution : Let

$$f\left(x, y, z, u_x, u_y, u_z\right) = xu_x + yu_y - u_z^{\,2} = 0 \qquad \text{... (4.50)}$$

be a given p.d.e. Where from equation (4.50) we find

$$f_{u_x} = x, f_{u_y} = y, f_{u_z} = 2u_z, f_x = u_x, f_y = u_y, f_z = 0 \qquad \text{... (4.51)}$$

Hence, the auxiliary equations (4.24) become

$$\frac{dx}{x} = \frac{dy}{y} = \frac{dz}{2u_z} = -\frac{du_x}{u_x} = -\frac{du_y}{u_y} = -\frac{du_z}{0}. \qquad \text{... (4.52)}$$

From which we find

$$du_z = 0 \Rightarrow u_z = C, \qquad \text{... (4.53)}$$

and $$\frac{dx}{x} = -\frac{du_x}{u_x} \Rightarrow \log x = -\log u_x + \log a \Rightarrow xu_x = a$$

$$\Rightarrow u_x = \frac{a}{x}. \qquad \text{... (4.54)}$$

Substituting these in (4.50) we get

$$a + yu_y - C^2 = 0 \Rightarrow u_y = \left(\frac{C^2 - a}{y}\right). \qquad \text{... (4.55)}$$

Substituting these values in

$$du = u_x dx + u_y dy + u_z dz \,,$$

we get

$$du = a\frac{dx}{x} + \left(C^2 - a\right)dy + cdz + b \,.$$

Integrating we get

$$u = a\log x + \left(C^2 - a\right)\log y + Cz + b$$

This is the required complete integral of equation (4.50).

Example 7 : Solve

$$z^2 u_x{}^2 u_y{}^2 u_z{}^2 + u_x{}^2 u_y{}^2 - u_z{}^2 = 0 \,.$$

Solution : Let

$$f\left(x, y, z, u_x, u_y, u_z\right) = z^2 u_x{}^2 u_y{}^2 u_z{}^2 + u_x{}^2 u_y{}^2 - u_z{}^2 = 0 \qquad \text{... (4.56)}$$

We first find two one-parameter family of p.d.e., which are compatible with (4.56).

From equation (4.56) we have

$$f_{u_x} = 2u_x z^2 u_y{}^2 u_z{}^2 + 2u_x u_y{}^2, f_{u_y} = 2u_y z^2 u_x{}^2 u_z{}^2 + 2u_x{}^2 u_y \,,$$

$$f_{u_z} = 2u_z z^2 u_x{}^2 u_y{}^2 - 2u_z, f_x = 0 = f_y, f_z = 2zu_x{}^2 u_y{}^2 u_z{}^2.$$

Hence the the auxiliary equations (4.24) become

$$\frac{dx}{2u_x z^2 u_y{}^2 u_z{}^2 + 2u_x u_y{}^2} = \frac{dy}{2u_y z^2 u_x{}^2 u_z{}^2 + 2u_x{}^2 u_y} = \frac{dz}{2u_z z^2 u_x{}^2 u_y{}^2 - 2u_z} =$$

$$= -\frac{du_x}{0} = -\frac{du_y}{0} = -\frac{du_z}{2zu_x{}^2 u_y{}^2 u_z{}^2} \,.$$

Therefore, the equation $du_x = 0$ and $du_y = 0$ give

$$u_x = a \text{ and } u_y = b \,.$$

Substituting these values in equation (4.56) we get

$$a^2b^2z^2{u_z}^2 + a^2b^2 - {u_z}^2 = 0$$

$$\Rightarrow {u_z}^2\left(1 - a^2b^2z^2\right) = a^2b^2 \Rightarrow u_z = \frac{ab}{\sqrt{1 - a^2b^2z^2}}.$$

Hence, the equation

$$du = u_x dx + u_y dy + u_z dz$$

reduces to

$$du = adx + bdy + \frac{ab}{\sqrt{1 - a^2b^2z^2}}dz.$$

Integrating we get

$$u = ax + by + \sin^{-1}(abz) + C.$$

Example 8 : Solve

$$u_z z\left(u_x + u_y\right) + x + y = 0.$$

Solution : Let

$$f\left(x, y, z, u_x, u_y, u_z\right) = u_z z\left(u_x + u_y\right) + x + y = 0. \qquad \text{... (4.57)}$$

$$\Rightarrow f_{u_x} = zu_z, f_{u_y} = zu_z, f_{u_z} = z\left(u_x + u_y\right), f_x = 1, f_y = 1, f_z = u_z\left(u_x + u_y\right).$$

Hence, the auxiliary equations are

$$\frac{dx}{zu_z} = \frac{dy}{zu_z} = \frac{dz}{z\left(u_x + u_y\right)} = -\frac{du_x}{1} = -\frac{du_y}{1} = -\frac{du_z}{u_z\left(u_x + u_y\right)}. \qquad \text{... (4.58)}$$

Consider the ratios

$$\frac{dz}{z\left(u_x + u_y\right)} = -\frac{du_z}{u_z\left(u_x + u_y\right)} \Rightarrow \frac{dz}{z} = -\frac{du_z}{u_z}.$$

Integrating we get

$$\log z + -\log u_z + \log a \Rightarrow zu_z = a$$

$$u_z = \frac{a}{z}. \qquad \text{... (4.59)}$$

Now consider the ratios

$$du_x = du_y.$$

Integrating we ger

$$u_x - u_y = b. \qquad \text{... (4.60)}$$

Now from (4.57) and (4.59) we have

$$u_x - u_y = \frac{x+y}{a}. \qquad \text{... (4.61)}$$

Solving (4.60) and (4.61) we get

$$\Rightarrow u_x = \frac{b}{2} - \frac{x+y}{2a}, \qquad \text{... (4.62)}$$

and
$$u_y = -\frac{b}{2} - \frac{x+y}{2a}. \qquad \text{... (4.63)}$$

Substituting for u_x, u_y, u_z in

$$du = u_x dx + u_y dy + u_z dz$$

we get
$$du = \left(\frac{b}{2} - \frac{x+y}{2a}\right)dx - \left(\frac{b}{2} - \frac{x+y}{2a}\right)dy + \frac{a}{z}dz$$

$$du = \frac{b}{2}dx - \frac{1}{2a}xdx - \frac{b}{2}dy - \frac{1}{2a}ydy - \frac{1}{2a}(ydx + xdy) + \frac{a}{z}dz.$$

Integrating we get
$$u = \frac{b}{2}x - \frac{1}{4a}x^2 - \frac{b}{2}y - \frac{y^2}{4a} - \frac{1}{2a}xy - \frac{a}{z^2} + C$$

i.e.
$$u = \frac{b}{2}(x-y) - \frac{1}{4a}(x^2 + y^2) - \frac{1}{2a}xy - \frac{a}{z^2} + C.$$

Jacobi's Method to solve a non-linear p.d.e. in two variables :

Consider the following non-linear partial differential equation in the form

$$f(x, y, z, p, q) = 0. \qquad \text{... (4.64)}$$

The solution of (4.64) is a relation between x, y and z. Let this relation be

$$u(x, y, z) = C \qquad \text{... (4.65)}$$

Then we have from (4.65) on differentiating w.r.t. x and y

$$p = -\frac{u_x}{u_z}, \quad q = -\frac{u_y}{u_z}.$$

On substituting these values in equation (4.64) we obtain a relation of the form

$$g(x, y, z, u_x, u_y, u_z) = 0. \qquad \text{... (4.66)}$$

This can be solved by Jacobi's method discussed earlier, which yields

$$u = f\left(x, y, z, a, b, c\right) \qquad \text{... (4.67)}$$

In this if we choose u = c, we get a complete integral of (4.64).

Example 9 : Find a complete integral of the equation

$$p^2 x + q^2 y = z$$

by Jacobi's method.

Solution : Let

$$f\left(x, y, z, p, q\right) = z - p^2 x + q^2 y = 0 \qquad \text{... (4.68)}$$

be a given non-linear partial differential equation.

Let

$$u\left(x, y, z\right) = C \qquad \text{... (4.69)}$$

be the solution of equation (4.68). Then on differentiating (4.69) w.r.t. x and then w.r.t. y we get respectively,

$$p = -\frac{u_x}{u_z}, \quad q = -\frac{u_y}{u_z} \cdot$$

Substituting this in equation (4.68) we get

$$x\left(\frac{u_x}{u_z}\right)^2 + y\left(\frac{u_y}{u_z}\right)^2 = z .$$

or $\qquad xu_x^{\,2} + yu_y^{\,2} = zu_z^{\,2}.$

Let

$$f\left(x, y, z, u_x, u_y, u_z\right) = xu_x^{\,2} + yu_y^{\,2} - zu_z^{\,2} = 0 . \qquad \text{... (4.70)}$$

The auxiliary equations are

$$\frac{dx}{f_{u_x}} = \frac{dy}{f_{u_y}} = \frac{dz}{f_{u_z}} = -\frac{du_x}{f_x} = -\frac{du_y}{f_y} = -\frac{du_z}{f_z} . \qquad \text{... (4.71)}$$

From equation (4.70) we find

$$f_{u_x} = 2xu_x, f_{u_y} = 2yu_y, f_{u_z} = -2zu_z , \; f_x = u_x^{\,2}, f_y = u_y^{\,2}, f_z = -u_z^{\,2} .$$

Therefore, equations (4.71) become

$$\frac{dx}{2xu_x} = \frac{dy}{2yu_y} = \frac{dz}{-2zu_z} = -\frac{du_x}{u_x^2} = -\frac{du_y}{u_y^2} = -\frac{du_z}{u_z^2}. \qquad \text{... (4.72)}$$

The two solutions of these equations are obtained by considering the ratios

$$\frac{dx}{2xu_x} = -\frac{du_x}{u_x^2} \Rightarrow \frac{dx}{x} + 2\frac{du_x}{u_x} = 0,$$

and
$$\frac{dy}{2yu_y} = -\frac{du_y}{u_y^2} \Rightarrow \frac{dy}{y} + 2\frac{du_y}{u_y} = 0.$$

Integrating we get

$$\log x + 2\log u_x = \log a \text{ and } \log y + 2\log u_y = \log b,$$

i.e.
$$xu_x^2 = a \text{ and } yu_y^2 = b.$$

Let
$$h_1\left(x, y, z, u_x, u_y, u_z, a\right) = xu_x^2 - a = 0, \qquad \text{... (4.73)}$$

and
$$h_2\left(x, y, z, u_x, u_y, u_z, b\right) = yu_y^2 - b = 0. \qquad \text{... (4.74)}$$

Which are compatible with (4.70). Solving (4.73) and (4.74) we obtain

$$u_x^2 = \frac{a}{x}, \; u_y^2 = \frac{b}{y}.$$

Substituting these values in the equation (4.70) we get

$$x\left(\frac{a}{x}\right) + y\left(\frac{b}{y}\right) = zu_z^2$$

$$\Rightarrow u_z^2 = \frac{1}{z}(a+b). \qquad \text{... (4.75)}$$

Consequently the equation

$$du = u_x dx + u_y dy + u_z dz$$

reduces to

$$du = \sqrt{\frac{a}{x}}dx + \sqrt{\frac{b}{y}}dy + \sqrt{\frac{a+b}{z}}dz.$$

Integrating we get

$$u = 2(ax)^{1/2} + 2(by)^{1/2} + 2((a+b)z)^{1/2} + C.$$

Writing u = C we get

$$((a+b)z)^{1/2} = -\left[(ax)^{1/2} + (by)^{1/2}\right]$$

or $$z^{1/2} = -\left[\left(\frac{ax}{a+b}\right)^{1/2} + \left(\frac{by}{a+b}\right)^{1/2}\right]$$

or $$z = \left[\left(\frac{ax}{a+b}\right)^{1/2} + \left(\frac{by}{a+b}\right)^{1/2}\right]^2. \quad \text{... (4.76)}$$

Which is the complete integral of the equation (4.68).

Example 10 : Solve the p.d.e. by Jacobi's Method

$$z^3 = pqxy.$$

Solution : Let

$$f(x, y, z, p, q) = z^3 - pqxy = 0 \quad \text{... (4.77)}$$

be a given non-linear partial differential equation.

Let $u(x, y, z) = C$ be its solution. ... (4.78)

Therefore, differentiating (4.78) w.r.t. x and y we get respectively

$$p = -\frac{u_x}{u_z}, q = -\frac{u_y}{u_z}.$$

Substituting these in equation (4.77) we get

$$f(x, y, z, u_x, u_y, u_z) = u_z^2 z^3 - u_x u_y xy = 0. \quad \text{... (4.79)}$$

$$\Rightarrow f_{u_x} = -xyu_y, f_{u_y} = -xyu_x, f_{u_z} = 2u_z z^3, f_x = -u_x u_y y, f_y = -u_x u_y x, f_z = 2z^3 u_z^2. \quad \text{... (4.80)}$$

Therefore, the auxiliary equations (4.71) become

$$\frac{dx}{-xyu_y} = \frac{dy}{-xyu_x} = \frac{dz}{2u_z z^3} = \frac{-du_x}{-yu_x u_y} = \frac{-du_y}{-u_x u_y x} = \frac{-du_z}{3z^2 u_z^2}. \quad \text{... (4.81)}$$

Consider the ratios

$$\frac{dx}{-xyu_y} = \frac{du_x}{u_x u_y y} \Rightarrow \frac{dx}{x} = -\frac{du_x}{u_x}.$$

Integrating we get

$$\log(xu_x) = \log a$$

$$xu_x = a \quad \text{or} \quad u_x = \frac{a}{x}. \qquad \text{... (4.82)}$$

Now consider the ratios

$$\frac{dy}{-xyu_x} = \frac{du_y}{xu_xu_y} \Rightarrow \frac{dy}{y} = -\frac{du_y}{u_y}.$$

Integrating we get

$$\log\left(yu_y\right) = \log b$$

$$\Rightarrow yu_y = b \;\; \text{or} \;\; u_y = \frac{b}{y}. \qquad \text{... (4.83)}$$

Using these values in equation (4.83) we get

or $$u_z = \frac{\sqrt{ab}}{z^{3/2}}. \qquad \text{... (4.84)}$$

Now substituting these values in the equation

$$du = u_x d_x + u_y dy + u_z dz$$

we get

$$du = a\frac{dx}{x} + b\frac{dy}{y} + \sqrt{ab}\,\frac{dz}{z^{3/2}}.$$

Integrating we get

$$u = a\log x + b\log y + \sqrt{ab}\left(-\frac{2}{\sqrt{z}}\right) + C.$$

Taking $u\left(x, y, z\right) = C$ we get

$$\frac{2\sqrt{ab}}{\sqrt{z}} = \log\left(x^a \cdot y^b\right).$$

$$x^a y^b = \exp\left(2\sqrt{\frac{ab}{z}}\right). \qquad \text{... (4.85)}$$

Which is the required complete integral.

Example 11 : Solve by Jacobi method the equation

$$pq = xz.$$

Solution : Let

$$f\left(x, y, z, p, q\right) = pq - xz = 0, \qquad \text{... (4.86)}$$

be a given non-linear partial differential equations.

Let $u\left(x, y, z\right) = C$ be its solution. ... (3.87)

Therefore, differentiating (4.87) w.r.t. x and y we get respectively

$$p = -\frac{u_x}{u_z} \text{ and } q = -\frac{u_y}{u_z}.$$

Substituting these values in (4.86) we get

$$f\left(x, y, z, u_x, u_y, u_z\right) = u_x u_y - xz{u_z}^2 = 0. \qquad \text{... (4.88)}$$

From equation (4.88) we find

$$f_{u_x} = u_y, f_{u_y} = u_x, f_{u_z} = -2xzu_z, f_x = z{u_z}^2, f_y = 0, f_z = -x{u_z}^2. \qquad \text{... (4.89)}$$

Hence the Jacobi's auxiliary equations (4.71) reduce to

$$\frac{dx}{u_y} = \frac{dy}{u_x} = \frac{dz}{-2xzu_z} = \frac{-du_x}{z{u_z}^2} = -\frac{du_y}{0} = \frac{-du_z}{-x{u_z}^2}.$$

Now the equation

$$du_y = 0 \Rightarrow u_y = a \qquad \text{... (4.90)}$$

The ratios

$$\frac{dz}{-2\cancel{x}zu_z} = \frac{-du_z}{-\cancel{x}{u_z}^2} \Rightarrow \frac{dz}{2z} = \frac{du_z}{-u_z},$$

$$\Rightarrow \log z = -2\log u_z + \log b,$$

$$\Rightarrow z{u_z}^2 = b. \qquad \text{... (4.91)}$$

Using (4.90) and (4.91) in (4.88) we get

$$au_x - xb = 0 \Rightarrow u_x = \left(\frac{b}{a}\right)x. \qquad \text{... (4.92)}$$

Substituting these values in the equation

$$du = u_x d_x + u_y dy + u_z dz$$

we get

$$du = \left(\frac{b}{a}\right)xdx + ady + \sqrt{b}\frac{1}{\sqrt{2}}dz.$$

Integrtaing we get

$$u=\left(\frac{b}{a}\right)\frac{x^2}{2}+ay+2\sqrt{b}\sqrt{z}+C.$$

$$\Rightarrow u=\frac{b}{2a}\cdot x^2+ay+2\sqrt{bz}+C.$$

Which is the required complete integral of (4.88). Writing u = C we get

$$z=\frac{1}{4b}\left(\frac{b}{2a}x^2+ay\right)^2$$

as a complete integral of equation (4.86).

Exercise :

1. Solve the partial differential equation by Jacobi method.

$$\left(p^2+q^2\right)y=qz.$$

2. Solve by Jacobi method.

$$p=\left(z+qy\right)^2.$$

UNIT - IV

THE CAUCHY PROBLEM

Introduction :

Given a partial differential equation and a curve in space, the Cauchy problem is to find an integral surface of the equation which contains the given curve.

Let a partial differentiable equation and a curve be given by

$$f(x, y, z, p, q) = 0, \quad \text{... (1.1)}$$

$$x = x_0(s), \;\; y = y_0(s), \;\; z = z_0(s). \qquad \forall s \in [a, b] \quad \text{... (1.2)}$$

Then the Cauchy problem is to find a solution

$$z = z(x, y)$$

of the partial differential equation (1.1) such that

$$z_0(s) = z(x_0(s), y_0(s)). \qquad \forall s \in [a, b].$$

In the unit 4, we find the integral surfaces through a given curve for a

1) linear partial differential equations,
2) non-linear partial differential equations,
3) and quasi-linear equations.

1. Integral Surfaces through a given curve for a Linear Partial Differential Equations.

Result : Discuss how a general solution may be used to determine the integral surface, which passes through a given curve.

Proof : Consider a linear partial differential equation in the form

$$Pp + Qq = R. \quad \text{... (1.3)}$$

The general solution of the equation (1.3) is given by

$$F(u, v) = 0, \quad \text{... (1.4)}$$

where F is an arbitrary function and $u(x, y, z) = C_1$, $v(x, y, z) = C_2$ are solutions of the equation

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}. \quad \text{... (1.5)}$$

This solution is a two parameter family of curves.

Let C be a given curve whose parametric equations are given by

$$x = x_0(s),\ \ y = y_0(s),\ \ z = z_0(s), \qquad \text{... (1.6)}$$

where s is a parameter (not necessarily the arc length) of the curve. Our aim is to find F such that the integral surface $F(u,v)=0$ contains the given curve C.

To obtain the integral surface containing the curve C, let us assume that, we can drive from equation (1.5) two relations of the form

$$u(x,y,z)=C_1 \text{ and } v(x,y,z)=C_2 \qquad \text{... (1.7)}$$

involving two arbitrary constants C_1 and C_2. Substituting $x = x_0(s),\ \ y = y_0(s),\ \ z = z_0(s)$ in these equations, we get

$$u\left(x_0(s), y_0(s), z_0(s)\right) = C_1,$$

and

$$v\left(x_0(s), y_0(s), z_0(s)\right) = C_2. \qquad \text{... (1.8)}$$

From this particular solutions, we can eliminate the parameter s to obtain the relation between C_1 and C_2 of the type

$$F(C_1, C_2) = 0. \qquad \text{... (1.9)}$$

Then the required integral surface $z = z(x,y)$ is obtained by eliminating C_1 and C_2 between equations (1.8) and (1.9).

Note : Sometimes the solution can also be obtained by assuming $v = G(u)$ and determining G.

Example 1 : Find the integral surface of the p.d.e.

$$(x-y)y^2p + (y-x)x^2q = \left(x^2+y^2\right)z$$

through the curve $xz = a^2, y = 0$.

Solution : Given p.d.e. is

$$(x-y)y^2p + (y-x)x^2q = \left(x^2+y^2\right)z \qquad \text{... (1.10)}$$

The integral surface of the equation (1.10) is generated by the integral curves of the auxiliary equations

$$\frac{dx}{(x-y)y^2} = \frac{dy}{(y-x)x^2} = \frac{dz}{\left(x^2+y^2\right)z}. \qquad \text{... (1.11)}$$

Consider the ratios

$$\frac{dx}{(x-y)y^2} = \frac{dy}{(y-x)x^2} \Rightarrow x^2dx + y^2dy = 0$$

Integrating we get

$$x^3 + y^3 = C_1.$$

Let $$u(x,y,z) = x^3 + y^3 = C_1. \qquad \text{... (1.12)}$$

Now consider the ratios

$$\frac{dx - dy}{(x-y)(x^2+y^2)} = \frac{dx}{(x^2+y^2)z} \Rightarrow \frac{dx-dy}{x-y} = \frac{dz}{z}.$$

Integrating we get

$$\log(x-y) = \log z + \log C_2,$$

i.e. $$x - y = zC_2,$$

or $$\frac{x-y}{z} = C_2.$$

Let $$v(x,y,z) = \frac{x-y}{z} = C_2. \qquad \text{... (1.13)}$$

It is given that the general surface represented by (1.12) and (1.13) passes through the curve

$$xz = a^2, y = 0.$$

The parametric representation of these equations are

$$x = as, \text{ y} = 0 \text{ and } z = \frac{a}{s}. \qquad \text{... (1.14)}$$

Substituting this in (1.12) and (1.13) we get

$$a^3 s^3 = C_1,$$

and $$\left(\frac{as - 0}{a}\right)s = C_2 \Rightarrow s^2 = C_2.$$

$$\Rightarrow a^6 C_2^{\ 3} = C_1^{\ 2}. \qquad \text{... (1.15)}$$

From equations (1.12), (1.13) and (1.15) we have

$$\left(x^3 + y^3\right)^2 = a^6\left(\frac{x-y}{z}\right)^3,$$

$$\Rightarrow z^3\left(x^3+y^3\right)^2 = a^6(x-y)^3. \qquad \text{... (1.16)}$$

This is the required integral surface of (1.10).

Example 2 : Find the integral surface of the differential equation

$$x(z+2)p+(xz+2yz+2y)q=z(z+1)$$

passing through the curve

$$x_0=s, y_0=0 \text{ and } z_0=2s.$$

Solution : Given p.d.e. is

$$x(z+2)p+(xz+2yz+2y)q=z(z+1). \qquad \text{... (1.17)}$$

The integral surface of the equation (1.17) is generated by the integral curves of the auxiliary equations

$$\frac{dx}{x(z+2)}=\frac{dy}{(xz+2yz+2y)}=\frac{dz}{z(z+1)}. \qquad \text{... (1.18)}$$

Each ratio of (1.18) is equal to

$$\frac{dx+dy}{xz+2x+xz+2yz+2y}=\frac{dx+dy}{2xz+2yz+2(x+y)},$$

$$=\frac{dx+dy}{2(x+y)(z+1)}.$$

Therefore, consider the ratios

$$\Rightarrow \frac{dx+dy}{x+y}=2\frac{dz}{z}.$$

Integrating we get $\quad \log(x+y)=2\log z+\log C_1,$

$$\Rightarrow \left(\frac{x+y}{z^2}\right)=C_1. \qquad \text{... (1.19)}$$

Let $\quad u(x,y,z)=\frac{x+y}{z^2}=C_1. \qquad \text{... (1.20)}$

Now consider the ratios

$$\frac{dx}{x(z+2)}=\frac{dz}{z(z+1)}$$

$$\Rightarrow \frac{dx}{x}=\frac{(z+2)dz}{z(z+1)}$$

$$\Rightarrow \frac{dx}{x} = \left(\frac{2}{z} - \frac{1}{z+1}\right)dz.$$

Integrating we get

$$\log x = 2\log z - \log(z+1) + \log C_2$$

$$\Rightarrow \frac{x(z+1)}{z^2} = C_2. \qquad \text{... (1.21)}$$

Let $$v(x,y,z) = \frac{x(z+1)}{z^2} = C_2.$$

It is given that the general surface represented by (1.19) and (1.21) passes through the curve

$$x_0 = s, y_0 = 0, z_0 = 2s.$$

Hence equations (1.19) and (1.21) become

$$\frac{s}{4s^2} = C_1 \Rightarrow \frac{1}{4s} = C_1$$

and $$\frac{1}{4s}(2s+1) = C_2.$$

$$\Rightarrow C_1(2s+1) = C_2 \Rightarrow 2s+1 = \frac{C_2}{C_1} \Rightarrow 2s = \frac{C_2}{C_1} - 1$$

$$2\left(\frac{1}{4C_1}\right) = \frac{C_2 - C_1}{C_1} \Rightarrow C_2 - C_1 = \frac{1}{2} \qquad \text{... (1.22)}$$

$$\Rightarrow \frac{x(z+1)}{z^2} - \frac{x+y}{z^2} = \frac{1}{2}$$

or $$\frac{1}{z^2}(zx - y) = \frac{1}{2}$$

or $$2(xz - y) = z^2.$$

Example 3 : Find the integral surface of

$$x^2p + y^2q + z^2 = 0$$

which passes through the parabola

$$xy = x + y, z = 1.$$

Solution : Given p.d.e. is

$$x^2p + y^2q = -z^2. \qquad \text{... (1.23)}$$

The integral surface of the equation (1.23) is generated by the integral curves of the auxiliary equation

$$\frac{dx}{x^2}=\frac{dy}{y^2}=\frac{dz}{-z^2}. \quad \text{... (1.24)}$$

Consider the ratios

$$\frac{dx}{x^2}=\frac{dy}{y^2}.$$

Integrating we get

$$\frac{1}{x}=\frac{1}{y}+C_1 \quad \text{... (1.25)}$$

Similarly, by considering the ratios

$$\frac{dy}{y^2}=\frac{-dz}{z^2},$$

we obtain

$$\frac{1}{y}+\frac{1}{z}=C_2. \quad \text{... (1.26)}$$

Given that the general surface represented by (1.25) and (1.26) passes through the curve

$$xy=x+y, z=1,$$

whose parametric equations are

$$y(x-1)=x \Rightarrow y=\frac{x}{x-1}$$

$$x=s, y=\frac{s}{s-1}, z=1. \quad \text{... (1.27)}$$

Substituting these in (1.25) and (1.26) we get

$$\frac{1}{s}-\frac{s-1}{s}=C_1 \Rightarrow \frac{2-s}{s}=C_1 \Rightarrow s=\frac{2}{1+C_1}$$

$$\frac{s-1}{s}+1=C_2 \Rightarrow \frac{2s-1}{s}=C_2 \Rightarrow \frac{\frac{4}{1+C_1}-1}{\frac{1}{1+C_1}}=C_2$$

i.e. $$\frac{4-1-C_1}{2}=C_2 \text{ or } 2C_2+C_1=3. \quad \text{... (1.28)}$$

Using equations (1.25) and (1.26) in the equation (1.28) we get

$$2\left(\frac{1}{y}+\frac{1}{z}\right)+\frac{1}{x}-\frac{1}{y}=3$$

or $$\frac{1}{y}+\frac{1}{x}+\frac{2}{z}=3. \qquad \text{... (1.29)}$$

This is the required integral surface.

Example 4 : Find the equation of the integral surface of the equation

$$x^3 p + y\left(3x^2+y\right)q = z\left(2x^2+y\right) \qquad \text{... (1.30)}$$

which passes through the curve $x_0 = 1,\ y_0 = s, z_0 = s(1+s)$.

Solution : The integral surface of the equation (1.30) is generated by the integral curves of the auxiliary equations

$$\frac{dx}{x^3} = \frac{dy}{y\left(3x^2+y\right)} = \frac{dz}{z\left(2x^2+y\right)}. \qquad \text{... (1.31)}$$

Each ratio of the equation $= \dfrac{-\frac{1}{x}dx+\frac{1}{y}dy-\frac{1}{z}dz}{-x^2+3x^2+y-2x^2-y}$

$$= \frac{-\frac{1}{x}dx+\frac{1}{y}dy-\frac{1}{z}dz}{0}$$

$$\Rightarrow -\frac{dx}{x}+\frac{dy}{y}-\frac{dz}{z}=0.$$

Integrating we get

$$-\log x+\log y-\log z=\log C_1$$

$$\Rightarrow \frac{y}{xz}=C_1.$$

Let $$u=\frac{y}{xz}=C_1. \qquad \text{... (1.32)}$$

Now consider the ratios

$$\frac{dx}{x^3}=\frac{dy}{y\left(3x^2+y\right)}$$

$$\Rightarrow \frac{(3x^2+y)dx}{x^3} = \frac{dy}{y} = \frac{(3x^2+y)dx+dy}{x^3+y} \qquad \text{... (1.33)}$$

Each ratio of (1.33)

$$= \frac{(3x^2+y)dx+dy+xdy}{x^3+y+xy}.$$

Consider

$$\frac{dy}{y} = \frac{(3x^2+y)dx+dy+xdy}{x^3+y+xy}$$

$$= \frac{3x^2dx+dy+(ydx+xdy)}{x^3+y+xy}$$

$$\frac{dy}{y} = \frac{3x^2dx+dy+d(xy)}{x^3+y+xy}$$

Integrating we get

$$\log y = \log(x^3+y+xy)+\log C_2$$

or $$\frac{x^3+y+xy}{y} = C_2.$$

Let $$v = \frac{x^3+y+xy}{y} = C_2. \qquad \text{... (1.34)}$$

Given that the general surface (1.32) and (1.34) passes through the curve

$$x_0 = 1, y_0 = s, z_0 = s(1+s).$$

$$\Rightarrow \frac{s}{s(1+s)} = C_1$$

$$1 = C_1(1+s) \Rightarrow s = \frac{1-C_1}{C_1}$$

and $$\frac{1+s+s}{s} = C_2 \Rightarrow 1+s2 = C_2 s$$

or $$1 = (C_2-2)s$$

$$\Rightarrow s=\frac{1}{C_2-2}\Rightarrow 1+s=\frac{1}{C_2-2}+1=\frac{C_2-1}{C_2-2}$$

Substituting in $1=C_1(1+s)$ we get

$$1=C_1\left(\frac{C_2-1}{C_2-2}\right)$$

$$\Rightarrow C_2-2=C_1C_2-C_1$$

or $$C_1C_2-C_1-C_2+2=0. \qquad \text{... (1.35)}$$

From equations (1.32), (1.34) and (1.35) we obtain

$$\Rightarrow \frac{x^3+xy}{xz}+\frac{2y-x^3-y-xy}{y}=0.$$

$$\Rightarrow \frac{x^2+y}{z}+\frac{y-x^3-xy}{y}=0$$

$$\Rightarrow \left(x^2+y\right)y+z\left(y-x^3-xy\right)=0$$

or $$\left(x^2+y\right)y-xz\left(x^2+y\right)+yz=0$$

$$\Rightarrow \left(x^2+y\right)\left(y-xz\right)+yz=0$$

or $$\left(x^2+y\right)\left(xz-y\right)-yz=0$$

or $$yz=\left(x^2+y\right)\left(xz-y\right). \qquad \text{... (1.36)}$$

This is the required integral surface.

Example 4 : Find the equation of the integral surface of the differential equation

$$2y(z-3)p+(2x-z)q=y(2x-3)$$

which passes through the circle.

$$z=0, x^2+y^2=2x.$$

Solution : Given p.d.e. is

$$2y(z-3)p+(2x-z)q=y(2x-3). \qquad \text{... (1.37)}$$

The integral surface of the given equation (1.37) is generated by the integral curves of the auxiliary equations

$$\frac{dx}{2y(z-3)}=\frac{dy}{2x-z}=\frac{dz}{y(2x-3)} \qquad \text{... (1.38)}$$

Each ratio $$= \frac{ydy - dz}{2yx - yz - 2xy + 3y}$$

$$= \frac{ydy - dz}{-y(z-3)}.$$

Therefore, consider the ratios

$$\frac{dx}{2y(z-3)} = \frac{ydy - dz}{-y(z-3)},$$

$$\Rightarrow dx = -2(ydy - dz).$$

Integrating we get

$$x = -y^2 + 2z + C_1.$$

or $$x + y^2 - 2z = C_1.$$

Let $$u = x + y^2 - 2z = C_1. \quad \text{... (1.39)}$$

Now consider the ratios

$$\frac{dx}{2y(z-3)} = \frac{dz}{y(2x-3)} \Rightarrow \frac{dx}{2(z-3)} = \frac{dz}{2x-3}$$

$$\Rightarrow (2x-3)\,dx = 2(z-3)\,dz$$

Integrating we get

$$x^2 - 3x = z^2 - 6z + C_2$$

or $$x^2 - z^2 - 3x + 6z = C_2.$$

Let $$v = x^2 - z^2 - 3x + 6z = C_2. \quad \text{... (1.40)}$$

Given that the general surface represented by (1.39) and (1.40) passes through the circle

$$z = 0, x^2 + y^2 = 2x,$$

or $$z = 0, x^2 - 2x + y^2 = 0,$$

i.e. $$z = 0, (x-1)^2 + y^2 = 1\cdot$$

The parametric representations of these equations are

$$x = 1 + \cos\theta, y = \sin\theta, z = 0.$$

Therefore, substituting these in (1.39) and (1.40) we get

$$1+\cos\theta+\sin^2\theta = C_1 . \qquad \text{... (1.42)}$$

and $$(1+\cos\theta)^2 - 3(1+\cos\theta) = C_2$$

or $$\Rightarrow \cos^2\theta - \cos\theta - 2 = C_2 . \qquad \text{... (1.43)}$$

Adding equations (1.42) and (1.43) we get

$$C_1 + C_2 = 0 \qquad \text{... (1.44)}$$

Thus eliminating C_1, C_2 between (1.39), (1.40) and (1.44) we get

$$x + y^2 - 2z + x^2 - z^2 - 3x + 6z = 0$$

or $$x^2 + y^2 - z^2 - 2x + 4z = 0 . \qquad \text{... (1.45)}$$

This is the required integral surface.

Example 5 : Find the integral surface of the linear partial differential equation

$$x\left(y^2+z\right)p - y\left(x^2+z\right)q = \left(x^2-y^2\right)z$$

which contains the straight line

$$x + y = 0, z = 1 .$$

Solution : The linear partial differential equation is given by

$$x\left(y^2+z\right)p - y\left(x^2+z\right)q = \left(x^2-y^2\right)z . \qquad \text{... (1.46)}$$

The integral surface of the equation (1.46) is generated by the integral curves of the auxiliary equations

$$\frac{dx}{x\left(y^2+z\right)} = \frac{dy}{-y\left(x^2+z\right)} = \frac{dz}{\left(x^2-y^2\right)z} . \qquad \text{... (1.47)}$$

Each ratio of (1.47)

$$= \frac{yzdx + xzdy + xydz}{xyz\left(\cancel{y^2} + \cancel{z} - \cancel{x^2} - \cancel{z} + \cancel{x^2} - \cancel{y^2}\right)}$$

$$\Rightarrow d\left(xyz\right) = 0 .$$

Integrating we get

$$xyz = C_1 .$$

Let $$u = xyz = C_1 . \qquad \text{... (1.48)}$$

Now each ratio of the equation (1.47)

$$= \frac{xdx + ydy - dz}{x^2\left(y^2+z\right) - y^2\left(x^2+z\right) - z\left(x^2-y^2\right)}$$

$$= \frac{xdx + ydy - dz}{0}$$

$$\Rightarrow xdx + ydy - dz = 0 .$$

Integrating we get

or $\quad x^2 + y^2 - 2z = C_2 .$

Let $\quad v = x^2 + y^2 - 2z = C_2 .$ ... (1.49)

It is given that the general surface given in (1.48) and (1.49) contains the straight line

$$x + y = 0, z = 1 .$$

The parametric equations of the straight line are

$$x = t, y = -t, z = 1 .$$

Therefore, substituting these in equations (1.48) and (1.49) we get

$$-t^2 = C_1,$$

$$2\left(t^2 - 1\right) = C_2 .$$

Eliminating the parameter t, we get

$$2C_1 + C_2 + 2 = 0 . \qquad ... (1.50)$$

Thus the required integral surface is obtained by eliminating C_1 and C_2 from (1.50). Elimination gives

$$2xyz + x^2 + y^2 - 2z + 2 = 0 \qquad ... (1.51)$$

Example 6 : Find the integral surface of the equation

$$(x + a)\, p + 2yq = 2z$$

passing through the initial data curve

$$x_0 = 1, \;\; y_0 = s, \;\; z_0 = \sqrt{s} .$$

Solution : The linear p.d.e. is given by

$$(x + 2)\, p + 2yq = 2z . \qquad ... (1.52)$$

The integral surface of the equation (1.52) is generated by the integral curves of the auxiliary equations

$$\frac{dx}{x+2} = \frac{dy}{2y} = \frac{dz}{2z} . \qquad ... (1.53)$$

Consider the ratios

$$\frac{dx}{x+2} = \frac{dy}{2y} .$$

Integrating we get

$$\log(x+2) = \frac{1}{2}\log y + \log C_1$$

$$\Rightarrow (x+2)^2 = yC_1 . \qquad \text{... (1.54)}$$

Let $$u = \frac{(x+2)^2}{y} = C_1 . \qquad \text{... (1.55)}$$

Now consider the ratio

$$\frac{dy}{2y} = \frac{dz}{2z}$$

Integrating we get

$$\log y = \log z + \log C_2$$

or $$y = zC_2 .$$

Let $$v = \frac{y}{z} = C_2 . \qquad \text{... (1.56)}$$

It is given that the general surfaces (1.55) and (1.56) passes through the initial data curve given by

$$x_0 = -1, y_0 = s \text{ and } z_0 = \sqrt{s} . \qquad \text{... (1.57)}$$

Therefore, substituting in (1.55) and (1.56) we get

$$\frac{1}{s} = C_1 \text{ and } \sqrt{s} = C_2$$

$$\Rightarrow C_2{}^2 = \frac{1}{C_1} . \qquad \text{... (1.58)}$$

Eliminating C_1 and C_2 from (1.58) we obtain

$$\Rightarrow \left(\frac{y}{z}\right)^2 = \frac{y}{(x+2)^2}$$

$$z = \sqrt{y}(x+2) . \qquad \text{... (1.59)}$$

This is the required integral surface.

Example 7 : Find the equation of the integral surface of the equation

$$\left(x^2+y^2\right)p + 2xyq = (x+y)z \qquad \text{... (1.60)}$$

which passes through the curve

$$x_0 = 0, y_0 = s^2, z_0 = -s$$

Solution : The general solution of equation (1.60) is obtained in the example (4) of unit 2 in the form.

$$\frac{x+y}{z} = C_1, \qquad \text{... (1.61)}$$

and

$$\frac{y^2 - x^2}{y} = C_2. \qquad \text{... (1.62)}$$

This surface passes through the given curve

$$x_0 = 0,\ \ y_0 = s^2,\ \ z_0 = -s. \qquad \text{... (1.63)}$$

Substituting these values in equations (1.61) and (1.62) we get

$$s = -C_1,$$

$$s^2 = C_2,$$

$$\Rightarrow C_2 = C_1^{\ 2}.$$

Consequently, from equations (1.61) and (1.62) we obtain

$$z^2\left(y^2 - x^2\right) = y(x+y)^2. \qquad \text{... (1.64)}$$

Given that the general surface (1.62) and (1.63) passes through the curve

$$x_0 = 0,\ \ y_0 = s^2,\ \ z_0 = -s.$$

Substituting these in equations (1.62) and (1.63) we get

$$C_1 = -s,\ \ C_2 = s^2,$$

$$\Rightarrow C_2 = C_1^{\ 2}.$$

Eliminating C_1 and C_2 we get

$$\Rightarrow y - \frac{x^2}{y} = \left(\frac{x+y}{z}\right)^2,$$

$$z^2\left(y^2 - x^2\right) = y(x+y)^2. \qquad \text{... (1.64)}$$

Example 8 : Find integral surface of

$$2x\left(y+z^2\right)p + y\left(2y+z^2\right)q = z^3$$

which passes through the curve

$$x_0 = s^2,\ \ y_0 = s,\ \ z_0 = 1.$$

Solution : We have obtained the general solution of the equation in the form (Refer example (2) of Unit 2)

$$u = \frac{x}{yz} = C_1, \qquad \text{... (1.65)}$$

and

$$v = \frac{z^2 - 2y}{yz} = C_2. \qquad \text{... (1.66)}$$

Given that the general surface (1.65) and (1.66) passes through the curve

$$x_0 = s^2,\ \ y_0 = s,\ \ z_0 = 1.$$

Substituting in (1.65) and (1.66) we get

$$s = C_1 \text{ and } \frac{1-2s}{s} = C_2$$

$$\Rightarrow C_1C_2 + 2C_1 = 1.$$

Using equations (1.65) and (1.66) we deminate C_1 and C_2 to get

$$yz - 2x = \frac{x}{yz}\left(z^2 - 2y\right)$$

$$\Rightarrow (yz - 2x)\,yz = x\left(z^2 - 2y\right). \qquad \text{... (1.67)}$$

Example 9 : Find the integral surface passing through the circle z = 1, $x^2 + y^2 = 1$ of the partial differential equation

$$(x-y)\,p + (y-x-z)\,q = z.$$

Solution : Let the linear partial differential equation be given by

$$(x-y)\,p + (y-x-z)\,q = z. \qquad \text{... (1.68)}$$

The auxiliary equations are

$$\frac{dx}{x-y} = \frac{dy}{y-x-z} = \frac{dz}{z}. \qquad \text{... (1.69)}$$

Each ratio of (1.69)

$$= \frac{dx+dy}{x-y+y-x-z} = \frac{dz}{z}$$

$$\Rightarrow dx + dy = -dz.$$

Integrating we get

$$x + y = z = C_1.$$

Let $$u = x + y + z = C_1. \qquad \text{... (1.70)}$$

Now each ratio of equation (1.69)

$$= \frac{dx - dy + dz}{x - y - y + x + z + z},$$

$$= \frac{dx - dy + dz}{2(x - y + z)}.$$

Consider the ratios

$$\frac{dz}{z} = \frac{dx - dy + dz}{2(x - y + z)}.$$

Integrating we get

$$2 \log z = \log(x - y + z) + \log C_2,$$

or $$x - y + z = C_2 z^2. \qquad \text{... (1.71)}$$

It is given that the general surface represented by (1.70) and (1.71) passes through the curve (circle).

$$z = 1, \;\; x^2 + y^2 = 1,$$

whose parametric equations are

$$x = \cos t, \;\; y = \sin t, \;\; z = 1. \qquad \text{... (1.72)}$$

Substituting this in (1.70) and (1.71) we get

$$\cos t + \sin t + 1 = C_1 \;\; \text{and} \;\; \cos t - \sin t + 1 = C_2$$

$$\Rightarrow \cos t = \frac{C_1 + C_2 - 2}{2} \;\; \text{and} \;\; \sin t = \frac{C_1 - C_2}{2}.$$

Hence $$\cos^2 t + \sin^2 t = 1 \Rightarrow \frac{1}{4}\left[(C_1 + C_2 - 2)^2 + (C_1 - C_2)^2\right] = 1$$

$$\Rightarrow C_1^2 + C_2^2 - 2(C_1 + C_2) = 0. \qquad \text{... (1.73)}$$

Substituting the values of C_1 and C_2 from equations (1.70) and (1.71) we get

$$(x + y + z)^2 + \left(\frac{x - y + z}{z^2}\right)^2 - 2\left(x + y + z + \frac{x - y + z}{z^2}\right) = 0,$$

$$\Rightarrow z^4 (x + y + z)^2 + (x - y + z)^2 - 2z^2\left[x - y + z + z^2 (x + y + z)\right] = 0,$$

i.e. $$z^4(x+y+z)^2+(x-y+z)^2-2z^2\left[(x-y+z)+z^2(x+y+z)\right]=0. \qquad \text{... (1.74)}$$

This is the required integral surface (particular solution) through the given circle.

Example 10 : Find the integral surface of the linear partial differential equation

$$xp+yq=z$$

which contains the circle

$$x^2+y^2+z^2=4, \quad x+y+z=2.$$

Solution : The given partial differential equation is

$$xp+yq=z. \qquad \text{... (1.75)}$$

The integral surface of the equation (1.75) is generated by the integral curves of the auxiliary equations

$$\frac{dx}{x}=\frac{dy}{y}=\frac{dz}{z}.$$

Consider the ratios

$$\frac{dx}{x}=\frac{dy}{y},$$

and integrating we get

$$\log x=\log y+\log C_1$$

$$\Rightarrow \frac{x}{y}=C_1. \qquad \text{... (1.76)}$$

Similarly, by considering the last two ratios we get

$$\frac{y}{z}=C_2. \qquad \text{... (1.77)}$$

Thus the integral surface of the equation (1.75) is

$$F\left(\frac{x}{y},\frac{y}{z}\right)=0. \qquad \text{... (1.78)}$$

It is given that this integral surface passes through the given curve

$$x^2+y^2+z^2=4, \qquad \text{... (1.79)}$$

$$x+y+z=2. \qquad \text{... (1.80)}$$

From equations (1.76) and (1.77) we find

$$y=\frac{x}{C_1} \quad \text{and} \quad z=\frac{x}{C_1C_2}.$$

Substituting this in equations (1.79) and (1.80) we get

$$x^2\left(1+\frac{1}{C_1^2}+\frac{1}{C_1^2C_2^2}\right)=4, \qquad \text{... (1.81)}$$

and

$$x\left(1+\frac{1}{C_1}+\frac{1}{C_1C_2}\right)=2. \qquad \text{... (1.82)}$$

From equations (1.81) and (1.82) we find

$$1+\frac{1}{C_1^2}+\frac{1}{C_1^2C_2^2}=\left(1+\frac{1}{C_1}+\frac{1}{C_1C_2}\right)^2$$

$$\Rightarrow \frac{1}{C_1}+\frac{1}{C_1C_2}+\frac{1}{C_1^2C_2}=0$$

$$\Rightarrow C_1C_2+C_1+1=0.$$

Now replacing $C_1=\frac{x}{y}$ and $C_2=\frac{y}{z}$ we get

$$xy+xz+yz=0. \qquad \text{... (1.83)}$$

This is the required integral surface of the given partial differential equation.

Exercise :

1. Find the integral surface of the equation

$$(2xy-1)p+(z-2x^2)q=2(x-yz)$$

which passes through the line $x_0(s)=1,\ y_0(s)=0$ and $z_0(s)=0$.

2. Integral surfaces through a given curve for a non-linear Partial Differential Equations

Result : Discuss the method of finding the integral surface of a non-linear partial differential equation.

Proof : Let

$$f(x,y,z,p,q)=0 \qquad \text{... (2.1)}$$

be a given non-linear partial differential equation. By usual (Charpit's) method we find its complete integral. Let

$$F(x,y,z,a,b)=0 \qquad \text{... (2.2)}$$

be a complete integral of equation (2.1), which involves two arbitrary constants 'a' and 'b'.

Let C be a given curve whose parametric equations are given by

$$x = x_0(s), \quad y = y_0(s), \quad z = z_0(s), \qquad \text{... (2.3)}$$

where s is aparameter of the curve. Our aim is to find the integral surface of the given partial differential equation (2.1) which contains the given curve (2.3).

We expect that, this solution to be an envelope of one parameter subfamily of (2.2). This envelope contains the curve C. This requires that

$$F\left(x_0(s), y_0(s), z_0(s), a, b\right) = 0. \qquad \text{... (2.4)}$$

Differentiating this with respect to s we get

$$\frac{\partial F}{\partial s}\left(x_0(s), y_0(s), z_0(s), a, b\right) = 0. \qquad \text{... (2.5)}$$

Thus we have two relations (2.4) and (2.5) from which we eliminate s to obtain the relation between 'a' and 'b' such as

$$\psi(a, b) = 0. \qquad \text{... (2.6)}$$

Factorizing this we get

$$b = \phi_1(a), \quad b = \phi_2(a) \qquad \text{... (2.7)}$$

Each one of the relations (2.7) defines a one parameter subfamily of the complete integral (2.2). The envelope of each of these subfamilies if it exists, is an integral surface of the equation (2.1).

Note : The solution may not be unique.

Example 1 : Find a complete integral of the equation

$$\left(p^2 + q^2\right)x = pz$$

and the integral surface which passes through the curve

$$C : x_0 = 0, \quad y_0 = s^2, \quad z_0 = 2s.$$

Solution : Let

$$f(x, y, z, p, q) = \left(p^2 + q^2\right)x - pz = 0. \qquad \text{... (2.8)}$$

be a given non-linear p.d.e. To find its complete integral, we know the Charpit's auxiliary equations are

$$\frac{dx}{f_p} = \frac{dy}{f_q} = \frac{dz}{pf_p + qf_q} = -\frac{dp}{f_x + pf_z} = -\frac{dq}{f_y + qf_z}. \qquad \text{... (2.9)}$$

Where from equation (2.8) we find

$$f_p = 2px - z, \quad f_q = 2qx, \quad f_x = p^2 + q^2, \quad f_y = 0, \quad f_z = -p.$$

Consequently, equation (2.9) becomes

$$\frac{dx}{2px-z}=\frac{dy}{2qx}=\frac{dz}{pz}=-\frac{dp}{q^2}=\frac{dq}{pq}. \qquad \text{... (2.10)}$$

Considering the ratios

$$-\frac{dp}{q^2}=\frac{dq}{pq}$$

$$\Rightarrow pdp+qdq=0.$$

Integrating we get

$$\frac{p^2}{2}+\frac{q^2}{2}=\text{ Constant}$$

or

$$p^2+q^2=a^2. \qquad \text{... (2.11)}$$

Substituting this in (2.8) we get

$$a^2x=pz\Rightarrow p=\frac{a^2x}{z}.$$

The equation (2.11) gives

$$q^2=a^2-\left(\frac{a^2x}{z}\right)^2$$

$$\Rightarrow q=\frac{a}{z}\sqrt{z^2-a^2x^2}. \qquad \text{... (2.12)}$$

Substituting these in equation

$$dz=pdx+qdy$$

we get

$$dz=\frac{a^2x}{z}dx+\frac{a}{z}\sqrt{z^2-a^2x^2}dy$$

$$\Rightarrow zdz=a^2xdx+a\sqrt{z^2-a^2x^2}dy$$

$$\Rightarrow\frac{zdz-a^2xdx}{\sqrt{z^2-a^2x^2}}=ady\Rightarrow d\left(\sqrt{z^2-a^2x^2}\right)=ady.$$

Integrating we get

$$\sqrt{z^2-a^2x^2}=ay+b$$

or $$z^2 - a^2x^2 = (ay+b)^2$$

$$\Rightarrow z^2 = a^2x^2 + (ay+b)^2. \qquad \text{... (2.13)}$$

This is the required complete integral. Given that this complete integral passes through the curve

$$C : x_0 = 0,\ \ y_0 = s^2,\ \ z_0 = 2s.$$

$$\Rightarrow 4s^2 = (as^2+b)^2. \qquad \text{... (2.14)}$$

Differentiating (2.14) w.r.t. s we get

$$8s = 2(as^2+b)\cdot 2sa$$

$$\Rightarrow 2 = a(as^2+b). \qquad \text{... (2.15)}$$

Eliminating s between (2.14) and (2.15) we get

$$4s^2 = \frac{4}{a^2} \Rightarrow s^2 = \frac{1}{a^2}$$

$$\Rightarrow 2 = \left(a^2\cdot\frac{1}{a^2} + ba\right)$$

$$\Rightarrow b = \frac{1}{a} \qquad \text{... (2.16)}$$

Substituting this in (2.13) we obtain one-parameter subfamily of the complete integral in the form

$$z^2 = a^2x^2 + \left(ay + \frac{1}{a}\right)^2$$

$$a^2z^2 = a^4x^2 + (a^2y+1)^2$$

$$\Rightarrow a^2z^2 = a^4(x^2+y^2) + 2a^2y + 1$$

$$\Rightarrow a^4(x^2+y^2) + a^2(2y - z^2) + 1 = 0. \qquad \text{... (2.17)}$$

To find the envelope of (2.17) differentiate (2.17) w.r.t. 'a' we get

$$2a^2(x^2+y^2) + (2y - z^2) = 0. \qquad \text{... (2.18)}$$

Eliminating 'a' between (2.17) and (2.18) we get the required envelope of one-parameter subfamily as

$$\frac{(2y-z^2)^2}{4(x^2+y^2)^2}(x^2+y^2) - \frac{(2y-z^2)}{2(x^2+y^2)}(2y-z^2) + 1 = 0$$

or $$\left(2y-z^2\right)^2-2\left(2y-z^2\right)^2+4\left(x^2+y^2\right)=0,$$

or $$\left(z^2-2y\right)^2=4\left(x^2+y^2\right).$$

$$\Rightarrow z^2=2\left(y\pm\sqrt{x^2+y^2}\right),$$

$$\Rightarrow z^2=2\left(y+\sqrt{x^2+y^2}\right). \qquad \text{... (2.19)}$$

(by discarding the negative sign as $\sqrt{x^2+y^2}\geq y$)

This is the required integral surface of (2.8).

Example 2 : Find a complete integral of the equation

$$\left(p^2+q^2\right)x=pz$$

and the integral surface passing through the parabola $x=0, z^2=4y$.

Solution : The complete integrate of the p.d.e

$$\left(p^2+q^2\right)x=pz \qquad \text{... (2.20)}$$

is given by (refer earlier example)

$$z^2=a^2x^2+\left(ay+b\right)^2. \qquad \text{... (2.21)}$$

Given that this passes through the parabola C

$$x=0, z^2=4y,$$

whose parametric equations are

$$x=0, y=t, z^2=4t$$

$$\Rightarrow 4t=(at+b)^2$$

$$\Rightarrow a^2t^2+(2ab-4)t+b^2=0$$

For real roots we must have

$$\sqrt{b^2-4aC}=0$$

$$\Rightarrow (2ab-4)^2-4a^2b^2=0$$

or $$(ab-2)^2=a^2b^2 \Rightarrow a^2b^2-4ab+4=a^2b^2$$

$$\Rightarrow ab=1 \qquad \text{... (2.22)}$$

or $$b = \frac{1}{a}.$$

Substituting this in (2.21) we obtain the equation of the required integral surface in the form

$$z^2 = 2\left(y + \sqrt{x^2 + y^2}\right) \text{ or } \left(2y - z^2\right)^2 = 4\left(x^2 + y^2\right).$$

Exercise 3 : Find the integral surface of $z = p^2 - q^2$ which passes through the curve

$$4z + x^2 = 0, y = 0.$$

Solution : Let

$$f\left(x, y, z, p, q\right) = p^2 - q^2 - z = 0 \qquad \text{... (2.23)}$$

be the given non-linear partial differential equation. To find the complete integral we have from Charpit's auxiliary equations

$$\frac{dx}{2p} = \frac{dy}{-2q} = \frac{dz}{2p^2 - 2q^2} = -\frac{dp}{0 - p} = \frac{-dq}{-q}.$$

Consider the ratios

$$\frac{dx}{2p} = \frac{dp}{p} \Rightarrow dx = 2dp$$

Integrating we get

$$x = 2p + a \quad \text{or} \quad p = \frac{1}{2}(x - a). \qquad \text{... (2.24)}$$

Substituting this in (2.23) we get

$$q^2 = p^2 - z \Rightarrow q^2 = \left(\frac{x-a}{2}\right)^2 - z$$

or $$q = \frac{\sqrt{(x-a)^2 - 4z}}{2} \Rightarrow q = \frac{1}{2}\sqrt{(x-a)^2 - 4z}. \qquad \text{... (2.25)}$$

Substituting these values in the equation

$$dz = pdx + qdy$$

we get

$$dz = \left(\frac{x-a}{2}\right)dx + \frac{1}{2}\sqrt{(x-a)^2 - 4z} \cdot dy$$

$$\Rightarrow dz - \left(\frac{x-a}{2}\right)dx = \frac{1}{2}\sqrt{(x-a)^2 - 4z} \cdot dy$$

$$\Rightarrow \frac{2dz - (x-a)\,dx}{\sqrt{(x-a)^2 - 4z}} = dy$$

$$\Rightarrow -d\left[\sqrt{(x-a)^2 - 4z}\right] = dy$$

Integrating we get

$$\sqrt{(x-a)^2 - 4z} + y = b$$

or $$(x-a)^2 - 4z = (y+b)^2. \qquad \text{... (2.26)}$$

Thisis the required complete integral. It is given that this integral passes through the curve

$$x_0 = s,\ \ y_0 = 0,\ \ z_0 = -\frac{s^2}{4}.$$

Hence equation (2.26) gives

$$(s-a)^2 + s^2 = b^2 \qquad \text{... (2.27)}$$

Differentiating (2.27) w.r.t. s we get

$$2(s-a) + 2s = 0$$

$$\Rightarrow a = 2s \cdot \qquad \text{... (2.28)}$$

Eliminating s between (2.27) and (2.28), we obtain

$$a = \sqrt{2}b \cdot$$

Substituting this in (2.26) we get

$$\left(x - \sqrt{2}b\right)^2 - 4z = (2s+b)^2, \qquad \text{... (2.29)}$$

which is the one parameter of subfamily of complete integral.

Differentiating (2.29) w.r.t. b we get

$$2\left(x - \sqrt{2}b\right)\cdot\left(\sqrt{2}\right) = 2\left(y+b\right),$$

$$\Rightarrow \left(\frac{y+b}{x-\sqrt{2}b}\right) = -\sqrt{2}. \qquad \text{... (2.30)}$$

Eliminating b between (2.29) and (2.30) we get

$$\frac{-4z}{\left(x-\sqrt{2}b\right)^2}=2,$$

$$\Rightarrow -2z=\left(x+\sqrt{2}y\right)^2. \qquad \text{... (2.31)}$$

Example 4 : Find the complete integral of the equation

$$p^2x+qy-z=0$$

and derive the equation of the integral surface containing the line y = 1 and x + z = 0.

Solution : Let

$$f\left(x,y,z,p,q\right)=p^2x+qy-z=0 \qquad \text{... (2.32)}$$

be a given non-linear p.d.e.

To find its complete integral, the Charpit's auxiliary equations give

$$\frac{dx}{2px}=\frac{dy}{y}=\frac{dz}{2p^2x+qy}=-\frac{dp}{-p\left(p-1\right)}=\frac{dq}{0}. \qquad \text{... (2.33)}$$

From which we obtain

$$dq=0 \text{ or } q=a \qquad \text{... (2.34)}$$

Using (2.34) in (2.32) we get

$$xp^2=z-ay \Rightarrow p=\left(\frac{z-ay}{x}\right)^{1/2}. \qquad \text{... (2.35)}$$

Substituting (2.34) and (2.35) in the equation

$$dz=pdx+qdy$$

we get

$$dz=\left(\frac{z-ay}{x}\right)^{1/2}dx+ady$$

$$\Rightarrow \frac{dz-ady}{\sqrt{z-ay}}=\frac{dx}{\sqrt{x}}.$$

On integrating we get

$$\sqrt{z-ay}=\sqrt{2}+\sqrt{b}.$$

Squaring we get

$$z-ay=x+b+2\sqrt{xb}$$

or $$z-ay-x-b=2\sqrt{xb}\,.$$

Squaring we get

$$(ay-z+x+b)^2=4xb\,. \qquad \text{... (2.36)}$$

Which is the required complete integral.

Given that this complete integral passes through the curve

$$C: y=1, x+z=0\,.$$

i.e. $$y=1,\ \ x=t,\ \ z=-t\,.$$

On substituting this in (2.36) we get

$$(a+t+t+b)^2=4bt$$

$$(a+b+2t)^2=4bt\,. \qquad \text{... (2.37)}$$

Differentiating (2.37) w.r.t. t we get

$$4(a+b+2t)=4b$$

$$\Rightarrow a+b+2t=b$$

$$\Rightarrow a=-2t \ \text{ and } \ b=4t$$

$$\Rightarrow b=-2a$$

or $$2a+b=0\,. \qquad \text{... (2.38)}$$

On substituting b = – 2a in equation (2.36) we get one-parameter subfamily of the complete integral of p.d.e. (2.32) in the form

$$(ay-z+x-2a)^2=4x(-2a)$$

$$(ay-z+x-2a)^2=-8ax\,. \qquad \text{... (2.39)}$$

Differentiate equation (2.39) w.r.t. a we get

$$2(ay-z+x-2a)(y-2)=-8x$$

or $$(y-2)(ay-z+x-2a)=-4x\,. \qquad \text{... (2.40)}$$

The envelope is obtained by eliminating a between equations (2.39) and (2.40) we get

$$xy=z(y-2)\,. \qquad \text{... (2.41)}$$

Which is the required integral surface of equation (2.32).

Exercise :

1. Find the integral surface of

$$x^2 p + y^2 q + z^2 = 0$$

which passes through the hyperbola

$$xy = x + y, z = 1.$$

3. Integral Surfaces through a given curve by a method of Characteristics

In this section we shall discuss the method of characteristics to find the integral surface of a semi-linear and quasi-linear partial differential equations.

(a) Semi-linear Partial Differential Equations :

Consider a semi-linear partial differential equation given by

$$P(x, y)p + Q(x, y)q = R(x, y, z). \qquad \text{... (3.1)}$$

The expression on the left hand side of the equation (3.1) is called the directional derivative of z (x, y) in the direction of $(P(x, y), Q(x, y))$ at the point (x, y).

The one parameter family of curves in the xy plane is characterized by the ordinary differential equation

$$\frac{dy}{dx} = \frac{Q(x, y)}{P(x, y)}. \qquad \text{... (3.2)}$$

Or the system of ordinary differential equations

$$\frac{dx}{dt} = P(x, y), \frac{dy}{dt} = Q(x, y). \qquad \text{... (3.3)}$$

These curves have the property that along them z (x, y) will satisfy the ordinary differential equation

$$\frac{dz}{dx} = z_x + z_y \frac{dy}{dx}$$

$$= z_x + z_y \frac{Q(x, y)}{P(x, y)}$$

$$\Rightarrow \frac{dz}{dx} = \frac{z_x P(x, y) + z_y Q(x, y)}{P(x, y)} \qquad \text{... (3.4)}$$

or

$$\frac{dz}{dt} = z_x \frac{dx}{dt} + z_y \frac{dy}{dt}$$

$$\Rightarrow \frac{dz}{dt} = z_x P(x, y) + z_y Q(x, y) \qquad \text{... (3.5)}$$

$$\Rightarrow \frac{dz}{dt} = P(x,y)p + Q(x,y)q,$$

where $p = z_z$ and $q = z_y$

$$\Rightarrow \frac{dz}{dt} = R(x,y,z). \quad \text{... (3.6)}$$

The one parameter family of curves defined by equation (3.3) are called the characteristics curves of the partial differential equation (3.1) and the equation is called characteristic equation.

Let (x_0, y_0) be a point in the xy plane. By the existence and uniqueness of the solution of the initial value problem for the ordinary differential equation, (3.3) will define a unique characteristic curve (say)

$$x(t) = x(x_0, y_0, t), y(t) = y(x_0, y_0, t), \quad \text{... (3.7)}$$

such that $x(0) = x_0$ and $y(0) = y_0$.

If z_0 is the value for $z(x,y)$ at (x_0, y_0) then the equation (3.7) determines a unique solution z as

$$z = z(x_0, y_0, t).$$

Thus z (x, y) is uniquely determined along the characteristic passing through the point (x_0, y_0) if we know z (x, y) at (x_0, y_0).

Example 1 : Solve the equation

$$xz_y - yz_x = z$$

with the initial condition

$$z(x,0) = f(x), x \geq 0.$$

Solution : Let the equation be

$$xz_y - yz_x = z. \quad \text{... (3.8)}$$

We know the characteristic curves of (3.8) are given by the equation

$$\frac{dy}{dx} = \frac{Q(x,y)}{P(x,y)} \quad \text{... (3.9)}$$

where

$$P(x,y) = -y, \quad Q(x,y) = x, \quad R(x,y,z) = z.$$

Hence equation (3.9) gives

$$\frac{dy}{dx} = -\frac{x}{y}. \quad \text{... (3.10)}$$

This equation has solution given by

$$x^2 + y^2 = C^2. \quad \text{... (3.11)}$$

These curves have the property that along them the function z (x, y) satisfies the ordinary differential equation

$$\frac{dz}{dx} = z_x + z_y \frac{dy}{dx}$$

$$= z_x + z_y \left(-\frac{x}{y}\right) \quad \text{by equation (3.10)}$$

$$= \frac{yz_x - xz_y}{y}$$

$$\frac{dz}{dx} = -\frac{z}{y}. \quad \text{by equation (3.8)}$$

i.e.

$$\frac{dz}{dx} = -\frac{z}{\sqrt{c^2 - x^2}} \quad \text{by equation (3.11)}$$

$$\Rightarrow \frac{dz}{z} = -\frac{dx}{\sqrt{c^2 - x^2}}.$$

Integrating we get

$$\log z = -\int \frac{dx}{\sqrt{c^2 - x^2}} + \log k$$

$$\log z = -\sin^{-1}\left(\frac{x}{c}\right) + \log k$$

or

$$z = k(c)e^{-\sin^{-1}\left(\frac{x}{c}\right)}, \quad \text{... (3.12)}$$

where the constant of integration k may depend on c.

Therefore we write the general solution of the equation as

$$z = k\left(x^2 + y^2\right)e^{-\sin^{-1}\left(\frac{x}{c}\right)}. \quad \text{... (3.13)}$$

Now applying the initial conditions, we get

$$z(x,0) = f(x), \quad x \geq 0$$

we have

$$\sin^{-1}\left(\frac{x}{c}\right) = \sin^{-1}\left(\frac{x}{\sqrt{x^2+y^2}}\right)$$

$$= \sin^{-1}\left(\frac{x}{x}\right) = \sin^{-1}(1) = \frac{\pi}{2}$$

$$\Rightarrow z(x,0) = f(x) = k(x^2)e^{-\pi/2}$$

$$\Rightarrow k(x^2) = f(x)e^{\pi/2}$$

or $$k(x) = f(\sqrt{x}) \cdot e^{\pi/2} \qquad \text{... (3.14)}$$

Hence the general solution (3.13) becomes

$$z(x,y) = f\left(\sqrt{x^2+y^2}\right)e^{\pi/2 - \sin^{-1}\left(\frac{x}{c}\right)}. \qquad \text{... (3.15)}$$

Example 2 : Solve the equations

$$z_x + z_y = z^2$$

with the initial condition $z(x,0) = f(x)$.

Solution : We are given that

$$z_x + z_y = z^2. \qquad \text{... (3.16)}$$

We know the characteristic curves of the p.d.e (3.16) are given by the equation

$$\frac{dy}{dx} = \frac{Q(x,y)}{P(x,y)}, \qquad \text{... (3.17)}$$

where

$$Q(x,y) = 1, \quad P(x,y) = 1. \qquad \text{... (3.18)}$$

$$\Rightarrow \frac{dy}{dx} = 1.$$

whose solution is

$$x - y = C. \qquad \text{... (3.19)}$$

These one parameter family of curves have the property that along them the function z (x, y) must satisfy the ordinary differential equation

$$\frac{dz}{dx} = z_x + z_y \frac{dy}{dx}$$

$$= z_x + z_y \cdot 1 \qquad \text{by equation (3.18)}$$

$$\frac{dz}{dx} = z^2. \qquad \text{by equation (3.16)}$$

$$\Rightarrow \frac{dz}{z^2} = dx$$

Integrating we get

$$-\frac{1}{z} = x + k \Rightarrow z = \frac{-1}{x + k(c)}, \qquad .... (3.20)$$

where the constant of integration k may be a function of C.

$$\Rightarrow z(x, y) = -\frac{1}{x + k(x - y)}. \qquad ... (3.21)$$

Now applying the initial condition

$$z(x,0) = f(x)$$

we get
$$\Rightarrow z(x,0) = f(x) = -\frac{1}{x + k(x)}$$

$$\Rightarrow f(x)[x + k(x)] = -1,$$

$$xf(x) + f(x)k(x) = -1$$

$$\Rightarrow f(x) \cdot k(x) = -1 - xf(x)$$

$$\Rightarrow k(x) = -\frac{(1 + xf(x))}{f(x)}$$

$$\Rightarrow k(x - y) = -\frac{[1 + (x - y)f(x - y)]}{f(x - y)}.$$

Substituting this in (3.21) we get

$$z(x, y) = \frac{-1}{x - \frac{[1 + (x - y)f(x - y)]}{f(x - y)}}$$

Simplifying we get

$$\Rightarrow z(x,y)=\frac{f(x-y)}{1-yf(x-y)}.$$

Which is the required solution.

(b) Quasi-linear Equation :

Consider a quasi-linear p.d.e. given by

$$P(x,y,z)z_x+Q(x,y,z)z_y=R(x,yz). \quad \text{... (3.22)}$$

We know its solution defines an integral surface $z=z(x,y)$ in the x, y, z space.

We know the direction ratios of the normal to this surface are given by $(z_x, z_y, -1)$.

Hence equation (3.22) states that the integral surface is such that at each point the line with direction ratios (P, Q, R) is tangent to the surface at that point. (Infact, any surface z = z (x, y) has the property that it is an integral surface iff the tangent plane contains the characteristic direction (P, Q, R) defined by the p.d.e at each point).

In the case of quasi-linear equation, the characteristic curves are a family of space curves whose tangent at each point coincides with the characteristic direction (P, Q, R) at that point. These are given by the following system of ordinary differential equations

$$\frac{dx}{P(x,y,z)}=\frac{dy}{Q(x,y,z)}=\frac{dz}{R(x,y,z)}=dt \text{ (say)}. \quad \text{... (3.23)}$$

or

$$\frac{dx}{dt}=P(x,y,z),\ \frac{dy}{dt}=Q(x,y,z),\ \frac{dz}{dt}=R(x,y,z). \quad \text{... (3.24)}$$

By the existence and uniqueness of the solution of IVP of a system of ordinary differential equations there passes a characteristic curve

$$x=x(x_0,y_0,z_0,t),\ y=y(x_0,y_0,z_0,t),\ z=z(x_0,y_0,z_0,t) \quad \text{... (3.25)}$$

through each point (x_0,y_0,z_0).

Hence there is a two parameter family of characteristic curves. (Two parameter family of characteristics are nothing but the curves of intersection of the surfaces $u(x,y,z)=C_1$ and $v(x,y,z)=C_2$).

Eliminating s and t from (3.25) we get the required integral surface.

Result : Every surface generated by a one parameter family of characteristics is an integral surface.

Proof : Let $z=z(x,y)$ be an integral surface. Take $P(x,y,z)$ be any point on the surface. Then the tangent to the characteristic curve passing through that point lies on the plane to the surface. Thus the tangent plane to the surface at each point contains the line with direction ratios (P, Q, R). Hence the surface is an integral surface.

Conversely : We prove that every integral surface is generated by a family of characteristic curves.

Consider an integral surface

$$z = z(x, y). \qquad \text{... (3.26)}$$

Let $x = x(t), y = y(t)$ be the solutions of the equations

$$\frac{dx}{dt} = P(x, y, z(x, y)),$$

$$\frac{dy}{dt} = Q(x, y, z(x, y)), \qquad \text{... (3.27)}$$

with the initial conditions $x = x_0, y = y_0$ at t = 0. The corresponding curve in 3-dimension is

$$x = x(t), y = y(t), z = (x(t), y(t)) \qquad \text{... (3.28)}$$

We see that this curve lies on the given integral surface (3.26).

Further,

$$\frac{dz}{dt} = z_x \frac{dx}{dt} + z_y \frac{dy}{dt}$$

$$= P(x, y, z) z_x + Q(x, y, z) z_y$$

$$= R(x, y, z)$$

$\Rightarrow$ The curve satisfies equations for characteristic curves. viz.

$$\frac{dx}{dt} = P(x, y, z), \frac{dy}{dt} = Q(x, y, z), \frac{dz}{dt} = R(x, y, z).$$

Therefore, integral surface (3.26) is generated by the characteristic curves.

Theorem : Consider the first order quasi-linear partial differential equation

$$P(x, y, z) z_x + Q(x, y, z) z_y = R(x, y, z)$$

where P, Q and R have continuous partial derivatives with respect to x, y and z and they do not vanish simultaneously. Let $z = z_0(s)$ be prescribed along the initial curve given by

$$\Gamma_0 : x = x_0(s), \quad y = y_0(s)$$

x_0, y_0 and z_0 being continuously differentiable functions. Further, for $a \le s \le b$, if

$$\frac{dy_0}{ds} P(x_0(s), y_0(s), z_0(s)) - \frac{dx_0}{ds} Q(x_0(s), y_0(s), z_0(s)) \neq 0,$$

then there exists a unique solution z (x, y) defined in some neighbourhood of the initial curve Γ_0, which satisfies the p.d.e. and the initial condition

$$z(x_0(s), y_0(s)) = z_0(s).$$

Proof : Consider the p.d.e

$$P(x,y,z)z_x + Q(x,y,z)z_y = R(x,y,z) \quad \text{... (3.29)}$$

where P, Q, R are continuous differentiable functions of x, y, z and do not vanish simultaneously.

Let $$x = x_0(s), y = y_0(s) \quad \text{... (3.30)}$$

be the initial data curve and

$$\frac{dy_0}{ds}P(x_0(s), y_0(s), z_0(s)) - \frac{dx_0}{ds}Q(x_0(s), y_0(s), z_0(s)) \neq 0. \quad \text{... (3.31)}$$

Claim : We prove that z (x, y) is a unique solution of p.d.e. (3.29) satisfying

$$z(x_0(s), y_0(s)) = z_0(s).$$

We know the integral surface of (3.29) are the family of space curves and are given by the system of ordinary differential equations

$$\frac{dx}{dt} = P(x,y,z), \quad \frac{dy}{dt} = Q(x,y,z), \quad \frac{dz}{dt} = R(x,y,z). \quad \text{... (3.32)}$$

We solve these equations to find a unique family of characteristics (through (x_0, y_0, z_0))

$$x = x(x_0, y_0, z_0, t) = x(s,t),$$

$$y = y(x_0, y_0, z_0, t) = y(s,t),$$

$$z = z(x_0, y_0, z_0, t) = z(s,t), \quad \text{... (3.33)}$$

where x, y, z have continuous derivatives w.r.t. the parameters s and t satisfying the initial conditions

$$x(s,o) = x_0(s), y(s,o) = y_0(s) \text{ and } z(s,o) = z_0(s)$$

We see from equations (3.33) that

$$\left.\frac{\partial(x,y)}{\partial(s,t)}\right|_{t=0} = \left.\begin{vmatrix} x_s & x_t \\ y_s & y_t \end{vmatrix}\right|_{t=0} = \left.(x_s y_t - y_s x_t)\right|_{t=0}$$

$$= (x_s Q - y_x P)_{t=0} \qquad \text{by equation (3.32)}$$

$$\neq 0. \qquad \text{due to admissibility conditions (3.31)}$$

Now solving equations (3.33) for s and t, we obtain the relation

Say $\phi(x,y) = z(s(x,y), t(x,y))$ ... (3.34)

At t = 0 we get

$$\phi(x_0, y_0) = z(s,o) = z_0(s)$$

This implies that $\phi(x,y)$ satisfies the initial condition. To prove $\phi(x,y)$ also satisfies the equation (3.29).

We consider,

$$P\phi_x + Q\phi_y = P[z_s s_x + z_t t_x] + Q[z_s s_y + z_t t_y]$$

$$= z_s(Ps_x + Qs_y) + z_t(Pt_x + Qt_y)$$

$$P\phi_x + Q\phi_y = z_s(s_x x_t + s_y y_t) + z_t(t_x x_t + t_y y_t).$$ due to equation (3.32)

However, by chain rule, we have

$$s_x x_t + s_y y_t = s_t = 0$$ (s and t are independent parameters)

and $$t_x x_t + t_y y_t = t_t = 1$$

Hence the above equation becomes,

$$P\phi_x + Q\phi_y = z_t = R(x,y,z)$$ by equation (3.32)

i.e. $$P\phi_x + Q\phi_y = R(x,y,z)$$

This shows that $x = \phi(x,y)$ satisfies the p.d.e. (3.29). Thus $z = \phi(x,y)$ is a solution of p.d.e. (3.29).

Uniqueness : Let $\phi(x,y)$ be not unique solution of (3.29). This means that there are two surfaces which intersect along the given initial curve. Through each point on the initial curve, there passes one and only one characteristic curve. Therefore, this characteristic curve has to be on both the surfaces. Hence the same family of characteristic curves which passes through each point of the initial curve lie on both the surfaces.

Hence both the surfaces must coincide as both are generated by the same family of characteristics curces.

This proves the uniqueness.

Example 1 : Solve the initial value problem for the quasi-linear equation

$$zz_x + z_y = 1$$

with the initial conditions

$$x = s, y = s, z = \frac{1}{2}s \text{ for } 0 \le s \le 1.$$

Solution : Given p.d.e. i.e. a quasi-linear p.d.e. given by

$$zz_x + z_y = 1, \quad \text{... (3.35)}$$

where $$P(x, y, z) = z, Q = 1, R = 1, \quad \text{... (3.36)}$$

subject to the initial conditions

$$x = x_0(s) = s, y = y_0(s) = s, z = z_0(s) = \frac{1}{2}s, \quad 0 \le s \le 1. \quad \text{... (3.37)}$$

We observe that

$$\frac{dy_0}{ds} P(x_0(s), y_0(s), z_0(s)) - \frac{dx_0}{ds} Q(x_0(s), y_0(s), z_0(s))$$

$$= \frac{1}{2}s - 1$$

$$\neq 0 \quad \text{for } 0 \le s \le 1$$

$\Rightarrow$ from the above theorem that there exists unique solution z (x,y) satisfies the p.d.e. and the initial condition. Hence we solve the equations.

We know the family of characteristic curves which generate the surface are the solution of the equations

$$\frac{dx}{dt} = P, \quad \frac{dy}{dt} = Q, \quad \frac{dz}{dt} = R.$$

i.e. $$\frac{dx}{dt} = z, \quad \frac{dy}{dt} = 1, \quad \frac{dz}{dt} = 1, \quad \text{... (3.38)}$$

with the initial conditions

$$x(s,0) = s, \quad y(s,0) = s, \quad z(s,0) = \frac{1}{2}s. \quad \text{... (3.39)}$$

From equation (3.38) we find

$$z = t + C_1, \quad y = t + C_2$$

and $\dfrac{dx}{dt} = t + C_1$ gives

$$x = \frac{t^2}{2} + C_1 t + tC_3.$$

Hence the family of characteristic curves through the initial data are found to be

$$x = \frac{t^2}{2} + \frac{1}{2}st + s,$$

$$y = t + s,$$

$$z = t + \frac{s}{2}. \qquad \text{... (3.39)}$$

Solving these equaqtions for s and t interms of x and y we obtain (from the first two equations of 3.39)

$$x = \frac{t^2}{2} + \frac{1}{2}t(y-t) + (y-t)$$

$$\Rightarrow x - y = t\left(\frac{1}{2}y - 1\right)$$

or

$$t = \frac{y-x}{1-\frac{y}{2}}. \qquad \text{... (3.40)}$$

Substituting this in y = t + s we get

$$s = y - \frac{y-x}{1-\frac{y}{2}}$$

$$\Rightarrow s = \frac{x - \frac{y^2}{2}}{1-\frac{y}{2}}. \qquad \text{... (3.41)}$$

Substituting the values of t and s from equations (3.40) and (3.41) in $z = t + \frac{s}{2}$ we get

$$z = \frac{2(y-x) + \left(x - \frac{y^2}{2}\right)}{(2-y)}. \qquad \text{... (3.42)}$$

This is the required solution of integral surface.

Example 2 : Solve the Chauchy problem for

$$2z_x + yz_y = z$$

when the initial data curve is

$$C : x_0 = s, y_0 = s^2, z_0 = s, \quad 1 \le s \le 2.$$

Solution : The partial differential equation is given by

$$2z_x + yz_y = z, \qquad \text{... (3.43)}$$

subject to the initial conditions

$$x_0 = s, y_0 = s^2, z_0 = s,\ 1 \le s \le 2. \qquad \text{... (3.44)}$$

Here $\quad P = 2, Q = y, R = z$.

Hence $\quad P(x_0, y_0, z_0) = 2, Q(x_0, y_0, z_0) = s^2, R(x_0, y_0, z_0) = s$.

Therefore, we observe that

$$\frac{dy_0}{ds} P(x_0, y_0, z_0) - \frac{dx_0}{ds} Q(x_0, y_0, z_0) = 4s - s^2,$$

$$\neq 0 \text{ for } 1 \le s \le 2. \qquad \text{... (3.45)}$$

The admissibility condition (3.45) implies that there exists unique solution z (x, y) satisfies the p.d.e. and the initial conditions.

We know the family of characteristic curves which generate the surface are the solutions of the equations

$$\frac{dx}{dt} = 2,\ \frac{dy}{dt} = y,\ \frac{dz}{dt} = z, \qquad \text{... (3.46)}$$

such that $\quad x_0 = s, y_0 = s^2, z_0 = s$.

Solving the equations (3.46) we obtain

$$x = 2t + C_1,\ \log y = t + \log C_2 \Rightarrow y = C_2 e^t,$$

and $\quad \log z = t + \log C_3 \Rightarrow z = C_3 e^t. \qquad \text{... (3.46a)}$

We have from equations (3.44) and (3.46a)

$$C_1 = s,\ C_2 = s^2,\ C_3 = s.$$

Thus the family of characteristic curves is found to be

$$x = s + 2t, \qquad \text{... (3.47)}$$

$$y = s^2 e^t, \qquad \text{... (3.48)}$$

$$z = se^t. \qquad \text{... (3.49)}$$

The solution is obtained by eliminating s and t between equations (3.47), (3.48) and (3.49).

Therefore, we see that

$$xz - y = s^2 e^t + 2ste^t - s^2 e^t,$$

$$xz - y = 2ste^t,$$

$$\frac{xz - y}{2z} = t.$$

Substituting in (3.47) we get

$$x - 2\left(\frac{xz - y}{2z}\right) = s,$$

$$\Rightarrow s = \frac{y}{z}.$$

Therefore the equation $z = se^t$ becomes

$$z = \left(\frac{y}{z}\right) exp\left(\frac{xz - y}{2z}\right),$$

or $$z^2 = yexp\left(\frac{xz - y}{2z}\right). \qquad \text{... (3.50)}$$

This is the required solution (integral surface).

Example 3 : Find the solution of the initial value problem for the quasi-linear equation

$$z_x - zz_y + z = 0 \quad \forall y \text{ and } x > 0$$

for the initial data curve

$$C : x_0 = 0, y_0 = s, z_0 = -2s, \qquad -\infty < s < \infty.$$

Solution : The quasi-linear p.d.e. is given by

$$z_x - zz_y = -z, \qquad \text{... (3.51)}$$

with the initial data curve

$$C : x_0 = 0, y_0 = s, z_0 = -2s.$$

Here P =1, Q = -z and R = – z.

Therefore $$P(x_0, y_0, z_0) = 1, \;\; Q(x_0, y_0, z_0) = 2s, \;\; R(x_0, y_0, z_0) = 2s.$$

We observe from the admissibility condition that

$$\frac{dy_0}{ds} P - \frac{dy_0}{ds} Q = 1 \neq 0. \qquad \forall s \qquad \text{... (3.52)}$$

This shows that there exists unique solution z (x, y) satisfies the p.d.e. and the initial conditions.

We know the family of characteristic curves which generate surface are the solutions of the equations

$$\frac{dx}{dt} = P, \;\; \frac{dy}{dt} = Q, \;\; \frac{dz}{dt} = R.$$

i.e. $$\frac{dx}{dt} = 1, \;\; \frac{dy}{dt} = -z, \;\; \frac{dz}{dt} = -z, \qquad \text{... (3.53)}$$

such that $x|_{t=0} = x_0 = 0, \quad y|_{t=0} = s, \quad z|_{t=0} = -2s.$

Solving these equations we get

$$x = t + C_1, \quad \log z = -t + \log C_2 \Rightarrow z = C_2 e^{-t},$$

and $$\frac{dy}{dt} = -C_2 e^{-t} \Rightarrow y = -C_2 \int e^{-t} dt + C_3,$$

$$\Rightarrow y = C_2 e^{-t} + C_3.$$

Hence the family of characteristic curves through the initial data curve is found to be

$$\left. \begin{array}{l} x = t, \\ y = -2se^{-t} + 3s, \quad and \\ z = -2se^{-t}. \end{array} \right\} \qquad \ldots (3.54)$$

The solution is obtain by eliminating t and s from equation (3.54). Thus we have

$$s = -\frac{y}{2e^{-x} - 3},$$

Substituting this value in $z = -2se^{-t}$ we get

$$z = \frac{2y}{2e^{-x} - 3} e^{-x},$$

$$\Rightarrow z = \frac{-2y}{3e^{x} - 2} \qquad \text{for } \log\left(\frac{2}{3}\right) > x \geq 0 \qquad \ldots (3.55)$$

This is the required solution (integral surface).

Note : The solution breaks down at $x = \log\frac{2}{3}$.

Example 4 : Find the integral surface for the differential equation

$$z\left(xz_x - yz_y\right) = y^2 - x^2$$

passing through the initial data curve (2s, s, s).

Solution : The quasi-linear p.d.e is given by

$$zxz_x - zyz_y = y^2 - x^2, \qquad \ldots (3.56)$$

subject to the initial conditions (2s, s, s)

i.e. $$x_0 = 2s, \quad y_0 = s, \quad z_0 = s. \qquad \ldots (3.57)$$

Here $P(x,y,z)=xz,\ \ Q(x,y,z)=-zy,\ \ R(x,y,z)=y^2-x^2$.

Therefore $P(x_0,y_0,z_0)=2s^2,\ \ Q(x_0,y_0,z_0)=-s^2,\ \ R(x_0,y_0,z_0)=s^2-4s^2=-3s^2$.

We observe from the admissibility condition that

$$\frac{dy_0}{ds}\cdot P-\frac{dx_0}{ds}\cdot Q=2s^2-2(-s)^2=4s^2\neq 0,\ \forall s>0. \quad \text{... (3.58)}$$

This shows that there exists unique solution z (x, y) satisfies the p.d.e. and the initial conditions.

We know the family of characteristic curves which generate surface are the solutions of the equations

$$\frac{dx}{dt}=P,\ \ \frac{dy}{dt}=Q,\ \ \frac{dz}{dt}=R.$$

i.e. $$\frac{dx}{dt}=xz,\ \ \frac{dy}{dt}=-zy,\ \ \frac{dz}{dt}=y^2-x^2, \quad \text{... (3.59)}$$

Satisfying the initial data (3.57). To solve equations (3.59) we write these equation as

$$\frac{dx}{xz}=\frac{dy}{-yz}=\frac{dz}{y^2-x^2}=dt. \quad \text{... (3.59a)}$$

$$\Rightarrow \frac{dy}{dx}=\frac{y}{x}.$$

Integrating we get

$$xy=C_1.$$

Now each ratio of equation (3.59a) $=\dfrac{xdx+ydy+zdz}{0}$

$$\Rightarrow xdx+ydy+zdz=0.$$

Integrating we get

$$x^2+y^2+z^2=C_2.$$

Hence the family of characteristic curves through the initial data (3.57) is given by

$$xy=2s^2,$$

$$x^2+y^2+z^2=6s^2.$$

If we choose

$$x=t\Rightarrow y=\frac{2s^2}{t}$$

and $$z^2 = 6s^2 - t^2 - 4s^2t^2.$$

Eliminating t and s we get the required integral surface

$$z^2 = 6s^2 - x^2 - x^2y^2x^{-2},$$

$$\Rightarrow z^2 = 3xy - x^2 - y^2.$$

This is a required integral surface.

Example 5 : Find the integral surface passing through $x = 1$, $z = y^2 + y$ of the equation

$$x^3z_x + y(3x^2 + y)z_y = z(2x^2 + y).$$

Solution : The given p.d.e. is

$$x^3z_x + y(3x^2 + y)z_y = z(2x^2 + y), \quad \text{... (3.60)}$$

subject to the initial conditions

$$x = 1, z = y^2 + y. \quad \text{... (3.61)}$$

Comparing (3.60) with the standard equation we have

$$P = x^3, Q = y(3x^2 + y), R = z(2x^2 + y). \quad \text{... (3.62)}$$

We choose the parameter $y_0 = s$

$$\Rightarrow z_0 = s^2 + s \Rightarrow z_0 = s(s+1).$$

Therefore, the initial data is

$$x_0 = 1, \ \ y_0 = s, \ \ z_0 = s(s+1). \quad \text{... (3.63)}$$

Hence we have

$$P(x_0, y_0, z_0) = 1, Q(x_0, y_0, z_0) = s(3+s), R(x_0, y_0, z_0) = s(s+1)(2+s).$$

We observe from the admissibility condition that

$$\frac{dy_0}{ds}P - \frac{dx_0}{ds}Q = 1 - 0 \neq 0.$$

This shows that there exists unique solution z (x, y) satisfying the p.d.e. and the initial conditions.

Thus the family of characteristic curves which generate surface are the solution of the equations

$$\frac{dx}{x^3} = \frac{dy}{y(3x^2 + y)} = \frac{dz}{z(2x^2 + y)} = dt \quad \text{... (3.64)}$$

$$\Rightarrow \frac{dx}{dt} = x^3, \frac{dy}{dt} = y(3x^2 + y), \frac{dz}{dt} = z(2x^2 + y), \quad \text{... (3.65)}$$

satisfying the initial data (3.63). To solve equations (3.65) consider the each ratio of equation (3.64)

$$= \frac{-\frac{1}{x}dx + \frac{1}{y}dy - \frac{1}{z}dz}{-x^2 + 3x^2 + y - 2x^2 - y} = \frac{-\frac{1}{x}dx + \frac{1}{y}dy - \frac{1}{z}dz}{0}$$

$$\Rightarrow -\frac{dx}{x} + \frac{dy}{y} - \frac{dz}{z} = 0 .$$

Integrating we get

$$-\log x + \log y - \log z = \log C_1$$

$$\Rightarrow \frac{y}{xz} = C_1 . \qquad \text{... (3.66)}$$

Now consider the ratios

$$\frac{dx}{x^3} = \frac{dy}{y\left(3x^2 + y\right)}$$

$$\Rightarrow \left(3x^2 + y\right)\frac{dx}{x^3} = \frac{dy}{y}$$

$$\Rightarrow 3x^2 y dx + y^2 dx = x^3 dy$$

$$\Rightarrow 3x^2 y dx - x^3 dy = -y^2 dx$$

$$\Rightarrow \frac{3x^2 y dx - x^3 dy}{y^2} = -dx \Rightarrow -dx = d\left(\frac{x^3}{y}\right).$$

Integrating we get

$$-x = \frac{x^3}{y} + C_2$$

or

$$\frac{x^3}{y} + x = C_2 . \qquad \text{... (3.67)}$$

Using the initial data (3.63) in equations (3.66) and (3.67) we get

$$C_1 = \frac{1}{s+1}, \qquad \text{... (3.68)}$$

$$C_2 = \frac{1}{s} + 1 . \qquad \text{... (3.69)}$$

Concequently, the family of characteristic curve is obtain by eliminating C_1 and C_2 from equations (3.66) and (3.67).

This gives $$\frac{y}{xz}=\frac{1}{s+1}, \qquad \text{... (3.70)}$$

and $$\frac{x^3}{y}+x=\frac{1}{s}+1. \qquad \text{... (3.71)}$$

If we choose $$x=t, \qquad \text{... (3.72)}$$

we get $$\frac{t^3}{y}+t=\frac{1}{s}+1,$$

$$\frac{t^3}{y}=\frac{1}{s}+1-t=\frac{1+s(1-t)}{s},$$

$$\frac{y}{t^3}=\frac{s}{1+s(1-t)} \Rightarrow y=\frac{st^3}{1+s(1-t)}. \qquad \text{... (3.73)}$$

Consequently, we have from equation (3.70)

$$\frac{st^3}{[1+s(1-t)]t}=\frac{z}{s+1} \Rightarrow z=\frac{s(s+1)t^2}{1+s(1-t)}. \qquad \text{... (3.74)}$$

Eliminating t and s between (3.72), (3.73), (3.74) we get

$$y=\frac{sx^3}{1+s(1-x)} \Rightarrow y[1+s(1-x)]=sx^3,$$

$$y+s(y-xy)=sx^3,$$

$$y=s\left[x^3-y+xy\right].$$

or $$\frac{y}{x^3-y+xy}=s. \qquad \text{... (3.75)}$$

Using (3.72) and (3.75) in (3.74) we get

$$z=\frac{x^2\left[\frac{y}{x^3-y+xy}\left(\frac{y}{x^3-y+xy}+1\right)\right]}{1+(1-x)\frac{y}{x^3-y+xy}}$$

$$z\left(x^3-y+xy\right)=y^2+x^2y$$

$$x^3z - x^2y + xyz - y^2 = yz$$

$$x^2(xz - y) + y(xz - y) = yz$$

$$(x^2 + y)(xz - y) = yz$$

This is the required integral surface.

(c) Non-Linear First Order Partial Differential Equations :

In this section we shall consider a method of finding integral surfaces of a non-linear partial differential equations of first order, which is based largely on geometrical ideas. The method was first developed by Cauchy and is called Cauchy's Method of Characteristics.

The method involves the following steps :

Step 1 : Let $f(x, y, z, p, q) = 0$ ... (3.76)

be a given p.d.e. and the initial data curve be

$$x = x_0(s),\ \ y = y_0(s),\ \ z = z_0(s). \quad \text{... (3.77)}$$

Using (3.76) and (3.77) determine the functions $p_0(s)$ and $q_0(s)$ such that

$$f(x_0(s),\ y_0(s),\ z_0(s),\ p_0(s),\ q_0(s)) = 0, \quad \text{... (3.78)}$$

and

$$\frac{dz_0}{ds} = p_0\frac{dx_0}{ds} + q_0\frac{dy_0}{ds} \quad \text{(These are called strip conditions)} \quad \text{... (3.79)}$$

Note : There could be several choices for $p_0(s)$ and $q_0(s)$. One can find unique solution for each such choice.

Step 2 : Once a choice for $p_0(s)$ and $q_0(s)$ is made (i.e. the initial strip is chosen) we can solve the following Cauchy characteristic equations

$$\frac{dx}{dt} = f_p,\ \ \frac{dy}{dt} = f_q,\ \ \frac{dz}{dt} = pf_p + qf_q,$$

$$\frac{dp}{dt} = -f_x - f_z p,\ \frac{dq}{dt} = -f_y - f_z q, \quad \text{... (3.80)}$$

subject to the initial conditions

$$x = x_0(s), y = y_0(s), z = z_0(s), p = p_0(s) \text{ and } q = q_0(s) \text{ at t} = 0. \quad \text{... (3.81)}$$

The corresponding characteristic curves $x = x(s,t), y = y(s,t), z = z(s,t)$ generate the required integral surface after eliminating s and t. Let the solution surface be in the $z = z(x, y)$.

The method is illustrated in the following example.

Example 1 : Determine the characteristics of the equation

$$z = p^2 - q^2$$

and find the integral surface which passes through the curve

$$x_0 = s,\ y_0 = 0,\ z_0 = -\frac{s^2}{4}. \qquad \text{(the parabola } 4z + x^2 = 0, y = 0 \text{).}$$

Solution : Let the given p.d.e. be denoted by

$$f(x, y, z, p, q) = p^2 - q^2 - z = 0. \qquad \text{... (3.82)}$$

The initial data curve is

$$x_0 = s,\ y_0 = 0,\ z_0 = -\frac{s^2}{4}. \qquad \text{... (3.83)}$$

To determine the function $p_0(s)$ and $q_0(s)$ we have the strip conditions.

$$f(x_0, y_0, z_0, p_0, q_0) = 0 \Rightarrow p_0{}^2 - q_0{}^2 + \frac{s^2}{4} = 0,$$

and $$\frac{dz_0}{ds} = p_0 \frac{dx_0}{ds} + q_0 \frac{dy_0}{ds} \Rightarrow -\frac{s}{2} = p_0. \qquad \text{... (3.84)}$$

$$\Rightarrow q_0{}^2 = \frac{s^2}{4} + \frac{s^2}{4} \Rightarrow q_0{}^2 = \frac{s^2}{2}$$

or $$q_0 = \frac{s}{\sqrt{2}}. \qquad \text{... (3.85)}$$

Now the Cauchy characteristic equations (3.80) become

$$\frac{dx}{dt} = 2p,$$

$$\frac{dy}{dt} = -2q,$$

$$\frac{dz}{dt} = 2p^2 - 2q^2,$$

$$\frac{dp}{dt} = p,$$

and $$\frac{dq}{dt} = q. \qquad \text{... (3.86)}$$

Thus we have

$$\frac{dx}{2p} = \frac{dy}{-2q} = \frac{dz}{2p^2 - 2q^2} = \frac{dp}{p} = \frac{dq}{q} = dt$$

The ratios $\frac{dx}{2p} = \frac{dp}{p}$ give

$$\Rightarrow x = 2p + C_1. \qquad \text{... (3.87)}$$

Now consider the ratios

$$\frac{dy}{-2q} = \frac{dq}{q} \Rightarrow y = -2q + C_2. \qquad \text{... (3.88)}$$

The conditions

$$\frac{dp}{p} = dt \Rightarrow \log p = t + \log C_3,$$

$$\Rightarrow p = C_3 e^t. \qquad \text{... (3.89)}$$

Similarly, $$\frac{dq}{q} = dt \Rightarrow \log q = t + \log C_4,$$

$$\Rightarrow q = C_4 e^t. \qquad \text{... (3.90)}$$

Now the equation $$\frac{dz}{2p^2 - 2q^2} = dt$$

gives $$dz = 2\left(C_3^2 e^{2t} - C_4^2 e^{2t}\right) dt,$$

$$dz = 2\left(C_3^2 - C_4^2\right) e^{2t} dt,$$

$$\Rightarrow z = \left(C_3^2 - C_4^2\right) e^{2t} + C_5. \qquad \text{... (3.91)}$$

Now on using the initial data (3.83) to (3.85) we have from equations (3.87) to (3.91) that

$$C_1 = s2, \quad C_2 = \sqrt{2}s, \quad C_3 = -\frac{s}{2}, \quad C_4 = \frac{s}{\sqrt{2}}, \quad C_5 = 0$$

Eliminating these constants, we have finally from equations (3.87) to (3.91)

$$x = -se^t + 2s, \qquad \text{... (3.92)}$$

$$y = \sqrt{2} \cdot s\left(1 - e^t\right), \qquad \text{... (3.93)}$$

$$p = -\frac{s}{2} e^t, \qquad \text{... (3.94)}$$

$$q = -\frac{s}{\sqrt{2}} e^t, \qquad \text{... (3.95)}$$

and
$$z = -\frac{s^2}{4} e^{2t}. \qquad \text{... (3.96)}$$

Solving (3.92) and (3.93) for s and e^t we get

$$s = \left(x - \frac{y}{\sqrt{2}} \right)$$

and
$$e^t = \frac{x - \sqrt{2}y}{x - \frac{y}{\sqrt{2}}}$$

Substituting in (3.96) we get

$$z = -\frac{1}{4}\left(x - \frac{y}{\sqrt{2}} \right)^2 \cdot \left(\frac{x - \sqrt{2}y}{x - \frac{y}{\sqrt{2}}} \right)$$

$$z = -\frac{1}{4}\left(x - \sqrt{2}y \right)^2. \qquad \text{... (3.97)}$$

This is the required integral surface.

Example 2 : Find by the method of characteristics, the integral surface of

$$pq = xy$$

which passes through the curve $z = x, y = 0$.

Solution : Let the given p.d.e. be

$$f(x, y, z, p, q) = pq - xy = 0, \qquad \text{... (3.98)}$$

and the initial data curve be

$$x_0(s) = s, y_0(s) = 0, z_0(s) = s. \qquad \text{... (3.99)}$$

Hence the equation $f(x_0, y_0, z_0, p_0, q_0) = 0$ becomes

$$p_0 q_0 - x_0 y_0 = 0$$

$$\Rightarrow p_0 q_0 - 0 = 0$$

$$\Rightarrow p_0(s) \cdot q_0(s) = 0 \qquad \text{... (3.100)}$$

Now the equation

$$\frac{dz_0}{ds} = p_0 \frac{dx_0}{ds} + q_0 \frac{dy_0}{ds}$$

becomes $$1 = p_0 \cdot 1 + q_0 \cdot 0 \qquad \text{... (3.101)}$$

$$\Rightarrow p_0 = 1. \qquad \text{(unique initial data)}$$

From equation (3.100)

$$\Rightarrow q_0 = 0. \qquad \text{... (3.102)}$$

Now the Cauchy characteristics equations (3.80) become

$$\frac{dx}{dt} = q, \quad \frac{dy}{dt} = p, \quad \frac{dz}{dt} = 2pq, \quad \frac{dp}{dt} = y, \quad \frac{dq}{dt} = x. \qquad \text{... (3.103)}$$

Thus from equations

$$\frac{dx}{dt} = q \text{ and } \frac{dq}{dt} = x,$$

we have $$\frac{d^2x}{dt^2} = \frac{dq}{dt} = x$$

$$\Rightarrow \frac{d^2x}{dt^2} - x = 0,$$

which has solution

$$x = ae^t + be^t \cdot \qquad \text{... (3.104)}$$

$$\Rightarrow q = \frac{dx}{dt} = ae^t - be^t. \qquad \text{... (3.105)}$$

Similarly, from equations

$$\frac{dy}{dt} = p, \quad \frac{dp}{dt} = y,$$

we have $$\frac{d^2y}{dt^2} = \frac{dp}{dt} = y$$

$$\Rightarrow \frac{d^2y}{dt^2} - y = 0 \Rightarrow y = ce^t + de^t. \qquad \text{... (3.106)}$$

Hence, $$p = \frac{dy}{dt} = ce^t - de^t \qquad \text{... (3.107)}$$

Therefore, $$\frac{dz}{2pq} = dt$$

$$\Rightarrow dz = 2\left(ce^t - de^t\right)\left(ae^t - be^{-t}\right)dt$$

$$dz = 2\left(ace^{2t} - bc - ad + bde^{-2t}\right)dt$$

$$\Rightarrow z = ace^{2t} - bde^{-2t} - 2(bc + ad)t + e \cdot \qquad \text{... (3.108)}$$

Using the initial data (3.99) to (3.102) at t = 0 we get

$$a = b = \frac{s}{2}, \qquad \text{... (3.109)}$$

$$c = -d = \frac{1}{2}, \qquad \text{...(3.100)}$$

and

$$s = ac - bd + e$$

$$\Rightarrow s = \frac{s}{4} + \frac{s}{4} + e \Rightarrow e = \frac{s}{2} \qquad \text{... (3.111)}$$

Finally, we have

$$x = s\left(\frac{e^t + e^{-t}}{2}\right) \Rightarrow x = s\cosh t, \qquad \text{... (3.112)}$$

$$y = \frac{e^t + e^{-t}}{2} \Rightarrow y = \sinh t, \qquad \text{... (3.113)}$$

$$z = \frac{s}{4}e^{2t} + \frac{s}{4}e^{-2t} - 2\left(\frac{s}{4} - \frac{s}{4}\right)t + \frac{s}{2},$$

$$z = \frac{s}{2}\left(\frac{e^{2t} + e^{-2t}}{2}\right) + \frac{s}{2}$$

$$\Rightarrow z = \frac{s}{2}[\cosh 2t + 1]$$

$$z = \frac{s}{2}\left[\cosh^2 t + \sinh^2 t + 1\right]$$

$$= \frac{s}{2}\left[\cosh^2 t + \cosh^2 t\right]$$

$$z = s\cosh^2 t \cdot \qquad \text{... (3.114)}$$

Now $$p=\frac{1}{2}\left(e^{t}+e^{-t}\right)\Rightarrow p=\cosh t\,. \qquad \text{... (3.115)}$$

$$q=s\left(\frac{e^{t}-e^{-t}}{2}\right)\Rightarrow q=s\sinh t\,. \qquad \text{... (3.116)}$$

Now eliminating s and t from (3.112), (3.113) and (3.114) we get

$$z^{2}=s^{2}\cosh^{4}t$$

$$=(s\cosh t)^{2}\cosh^{2}t$$

$$=x^{2}\left(1+\sinh^{2}t\right)$$

$$z^{2}=x^{2}\left(1+y^{2}\right). \qquad \text{... (3.117)}$$

This is the required integral surface through the given initial data curve.

Example 3 : Find the characteristics of the equation

$$pq=z$$

and determine the integral surface which passes through the parabola $x=0, y^2=z$.

Solution : Let the given p.d.e. be denoted by

$$f\left(x,y,z,p,q\right)=pq-z=0\,. \qquad \text{... (3.118)}$$

The initial data curve is

$$x_0=0, y_0=s, z_0=s^2\,. \qquad \text{... (3.119)}$$

To determine the functions $p_0(s)$ and $q_0(s)$, we have the strip conditions

$$f\left(x_0,y_0,z_0,p_0,q_0\right)=0$$

$$\Rightarrow p_0q_0-z_0=0\Rightarrow p_0q_0-s^2=0, \qquad \text{... (3.20)}$$

and $$\frac{dz_0}{ds}=p_0\frac{dx_0}{ds}+q_0\frac{dy_0}{ds}$$

$$\Rightarrow 2s=p_0(0)+q_0\Rightarrow q_0=2s\,. \qquad \text{... (3.121)}$$

Equation (3.120) $$\Rightarrow p_0=\frac{s}{2}. \qquad \text{... (3.122)}$$

Now the Cauchy characteristic equations (3.80) become

$$\frac{dx}{dt}=q\Rightarrow dx=C_2e^{t}dt\Rightarrow x=C_2e^{t}+C_3\,.$$

$$\frac{dy}{dt} = p \Rightarrow dy = C_1 e^t dt \Rightarrow y = C_1 e^t + C_4 ,$$

$$\frac{dz}{dt} = 2pq \Rightarrow \frac{dz}{z} = 2t \Rightarrow \log z = t^2 + \log C_5 ,$$

$$z = C_5 e^{t^2} ,$$

$$\frac{dp}{dt} = p \Rightarrow p = C_1 e^t ,$$

$$\frac{dq}{dt} = q \Rightarrow q = C_2 e^t . \qquad \text{... (3.123)}$$

Thus we have

$$\frac{dx}{q} = \frac{dy}{p} = \frac{dz}{2pq} = \frac{dp}{p} = \frac{dq}{q} = dt .$$

The ratios

$$\frac{dx}{q} = \frac{dq}{q} \Rightarrow x = q + C_1 , \qquad \text{... (3.124)}$$

and $$\frac{dy}{p} = \frac{dp}{p} \Rightarrow y = p + C_2 . \qquad \text{... (3.125)}$$

Also $$\frac{dp}{p} = dt \Rightarrow \log p = t + \log C_3$$

$$\Rightarrow p = C_3 e^t , \qquad \text{... (3.126)}$$

and $$\frac{dq}{q} = dt \Rightarrow \log q = t + \log C_4 \Rightarrow q = C_4 e^t . \qquad \text{... (3.127)}$$

Now $$\frac{dz}{2pq} = dt \Rightarrow dz = 2C_3 C_4 e^{2t} - dt$$

$$z = C_3 C_4 e^{2t} + C_5 . \qquad \text{... (3.128)}$$

Now using the initial data curve (3.119), (3.121) and (3.122) at t = 0, we have from (3.124)

$$0 = 2s + C_1 \Rightarrow C_1 = -2s ,$$

$$s = \frac{s}{2} + C_2 \Rightarrow C_2 = \frac{s}{2} ,$$

Equation (3.126) gives $\frac{s}{2} = C_3$.

Equation (3.127) gives $2s = C_4$,

and equation (3.128) gives $s^2 = \frac{s}{2}2s + C_5 \Rightarrow C_5 = 0$.

Finally we have on substituting the values of these constants in equations (3.124) to (3.128)

$$x = 2se^t - 2s, \quad \text{... (3.129)}$$

$$y = \frac{s}{2}e^t + \frac{s}{2} \Rightarrow y = \frac{s}{2}\left(e^t + 1\right), \quad \text{... (3.130)}$$

$$p = \frac{s}{2}e^t, \quad \text{... (3.131)}$$

$$q = 2se^t, \quad \text{... (3.132)}$$

and

$$z = s^2 e^{2t}. \quad \text{... (3.133)}$$

Now eliminating s and t between (3.129), (3.130) and (3.133) we have

$$\left.\begin{aligned} \frac{x}{2s} &= e^t - 1 \\ \frac{2y}{s} &= e^t + 1 \end{aligned}\right\} \text{ adding we get}$$

$$e^t = \frac{(x + 4y)}{4s}.$$

Putting this in (3.129) we get

$$x = 2s\left[\frac{x + 4y}{4s} - 1\right]$$

$$x = 2s\left(\frac{x + 4y - 4s}{4s}\right)$$

$$\Rightarrow 2x = x + 4y - 4s$$

or

$$4s = -x + 4y$$

Therefore,

$$s = \frac{4y - x}{4}$$

Hence

$$e^t = \frac{x + 4y}{4y - x}.$$

Substituting this in (3.133) we get

$$z=\left(\frac{4y-x}{4}\right)^2\left(\frac{x+4y}{4y-x}\right)^2$$

$$z=\left(\frac{x+4y}{4}\right)^2.$$

This is the required integral surface.

Example 4 : Find by the method of characteristics the integral surface of the equation

$$p^2x+qy-z=0$$

which passes through the initial data y = 1, x + z = 0.

Solution : Let the given p.d.e. be denoted by

$$f(x,y,z,p,q)=p^2x+qy-z=0. \qquad \text{... (3.134)}$$

The initial data curve is

$$x_0=s, y_0=1, z_0=-s. \qquad \text{... (3.135)}$$

To determine the functions $p_0(s)$ and $q_0(s)$, we have the strip conditions

$$f(x_0,y_0,z_0,p_0,q_0)=0 \Rightarrow p_0{}^2s+q_0+s=0, \qquad \text{... (3.136)}$$

and $$\frac{dz_0}{ds}=p_0\frac{dx_0}{ds}+q_0\frac{dy_0}{ds}\Rightarrow -1=p_0+q_0(0)\Rightarrow p_0=-1. \qquad \text{... (3.137)}$$

Therefore equation (3.136) give

$$s+q_0+s=0\Rightarrow q_0=-2s. \qquad \text{... (3.138)}$$

Now the Cauchy characteristic equations (3.80) reduce to

$$\frac{dx}{dt}=2px,$$

$$\frac{dy}{dt}=y,$$

$$\frac{dz}{dt}=2p^2x+qy,$$

$$\frac{dp}{dt}=-p^2+p,$$

$$\frac{dq}{dt}=-q+q\Rightarrow\frac{dq}{dt}=0. \qquad \text{... (3.139)}$$

The equation $\frac{dy}{dt} = y$ gives

$$\Rightarrow y = C_1 e^t \qquad \text{... (3.140)}$$

The equation $$\frac{dp}{dt} = -p(p-1) \Rightarrow -\frac{dp}{p(p-1)} = dt$$

$$\Rightarrow dp\left(\frac{1}{p} - \frac{1}{p-1}\right) = dt$$

Integrating we get

$$\log p - \log(p-1) = t + \log C_2$$

$$\Rightarrow \frac{p}{(p-1)} = C_2 e^t \Rightarrow p = \frac{C_2 e^t}{(C_2 e^t - 1)}. \qquad \text{... (3.141)}$$

The equation $$\frac{dq}{dt} = 0 \Rightarrow q = C_3. \qquad \text{... (3.142)}$$

The equation $$\frac{dx}{dp} = \frac{2px}{-p(p-1)} \Rightarrow \frac{dx}{dp} = -\frac{2x}{(p-1)}$$

or $$\frac{dx}{x} = -\frac{2dp}{p-1}.$$

Integrating we get

$$\log x = -2\log(p-1) + \log C_4$$

$$\Rightarrow x = C_4 (p-1)^{-2}. \qquad \text{... (3.143)}$$

From (3.141) $$p = \frac{C_2 e^t}{C_2 e^t - 1} \Rightarrow p - 1 = \frac{C_2 e^t}{C_2 e^t - 1} - 1$$

$$\Rightarrow p - 1 = \frac{1}{C_2 e^t - 1} \Rightarrow (p-1)^{-2} = (C_2 e^t - 1)^2$$

Hence $$x = C_4 (C_2 e^t - 1)^2. \qquad \text{... (3.144)}$$

Now the equation $\frac{dz}{dt} = 2p^2 x + qy$ becomes

$$\frac{dz}{dt} = 2C_4\left(\frac{C_2e^t}{C_2e^t - 1}\right)^2 \left(C_2e^t - 1\right)^2 + C_3C_1e^t$$

$$\frac{dz}{dt} = 2C_2{}^2C_4e^{2t} + C_1C_3e^t$$

$$\Rightarrow dz = 2C_2{}^2C_4\left(e^{2t}dt\right) + C_1C_3\left(e^t dt\right)$$

Integrating we get

$$z = C_2{}^2C_4e^{2t} + C_1C_3e^t + C_5. \qquad \text{... (3.145)}$$

Now using the initial data curve

$$x_0 = s, y_0 = 1, z_0 = -s, p_0 = -1 \text{ and } q_0 = -2s,$$

we have from equation (3.140) $\Rightarrow C_1 = 1$.

From equation (3.141) we have $-1 = \dfrac{C_2}{C_2 - 1} \Rightarrow C_2 = -C_2 + 1$.

From equation (3.142) we find $C_3 = -2s$.

From equation (3.143) we have $s = C_4(-1-1)^{-2} \Rightarrow s = \dfrac{C_4}{4} \Rightarrow C_4 = 4s$.

From equation (3.145) we find $C_5 = 0$.

Thus the family of characteristic curves are given by

$$x = C_4\left(C_2e^t - 1\right)^2$$

$$\Rightarrow x = s\left(e^t - 2\right)^2,$$

$$y = e^t, \quad z = se^{2t} - 2se^t, \qquad \text{.... (3.146)}$$

$$p = \frac{\frac{1}{2}e^t}{\left(\frac{1}{2}e^t - 1\right)} \Rightarrow p = \frac{e^t}{e^t - 2} \text{ and } q = -2s.$$

Solving the equations (3.146) for s and t we get

$$s = \frac{x}{(y-2)^2} \text{ and } e^t = y.$$

Putting this in the expression for z we get

$$z = \frac{x}{(y-2)^2} y^2 - \frac{2xy}{y-2} \quad \text{or} \quad \Rightarrow z = \frac{xy}{x-y}.$$

This is the required integral surface.

Example 5 : Find the integral surface of the equation

$$z = p^2 - 3q^2$$

passing through $C : x_0 = s, y_0 = 0, z_0 = s^2$.

Show that there are two possible initial strips $p_0 = 2s$, $q_0 = \pm s$.

Solution : Let the given non-linear p.d.e. be given by

$$f(x, y, z, p, q) = p^2 - 3q^2 - z = 0, \quad \text{... (3.147)}$$

with the initial conditions

$$x_0 = s, y_0 = 0, z_0 = s^2. \quad \text{... (3.148)}$$

To determine initial strip, we have the strip conditions

$$f(x_0, y_0, z_0, p_0, q_0) = 0,$$

$$\Rightarrow p_0^{\,2} - 3q_0^{\,2} - z_0 = 0 \Rightarrow p_0^{\,2} - 3q_0^{\,2} = s^2, \quad \text{... (3.149)}$$

and $$\frac{dz_0}{ds} = p_0 \frac{dx_0}{ds} + q_0 \frac{dy_0}{ds}$$

$$\Rightarrow p_0 = 2s. \quad \text{... (3.150)}$$

Hence equation (3.149) gives $q_0 = \pm s$.

Hence the initial strip is

$$x_0 = s, y_0 = 0, z_0 = s^2, p_0 = 2s, q_0 = \pm s. \quad \text{... (3.151)}$$

Now the Cauchy characteristic equations (3.80) are given by

$$\frac{dx}{dt} = 2p,$$

$$\frac{dy}{dt} = -6q,$$

$$\frac{dz}{dt} = 2p^2 - 6q^2,$$

$$\frac{dp}{dt} = p,$$

$$\frac{dq}{dt} = q .$$

$$\Rightarrow \frac{dx}{dp} = 2 .$$

Integrating we get $x = 2p + C_1 .$ ... (3.152)

Similarly, $\frac{dy}{dq} = -6 \Rightarrow dy = -6dq$ ... (3.153)

$$\Rightarrow y = -6q + C_2$$

$$\frac{dp}{dt} = p \Rightarrow \log p = t + \log C_3$$

$$\Rightarrow p = C_3 e^t$$

Hence equation (3.152) gives

$$x = 2C_3 e^t + C_1 . \quad \text{... (3.154)}$$

Now $\frac{dq}{dt} = q \Rightarrow q = C_4 e^t .$ ... (3.155)

Consequently, equation (3.153) gives

$$y = -6C_4 e^t + C_2 .$$

Now $\frac{dz}{dt} = 2p^2 - 6q^2 = 2z ,$

$$\Rightarrow \frac{dz}{z} = 2dt .$$

Integrating we get

$$\Rightarrow z = C_5 e^{2t} . \quad \text{... (3.156)}$$

Now using the initial data curve (3.151) we have from above equations (3.152) to (3.156)

$$s = 4s + C_1 \Rightarrow C_1 = -3s ,$$

$$0 = -6s + C_2 \Rightarrow C_2 = 6s ,$$

$$2s = C_3 \Rightarrow C_3 = 2s ,$$

$$s = C_4 \Rightarrow C_4 = s ,$$

$$s^2 = C_5 \Rightarrow C_5 = s^2 .$$

Therefore, the characteristic curves are given by

$$\left.\begin{aligned} &x = 4se^t - 3s \Rightarrow x = 4s\left(e^t - 1\right) + s, \\ &y = -6se^t + 6s \Rightarrow y = -6s\left(e^t - 1\right), \\ &z = s^2 e^{2t}, \\ &p = 2se^t, \\ &q = se^t. \end{aligned}\right\} \qquad \text{.... (3.157)}$$

The integral surface is obtained by eliminating s and t from (3.157) we get

$$z = \left(\frac{3x+2t}{3}\right)^2 \left[1 - \frac{y}{2(3x+2y)}\right]^2$$

$$z = \frac{9(2x+2y)^2}{36} \Rightarrow z = \frac{(2x+y)^2}{4}. \qquad \text{... (3.158)}$$

Exercise :

1. Find the characteristics of the equation $pq = z$ and determine the integral surface which passes through the straight line

$$x = 1,\ z = y.$$

2. Find the characteristics of the equation

$$p^2 + q^2 = 2$$

and determine the integral surface which passes through $x = 0, z = y$.

UNIT - V

SECOND ORDER PARTIAL DIFFERENTIAL EQUATIONS

Introduction :

Partial differential equations of second order describe the physical behaviour of many practical situations in science and engineering. We will see how such second order partial differential equations arise in physics and engineering mathematics. Further, in many situations a given partial differential equation of second order is difficult to solve, hence in this unit we classify the second order partial differential equation in to elliptic, parabolic and hyperbolic forms by transforming it into canonical form. The idea of reducing the given partial differential equation to a canonical form is that the transformed equation assumes a simple form so that the subsequent analysis of solving the equation is easy. We also discuss the methods of separation of variables of solving second order partial differential equation.

Definition : A semi-linear second order partial differential equation is expressed in the form.

$$Ru_{xx} + Su_{xy} + Tu_{yy} + g\left(x, y, u, u_x, u_y\right) = 0, \quad \text{... (1.1)}$$

where R, S, T are continuous functions of x and y only and $R^2 + S^2 + T^2 \neq 0$, u is a dependent variable. Equation (1.1) can also be written as

$$Rr + Ss + Tt + g\left(x, y, u, u_x, u_y\right) = 0, \quad \text{... (1.1a)}$$

where $r = u_{xx}$, $s = u_{xy}$, $t = u_{yy}$.

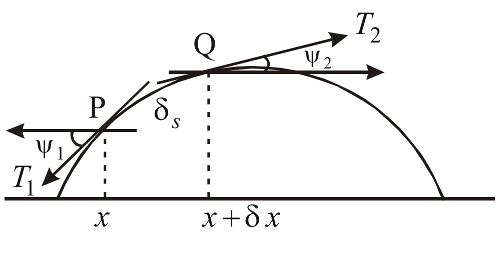
Solution of the Equation :

Definition : A function $u = f(x, y)$ is said to be a regular solution of equation (1.1) in a domain $D \subset \mathbb{R} \times \mathbb{R}$ iff $f(x, y) \in$ on D and the function and its derivatives satisfy equation (1.1) identically.

Origin of Partial Differential Equation :

One dimensional wave equation :

Result : Derive an equation governing small transverse vibrations of an elastic string.



Proof : Let an elastic string be stretched to a length ℓ and then fixed at the end points. Let the string be distorted and further let at time t = 0 it be released and allowed to vibrate. Our aim is to obtain the equation which governs the deflection $y(x, t)$ at any point x after any time t > 0.

Let y = y (x, t) be the displacement from the mean position (x-axis) of a string at time t at point x. Let δs be the small portion of the string between two points P amd Q. We assume that the string is homogeneous (i.e. mass per unit length is constant) perfectly elastic (i.e. does not offer any resistence on bending) and weight of the string is neglected (i.e. action of the gravitational force on the string is neglected).

In order to find the differential equation which describes the motion of string, we consider the forces acting on the portion δs. Let T_1 and T_2 be the tensions at points P and Q respectively acting along the tangential direction. Since there is no motion of the string in the horizontal direction, therefore, the horizontal components of the tensions will be constants.

$$\Rightarrow T_1 \cos\psi_1 = T_2 \cos\psi_2 = \text{constant} = T \text{ (say).} \qquad \text{... (1.2)}$$

The resultant vertical force acting on the portion PQ is

$$T_2 \sin\psi_2 - T_1 \sin\psi_1 .$$

Hence, the equation of motion is given by

Force = Mass • Acceleration

$$\Rightarrow T_2 \sin\psi_2 - T_1 \sin\psi_1 = \rho\delta s \cdot \frac{\partial^2 y}{\partial t^2}, \qquad \text{... (1.3)}$$

where ρ is the density of the string and $\rho\delta s$ is the mass of the portion PQ and $\frac{\partial^2 y}{\partial t^2}$ is the acceleration in the vertical direction.

We write from equation (1.3) that

$$\frac{T_2 \sin\psi_2 - T_1 \sin\psi_1}{T} = \rho \frac{\delta s}{T} \cdot \frac{\partial^2 y}{\partial t^2}$$

$$\Rightarrow \frac{T_2 \sin\psi_2}{T_2 \cos\psi_2} - \frac{T_1 \sin\psi_1}{T_1 \cos\psi_1} = \rho \frac{\delta s}{T} \cdot \frac{\partial^2 y}{\partial t^2}$$

or

$$\tan\psi_2 - \tan\psi_1 = \rho \frac{\delta s}{T} \cdot \frac{\partial^2 y}{\partial t^2}. \qquad \text{... (1.4)}$$

Since $\tan\psi_1$ and $\tan\psi_2$ are the slopes of the curve of the string at points P and Q respectively, therefore we have by definition.

$$\tan\psi_1 = \left(\frac{\partial y}{\partial x}\right)_P = \left(\frac{\partial y}{\partial x}\right)_x \text{ and}$$

$$\tan\psi_2 = \left(\frac{\partial y}{\partial x}\right)_Q = \left(\frac{\partial y}{\partial x}\right)_{x+\delta x}.$$

Hence equation (1.4) beomes

$$\left(\frac{\partial y}{\partial x}\right)_{x+\delta x}-\left(\frac{\partial y}{\partial x}\right)_{x}=\rho\frac{\delta s}{T}\frac{\partial^2 y}{\partial t^2}.$$

In the limiting case as $\delta x \to 0$ i.e. $Q \to P$, we have

$$\delta s=\delta x,$$

therefore, we write

$$\lim_{\delta x\to 0}\left[\frac{\left(\frac{\partial y}{\partial x}\right)_{x+\delta x}-\left(\frac{\partial y}{\partial x}\right)_{x}}{\delta x}\right]=\frac{\rho}{T}\frac{\partial^2 y}{\partial t^2},$$

$$\Rightarrow \frac{\partial^2 y}{\partial x^2}=\frac{\rho}{T}\frac{\partial^2 y}{\partial t^2},$$

$$\Rightarrow \frac{\partial^2 y}{\partial x^2}=\frac{1}{c^2}\frac{\partial^2 y}{\partial t^2}, \qquad \text{... (1.5)}$$

where $c^2=\frac{T}{\rho}$, and c represents the speed of the wave propagation. Equation (1.5) is called the one dimensional wave equation.

Heat Conduction Equation :

Result : Derive the second order partial differential equation which describes the temperature distribution in a homogeneous isotropic solid.

Note : Homogeneous means distribution of material is uniform, isotropic means the material properties are the same in all directions.

Specific heat of the solid means the amount of heat absorbed by the matter per unit mass per unit rise in temperature.

Density of the solid means mass per unit volume.

Proof : Consider a homogeneous isotropic solid and V be any arbitrary volume inside the solid bounded by a surface S. Let δV be a volume element. We denote

c : the specific heat of the solid,

ρ : the density of the solid and,

u : the temperature which is a function of position and time.

Hence the heat energy stored in the volume element δV is equal to $c\rho u\delta V$.

Hence the total heat energy in the volume V is given by $\iiint_V c\rho u\,dV$ ... (1.6)

If δS is a surface element, then the heat flow across $\delta S = k \cdot \nabla u \bar{n} \delta S$ ... (1.7)

where $\bar{n}$ is the outward normal to the surface S,

k - the thermal conductivity of the solid.

Hence the total flux across $S = \iint_S k \nabla u \bar{n} dS$. ... (1.8)

Using the Gauss-Divergence theorem, we write

Total flux across $S = \iiint_V \nabla \cdot (k \nabla u) dV$ ... (1.9)

Since the rate of change of heat energy in V is equal to the flux of heat energy across S. Therefore from equations (1.6) and (1.9) we have

$$\frac{\partial}{\partial t} \iiint_V c\rho u dV = \iiint_V \nabla \cdot (k \nabla u) dV,$$

$$\Rightarrow \iiint_V \left[\frac{\partial}{\partial t}(c\rho u) - \nabla \cdot (k \nabla u) \right] dV = 0. \quad \text{... (1.10)}$$

Since V is an arbitrary volume, we have therefore

$$\frac{\partial}{\partial t}(c\rho u) - \nabla (k \nabla u) = 0,$$

$$\Rightarrow c\rho \frac{\partial u}{\partial t} - \nabla (k \nabla u) = 0.$$

If the thermal conductivity k is constant through out the body, then we have

$$c\rho \frac{\partial u}{\partial t} - k \nabla^2 u = 0.$$

or $$\frac{\partial u}{\partial t} = K \nabla^2 u, \quad \text{... (1.11)}$$

where $K = \frac{k}{c\rho}$ represents the heat conductivity, and

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}.$$

Equation (1.11) is the required heat conduction equation.

Note : One dimensional heat equation is given by

$$\frac{\partial u}{\partial t} = K \frac{\partial^2 u}{\partial x^2}. \quad \text{... (1.12)}$$

There are some equations arise in physics. One of the most important partial differential equations in Physics is the Laplace equation given by

$$\nabla^2 u = 0$$

i.e. $$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}=0. \qquad \text{... (1.13)}$$

Note : The heat equation (1.11) reduces to Laplace equation when the temperature u does not change with time t.

i.e. $\frac{\partial u}{\partial t}=0$ then equation (1.11) becomes

$$\nabla^2 u = 0 \cdot$$

Note : 2-dimensional Laplace equation is given by

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0. \qquad \text{... (1.14)}$$

Classification of second order Partial Differential Equation :

Result : By a suitable change of the independent variables, show that a second order partial differential equation

$$Rr+Ss+Tt+g\left(x,y,u,u_x,u_y\right)=0$$

can be reduced to one of the canonical forms on the basis of

$$S^2-4RT>0,\;\; S^2-4RT=0,\;\; S^2-4RT<0 \cdot$$

Proof : A semi-linear second order partial differential equation can also be written as

$$Lu+g\left(x,y,u,u_x,u_y\right)=0, \qquad \text{... (1.15)}$$

where $$L=R\frac{\partial^2}{\partial x^2}+S\frac{\partial^2}{\partial x\partial y}+T\frac{\partial^2}{\partial y^2}, \qquad \text{... (1.16)}$$

(x, y) are independent variables and u the dependent. We change the independent variables x, y to new independent variables ξ, η by means of the transformation equations

$$\xi=\xi(x,y),\;\; \eta=\eta(x,y), \qquad \text{... (1.17)}$$

where $$J=\frac{\partial(\xi,\eta)}{\partial(x,y)}=\begin{vmatrix}\xi_x & \xi_y\\ \eta_x & \eta y\end{vmatrix}=\xi_x\eta_y-\xi_y\eta_x\neq 0.$$

Then by using the chain rule of partial differentiation we obtain,

$$u_x = u_\xi \xi_x + u_\eta \eta_x ,$$

$$u_y = u_\xi \xi_y + u_\eta \eta_y ,$$

$$u_{xy} = u_{\xi\xi}\xi_x\xi_y + u_{\xi\eta}\xi_x\eta_y + u_\xi \xi_{xy} + u_{\eta\xi}\xi_y\eta_x + u_{\eta\eta}\eta_x\eta_y + u_\eta \eta_{xy} .$$

Similarly, we find

$$u_{xx} = u_{\xi\xi}\xi_x\xi_x + u_{\xi\eta}\xi_x\eta_x + u_\xi \xi_{xx} + u_{\eta\xi}\xi_x\eta_x + u_{\eta\eta}\eta_x\eta_x + u_\eta \eta_{xx} ,$$

$$u_{yy} = u_{\xi\xi}\xi_y\xi_y + u_{\xi\eta}\xi_y\eta_y + u_\xi \xi_{yy} + u_{\eta\xi}\xi_y\eta_y + u_{\eta\eta}\eta_y\eta_y + u_\eta \eta_{yy} .$$

Hence the operator (1.16) becomes

$$\begin{aligned} Lu = R\left[u_{\xi\xi}\xi_x^2 + 2u_{\xi\eta}\xi_x\eta_x + u_{\eta\eta}\eta_x^2 + u_\xi\xi_{xx} + u_\eta\eta_{xx}\right] + \\ +S\left[u_{\xi\xi}\xi_x\xi_y + u_{\xi\eta}\left(\xi_x\eta_y + \xi_y\eta_x\right) + u_{\eta\eta}\eta_x\eta_y + u_\xi\xi_{xy} + u_\eta\eta_{xy}\right] + \\ +T\left[u_{\xi\xi}\xi_y^2 + 2u_{\xi\eta}\xi_y\eta_y + u_{\eta\eta}\eta_y^2 + u_\xi\xi_{yy} + u_\eta\eta_{yy}\right]. \end{aligned}$$

We write this equation as

$$\begin{aligned} Lu = u_{\xi\xi}\left(R\xi_x^2 + S\xi_x\xi_y + T\xi_y^2\right) + u_{\xi\eta}\left[2R\xi_x\eta_x + S\left(\xi_x\eta_y + \xi_y\eta_x\right) + 2T\xi_y\eta_y\right] + \\ +u_{\eta\eta}\left(R\eta_x^2 + S\eta_x\eta_y + T\eta_y^2\right) + R\left(u_\xi\xi_{xx} + u_\eta\eta_{xx}\right) + \\ +S\left(u_\xi\xi_{xy} + u_\eta\eta_{xy}\right) + T\left(u_\eta\xi_{yy} + u_\xi\eta_{yy}\right) \end{aligned}$$

Substituting this in equation (1.15) we get

$$A\left(\xi_x,\xi_y\right)u_{\xi\xi} + 2B\left(\xi_x,\xi_y;\eta_x,\eta_y\right)u_{\xi\eta} + A\left(\eta_x,\eta_y\right)u_{\eta\eta} = G\left(\xi,\eta,u,u_\xi,u_\eta\right), \quad \text{... (1.18)}$$

where,

$$A(u,v) = Ru^2 + Suv + Tv^2 , \quad \text{... (1.19)}$$

$$B\left(u_1,v_1;u_2,v_2\right) = Ru_1u_2 + \frac{1}{2}S\left(u_1v_2 + u_2v_1\right) + Tv_1v_1 , \quad \text{... (1.20)}$$

and A, B satisfy the equation

$$A\left(\xi_x,\xi_y\right)\cdot A\left(\eta_x,\eta_y\right) - B^2\left(\xi_x,\xi_y;\eta_x,\eta_y\right) = \frac{1}{4}\left(4RT - S^2\right)\left(\xi_x\eta_y - \xi_y\eta_x\right)^2 . \quad \text{... (1.21)}$$

We see that the transformed equation (1.18) has the same form as that of the original equation (1.15) under the transformation (1.17). Since the classification of (1.15) depends on $S^2 - 4RT$; therefore we choose the new independent variables ξ and η so that the equation (1.18) takes the simplest possible form. Thus the equation (1.18) will reduce to its simplest integrable form if the discriminant $S^2 - 4RT$ of the quadratic equation

$$R\lambda^2 + S\lambda + T = 0 \quad \text{... (1.22)}$$

is either positive, zero or negative every where.

Case (i) : Let $S^2 - 4RT > 0$.

In this case the roots λ_1, λ_2 of the equation (1.22) will be real and distinct. Thus we choose ξ and η such that

$$\xi_x = \lambda_1 \xi_y, \qquad \text{... (1.23)}$$

and

$$\eta_x = \lambda_2 \eta_y. \qquad \text{... (1.24)}$$

These are the first order partial differential equations for ξ and η.

Solving equation (1.23) by Lagrange's method, we have

$$\frac{dx}{1} = \frac{-dy}{-\lambda_1} = \frac{dt}{0}$$

$$\Rightarrow \frac{dy}{dx} \neq \lambda_1(x, y) = 0. \qquad \text{... (1.25)}$$

Similarly, from equation (1.24) we find

$$\frac{dy}{dx} + \lambda_2(x, y) = 0. \qquad \text{... (1.26)}$$

If $f_1(x, y) = C_1$ and $f_2(x, y) = C_2$ are the solutions of the ordinary differential equations (1.25) and (1.26) respectively, and are called the characteristic curves of the equation (1.15), then we choose

$$\xi = f_1(x, y),$$

and

$$\eta = f_2(x, y). \qquad \text{... (1.27)}$$

The variables ξ, η are called the characteristic variables. For this choice of ξ and η we have

$$A(\xi_x, \xi_y) = R\xi_x^2 + S\xi_x\xi_y + T\xi_y^2$$

$$= R\lambda_1^2\xi_y^2 + S\lambda_1\xi_y^2 + T\xi_y^2$$

$$\Rightarrow A(\xi_x, \xi_y) = (R\lambda_1^2 + S\lambda_1 + T)\xi_y^2.$$

As λ_1 is a root of equation (1.22), we have therefore

$$A(\xi_x, \xi_y) = 0.$$

Similarly, we show that

$$A(\eta_x, \eta_y) = \eta_y^2(R\lambda_2^2 + S\lambda_2 + T) = 0,$$

$$\Rightarrow A = 0.$$

Consequently, equation (1.18) reduces to

$$2B\left(\xi_x,\xi_y;\eta_x,\eta_y\right)u_{\xi\eta}=G\left(\xi,\eta,u,u_\xi,u_\eta\right). \quad \text{... (1.28)}$$

Since A = 0 and $S^2-4RT>0$, then from equation (1.21) we have

$$B^2>0 \Rightarrow B\neq 0\cdot$$

Thus we have from equation (1.28)

$$u_{\xi\eta}=\frac{G}{2B}=\phi\left(\xi,\eta,u,u_\xi,u_\eta\right),$$

or

$$u_{\xi\eta}=\phi\left(\xi,\eta,u,u_\xi,u_\eta\right). \quad \text{... (1.29)}$$

This is the desired canonical form of the equation (1.15). This form (1.29) is called hyperbolic form of equation (1.15).

Case (ii) : Let $S^2-4RT=0$

In this case the roots of the equation (1.22) are equal say $\lambda_1=\lambda_2=\lambda$

We choose ξ such that

$$\xi_x=\lambda\xi_y,$$

and η to be any arbitrary function of x and y independent of ξ. This is the Lagranges form of the equation, solving we obtain

$$\xi=f\left(x,y\right),$$

where $f\left(x,y\right)=C$ is a solution of the equation

$$\frac{dy}{dx}+\lambda\left(x,y\right)=0.$$

Since

$$A\left(\xi_x,\xi_y\right)=\xi_y^2\left(R\lambda^2+S\lambda+T\right)=0, \qquad \text{due to (1.22)}$$

$$\Rightarrow A\left(\xi_x,\xi_y\right)=0,$$

and $A\left(\eta_x,\eta_y\right)\neq 0$, otherwise η would be function of ξ. Hence from equation (1.21) we have B = 0. Putting these values in equation (1.18) we get

$$A\left(\eta_x,\eta_y\right)u_{\eta\eta}=G\left(\xi,\eta,u,u_\xi,u_\eta\right)$$

or

$$u_{\eta\eta}=\phi\left(\xi,\eta,u,u_\xi,u_\eta\right). \quad \text{... (1.30)}$$

This is the desired canonical form of the equation (1.15) and is called the parabolic form of (1.15).

Case (iii) : Let $S^2 - 4RT < 0$

In this case the roots of the quadratic equation (1.22) are complex. We choose ξ and η as in the case (i), so that

$$A(\xi_x, \xi_y) = 0 = A(\eta_x, \eta_y),$$

and equation (1.18) reduces to

$$u_{\xi\eta} = \phi(\xi, \eta, u, u_\xi, u_\eta). \qquad \text{... (1.31)}$$

This is similar to equation (1.29) except that the variables ξ, η are not real but are the complex conjugates. Hence to obtain a real canonical form we make the transformation

$$\alpha = \frac{1}{2}(\xi + \eta),$$

and

$$\beta = -\frac{1}{2}(\xi - \eta). \qquad \text{... (1.32)}$$

Hence by using the chain rule of partial differentiation we have,

$$u_\eta = u_\alpha \alpha_\eta + u_\beta \beta_\eta .$$

Using (1.32) we find

$$\Rightarrow u_\eta = \frac{1}{2}(u_\alpha + iu_\beta)$$

and

$$u_{\xi\eta} = (u_\eta)_\alpha \alpha_\xi + (u_\eta)_\beta \beta_\xi$$

$$u_{\xi\eta} = \frac{1}{4}(u_{\alpha\alpha} + iu_{\beta\alpha}) - \frac{i}{4}(u_{\alpha\beta} + iu_{\beta\beta})$$

$$\Rightarrow u_{\xi\eta} = \frac{1}{4}(u_{\alpha\alpha} + u_{\beta\beta})$$

Hence equation (1.31) becomes

$$u_{\alpha\alpha} + u_{\beta\beta} = \phi(\alpha, \beta, u, u_\alpha, u_\beta). \qquad \text{... (1.33)}$$

This is the required real canonical form and is called an elliptic form of partial differential equation.

Thus we define the three types of canonical forms as follows :

Definition : A partial differential equation of second order viz.

$$Rr + Ss + Tt + g(x, y, u, u_x, u_y) = 0$$

is said to

(i) hyperbolic if $S^2 - 4RT > 0$ and the corresponding canonical form is given by

$$u_{\xi\eta} = \phi\left(\xi, \eta, u, u_\xi, u_\eta\right),$$

(ii) parabolic if $S^2 - 4RT = 0$, and the corresponding canonical form is

$$u_{\eta\eta} = \phi\left(\xi, \eta, u, u_\xi, u_\eta\right),$$

(iii) elliptic if $S^2 - 4RT < 0$ and the corresponding canonical form is

$$u_{\alpha\alpha} + u_{\beta\beta} = \phi\left(\alpha, \beta, u, u_\alpha, u_\beta\right).$$

Example 1 : Show that

$$A\left(\xi_x, \xi_y\right) A\left(\eta_x, \eta_y\right) - B^2\left(\xi_x, \xi_y; \eta_x, \eta_y\right) = \frac{1}{4}\left(4RT - S^2\right)\left(\xi_x \eta_y - \xi_y \eta_x\right)^2,$$

where $A\left(\xi_x, \xi_y\right) = R\xi_x^2 + S\xi_x\xi_y + T\xi_y^2$, $A\left(\eta_x, \eta_y\right) = R\eta_x^2 + S\eta_x\eta_y + T\eta_y^2$,

and $B\left(\xi_x, \xi_y; \eta_x, \eta_y\right) = R\xi_x\eta_x + \frac{1}{2} S\left(\xi_x\eta_y + \xi_y\eta_x\right) + T\xi_y\eta_y$.

Solution : Consider

$$A\left(\xi_x, \xi_y\right) A\left(\eta_x, \eta_y\right) - B^2\left(\xi_x, \xi_y; \eta_x, \eta_y\right) = R^2\xi_x^2\eta_x^2 + RS\xi_x^2 \cdot \eta_x\eta_y + RT\xi_x^2\eta_y^2 + RS\eta_y^2\xi_x\xi_y +$$

$$+S^2\xi_x\xi_y\eta_x\eta_y + ST\xi_x\xi_y\eta_y^2 + RT\eta_x^2\xi_y^2 + ST\xi_y^2\eta_x\eta_y$$

$$+T^2\xi_y^2\eta_y^2 - R^2\xi_x^2\eta_x^2 - \frac{1}{4}S^2\left(\xi_x^2\eta_y^2 + \xi_y^2\eta_x^2 + 2\xi_x\xi_y\eta_x\eta_y\right) -$$

$$-T^2\xi_y^2\eta_y^2 - \cdot RS\xi_x\eta_x\left(\xi_x\eta_y + \xi_y\eta_x\right) - 2RT\xi_x\xi_y\eta_x\eta_y -$$

$$-ST\xi_y\eta_y\left(\xi_x\eta_y + \xi_y\eta_x\right)$$

$$\Rightarrow A\left(\xi_x, \xi_y\right) A\left(\eta_x, \eta_y\right) - B^2\left(\xi_x, \xi_y; \eta_x, \eta_y\right) = -\frac{1}{4}S^2\left(\xi_x^2\eta_y^2 + \xi_y^2\eta_x^2 - 2\xi_x\xi_y\eta_x\eta_y\right) +$$

$$+RT\left(\xi_x^2\eta_y^2 + \eta_x^2\xi_y^2 - 2\xi_x\eta_x\xi_y\eta_y\right)$$

$$\Rightarrow A\left(\xi_x, \xi_y\right) A\left(\eta_x, \eta_y\right) - B^2\left(\xi_x, \xi_y; \eta_x, \eta_y\right) = -\frac{1}{4}\left(S^2 - 4RT\right)\left(\xi_x\eta_y - \xi_y\eta_x\right)^2$$

Example 2 : Reduce the equation $u_{xx} - x^2 u_{yy} = 0$ to a canonical form.

Solution : The equation $u_{xx} - x^2 u_{yy} = 0$

can be written as $\qquad r - x^2 t = 0\cdot$ ... (1.34)

Comparing this with the standard form we have

$$R = 1,\quad S = 0,\quad T = -x^2\,.$$

Hence we see that

$$S^2 - 4RT = 4x^2 > 0\cdot$$

$\Rightarrow$ Equation (1.34) is hyperbolic. Therefore, the quadratic equation

$$R\lambda^2 + S\lambda + T = 0,$$

become

$$\lambda^2 - x^2 = 0$$

$$\Rightarrow \lambda = \pm x\,.$$

Let $\lambda_1 = x$, and $\lambda_2 = -x$ be its roots. Hence the ordinary differential equations

$$\frac{dy}{dx} + \lambda_1 = 0 \text{ and } \frac{dy}{dx} + \lambda_2 = 0,$$

become

$$\frac{dy}{dx} + x = 0 \text{ and } \frac{dy}{dx} - x = 0.$$

Integrating we get

$$y + \frac{x^2}{2} = C_1 \text{ and } y - \frac{x^2}{2} = C_2.$$

Therefore, we choose the new independent variables ξ and η in the form

$$\xi = y + \frac{x^2}{2} \text{ and} \qquad \text{... (1.35)}$$

$$\eta = y - \frac{x^2}{2}. \qquad \text{... (1.36)}$$

Now by changing the independent variables x, y as new independent variables ξ and η, we obtain by using the chain rule of partial differention

$$u_x = u_\xi \xi_x + u_\eta \eta_x,$$

$$u_y = u_\xi \xi_y + u_\eta \eta_y,$$

and $$u_{xx} = u_{\xi\xi}\xi_x\xi_y + 2u_{\xi\eta}\xi_x\eta_x + u_{\eta\eta}\eta_x\eta_x + u_\xi\xi_{xx} + u_\eta\eta_{xx},$$

$$u_{yy} = u_{\xi\xi}\xi_y\xi_y + 2u_{\xi\eta}\xi_y\eta_y + u_{\eta\eta}\eta_y\eta_y + u_\xi\xi_{yy} + u_\eta\eta_{yy},$$

where from equations (1.35) and (1.36) we obtain

$$\xi_x = x,\ \ \xi_y = 1,\ \ \xi_{xx} = 1,\ \ \eta_{xx} = -1,\ \ \eta_x = -x,\ \ \eta_y = 1,\ \ \xi_{xy} = 0,\ \ \xi_{yy} = 0,$$

$$u_{xx} = u_{\xi\xi}x^2 + 2u_{\xi\eta}(-x)^2 + u_{\eta\eta}\left(x^2\right) + u_\xi(1) + u_\eta(-1)$$

$$\Rightarrow u_{xx} = x^2\left(u_{\xi\xi} - 2u_{\xi\eta} + u_{\eta\eta}\right) + u_\xi - u_\eta,$$

and $$u_{yy} = u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta}$$

Substituting this in equation (1.34) we get

$$x^2\,\cancel{u_{\xi\xi}} - 2x^2u_{\xi\eta} + x^2\,\cancel{u_{\eta\eta}} + u_\xi - u_\eta - x^2\,\cancel{u_{\xi\xi}} - 2x^2u_{\xi\eta} - x^2\,\cancel{u_{\eta\eta}} = 0$$

$$-4x^2u_{\xi\eta} = -\left(u_\xi - u_\eta\right)$$

or $$u_{\xi\eta} = \frac{1}{4(\xi-\eta)}\left(u_\xi - u_\eta\right), \qquad \text{for } x^2 = \xi - \eta.$$

This is a required hyperbolic canonical form.

Example 3 : Reduce the equation

$$y^2u_{xx} - 2xyu_{xy} + x^2u_{yy} = \frac{y^2}{x}u_x + \frac{x^2}{y}u_y$$

into canonical form and hence solve it.

Solution : Given equation can be written as

$$y^2r - 2xys + x^2t = \frac{y^2}{x}u_x + \frac{x^2}{y}u_y. \qquad \text{... (1.37)}$$

Comparing this with the standard form, we have

$$R = y^2,\ \ S = -2xy,\ \ T = x^2.$$

We observe that

$$S^2 - 4RT = 4x^2y^2 - 4x^2y^2 = 0.$$

Hence equation (1.37) is parabolic.

Hence the roots of the quadratic equation $R\lambda^2 + S\lambda + T = 0$ become

$$y^2\lambda^2 - 2xy\lambda + x^2 = 0$$

$$\Rightarrow (y\lambda - x)^2 = 0$$

$$\Rightarrow \lambda = \frac{x}{y}. \qquad \text{twice}$$

Hence the solution of the ordinary differential equation

$$\frac{dy}{dx} + \lambda(x, y) = 0 \Rightarrow \frac{dy}{dx} + \frac{x}{y} = 0$$

is given by

$$y^2 + x^2 = c^2. \qquad \text{... (1.38)}$$

Now we choose the new independent variable ξ such that

$$\xi = x^2 + y^2.$$

Let us choose $\eta = x^2 - y^2$ (choice is arbitrary)

Therefore, we have

$$\xi_x = 2x, \quad \xi_y = 2y, \quad \eta_x = 2x, \quad \eta_y = -2y,$$

$$\xi_{xx} = 2, \quad \xi_{yy} = 2, \quad \eta_{xx} = 2, \quad \eta_{yy} = -2,$$

and $\qquad \xi_{xy} = 0 = \eta_{xy}.$

Thus by changing the independent variables (x, y) to (ξ, η) we obtain

$$u_x = 2x(u_\xi + u_n),$$

$$u_y = 2y(u_\xi - u_n),$$

$$\Rightarrow u_{xy} = 2x\left(u_{\xi\xi}2y + u_{\xi\eta}(-2y) + u_{\eta\xi}2y + u_{\eta\eta}(-2y)\right),$$

and

$$u_{xx} = u_{\xi\xi}(4x^2) + 2u_{\xi\eta}(4x^2) + u_{\eta\eta}(4x^2) + u_\xi(2) + u_\eta(2),$$

$$u_{yy} = u_{\xi\xi}(4y^2) + 2u_{\xi\eta}(-4y^2) + u_{\eta\eta}(4y^2) + u_\xi(2) + u_\eta(-2).$$

Hence equation (1.37) becomes.

$$4x^2y^2\left(\cancel{u_{\xi\xi}} + 2\cancel{u_{\xi\eta}} + u_{\eta\eta}\right) + 2y^2(u_\xi + u_\eta) -$$

$$-2xy(4xy)\left(\cancel{u_{\xi\xi}} - u_{\eta\eta}\right) + 4x^2y^2\left(\cancel{u_{\xi\xi}} - \cancel{2u_{\xi\eta}} + u_{\eta\eta}\right) + 2x^2(u_\xi - u_\eta) =$$

$$= \frac{y^2}{x}2x(u_\xi + u_\eta) + \frac{x^2}{y}2y(u_\xi - u_\eta)$$

$$\Rightarrow 16x^2y^2u_{\eta\eta} = 2u_\xi\left(-y^2 - x^2 + y^2 + x^2\right) + 2u_\eta\left(-y^2 + x^2 + y^2 - x^2\right)$$

$$\Rightarrow u_{\eta\eta} = 0. \qquad \text{... (1.39)}$$

Which is required parabolic canonical form. This is a homogeneous second order p.d.e. with constant coefficients.

Integrating w.r.t. η we get

$$\frac{\partial u}{\partial \eta} = f(\xi)$$

$$\Rightarrow u = f(\xi)\eta + g(\xi)$$

or $$u(x,y) = f\left(x^2 + y^2\right)\left(x^2 - y^2\right) + g\left(x^2 + y^2\right),$$

where f and g are arbitrary.

Example 4 : Reduce the equation

$$u_{xx} + x^2u_{yy} = 0$$

to a canonical form.

Solution : Given second order p.d.e. can also be written as

$$r + x^2t = 0 \qquad \text{... (1.40)}$$

Comparing this equation with the standard form we get

$$R = 1, \quad S = 0, \quad T = x^2.$$

We notice that

$$S^2 - 4RT = -4x^2 < 0.$$

Hence the equation (1.40) is elliptic. Hence the quadratic equation $R\lambda^2 + S\lambda + T = 0$ becomes

$$\lambda^2 + x^2 = 0$$

It has roots

$$\lambda = \pm ix$$

Let $\lambda_1 = ix$ and $\lambda_2 = -ix$ be the complex roots. Hence the solutions of the first order ordinary differential equations

$$\frac{dy}{dx} + \lambda_1(x,y) = 0 \text{ and } \frac{dy}{dx} + \lambda_2(x,y) = 0$$

i.e. $$\frac{dy}{dx} + ix = 0 \text{ and } \frac{dy}{dx} - ix = 0,$$

are given by

$$y + i\frac{x^2}{2} = \text{constant} \quad \text{and} \quad y - i\frac{x^2}{2} = \text{constant}.$$

We write this as

$$-iy + \frac{x^2}{2} = C_1 \text{ and } iy + \frac{x^2}{2} = C_2. \qquad \dots (1.41)$$

Therefore we choose the independent variables ξ and η such that

$$\xi = iy + \frac{x^2}{2},$$

and $$\eta = -iy + \frac{x^2}{2}.$$

To obtain the real canonical form, further we make the transformations

$$\alpha = \frac{1}{2}(\xi + \eta) = \frac{x^2}{2} \Rightarrow \alpha_x = x, \alpha_y = 0,$$

and $$\beta = \frac{1}{2}i(\eta - \xi) = y \Rightarrow \beta_x = 0, \beta_y = 1,$$

$$\alpha_{xx} = 1, \beta_{yy} = 0.$$

Hence we obtain

$$u_x = u_\alpha \alpha_x + u_\beta \beta_x = u_\alpha x + u_\beta (0)$$

$$u_x = xu_\alpha.$$

$$u_y = u_\alpha \alpha_y + u_\beta \beta_y = u_\alpha (0) + u_\beta \cdot 1$$

$$u_y = u_\beta.$$

$$\Rightarrow u_{xx} = x\left[u_{\alpha\alpha}\alpha_x + u_{\alpha\beta}\beta_x\right] + u_\alpha$$

$$= x\left(xu_{\alpha\alpha}\right) + u_\alpha$$

$$u_{xx} = x^2 u_{\alpha\alpha} + u_\alpha.$$

Similarly, $$u_{yy} = u_{\beta\alpha}\alpha_y + u_{\beta\beta}\beta_y = u_{\beta\beta}$$

$$u_{yy} = u_{\beta\beta}.$$

Hence the equation (1.40) becomes

$$x^2 u_{\alpha\alpha} + u_\alpha + x^2 u_{\beta\beta} = 0$$

or $$u_{\alpha\alpha} + u_{\beta\beta} = -\frac{1}{2\alpha} u_{\alpha} \qquad \text{... (1.42)}$$

$$x^2 = 2\alpha \cdot$$

Which is the desired elliptic canonical form.

Example 5 : Reduce the equation

$$(n-1)^2 u_{xx} - y^{2n} u_{yy} = ny^{2n-1} u_y$$

to a canonical form and hence find its general solution

Solution : The given second order p.d.e. is

$$(n-1)^2 u_{xx} - y^{2n} u_{yy} = ny^{2n-1} u_y, \qquad \text{... (1.43)}$$

where in this case

$$R = (n-1)^2, \quad S = 0, \quad T = -y^{2n} \cdot$$

Case (i) When n= 1, we see that $S^2 - 4RT = 0$. Hence equation (1.43) reduces to

$$u_{yy} = -\frac{1}{y} u_y, \qquad \text{... (1.44)}$$

which is in the parabolic canonical form.

Case (ii) n > 1.

Then we see that $S^2 - 4RT = 4(n-1)^2 y^{2n} > 0$. Hence equation (1.43) is hyperbolic.

Hence the quadratic equation $R\lambda^2 + S\lambda + T = 0$

i.e. $$(n-1)^2 \lambda^2 - y^{2n} = 0$$

has roots

$$\lambda = \pm \frac{y^n}{n-1}.$$

Let $\lambda_1 = \frac{y^n}{n-1}$, and $\lambda_2 = -\frac{y^n}{n-1}$ be roots of the equation. Hence the first order ordinary differential equations

$$\frac{dy}{dx} + \lambda_1(x, y) = 0 \text{ and } \frac{dy}{dx} + \lambda_2(x, y) = 0,$$

become

$$\frac{dy}{dx}+\frac{y^n}{(n-1)}=0 \text{ and } \frac{dy}{dx}-\frac{y^n}{n-1}=0.$$

We write these equations as

$$(n-1)\frac{dy}{y^n}+dx=0 \text{ and } (n-1)\frac{dy}{y^n}-dx=0.$$

The solutions of these equations are given by

$$-y^{-n+1}+x=C_1 \text{ and } y^{-n+1}+x=C_2.$$

These are called the characteristic curves of the equation. Therefore we choose the independent variables ξ and η (which are called characteristic variable) such that

$$\xi = x-\frac{1}{y^{n-1}} \Rightarrow \xi_x = 1,\quad \xi_y = \frac{(n-1)}{y^n},$$

and
$$\eta = x+\frac{1}{y^{n-1}} \Rightarrow \eta_x = 1,\quad \eta_y = -\frac{(n-1)}{y^n},$$

and
$$\xi_{xx}=0,\quad \xi_{xy}=0,\quad \eta_{xy}=0,\quad \eta_{xx}=0,$$

$$\xi_{yy}=-n(n-1)\frac{1}{y^{n+1}},\quad \eta_{yy}=n(n-1)\frac{1}{y^{n+1}}.$$

Hence we obtain

$$u_x = u_\xi + u_\eta,$$

$$u_y = u_\xi (n-1)\frac{1}{y^n}+u_\eta\left[-\frac{(n-1)}{y^n}\right] \Rightarrow u_y = \frac{(n-1)}{y^n}\left[u_\xi - u_\eta\right],$$

$$u_{xx} = u_{\xi\xi}+u_{\xi\eta}+u_{\eta\xi}+u_{\eta\eta},$$

and
$$u_{yy} = u_{\xi\xi}\frac{(n-1)^2}{y^{2n}}+u_{\xi\eta}\left[-\frac{(n-1)^2}{y^{2n}}\right]-$$

$$-u_\xi (n-1)\frac{n}{y^{n+1}}-u_{\eta\xi}\frac{(n-1)^2}{y^{2n}}+u_{\eta\eta}\frac{(n-1)^2}{y^{2n}}+u_\eta\frac{n(n-1)}{y^{n+1}},$$

$$\Rightarrow u_{yy} = \frac{(n-1)^2}{y^{2n}}\left[u_{\xi\xi}-2u_{\xi\eta}+u_{\eta\eta}\right]-\frac{n(n-1)}{y^{n+1}}\left(u_\xi - u_\eta\right).$$

Substituting these values in equation (1.43) we get

$$(n-1)^2\left[u_{\xi\xi}+2u_{\xi\eta}+u_{\eta\eta}\right]-y^{2n}\left\{\frac{(n-1)^2}{y^{2n}}\left[u_{\xi\xi}-2u_{\xi\eta}+u_{\eta\eta}\right]\right\}+$$

$$+n(n-1)\frac{y^{2n}}{y^{n+1}}\left(u_\xi-u_\eta\right)=ny^{2n-1}\frac{(n-1)}{y^n}\left(u_\xi-u_\eta\right)$$

$$\Rightarrow 4(n-1)^2 u_{\xi\eta}=0$$

$$\Rightarrow u_{\xi\eta}=0 \qquad \text{for } n>1.$$

i.e.
$$\frac{\partial^2 u}{\partial\xi\partial\eta}=0. \qquad \text{... (1.45)}$$

This is the required hyperbolic canonical form. To find its solution, we integrate equation (1.45) w.r.t. η to get

$$\frac{\partial u}{\partial \xi}=\phi_1(\xi),$$

where $\phi(\xi)$ is a function of ξ. Integrating again w.r.t. ξ, we get

$$u=\int\phi_1(\xi)\,d\xi+g(\eta).$$

We write this as

$$u(x,y)=f(\xi)+g(\eta),$$

where
$$f(\xi)=\int\phi_1(\xi)\,d\xi.$$

i.e.
$$u(x,y)=f\left(x-y^{1-n}\right)+g\left(x+y^{1-n}\right), \qquad \text{... (1.46)}$$

where *f* and g are arbitrary. This is the required general solution of the equation (1.43).

Example 6 : Classify the equation

$$u_{xx}-2\sin x u_{xy}-\cos^2 x u_{yy}-\cos x u_y=0.$$

Reduce it to the canonical form and obtain its general solution.

Solution : The given partial differential equation is

$$u_{xx}-2\sin x u_{xy}-\cos^2 x x u_{yy}-\cos x u_y=0, \qquad \text{... (1.47)}$$

where
$$R=1,\quad S=-2\sin x,\quad T=-\cos^2 x.$$

We see that

$$S^2-4RT=4\sin^2 x+4\cos^2 x=4>0$$

$\Rightarrow$ The equation (1.47) is hyperbolic. Hence the quadratic equation $R\lambda^2 + S\lambda + T = 0$ becomes

$$\lambda^2 - 2\sin x\lambda - \cos^2 x = 0$$

It has roots $\lambda = \sin x \pm 1$. Let $\lambda_1 = \sin x + 1$ and $\lambda_2 = \sin x - 1$. Hence the first order ordinary equation.

$$\frac{dy}{dx} + \lambda_1(x, y) = 0 \text{ and } \frac{dy}{dx} + \lambda_2(x, y) = 0,$$

become $\frac{dy}{dx} + \sin x + 1 = 0$ and $\frac{dy}{dx} + \sin x - 1 = 0$.

Solutions of these equations are obtained by integrating

$$y + x - \cos x = C_1, \; y - \cos x - x = C_2 .$$

So that we choose the independent variables ξ and η such that

$$\xi = y - \cos x + x, \; \eta = y - \cos x - x .$$

From this we find $\xi_x = \sin x + 1, \; \xi_y = 1, \; \xi_{xx} = \cos x, \; \xi_{xy} = 0, \; \xi_{yy} = 0,$

$$\eta_x = \sin x - 1, \; \eta_y = 1, \; \eta_{xx} = \cos x, \; \eta_{xy} = 0 = \eta_{yy} .$$

Hence we obtain

$$u_x = u_\xi (1 + \sin x) + u_\eta (\sin x - 1),$$

$$u_{xx} = u_{\xi\xi} (1 + \sin x)^2 + (1 + \sin x) u_{\xi\eta} (\sin x - 1) + u_\xi \cos x +$$

$$+ (\sin x - 1) u_{\eta\xi} (\sin x + 1) + (\sin x - 1)^2 u_{\eta\eta} + u_\eta \cos x,$$

$$u_{xx} = (1 + \sin x)^2 u_{\xi\xi} + 2(\sin^2 x - 1) u_{\xi\eta} + (\sin x - 1)^2 u_{\eta\eta} + \cos x (u_\xi + u_\eta),$$

$$\Rightarrow u_{xy} = u_{\xi\xi} (1 + \sin x) + u_{\xi\eta} 2\sin x + (\sin x - 1) u_{\eta\eta}$$

and $u_y = u_\xi + u_\eta$

$$u_{yy} = u_{\xi\xi} + u_{\xi\eta} + u_{\eta\xi} + u_{\eta\eta} \Rightarrow u_{yy} = u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta} .$$

Substituting these values in equation (1.47) we get

$$(1 + \sin x)^2 u_{\xi\xi} + 2(\sin^2 - 1) u_{\xi\eta} + (\sin x - 1)^2 u_{\eta\eta} + \cos x (u_\xi + u_\eta) -$$

$$-2\sin x\left[(1+\sin x)u_{\xi\xi}+2\sin x u_{\xi\eta}+(\sin x-1)u_{\eta\eta}\right]-$$

$$-\cos^2 x\left[u_{\xi\xi}+2u_{\xi\eta}+u_{\eta\eta}\right]-\cos x\left(u_{\xi}+u_{\eta}\right)=0$$

On simplifying we obtain

$$u_{\xi\eta}=0$$

or $$\frac{\partial^2 u}{\partial\xi\partial\eta}=0. \qquad \text{... (1.48)}$$

This is the required hyperbolic canonical form of the given p.d.e. Clearly its solution is

$$u(x,y)=f(\xi)+g(\eta)$$

or $$u(x,y)=f(y-\cos x+x)+g(y-\cos x-x). \qquad \text{... (1.49)}$$

This is the required general solution of (1.47).

Example 7 : Reduce the equation

$$x^2u_{xx}-y^2u_{yy}=0$$

into canonical form

Solution : Given equation is

$$x^2u_{xx}-y^2u_{yy}=0, \qquad \text{... (1.50)}$$

where $R=x^2,\ \ S=0,\ \ T=-y^2$.

We see that $S^2-4RT=+4x^2y^2>0$.

$\Rightarrow$ The p.d.e. of second order (1.50) is hyperbolic. Now the quadratic equation

$$R\lambda^2+S\lambda+T=0$$

becomes $$x^2\lambda^2-y^2=0.$$

It has roots $$\lambda=\pm\frac{y}{x}.$$

Let $\lambda_1=\frac{y}{x}$ and $\lambda_2=-\frac{y}{x}$. Consider the first order ordinary differential equations

$$\frac{dy}{dx}+\lambda_1(x,y)=0 \text{ and } \frac{dy}{dx}+\lambda_2(x,y)=0.$$

$$\Rightarrow\frac{dy}{dx}+\frac{y}{x}=0 \text{ and } \frac{dy}{dx}-\frac{y}{x}=0,$$

or $$\frac{dy}{y}+\frac{dx}{x}=0 \text{ and } \frac{dy}{y}-\frac{dx}{x}=0.$$

Integrating we get

$$\log y+\log x=\log C_1 \text{ and } \log y-\log x=\log C_2$$

i.e. $$xy=C_1 \text{ and } \frac{y}{x}=C_2.$$

Hence we choose the independent variables ξ and η such that

$$\xi=xy \text{ and } \eta=\frac{y}{x}\Rightarrow \xi_x=y,\quad \xi_y=x,\quad \xi_{xx}=0=\xi_{yy},$$

$$\eta_x=-\frac{y}{x^2},\ \eta_y=\frac{1}{x},\ \eta_{xx}=-\frac{2y}{x^3},\ \eta_{yy}=0,\ \eta_{yx}=-\frac{1}{x^2}.$$

Hence we obtain

$$u_x=u_\xi y+u_\eta\left(-\frac{y}{x^2}\right),\quad u_y=u_\xi x+u_\eta\frac{1}{x},$$

$$u_{xx}=y^2u_{\xi\xi}+yu_{\xi\eta}\left(-\frac{y}{x^2}\right)-\frac{y}{x^2}\left(u_{\eta\xi}y+u_{\eta\eta}\left(-\frac{y}{x^2}\right)\right),$$

$$u_{xx}=y^2u_{\xi\xi}-\frac{2y^2}{x^2}u_{\xi\eta}+\frac{y^2}{x^4}u_{\eta\eta}+u_\eta\left(\frac{2y}{x^3}\right),$$

and $$u_{yy}=x^2u_{\xi\xi}+2u_{\xi\eta}+\frac{1}{x^2}u_{\eta\eta}.$$

Hence the given p.d.e. (1.50) becomes

$$-4y^2u_{\xi\eta}+\frac{2y}{x}u_\eta=0.$$

or $$u_{\xi\eta}-\frac{1}{2xy}u_\eta=0.$$

i.e. $$u_{\xi\eta}-\frac{1}{2\xi}u_\eta=0. \qquad \text{... (1.51)}$$

Which is required hyperbolic canonical form.

Example 8 : Reduce the equation

$$4u_{xx} - 4u_{xy} + 5u_{yy} = 0 \text{ to canonical form.}$$

Solution : Let $4u_{xx} - 4u_{xy} + 5u_{yy} = 0$, ... (1.52)

where S = – 4, R = 4, T = 5.

Therefore, $S^2 - 4Rt = 16 - 80 < 0$

Therefore equation (1.52) is elliptic. The quadratic equation $R\lambda^2 + S\lambda + T = 0$ becomes

$$4\lambda^2 - 4\lambda + 5 = 0 \cdot$$

This has roots

$$\lambda = \frac{1}{2}(1 \pm 2i).$$

We choose $\lambda_1 = \frac{1}{2} + i$ and $\lambda_2 = \frac{1}{2} - i$.

Hence the ordinary differential equations $\frac{dy}{dx} + \lambda_1 = 0$ and $\frac{dy}{dx} + \lambda_2 = 0$ become

$$\frac{dy}{dx} + \left(\frac{1}{2} + i\right) = 0 \text{ and } \frac{dy}{dx} + \left(\frac{1}{2} - i\right) = 0.$$

Integrating we get

$$y + \left(\frac{1}{2} + i\right)x = C_1 \text{ and } y + \left(\frac{1}{2} - i\right)x = C_2,$$

i.e. $x + 2y + 2ix = C_1$,

and $x + 2y - 2ix = C_2$.

We choose $\xi = x + 2y + 2ix$,

and $\eta = x + 2y - 2ix$.

Therefore, to obtain real canonical form, we consider the transformation

$$\alpha = \frac{1}{2}(\xi + \eta) \text{ and } \beta = \frac{i}{2}(\xi - \eta).$$

$$\Rightarrow \alpha = x + 2y \text{ and } \beta = -2x.$$

$$u_x = u_\alpha \alpha_x + u_\beta \beta_x,$$

$$\Rightarrow u_x = u_\alpha - 2u_\beta,$$

$$u_y = u_\alpha \alpha_y + u_\beta \beta_y \Rightarrow u_y = 2u_\alpha ,$$

$$u_{xx} = u_{\alpha\alpha} + u_{\alpha\beta}(-2) + u_\alpha(0) + u_{\beta\alpha}(-2) + u_{\beta\beta}(-2)^2 + u_\beta(0)$$

$$u_{xx} = u_{\alpha\alpha} - 4u_{\alpha\beta} + 4u_{\beta\beta} ,$$

$$u_{xy} = 2u_{\alpha\alpha} + u_{\alpha\beta} , \qquad \text{... (1.53)}$$

$$u_{xy} = 2u_{\alpha\alpha} - 4u_{\alpha\beta} ,$$

and $$u_{yy} = 4u_{\alpha\alpha} .$$

Substituting these values in equation (1.52) we get

$$4u_{\alpha\alpha} - \cancel{16u_{\alpha\beta}} + 16u_{\beta\beta} - 8u_{\alpha\alpha} + \cancel{16u_{\alpha\beta}} + 20u_{\alpha\alpha} = 0 .$$

$$\Rightarrow u_{\alpha\alpha} + u_{\beta\beta} = 0 .$$

This is required elliptic canonical form of the equation.

Example 8 : Reduce the equation

$$e^{2x}u_{xx} + 2e^{x+y}u_{xy} + e^{2y}u_{yy} = 0$$

into canonical form

Solution : The second order p.d.e. is given by

$$e^{2x}u_{xx} + 2e^{x+y}u_{xy} + e^{2y}u_{yy} = 0 , \qquad \text{... (1.54)}$$

where $$R = e^{2x}, \ S = 2e^{x+y}, \ T = e^{2y} .$$

We observe that

$$S^2 - 4RT = 4e^{2(x+y)} - 4e^{2(x+y)} = 0 \cdot$$

$\Rightarrow$ The equation (1.54) is parabolic. Hence the quadratic equation $R\lambda^2 + S\lambda + T = 0$ becomes

i.e. $$e^{2x}\lambda^2 + 2e^{x+y}\lambda + e^{2y} = 0$$

$$\Rightarrow \left(e^x\lambda + e^y\right)^2 = 0$$

$$\Rightarrow e^x\lambda + e^y = 0 \cdot$$

$$\Rightarrow \lambda = -e^{y-x} \cdot$$

Hence the ordinary differential equation

$$\frac{dy}{dx} + \lambda(x, y) = 0 ,$$

becomes $$\frac{dy}{dx} - e^{y-x} = 0.$$

We write this equation as

$$e^{-y}dy - e^{-x}dx = 0.$$

On integrating we obtain its solution as

$$-e^{-y} + e^{-x} = C_1. \quad \text{... (1.55)}$$

Now we choose the independent variables ξ and η such that

$$\xi = e^{-x} - e^{-y} \text{ and } \eta \text{ is arbitrary.} \quad \text{... (1.56)}$$

We choose $$\eta = e^{-x} + e^{-y}. \quad \text{... (1.57)}$$

From these equations we find

$$\xi_x = -e^{-x}, \xi_y = e^{-y}, \eta_x = -e^{-x}, \eta_y = -e^{-y},$$

$$\xi_{xx} = e^{-x}, \xi_{yy} = -e^{-y}, \eta_{xx} = e^{-x}, \eta_{yy} = e^{-y},$$

and $$\xi_{xy} = 0, \; \eta_{xy} = 0.$$

Therefore, we find

$$u_x = u_\xi\left(-e^{-x}\right) + u_\eta\left(-e^{-x}\right)$$

$$\Rightarrow u_x = -e^{-x}\left(u_\xi + u_\eta\right),$$

and $$u_y = u_\xi e^{-y} + u_\eta\left(-e^{-y}\right) \Rightarrow u_y = e^{-y}\left(u_\xi - u_\eta\right).$$

Now $$u_{xx} = e^{-x}\left(u_\xi + u_\eta\right) - e^{-x}\left[u_{\xi\xi}\left(-e^{-x}\right) + u_{\xi\eta}\left(-e^{-x}\right) + u_{\eta\xi}\left(-e^{-x}\right) + u_{\eta\eta}\left(-e^{-x}\right)\right]$$

$$u_{xx} = e^{-x}\left(u_\xi + u_\eta\right) + e^{-2x}\left(u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta}\right),$$

$$u_{xy} = -e^{-(x+y)}\left[u_{\xi\xi} - u_{\eta\eta}\right],$$

and $$u_{yy} = e^{-y}\left[u_{\xi\xi}e^{-y} + u_{\xi\eta}\left(-e^{-y}\right) - u_{\eta\xi}e^{-y} + u_{\eta\eta}e^{-y}\right] - e^{-y}\left(u_\xi - u_\eta\right)$$

$$u_{yy} = e^{-2y}\left[u_{\xi\xi} - 2u_{\xi\eta} + u_{\eta\eta}\right] - e^{-y}\left(u_\xi - u_\eta\right).$$

Substituting these in given p.d.e. (1.54) we get

$$e^{2x}e^{-x}\left(u_\xi+u_\eta\right)+\left(u_{\xi\xi}+2u_{\xi\eta}+u_{\eta\eta}\right)+2e^{x+y}\left(-e^{-(x+y)}\right)\left[u_{\xi\xi}-u_{\eta\eta}\right]+$$

$$+e^{2y}\left[e^{-2y}\left(u_{\xi\xi}-2u_{\xi\eta}+u_{\eta\eta}\right)-e^{-y}\left(u_\xi-u_\eta\right)\right]=0,$$

$$\Rightarrow 4u_{\eta\eta}=u_\xi\left(e^y-e^x\right)-u_\eta\left(e^x+e^y\right)$$

Solving equations (1.56) and (1.57) we find

$$e^x=\frac{2}{\xi+\eta} \text{ and } e^y=\frac{2}{\eta-\xi}.$$

Thus we obtain

$$\Rightarrow u_{\eta\eta}=\frac{4}{\eta^2-\xi^2}\left[\xi u_\xi-\eta u_\eta\right]. \qquad \text{... (1.58)}$$

This is the required parabolic canonical form.

Exercise :

1. Reduce the equation

   $u_{xx}+2u_{xy}+u_{yy}=0$

   into the canonical form and hence solve it.

2. Reduce the equation

   $\sin^2 xu_{xx}+2\cos xu_{xy}-u_{yy}=0$

   into canonical form.

3. Find the characteristics of the equation

   $u_{xx}+2u_{xy}+\sin^2 xu_{yy}+u_y=0$

   when it is of hyperbolic form.

4. Reduce the equation to a canonical form

   $\left(1+x^2\right)u_{xx}+\left(1+y^2\right)u_{yy}+xu_x+yu_y=0.$

2. One Dimensional Wave Equation :

1. Vibration of an infinite string (both ends are not fixed)

Result : Obtain DAlembert's solution of the one dimensional wave equation which describes the vibrations of an infinite string.

Proof : We know the vibrations of a string is governed by the second order partial differential euqtion given by

$$y_{xx} = \frac{1}{c^2} y_{tt}, \quad -\infty < x < \infty, \qquad \text{... (2.1)}$$

where y (x, t) is the deflection of the string.

Since the string is infinite boundaries of the string are not fixed. If *f (x)* is the initial deflection (mean position) of the string and *g (x)* the initial velocity of the string, then the function *y (x, t)* is required to satisfy the initial conditions

$$y(x,0) = f(x), \qquad \text{... (2.2)}$$

(this gives initial position of the string)

and $$y_t(x,0) = g(x), \; -\infty < x < \infty. \qquad \text{... (2.3)}$$

(this gives the initial velocity of the string.)

Thus our problem is to find the solution of the one-dimensional wave equation (2.1) satisfying the initial conditions (2.2) and (2.3). We first reduce the equation (2.1) into canonical form by changing the independent variables (x, t) into the new independent variables (Characteristic variables) ξ and η by using the transformation equations

$$\xi = x - ct, \qquad \text{... (2.4)}$$

and $$\eta = x + ct, \qquad \text{... (2.5)}$$

where $$\xi_x = 1, \quad \xi_t = -c, \quad \xi_{xx} = 0, \quad \xi_{xt} = 0, \quad \xi_{tt} = 1,$$

$$\eta_x = 1, \quad \eta_t = c, \quad \eta_{xx} = 0, \quad \eta_{xt} = 0, \quad \eta_{tt} = 0.$$

Also by using the chain rule of partial differentiation, we find,

$$y_x = y_\xi + y_\eta,$$

$$y_t = -c\left(y_\xi - y_\eta\right),$$

$$\Rightarrow y_{xx} = y_{\xi\xi} + 2y_{\xi\eta} + y_{\eta\eta},$$

and $$y_{tt} = c^2\left(y_{\xi\xi} - 2y_{\xi\eta} + y_{\eta\eta}\right),$$

Substituting these values in euqtion (2.1) we get

$$y_{\xi\xi} + 2y_{\xi\eta} + y_{\eta\eta} = y_{\xi\xi} - 2y_{\xi\eta} + y_{\eta\eta}$$

$$\Rightarrow y_{\xi\eta} = 0. \qquad \text{... (2.6)}$$

This is the required canonical form of the equation (2.1).

Now integrating equation (2.6) we obtain

$$y(x,t) = F(\xi) + G(\eta).$$

Replacing ξ and η as defined in (2.4) and (2.5) we get

$$y(x,t) = F(x-ct) + G(x+ct), \qquad \text{... (2.7)}$$

where F and G are arbitrary functions. Equation (2.7) is the general solution of the one dimensional wave equation. The two terms in equation (2.7) can be interpreted as waves travelling to the right and left respectively with velocity c.

The solution (2.7) is required to staisfy the initial conditions (2.2) and (2.3). Hence we have

$$y(x,0) = f(x) = F(x) + G(x). \qquad \text{... (2.8)}$$

Now differentiating equation (2.7) with respect to t we get

$$y_t(x,t) = -cF'(x-ct) + cG'(x+ct)$$

$$\Rightarrow y_t(x,0) = g(x) = -cF'(x) + cG'(x). \qquad \text{... (2.9)}$$

On integrating equation (2.9) between x_0 to x we get

$$\frac{1}{c}\int_{x_0}^{x} g(x)\,dx = -F(x) + G(x). \qquad \text{... (2.10)}$$

Adding and subtracting equations (2.8) and (2.10) we get respectively

$$G(x) = \frac{1}{2c}\left(cf(x) + \int_{x_0}^{x} g(x)dx \right),$$

and

$$F(x) = \frac{1}{2c}\left(cf(x) + \int_{x_0}^{x} g(x)dx \right).$$

Substituting these values in equation (2.7) we get

$$y(x,t) = \frac{1}{2c}\left[cf(x-ct) - \int_{x_0}^{x-ct} g(s)ds \right] + \frac{1}{2c}\left[cf(x+ct) + \int_{x_0}^{x+ct} g(s)ds \right],$$

$$\Rightarrow y(x,t) = \frac{1}{2}\left[f(x-ct) + f(x+ct) \right] + \frac{1}{2c}\left[\int_{x-ct}^{x_0} g(s)ds + \int_{x_0}^{x+ct} g(s)ds \right],$$

$$\Rightarrow y(x,t)=\frac{1}{2}\left[f(x-ct)+f(x+ct)\right]+\frac{1}{2c}\int_{x-ct}^{x+ct} g(s)ds. \qquad \text{... (2.11)}$$

where $f \in c^2$, $g \in C^1$ so that $y(x,t) \in C^2$ function. This is called the d'Alembert's solution which describes the vibrations of an infinite string at any point x and at any time t.

Note : x – ct = constant and x + ct = constant are called the characteristic curves of one dimensional wave equation.

Note : If the string is released from rest then g (x) = 0, so that the solution (2.11) becomes

$$y(x,t)=\frac{1}{2}\left(f(x-ct)+f(x-ct)\right).$$

Physical Meaning of the solution of the wave equation

We know the general solution of one dimensional wave equation (2.1) is given by

$$y(x,t)=F(x-ct)+G(x+ct). \qquad \text{... (2.12)}$$

Consider $$u_1(x,t)=F(x-ct). \qquad \text{... (2.13)}$$

Hence the initial wave profile (shape) is given by

$$u_1(x,0)=F(x).$$

Now at time $t=\frac{1}{c}$, we have from (2.13) that

$$u_1\left(x,\frac{1}{c}\right)=F(x-1).$$

$\Rightarrow$ In time $t=\frac{1}{c}$, the wave has travelled through a distance of 1-unit. Further, if we put $x'=x-1$, then we have

$$F(x-1)=F(x').$$

This implies that the original shape of the wave is retained even if the origin is shifted by one unit along the x-axis.

Now at time $t=\frac{2}{c}$, we have from equation (2.13)

$$u_1\left(x,\frac{2}{c}\right)=F(x-2),$$

$\Rightarrow$ the wave has travelled through a distance of 2 units at time $t=\frac{2}{c}$.

Thus in particular, at t = 1, we have from equation (2.13),

$$u_1(x,1) = F(x-c)$$

$\Rightarrow$ in one unit of time the profile has moved c units to the right.

$\Rightarrow$ c is the speed of propagation.

Similarly, we conclude that the equation

$$u_2(x,t) = G(x+ct)$$

represents a wave profile travelling to the left with speed c along x-axis. Thus the general solution (2.12) of the one dimensional wave equation represents the superposition of two arbitrary wave profiles, both of which are travelling with a common speed but in the opposite direction along the x-axis.

3. Vibrations of a Semi-infinite String (one end point is fixed)

Result : Obtain d'Alembert's solution of the one dimensional wave equation which describes the vibrations of a semi-infinite string.

Proof : The vibration of a string is governed by the second order one dimensional wave equation

$$y_{xx} = \frac{1}{c^2} y_{tt}, \quad 0 < x < \infty, \ t > 0, \qquad \text{... (3.1)}$$

where y (x, t) represents the deflection of the string at any point x and at any time t. Since the string is semi-infinite i.e. one end of the string x = 0 is kept fixed for all time. If u (x) and v (x) are the initial deflection and the initial velocity of the string, then the function y (x, t) is required to satisfy the initial conditions.

$$y(x,0) = u(x), \ x > 0 \cdot \qquad \text{... (3.2)}$$

This equation describes initial position of the string and

$$y_t(x,0) = v(x). \qquad \text{... (3.3)}$$

This describes initial velocity at point x. The deflection y (x, t) has to satisfy the boundary conditions

$$y(0,t) = 0, \qquad \forall t \geq 0 \cdot \qquad \text{... (3.4)}$$

This shows there is no deflection at fixed point x = 0 at any time t,

and $$y_t(0,t) = 0, \qquad \text{... (3.5)}$$

showing that velocity at fixed point x = 0 is zero.

Our aim is to find the solution of equation (3.1) satisfying the conditions (3.2) to (3.5).

We know the d Alembert's solution of one dimensional wave equation is given by

$$y(x,t) = \frac{1}{2}[u(x-ct) + u(x+ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} v(x)ds \, . \qquad \text{... (3.6)}$$

However, this solution cannot be used for the given initial value problem, since $u(x-ct)$ has no meaning for values $t > \frac{x}{c}$. Therefore we modify our semi-infinite string problem to an infinite string problem. Thus our problem is to find deflection of an infinite string subject to the initial conditions.

$$y(x,0) = U(x)$$

$$y_t(x,0) = V(x), \qquad -\infty < x < \infty,$$

where
$$U(x) = \begin{cases} u(x) & \text{if } x \geq 0, \\ -u(-x) & \text{if } x \leq 0, \end{cases} \qquad \text{... (3.7)}$$

and
$$V(x) = \begin{cases} v(x) & \text{if } x \geq 0, \\ -v(-x) & \text{if } x \leq 0. \end{cases} \qquad \text{... (3.8)}$$

We notice that U and V are odd functions of x. Thus the solution of equation (3.1) subject to the conditions (3.7) and (3.8) is given by d Alembert's solution

$$y(x,t) = \frac{1}{2}[U(x-ct) + U(x+ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} V(s)ds \qquad \text{... (3.9)}$$

Now we will show that the solution (3.9) is also a solution of equation (3.1) subject to the conditions (3.2) to (3.5). For this, we simply prove that the solution (3.9) satisfies the initial and boundary conditions (3.2) to (3.5). Therefore, from equation (3.9) we have at t = 0 and x > 0.

$$y(x,0) = \frac{1}{2}[U(x) + U(x)] + \frac{1}{2c}\int_{x}^{x} V(s)ds, \quad x > 0$$

$$\Rightarrow y(x,0) = u(x), \, x > 0, \qquad \text{due to (3.7).} \qquad \text{... (3.10)}$$

Now from equation (3.9) we find after differentiating (3.9) with respect to t.

$$y_t(x,t) = \frac{1}{2}[-cU'(x-ct) + cU'(x+ct)] + \frac{1}{2c}\frac{\partial}{\partial t}\int_{x-ct}^{x+ct} V(s)ds.$$

We use the formula

$$\frac{\partial}{\partial t}\int_{x-ct}^{x+ct} V(s)\,ds = \int_{x-ct}^{x+ct} \frac{\partial}{\partial t} V(s)\,ds + V(x+ct)\frac{\partial}{\partial t}(x+ct) - V(c-ct)\frac{\partial}{\partial t}(x-ct).$$

$$\Rightarrow y_t(x,t)=\frac{1}{2}\left[-cU'(x-ct)+cU'(x+ct)\right]+\frac{1}{2c}\left[cV(x+ct)+cV(x-ct)\right]. \quad \text{... (3.11)}$$

At t = 0, this gives

$$y_t(x,0)=\frac{1}{2}\left[-cU'(x)+cU'(x)\right]+\frac{1}{2}\left(V(x)+V(x)\right)$$

$$y_t(x,0)=V(x). \qquad \text{for } x>0 \quad \text{... (3.12)}$$

Now from equation (3.9) we have at x = 0

$$y(0,t)=\frac{1}{2}\left[U(-ct)+U(ct)\right]+\frac{1}{2c}\int_{-ct}^{ct}V(s)ds.$$

Since V is an odd function, this implies the integral $\int_{-ct}^{ct}V(s)ds$ vanishes.

$$\Rightarrow y(0,t)=\frac{1}{2}\left[U(-ct)+U(ct)\right].$$

Using equation (3.7) we get

$$y(0,t)=\frac{1}{2}\left[-u(ct)+u(ct)\right],$$

$$\Rightarrow y(0,t)=0. \quad \text{... (3.13)}$$

Now from equation (3.11) we find for x = 0, and t > 0

$$y_t(0,t)=\frac{1}{2}\left[-cU'(-ct)+cU'(ct)\right]+\frac{1}{2}\left[V(ct)+V(-ct)\right].$$

Using equations (3.7) and (3.8) we find for x $\leq$ 0

$$U'(x)=-u'(-x)(-1),$$

$$\Rightarrow U'(x)=u'(-x),$$

$$\Rightarrow U'(-ct)=u'(ct).$$

Similarly, for $x \geq 0$, $\quad U'(x)=u'(x)$

$$\Rightarrow U'(ct)=u'(ct).$$

Hence we get

$$y_t(0,t)=\frac{1}{2}\left[-cu'(ct)+cu'(ct)\right]+\frac{1}{2}\left[v(ct)-v(ct)\right].$$

$$\Rightarrow y_t(0,t)=0. \quad \text{... (3.14)}$$

Thus we have proved that the d'Alembert's solution (3.9) also satisfies the initial and the boundary conditions (3.2) to (3.5). This proves the D'Alembert's solution (3.9) is the desired solution of the one dimensional wave equation (3.1) subject to the conditions (3.2) to (3.5).

Note : In particular, if the string is released from rest i.e. v(x) = 0 then the solution (3.9) becomes

$$y(x,t)=\frac{1}{2}[u(x-ct)+u(x+ct)], \quad \text{for } x \geq ct,$$

$$=\frac{1}{2}\left[\frac{1}{2}u(x+ct)-u(ct-x)\right], \text{ for } x \leq ct.$$

4. Vibrations of a string of finite length

Result : Show that the d'Alembert's solution of the one domensional wave equation which describes the vibrations of a finite length string is given by

$$y(x,t)=\sum_{n=1}^{\infty}u_n \sin\left(\frac{n\pi x}{\ell}\right)\cos\left(\frac{n\pi ct}{\ell}\right)+\frac{\ell}{\pi c}\sum_{n=1}^{\infty}\frac{v_n}{n}\sin\left(\frac{n\pi x}{\ell}\right)\sin\left(\frac{n\pi ct}{\ell}\right).$$

Proof : Let a string be of length ℓ. The vibrations of a string is goverened by the second order partial differential equation given by

$$y_{xx}=c^2 y_{tt}, \; 0<x<\ell, \; t>0. \qquad \text{... (4.1)}$$

Since the string is finite, hence both the ends of the string are fixed for all time. Therefore the function y (x, t) must satisfy the initial conditions

$$y(x,0)=u(x), \qquad \text{... (4.2)}$$

$$y_t(x,0)=v(x), \quad 0 \leq x \leq \ell,$$

where u (x) represents the initial position of the string and v (x) represents the initial velocity of the string. The deflection of the string y (x, t) also satisfy the boundary conditions.

$$y(0,t)=y(\ell,t)=0, \qquad \text{... (4.3)}$$

$\Rightarrow$ there is no deflection at the end points of the string at any time t > 0,

and $$y_t(0,t)=y_t(\ell,t)=0. \qquad \text{... (4.4)}$$

This shows the velocity of the string at end points at any time t is zero.

The d'Alembert's solution of equation (4.1) is given by

$$y(x,t)=\frac{1}{2}[u(x-ct)+u(x+ct)]+\frac{1}{2c}\int_{x-ct}^{x+ct} v(s)ds. \qquad \text{... (4.5)}$$

However, this solution cannot be used for given initial value problem as u (x – ct) has no meaning for values $t > \frac{x}{c}$. Hence we convert our problem into a problem of vibrations of an infinite string by extending our data.

Thus we consider the vibrations of an infinite string subject to the initial conditions.

$$y(x,0) = U(x),$$

and $$y_t(x,0) = V(x), \quad \text{... (4.6)}$$

where $$U(x) = \begin{cases} u(x), & \text{if } 0 \le x \le \ell \\ -u(-x), & \text{if } -\ell \le x \le 0 \end{cases} \quad \text{... (4.7)}$$

and $U(x+2r\ell) = U(x)$, if $-\ell \le x \le \ell$, $r = \pm 1, \pm 2, \ldots,$

and $$V(x) = \begin{cases} v(x), & \text{if } 0 \le x \le \ell \\ -v(-x), & \text{if } -\ell \le x \le 0 \end{cases} \quad \text{... (4.8)}$$

and $V(x+2r\ell) = V(x)$, $-\ell \le x \le \ell$, $r = \pm 1, \pm 2, \ldots$

This shows that U (x) and V (x) are odd functions of x and are periodic with period 2 ℓ.

Hence the deflection of the string given in (4.5) subject to the conditions (4.6) to (4.8) becomes

$$y(x,t) = \frac{1}{2}\left[U(x-ct) + U(x+ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} V(s)\,ds. \quad \text{... (4.9)}$$

We assume U (x) and V (x) can be expanded into a Fourier series in $(-\ell, \ell)$. Since U (x) and V (x) are odd functions, it contains only sine terms.

Thus we have,

$$U(x) = \sum_{n=1}^{\infty} u_n \sin\left(\frac{n\pi x}{\ell}\right), \quad \text{... (4.10)}$$

and $$V(x) = \sum_{n=1}^{\infty} v_n \sin\left(\frac{n\pi x}{\ell}\right), \quad \text{... (4.11)}$$

where the Fourier constants u_n and v_n are given by

$$u_n = \frac{2}{\ell}\int_0^{\ell} u(s)\sin\left(\frac{n\pi s}{\ell}\right)ds, \quad \text{... (4.12)}$$

and $$v_n = \frac{2}{\ell}\int_0^{\ell} v(s)\sin\left(\frac{n\pi s}{\ell}\right)ds. \quad \text{... (4.13)}$$

Using equation (4.10) we find

$$\frac{1}{2}\left[U(x-ct)+U(x+ct)\right]=\frac{1}{2}\sum_{n=1}^{\infty}u_n\left[\sin\left(\frac{n\pi}{\ell}(x-ct)\right)+\sin\left(\frac{n\pi}{\ell}(x+ct)\right)\right].$$

Since $\sin A+\sin B=2\sin\left(\frac{A+B}{2}\right)\cdot\cos\left(\frac{A-B}{2}\right)$

$$\Rightarrow\frac{1}{2}\left[U(x-ct)+U(x+ct)\right]=\sum_{n=1}^{\infty}u_n\sin\left(\frac{n\pi x}{\ell}\right)\cdot\cos\left(\frac{n\pi}{\ell}ct\right). \quad \text{... (4.14)}$$

Similarly, on using (4.11) we find

$$\frac{1}{2c}\int_{x-ct}^{x+ct}V(s)ds=\frac{1}{2c}\int_{x-ct}^{x+ct}\sum_{n=1}^{\infty}v_n\sin\left(\frac{n\pi}{\ell}s\right)ds$$

$$=\frac{1}{2c}\sum_{n=1}^{\infty}v_n\int_{x-ct}^{x+ct}\sin\left(\frac{n\pi}{\ell}s\right)ds$$

$$=\frac{1}{2c}\sum_{n=1}^{\infty}v_n\left[-\frac{\ell}{n\pi}\cos\left(\frac{n\pi}{\ell}s\right)\Bigg|_{x-ct}^{x+ct}\right]$$

$$=-\frac{\ell}{2\pi c}\sum_{n=1}^{\infty}\frac{v_n}{n}\left[\cos\left(\frac{n\pi}{\ell}(x+ct)\right)-\cos\left(\frac{n\pi}{\ell}(x-ct)\right)\right].$$

Using the formula

$$\cos A-\cos B=2\sin\left(\frac{A+B}{2}\right)\cdot\sin\left(\frac{B-A}{2}\right)$$

we find,

$$\frac{1}{2c}\int_{x-ct}^{x+ct}V(s)ds=\frac{\ell}{2\pi c}\sum_{n=1}^{\infty}\frac{v_n}{n}\left[\sin\left(\frac{n\pi}{\ell}x\right)\cdot\sin\left(\frac{n\pi}{\ell}ct\right)\right] \quad \text{... (4.15)}$$

Using equations (4.14) and (4.15) in equation (4.9), we readily obtain

$$y(x,t)=\sum_{n=1}^{\infty}u_n\sin\left(\frac{n\pi}{\ell}x\right)\cdot\cos\left(\frac{n\pi ct}{\ell}\right)+\frac{\ell}{\pi c}\sum_{n=1}^{\infty}\frac{v_n}{n}\sin\left(\frac{n\pi}{\ell}x\right)\sin\left(\frac{n\pi ct}{\ell}\right). \quad \text{... (4.16)}$$

Differentiating equation (4.16) with respect to t we get

$$y_t(x,t)=-\frac{\pi c}{\ell}\sum_{n=1}^{\infty}nu_n\sin\left(\frac{n\pi}{\ell}x\right)\sin\left(\frac{n\pi ct}{\ell}\right)+\left(\frac{\ell}{\pi c}\right)\left(\frac{\pi c}{\ell}\right)\sum_{n=1}^{\infty}v_n\sin\left(\frac{n\pi}{\ell}x\right)\cos\left(\frac{n\pi ct}{\ell}\right).$$

$$\Rightarrow y_t(x,t) = -\frac{\pi c}{\ell}\sum_{n=1}^{\infty} n u_n \sin\left(\frac{n\pi}{\ell}x\right)\sin\left(\frac{n\pi ct}{\ell}\right) + \sum_{n=1}^{\infty} v_n \sin\left(\frac{n\pi}{\ell}x\right)\cos\left(\frac{n\pi ct}{\ell}\right). \quad \text{... (4.17)}$$

Now we easily check that the d'Alembert's solution (4.16) satisfies the initial and boundary conditions. Thus from equation (4.16) we find at t = 0

$$y(x,0) = \sum_{n=1}^{\infty} u_n \sin\left(\frac{n\pi}{\ell}x\right) = U(x), \qquad \text{by (4.10)} \qquad \text{... (4.18)}$$

and $$y(0,t) = 0 = y(\ell,t). \qquad \text{as } \sin 0 = \sin n\pi = 0$$

Now from equation (4.17), we find at t = 0

$$y_t(x,0) = \sum_{n=1}^{\infty} v_n \sin\left(\frac{n\pi}{\ell}x\right) = V(x), \qquad \text{due to (4.11)}$$

and $$y_t(0,t) = 0 = y_t(\ell,t). \qquad \text{... (4.19)}$$

Thus we see from equations (4.18) and (4.19) that the d'Alembert's solution (4.16) satisfies the initial and the boundary conditions identically. Hence equation (4.16) is the desired solution of the one-dimensional wave equation (4.1).

5. Vibrations of a string of finite length (Method of Separation of Variables)

Introduction : Among the many methods that are available for the solutions of a second order partial differential equation, the method of separation of variables is a powerful method which is applicable in certain circumstances. We will apply the method to find the solutions of one dimensional wave equation. The method will also be used to solve Heat and Laplace equations in the Units 6 and 7 below :

Result : By separable variable method find the solution of

$$y_{tt} = c^2 y_{xx}, \qquad 0 < x < \ell,\ t > 0. \qquad \text{... (5.1)}$$

subject to the conditions that

$$\begin{aligned} y(x,0) &= f(x), \quad 0 \le x \le \ell \\ y_t(x,0) &= g(x), \quad 0 \le x \le \ell \end{aligned} \qquad \text{... (5.2)}$$

and $$y(0,t) = y(\ell,t) = 0, \qquad \text{... (5.3)}$$

where f (x) and g (x) are initial displacement and velocity of the string.

Proof : We assume the method of separation of variables to find the vibration in a string which is governed by the equation (5.1).

Therefore, let $$y(x,t) = X(x)T(t) \qquad \text{... (5.4)}$$

be the solution of equation (5.1)

$$\Rightarrow y_x = X'(x)T(t),\; y_{xx} = X''(x)T(t).$$

Similarly, $y_{tt} = X(x)T''(t)$.

Substituting this in equation (5.1) we get

$$XT'' = c^2 X''T,$$

$$\Rightarrow \frac{X''}{X} = \frac{T''}{c^2 T}. \qquad \dots (5.5)$$

We see that the left handside is a function of x and the right hand side is a function t alone.

Equation (5.5) shows that each side must be a constant say λ.

$$\Rightarrow X'' = \lambda X \quad \text{and} \quad T'' = c^2 \lambda T$$

or $X'' - \lambda X = 0$ and $T'' - c^2 \lambda T = 0$, ... (5.6)

where λ may be zero, positive or negative. From the boundary condition

$$y(0,t) = 0 \Rightarrow X(0)T(t) = 0$$

$$\Rightarrow X(0) = 0. \qquad \text{as } T(t) \neq 0$$

Similarly, $y(\ell,t) = 0 \Rightarrow X(\ell) = 0$.

Thus our problem is

$$X'' - \lambda X = 0 \qquad \dots (5.7)$$

such that $X(0) = X(\ell) = 0$.

Case (i) : $\lambda = 0$

The solution of the equation (5.7) in this case is

$$X = Ax + B \qquad \dots (5.8)$$

The boundary conditions $X(0) = 0$ and $X(\ell) = 0$ give

$$A = 0 \text{ and } B = 0.$$

Consequently, we get X (x) = 0 as the solution of equation (5.7). This is a trivial solution hence we drop it.

Case (ii) : Let $\lambda > 0$

Let $\lambda = \alpha^2$, where α is positive or negative. In this case the solution of equation (5.7) is given by

$$X(x) = A \cdot e^{-\alpha x} + B^{\alpha} e^{x}. \qquad \dots (5.9)$$

To determine the constants A and B, we use the boundary conditions

$$X(0)=0 \Rightarrow A+B=0,$$

and $$X(\ell)=0 \Rightarrow Ae^{-\alpha\ell}+Be^{\alpha\ell}=0$$

$$\Rightarrow A+Be^{2\alpha\ell}=0\cdot$$

or $$B\left[1-e^{2\alpha\ell}\right]=0$$

$$\Rightarrow B=0 \Rightarrow A=0$$

Hence for $\lambda=0$ and $\lambda>0$, the solutions (5.8) and (5.9) do not constitute the solution of the wave equation (5.1).

Case (iii) : Let $\lambda<0$

Let $\lambda=-\alpha^2$

In this case the solution of equation (5.7) is given by

$$X(x)=A\cos\alpha x+B\sin\alpha x\,. \qquad \text{... (5.10)}$$

Now the boundary conditions

$$X(0)=0 \Rightarrow 0=A$$

$$\Rightarrow A=0$$

and $$X(\ell)=0 \Rightarrow 0=B\sin(\alpha\ell)$$

Now if B = 0 then we have $y(x,t)=0$ is again a trivial solution of equation (5.1).

Therefore we assume $B\neq 0$

$$\Rightarrow \sin(\alpha\ell)=0$$

$$\Rightarrow \alpha\ell=n\pi,\ \text{n} = 1, 2, 3, ...,$$

or $$\alpha=\frac{n\pi}{\ell},\ \text{n} = 1, 2, ...,$$

For each value of n = 1, 2, ..., let $\alpha=\alpha_n$.

Thus $$\alpha_n=\frac{n\pi}{\ell}\ ,\ \text{n} = 1, 2, ... \qquad \text{... (5.11)}$$

These α_n are called eigen values of the equation (5.1) and the corresponding functions $\sin\left(\frac{n\pi}{\ell}x\right)$ are called eigen functions. Hence the solution (5.10) can be denoted by

$$X_n=B_n\sin\left(\frac{n\pi}{\ell}x\right),\ \text{n} = 1, 2, ... \qquad \text{... (5.12)}$$

Similarly, for $\alpha_n = \frac{n\pi}{\ell}$, the solution of other equation, $T''+c^2\alpha^2T=0$ is given by

$$T_n(t) = C_n \cos\left(\frac{n\pi x}{\ell}t\right) + D_n \sin\left(\frac{n\pi c}{\ell}t\right), \qquad \text{... (5.13)}$$

where C_n and D_n are arbitrary constants. Hence, the solution (5.4) becomes

$$y_n(x,t) = \left[a_n \cos\left(\frac{n\pi ct}{\ell}\right) + b_n \sin\left(\frac{n\pi ct}{\ell}\right)\right]\sin\left(\frac{n\pi\lambda}{\ell}\right), \qquad \text{... (5.14)}$$

where $a_n = C_nB_n$ and $b_n = D_nB_n$.

By the principle of superposition, the series

$$y(x,t) = \sum_{n=1}^{\infty} y_n(x,t) = \sum_{n=1}^{\infty}\left[a_n \cos\left(\frac{n\pi ct}{\ell}\right) + b_n \sin\left(\frac{n\pi ct}{\ell}\right)\right]\sin\left(\frac{n\pi\lambda}{\ell}\right) \qquad \text{... (5.15)}$$

if it converges, is also a solution of equation (5.1) satisfying the boundary conditions (5.3). We choose a_n and b_n such that y (x, t) in (5.15) satisfies the initial conditions (5.2).

Therefore the initial condition $y(x,0) = f(x)$ gives

$$f(x) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi}{\ell}\lambda\right), \quad 0 < x < \ell \qquad \text{... (5.16)}$$

Now differentiating equation (5.15) with respect to t we get

$$y_t(x,t) = \frac{\pi c}{\ell}\sum_{n=1}^{\infty}\left[-na_n \sin\left(\frac{n\pi}{\ell}ct\right) + nb_n \cos\left(\frac{n\pi ct}{\ell}\right)\right]\sin\left(\frac{n\pi}{\ell}x\right), \qquad \text{... (5.17)}$$

Thus at t = 0, we have

$$y_t(x,0) = g(x),$$

$$\Rightarrow g(x) = \sum_{n=1}^{\infty} b_n\left(\frac{n\pi n}{\ell}\right)\sin\left(\frac{n\pi}{\ell}x\right), \quad 0 < x < \ell. \qquad \text{... (5.18)}$$

Equations (5.16) and (5.18) show that *f (x)* and *g (x)* are expanded in a half range sine series. Therefore a_n and b_n are coefficients of the half range sine series of f (x) and g (x) respectively.

$$\Rightarrow a_n = \frac{2}{\ell}\int_0^{\ell} f(x)\sin\left(\frac{n\pi x}{\ell}\right)dx, \qquad \text{... (5.19)}$$

and

$$b_n = \frac{2}{n\pi c}\int_0^{\ell} g(x)\sin\left(\frac{n\pi x}{\ell}\right)dx. \qquad \text{... (5.20)}$$

Thus the solution of one dimensional wave equation (5.1) subject to the conditions (5.2) and (5.3) is given by the equation (5.15) with the coefficients a_n and b_n given in equations (5.19) and (5.20) respectively.

Note : When initial velocity of the string is zero. i.e. if g (x) = 0, then we have $b_n = 0$. In this case the solution (5.15) becomes

$$y(x,y) = \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi ct}{\ell}\right) \cdot \sin\left(\frac{n\pi x}{\ell}\right), \qquad \text{... (5.21)}$$

with

$$a_n = \frac{2}{\ell}\int_0^{\ell} f(x)\sin\left(\frac{n\pi x}{\ell}\right)dx. \qquad \text{... (5.22)}$$

Example 1 : A tightly stretched string with fixed end points x = 0 and x = 1 is initially in a position given by

$$y(x,0) = x(1-x)$$

It is released from rest from this position. Find the displacement y (x, t) at any time t.

Solution : We know the vibration of a string is governed by the second order p.d.e. given by

$$y_{tt} - c^2 y_{xx} = 0,\ 0 < x < 1,\ t > 0, \qquad \text{... (5.23)}$$

subject to the initial conditions

$$y(0,t) = y(1,t) = 0, \qquad \forall t \qquad \text{... (5.24)}$$

and

$$y(x,0) = x(1-x). \qquad \text{... (5.25)}$$

Also the initial velocity of the string is given by

$$y_t(x,0) = 0. \qquad \text{... (5.26)}$$

By variable separable method, the solution of (5.23) is given by

$$y(x,t) = (A\cos(\alpha x) + B\sin(\alpha x))(C\cos(\alpha ct) + D\sin(\alpha ct)). \qquad \text{... (5.27)}$$

The boundary conditions

$$y(0,t) = 0 \Rightarrow 0 = A. \qquad \text{for } C \neq 0, D \neq 0$$

Also from (5.27) we have

$$y_t(x,t) = \alpha c[A\cos(\alpha x) + B\sin(\alpha x)][-C\sin(\alpha ct) + D\cos(\alpha ct)]$$

Therefore, the condition $y_t(x,0) = 0 \Rightarrow D = 0$.

Therefore, equation (5.27) becomes

$$y(x,t) = B\sin(\alpha x)[C\cos(\alpha ct)]. \qquad \text{... (5.28)}$$

Now the condition $y(1,t) = 0$ gives

$$0 = B\sin(\alpha)\cdot C\cos(\alpha ct)$$

$$\Rightarrow B = 0 \text{ or } \sin\alpha = 0 \text{ for } C \neq 0$$

If B = 0 then we have only trivial solution of (5.23).

Therefore, we assume $B \neq 0$

$$\Rightarrow \sin\alpha = 0 \Rightarrow \alpha = n\pi,\ n = 1, 2, \ldots\ldots$$

Let $\alpha_n = n\pi$, for n = 1, 2, 3, ...

Therefore, corresponding to each n, the solution (5.28) becomes

$$y_n(x,t) = a_n \sin(n\pi x)\cos(n\pi ct), \quad \text{for } a_n = B_n C_n. \qquad \ldots (5.29)$$

By superposition principle, the most general solution of equation (5.23) is given by

$$y(x,t) = \sum_{n=1}^{\infty} y_n(x,t) = \sum_{n=1}^{\infty} a_n \sin(n\pi x)\cos(n\pi ct), \qquad \ldots (5.30)$$

where the constant a_n is determined by using the condition

$$y(x,0) = x(1-x).$$

Therefore, from (5.30) we have

$$y(x,0) = x(1-x) = \sum_{n=1}^{\infty} a_n \sin(n\pi x)\cdot \qquad 0 \le x \le 1 \qquad \ldots (5.31)$$

We see from equation (5.31) that $f(x) = x(1-x)$ is expressed in the Fourier sine series. Hence the corresponding Fourier constant a_n is given by

$$a_n = 2\int_0^1 x(1-x)\sin(n\pi x)\,dx$$

$$= 2\left[\int_0^1 x\sin(n\pi x)\,dx - \int_0^1 x^2\sin(n\pi x)\,dx\right].$$

Integrating by parts we get

$$a_n = 2\left[-\frac{x}{n\pi}\cos(n\pi x)\bigg|_0^1 + \int_0^1 \frac{1}{n\pi}\cos(n\pi x)\,dx + \frac{x^2}{n\pi}\cos(n\pi x)\bigg|_0^1 - \int_0^1 \frac{2x}{n\pi}\cos(n\pi x)\,dx\right]$$

$$a_n = \frac{4}{n^2\pi^2}\left[-\frac{\cos(n\pi x)}{n\pi}\bigg|_0^1\right]$$

$$a_n = -\frac{4}{n^3\pi^3}\left[(-1)^n - 1\right]$$

$$a_n = \frac{8}{n^3\pi^3}, \text{ for n is odd,}$$

$$a_n = 0 \text{ for n even.}$$

Substituting this in (5.30) we get

$$y(x,t) = \frac{8}{\pi^3}\sum_{n=1}^{\infty}\frac{1}{n^3}\sin(n\pi x)\cdot\cos(n\pi ct). \quad \text{... (5.32)}$$

Example 2 : Solve

$$y_{tt} - c^2 y_{xx} = 0, \qquad o < x < 1,\ t > 0, \quad \text{... (5.33)}$$

$$y(0,t) = y(1,t) = 0, \quad \text{... (5.34)}$$

$$y(x,0) = 0, \qquad 0 \le x \le 1, \quad \text{... (5.35)}$$

$$y_t(x,0) = x^2, \qquad 0 \le x \le 1. \quad \text{... (5.36)}$$

Solution : Let $y(x,t) = X(x)T(t)$ ...(5.37)

be a solution of equation (5.33) and is given by

$$y(x,t) = \left(A\cos(\alpha x) + B\sin(\alpha x)\right)\left(C\cos(\alpha ct) + D\sin(\alpha ct)\right). \quad \text{... (5.38)}$$

The boundary condition $y(0,t) = 0 \Rightarrow A = 0$ for $C \ne 0, D \ne 0$.

Also the condition

$$y(x,0) = 0 \Rightarrow 0 = B\sin(\alpha x)(C) \Rightarrow C = 0 \text{ for } B \ne 0.$$

Therefore, the solution (5.38) implies

$$y(x,t) = B\sin(\alpha x)\left(D\sin(\alpha ct)\right). \quad \text{... (5.39)}$$

Now the condition

$$y(1,t) = 0 \Rightarrow 0 = B\sin(\alpha)\cdot D\sin(\alpha ct)$$

$$\Rightarrow B = 0 \text{ or } \sin\alpha = 0 \text{ for } D \ne 0$$

If B = 0 we have trivial solution of equation (5.33).

Therefore we assume

$$B \ne 0 \Rightarrow \sin\alpha = 0 \Rightarrow \alpha = n\pi, \text{ n} = 1, 2, ...$$

Let $\alpha_n = n\pi$, n = 1, 2, ...

Therefore, corresponding to each n the solution (5.39) becomes

$$y_n(x,t) = a_n \sin(n\pi x)\sin(n\pi ct). \qquad \text{... (5.40)}$$

Therefore, by superposition principle, the most general solution of (5.33) is given by

$$y(x,t) = \sum_{n=1}^{\infty} y_n(x,t) = \sum_{n=1}^{\infty} a_n \sin(n\pi x)\sin(n\pi ct), \qquad \text{... (5.41)}$$

where the constant a_n is to be determined.

From equation (5.41) we have

$$y_t(x,t) = \sum_{n=1}^{\infty} a_n \sin(n\pi x)\cdot(n\pi c)\cdot\cos(n\pi ct). \qquad \text{... (5.42)}$$

Therefore, the condition $y_t(x,0) = x^2$ gives

$$x^2 = \pi c \sum_{n=1}^{\infty} a_n n \sin(n\pi x), \qquad 0 \le x \le 1. \qquad \text{... (5.43)}$$

This shows that x^2 is expressed in the Fourier sine series. Hence the corresponding Fourier constant na_n is given by

$$\pi c n a_n = 2\int_0^1 x^2 \sin(n\pi x)dx.$$

Integrating by parts we get

$$\pi c n a_n = 2\left[\frac{x^2}{n\pi}\cos(n\pi x)\Big|_0^1 - \frac{1}{n\pi}\int_0^1 2x\cos(n\pi x)\,dx\right],$$

$$= 2\left[\frac{1}{n\pi}\cos(n\pi) - \frac{2}{n\pi}\left\{\frac{x}{n\pi}\sin(n\pi x)\Big|_0^1 - \int_0^1 \frac{1}{n\pi}\sin(n\pi x)\,dx\right\}\right],$$

$$= 2\left[\frac{1}{n\pi}(-1)^n + \frac{2}{(n\pi)^2}\left(-\frac{\cos(n\pi x)}{n\pi}\right)_0^1\right],$$

$$= 2\left[\frac{1}{n\pi}(-1)^n - \frac{2}{(n\pi)^3}\left((-1)^n - 1\right)\right],$$

$$\pi c n a_n = \frac{2(-1)^n}{n\pi} - \frac{4}{n^3\pi^3}\left((-1)^n - 1\right),$$

$$a_n = \frac{2(-1)^n}{n^2\pi^2 c} - \frac{4}{cn^4\pi^4}\left((-1)^n - 1\right). \qquad \text{... (5.44)}$$

Substituting this in (5.41) we get

$$y(x,t)=\sum_{n=1}^{\infty}\left\{\frac{2(-1)^n}{n^2\pi^2 c}-\frac{4}{n^4\pi^4 c}\left((-1)^n-1\right)\right\}\sin(n\pi x)\sin(n\pi ct). \qquad \dots (5.45)$$

This is the required solution.

Uniqueness of Solution of Wave Equation :

Theorem : Show that the solution u (x, t) of the equation

$$u_{tt}-c^2u_{xx}=F(x,t),\quad 0<x<\ell,\quad t>0,$$

$$u(x,0)=f(x),\quad 0\le x\le \ell,$$

$$u_t(x,0)=g(x),\quad 0\le x\le \ell,$$

$$u(0,t)=u(\ell,t)=0,\quad t\ge 0,$$

if it exists, is unique.

Proof : Let there be two solutions $u_1(x,t)$ and $u_2(x,t)$ of the equation

$$u_{tt}-c^2u_{xx}=F(x,t),\qquad 0<x<\ell,\ t>0, \qquad \dots (5.46)$$

satisfing the conditions

$$u(x,0)=f(x),\qquad 0\le x\le \ell, \qquad \dots (5.47)$$

$$u_t(x,0)=g(x),$$

and $$u(0,t)=u(\ell,t)=0.\qquad t\ge 0 \qquad \dots (5.48)$$

$$\Rightarrow \frac{\partial^2 u_1}{\partial t^2}-c^2\frac{\partial^2 u_1}{\partial x^2}=F(x,t),$$

and $$\frac{\partial^2 u_2}{\partial t^2}-c^2\frac{\partial^2 u_2}{\partial x^2}=F(x,t)$$

Subtracting these equations we get

$$\frac{\partial^2 (u_1-u_2)}{\partial t^2}-c^2\frac{\partial^2}{\partial x^2}(u_1-u_2)=0.$$

Also $$u_1(x,0)=f(x) \text{ and } u_2(x,0)=f(x)$$

$$\Rightarrow (u_1-u_2)(x,0)=0$$

Also $\frac{\partial u_1}{\partial t}(x,0) = g(x)$ and $\frac{\partial u_2}{\partial t}(x,0) = g(x)$

$$\Rightarrow \frac{\partial}{\partial t}(u_1 - u_2)(x,0) = 0.$$

This shows that the function $v = u_1 - u_2$ satisfies the corresponding partial differential equation

$$\frac{\partial^2 v}{\partial t^2} - c^2 \frac{\partial^2 v}{\partial x^2} = 0, \qquad 0 < x < \ell,\ t > 0 \qquad \text{... (5.49)}$$

subject to the conditions

$$v(x,0) = 0, \qquad o \le x \le \ell, \qquad \text{... (5.50)}$$

$$v_t(x,0) = 0, \qquad o \le x \le \ell, \qquad \text{... (5.51)}$$

and $v(0,t) = v(\ell,t) = 0, \qquad t \ge 0.$

Claim : We prove v = 0 i.e. $u_1(x,t) = u_2(x,t)$.

Therefore, consider

$$E(t) = \frac{1}{2}\int_0^\ell \left(c^2 v_x^2 + v_t^2\right) dx \qquad \text{... (5.52)}$$

Since v (x, t) is twice differentiable, we see that E (t) is a differentiable function of t.

Hence $\frac{dE}{dt} = \frac{1}{2}\int_0^\ell 2\left(c^2 v_x v_{xt} + v_t v_{tt}\right) dx$

$$\frac{dE}{dt} = \int_0^\ell v_t v_{tt} dx + \int_0^\ell c^2 v_x v_{xt} dx.$$

Integrating the second integral by parts we get

$$= \int_0^\ell v_t v_{tt} dx + \left(c^2 v_x v_t\right)_0^\ell - \int_0^\ell c^2 v_t v_{xx} dx.$$

However, from equation (5.51) we have

$$v(0,t) = 0 \Rightarrow v_t(0,t) = 0 \quad \forall t,\ t \ge 0,$$

and $v(\ell,t) = 0 \Rightarrow v_t(\ell,t) = 0 \quad \forall t \ge 0$

Hence we have

$$\frac{dE}{dt} = \int_0^{\ell} v_t \left(v_{tt} - c^2 v_{xx} \right) dx .$$

From equation (5.49) we have

$$v_{tt} - c^2 v_{xx} = 0 .$$

$$\Rightarrow \frac{dE}{dt} = 0 .$$

$$\Rightarrow E = \text{Constant} \cdot \qquad \text{... (5.53)}$$

However, $v(x,0) = 0 \Rightarrow v_x(x,0) = 0$ and $v_t(x,0) = 0$, therefore, from equation (5.52) we have

$$E(0) = \frac{1}{2} \int_0^{\ell} \left(c^2 v_x^2(x,0) + v_t^2(x,0) \right) dx \qquad \text{... (5.54)}$$

$$E(0) = 0 ,$$

Therefore $\qquad E \equiv 0 \cdot$

Hence from equation (5.52) we have

$$v_x(x,t) = 0, v_t(x,t) = 0 \ \forall \ t > 0, \ 0 < x < \ell .$$

This is possible only if v (x, t) = constant. The condition (5.50) gives

$$v(x,0) = 0 \Rightarrow \text{constant} = 0$$

$$\Rightarrow v(x,t) = 0$$

$$\Rightarrow u_1(x,t) = u_2(x,t),$$

which proves the uniqueness of the solution of the wave equation.

Remarks : The solution of the problem of vibrations of a string of finite length is also unique, as it is a special case of the problem when $F(x,t) = 0$.

Example 3 : A tightly stretched string with fixed end points x = 0 and x = ℓ is initially in a position given by

$$y(x,0) = y_0 \sin^3 \left(\frac{\pi x}{\ell} \right).$$

It is released from rest from this position find the displacement y(x, t) at any time t.

Solution : We know vibrations in a string are governed by the second order p.d.e. given by

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2} , \qquad 0 < x < \ell, \ t > 0, \qquad \text{... (5.55)}$$

such that $$y(0,t) = 0 = y(\ell,t),\ \forall t, \quad \text{... (5.56)}$$

and $$y(x,0) = y_0 \sin^3\left(\frac{\pi x}{\ell}\right). \quad \text{... (5.57)}$$

It is also given that the initial velocity of the string is zero.

$$\Rightarrow y_t(x,0) = 0. \quad \text{... (5.58)}$$

We know by separable variable method the solution of equation (5.55) is given by

$$y(x,t) = (A\cos(\alpha x) + B\sin\alpha x)(C\cos(\alpha ct) + D\sin(\alpha ct)). \quad \text{... (5.59)}$$

The boundary condition

$$y(0,t) = 0 \Rightarrow A = 0.$$

Also from (5.59) we find

$$y_t(x,t) = \alpha c[A\cos(\alpha x) + B\sin(\alpha x)][-C\sin(\alpha ct) + D\cos(\alpha ct)]$$

Hence, $$y_t(x,0) = 0 \Rightarrow D = 0.$$

Hence solution (5.59) becomes

$$y(x,t) = BC\sin(\alpha x)\cdot\cos(\alpha ct).$$

Now the condition

$$y(\ell,t) = 0 \Rightarrow 0 = B\sin(\alpha\ell)\cdot C\cos(\alpha ct) \quad \text{... (5.60)}$$

$$\Rightarrow B = 0, \qquad \text{for } C \neq 0,$$

or $$\sin(\alpha\ell) = 0, \qquad \text{for } C \neq 0,$$

If B = 0 we have trivial solution y (x, t) = 0. Therefore, we assume $B \neq 0 \Rightarrow \sin(\alpha\ell) = 0$,

$$\Rightarrow \alpha\ell = n\pi,\ n = 1, 2, 3, ...$$

Let $$\alpha_n = \left(\frac{n\pi}{\ell}\right),\ n = 1, 2, 3, ... \quad \text{... (5.61)}$$

These are called the eigen values of the equation. Hence the solution (5.60) becomes

$$y_n(x,t) = a_n \sin\left(\frac{n\pi x}{\ell}\right)\cdot\cos\left(\frac{n\pi ct}{\ell}\right). \quad \text{... (5.62)}$$

By the superposition principle, the most general solution of (5.55) is given by

$$y(x,t)=\sum_{n=1}^{\infty} y_n(x,t)=\sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{\ell}\right)\cdot\cos\left(\frac{n\pi ct}{\ell}\right), \quad \text{... (5.63)}$$

where the constant a_n is determined by using the condition that

$$y(x,0)=y_0\sin^3\left(\frac{\pi x}{\ell}\right).$$

$\Rightarrow$ From equations (5.57) and (5.63) we have

$$y(x,0)=y_0\sin^3\left(\frac{\pi x}{\ell}\right)=\sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{\ell}\right). \quad \text{... (5.64)}$$

We know $\quad \sin 3\theta = 3\sin\theta - 4\sin^3\theta$

$$\Rightarrow \sin^3\left(\frac{\pi x}{\ell}\right)=\frac{3\sin\left(\frac{\pi x}{\ell}\right)-\sin\left(\frac{3\pi x}{\ell}\right)}{4}.$$

Therefore, equation (5.64) becomes

$$y_0\left[\frac{3\sin\left(\frac{\pi x}{\ell}\right)-\sin\left(\frac{3\pi x}{\ell}\right)}{4}\right]=\sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{\ell}\right).$$

Comparing corresponding coefficients on both sides we get

$$a_1 = y_0\frac{3}{4},\ \ a_2=0,\ \ a_3=-\frac{1}{4}y_0,\ \ a_4=0....$$

Therefore, the solution (5.63) becomes

$$y(x,t)=\frac{3}{4}y_0\sin\left(\frac{\pi x}{\ell}\right)\cos\left(\frac{\pi ct}{\ell}\right)-\frac{y_0}{4}\sin\left(\frac{3\pi x}{\ell}\right)\cos\left(\frac{3\pi ct}{\ell}\right).$$

or $$y(x,t)=\frac{y_0}{4}\left[3\sin\left(\frac{\pi x}{\ell}\right)\cdot\cos\left(\frac{\pi ct}{\ell}\right)-\sin\left(\frac{3\pi x}{\ell}\right)\cos\left(\frac{3\pi ct}{\ell}\right)\right]. \quad \text{... (5.65)}$$

Example 4 : By separating the variables, show that one dimensional wave equation

$$\frac{\partial^2 y}{\partial x^2}=\frac{1}{c^2}\frac{\partial^2 y}{\partial t^2}$$

has solution solution of the form A $\exp(\pm\ in\, x \pm\ in\, ct)$, where A and n are constants. Hence show that the functions of the form

$$y(x,t)=\sum_r\left[A_r\cos\left(\frac{r\pi ct}{a}\right)+B_r\left(\frac{r\pi ct}{a}\right)\right]\sin\left(\frac{r\pi x}{a}\right)$$

where A_r and B_r are constants, satisfy the wave equation and the boundary conditions.

$$y(0,t)=0,\ \ y(a,t)=0\ \forall t.$$

Solution : One dimension wave equation is given by

$$\frac{\partial^2 y}{\partial x^2}=\frac{1}{c^2}\frac{\partial^2 y}{\partial t^2},\qquad 0<x<a \qquad \text{... (5.66)}$$

where the deflection y (x, t) satisfies the conditions

$$y(0,t)=0=y(a,t)\quad \forall t \qquad \text{... (5.67)}$$

Let $$y(x,t)=X(x)T(t) \qquad \text{... (5.68)}$$

be its solution. Therefore, we have

$$y_{xx}=X''(x)T(t)\ \text{and}\ y_{tt}=X(x)T''(t).$$

Therefore, equation (5.66) becomes

$$X''T=\frac{1}{c^2}X(x)T''$$

or $$\frac{X''}{X}=\frac{T''}{c^2T}=-n^2\ \text{(say)} \qquad \text{... (5.69)}$$

$$\Rightarrow X''+n^2X=0, \qquad \text{... (5.70)}$$

and $$T''+c^2n^2T=0. \qquad \text{... (5.71)}$$

Solving equations (5.70) and (5.71) we have

$$X=e^{\pm inx},$$

$$T=e^{\pm inct}.$$

Hence $$y(x,t)=A\cdot e^{\pm inx\pm inct}, \qquad \text{... (5.72)}$$

is a solution of equation (5.73), where A = constant. We can also write the solution of (5.70) and (5.71) as

$$X=A\cos(nx)+B\sin(nx)\ \text{and}\ T=C\cos(nct)+D\sin(nct).$$

Therefore, the solution of (5.66) is given by

$$y(x,t)=[A\cos(nx)+B\sin(nx)][C\cos(nct)+D\sin(nct)]. \qquad \text{... (5.73)}$$

Now applying the initial condition

$$y(0,t)=0 \Rightarrow 0=A,$$

and $$y(a,t)=0 \Rightarrow 0=B\sin(an)[C\cos(nct)+D\sin(nct)]$$

$$\Rightarrow B=0 \text{ or } \sin(an)=0 \quad \text{for} \quad C\neq 0,\ D\neq 0 \text{ as } T(t)\neq 0$$

If B = 0, we have trivial solution. Therefore, we assume $B\neq 0 \Rightarrow \sin(an)=0$

$$\Rightarrow an=r\pi,\ \text{r} = 1, 2, ...$$

or $$n=\left(\frac{r\pi}{a}\right),\ \text{r} = 1, 2, ...$$

Therefore, solution for each value of r, we have

$$y_r(x,t)=B_r\sin\left(\frac{r\pi x}{a}\right)\cdot\left[C\cos\left(\frac{\pi rct}{a}\right)+D\sin\left(\frac{\pi rct}{a}\right)\right]. \qquad ...(5.74)$$

By superposition principle, the most general solution is given by

$$y(x,t)=\sum_{r=1}^{\infty}y_r(x,t)=\sum_{r=1}^{\infty}\left[A_r\cos\left(\frac{\pi rct}{a}\right)+B_r\sin\left(\frac{\pi rct}{a}\right)\right]\sin\left(\frac{\pi rx}{a}\right), \qquad ...(5.75)$$

where A_r and B_r are constants.

Exercise :

1. Obtain the solution of the wave equation

$$u_{tt}=c^2u_{xx}$$

under the following conditions

$$u(0,t)=u(2,t)=0,$$

$$u(x,0)=\sin^3\left(\frac{\pi}{2}x\right),$$

$$u_t(x,0)=0.$$

2. The vibrations of an elastic string is governed by the partial differential equation

$$u_{tt} = u_{xx}$$

The length of the string is π and the ends are fixed. The initial velocity is zero and initial deflection is $u(x,0) = 2(\sin x + \sin 3x)$. Find the deflection u (x, t) of the vibrating string for t >0.

3. A string is fixed at two points ℓ apart and is stretched. The motion takes place by displacing the string in the form $y = a\sin\left(\frac{\pi x}{\ell}\right)$ from which it is released at time t. Show that the displacement of any point at a distance x from one end at time t is

$$y(x,t) = a\sin\left(\frac{\pi x}{\ell}\right)\cos\left(\frac{\pi ct}{\ell}\right).$$

HEAT CONDUCTION PROBLEM

In this unit we consider heat conduction problem in a rod with the following assumptions.

1. The rod is homogeneous.
2. It is sufficiently thin so that the heat is uniformly distributed over it cross section at a given time t.
3. The surface of the rod is insulated to prevent any loss of heat through the boundary.
4. $u(x,t)$ is the temperature at the point x at time t.

We know the temperature $u(x,t)$ in a rod is governed by the second order one dimensional. p.d..e.

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2},$$

satisfying some initial and boundary conditions.

Case 1 : Heat conduction - Finite rod.

Result : By separable variable method, find the temperature distribution in a rod of length ℓ satisfying the boundary conditions

$$u(0,t) = u(\ell,t) = 0, \quad t > 0 \text{ (end points of the rod are kept at } 0° \text{ C.)}$$

The initial temperature is $u(x,0) = f(x)$, $0 \le x \le \ell$.

The rod and its ends are perfectly insulated.

Or

By separable variable method, find the solution of the equation $\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}$, $0 < x < \ell, t > 0$

satisfying the conditions

$$u(0,t) = u(\ell,t) = 0, \quad t > 0$$

$$u(x,0) = f(x). \quad 0 \le x \le \ell$$

Solution : Let $u(x,t)$ be the temperature in a rod of length ℓ. We know the temperature distribution in rod is governed by the second order partial differential equation given by

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}, \qquad 0 < x < \ell, \; t > 0 \qquad \text{... (1.1)}$$

satisfying the boundary conditions

$$u(0,t) = u(\ell,t) = 0, \quad t > 0 \qquad \text{... (1.2)}$$

and $$u(x,0) = f(x), \qquad 0 \le x \le \ell, \qquad \text{... (1.3)}$$

where $f(x)$ is the initial temperature in the rod.

To find temperature in the rod at any instant t, by separable variable method, we assume the solution of the equation (1.1) in the form

$$u(x,t) = X(x)\mathrm{T}(t) \qquad \text{... (1.4)}$$

$$\Rightarrow u_t(x,t) = X(x)\mathrm{T}'(t),$$

and $$u_x(x,t) = X'(x)\mathrm{T}(t),$$

$$\Rightarrow u_{xx}(x,t) = X''(x)\mathrm{T}(t).$$

Hence equation (1.1) becomes

$$X(x)\mathrm{T}'(t) = kX''(x)\mathrm{T}(t)$$

$$\Rightarrow \frac{X''}{X} = \frac{\mathrm{T}'}{k\mathrm{T}} = \lambda \quad \text{(say)}, \qquad \text{... (1.5)}$$

where λ is a constant may be zero, positive or negative

$$\Rightarrow X'' - \lambda X = 0, \qquad \text{... (1.6)}$$

and $$\mathrm{T}' - \lambda k\mathrm{T} = 0. \qquad \text{... (1.7)}$$

Case (i) If $\lambda = 0$, then solutions of (1.6) and (1.7) are given by

$$X = Ax + B, \quad T = \mathbb{C}$$

The conditions (1.2) $\Rightarrow X(0) = X(\ell) = 0 \Rightarrow A = 0 = B.$

Consequently, $u(x,t) = 0$, which is a trivial solution of equation (1.1).

Case (ii) If $\lambda > 0$ say $\lambda = \alpha^2$

Therefore, solutions of equations (1.6) and (1.7) are

$$X(x) = Ae^{\alpha x} + Be^{-\alpha x} \quad \text{and} \quad \mathrm{T}(t) = Ce^{\alpha^2 kt}.$$

Now the conditions (1.2) $\Rightarrow X(0) = X(\ell) = 0$,

and
$$\left.\begin{array}{l}\Rightarrow A+B=0\\ Ae^{\alpha\ell}+Be^{\alpha\ell}=0\end{array}\right\}\Rightarrow A=B=0.$$

$$\Rightarrow u(x,t)=0.$$

Thus for $\lambda=0$ and $\lambda>0$ we have trivial solutions. Therefore, we assume

Case (iii) : $\lambda<0$ say $\lambda=-\alpha^2$, $\alpha>0$.

Therfore, equations (1.6) and (1.7) become

$$X''+\alpha^2X=0 \text{ and } T'+\alpha^2kT=0,$$

which have solutions

$$X(x)=A\cos\alpha x+B\sin\alpha x, \qquad \text{... (1.8)}$$

and
$$T=Ce^{-\alpha^2kt}. \qquad \text{... (1.9)}$$

Therefore, solution (1.4) becomes

$$u(x,t)=(A\cos\alpha x+B\sin\alpha x)C.e^{-\alpha^2kt}. \qquad \text{... (1.10)}$$

The boundary conditions (1.2) viz.

$$u(0,t)=u(\ell,t)=0\Rightarrow X(0)=0=X(\ell),$$

$$X(0)=0\Rightarrow A=0,$$

and
$$X(\ell)=0\Rightarrow \text{B sin }(\alpha\ell)=0.$$

If $B=0$ yields only trivial solutions. Therefore we assume $B\neq 0$

$$\Rightarrow \sin\alpha\ell=0,$$

$$\Rightarrow (\alpha\ell)=n\pi \ , \forall n=1,2,3,....$$

Let for each value of n=1,2,....

$$\alpha_n=\frac{n\pi}{\ell}, \quad n=1,2,... \qquad \text{... (1.11)}$$

These are called eigen values of the differential equation. Hence the solutions of (1.6) and (1.7) are respectively

$$X_n(x)=B_n\sin\left(\frac{n\pi x}{\ell}\right), \qquad \text{... (1.12)}$$

$$T_n(t)=C_n\exp\left(\frac{-n^2\pi^2}{\ell}kt\right). \qquad \text{... (1.13)}$$

These are called the corresponding eigen functions of the equations. Therefore we write from (1.4)

$$u_n(x,t) = B_n C_n \exp\left(\frac{-n^2\pi^2 kt}{\ell}\right)\sin\left(\frac{n\pi x}{\ell}\right).$$

or $$u_n(x,t) = a_n \exp\left(\frac{-n^2\pi^2 kt}{\ell^2}\right)\sin\left(\frac{n\pi}{\ell}x\right), \text{ for } a_n = B_n C_n \qquad \ldots (1.14)$$

Thus by the principle of superposition, we have

$$u(x,t) = \sum_{n=1}^{\infty} u_n(x,t)$$

is also solution of (1.1).

$$\Rightarrow u(x,t) = \sum_{n=1}^{\infty} a_n \exp\left(\frac{-n^2\pi^2 kt}{\ell^2}\right)\sin\left(\frac{n\pi}{\ell}x\right), \qquad \ldots (1.15)$$

if it converges, is also a solution of (1.1) satisfying the boundary conditions. That the initial temperature in the rod is given by

$$u(x,0) = f(x).$$

$$\Rightarrow f(x) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi}{\ell}x\right), \quad 0 \le x \le \ell. \qquad \ldots (1.16)$$

This is a Fourier series expansion of $f(x)$, where the Fourier constant a_n is given by

$$a_n = \frac{2}{\ell}\int_o^{\ell} f(x)\sin\left(\frac{n\pi x}{\ell}\right)dx. \qquad \ldots (1.17)$$

Thus equation (1.15) is a solution of the equation (1.1), where the constant a_n is given in equation (1.17).

Example 1 : Solve $u_t = u_{xx}$, $\quad 0 < x < \ell,\ t > 0$,

$u(0,t) = u(\ell,t) = 0,$

$u(x,0) = x(\ell,t), \quad 0 \le x \le \ell.$

OR

A heat flow in a rod of length 10 cm of homogeneous material is governed by the p.d.e $u_t = c^2 u_{xx}$. The ends of the rod are kept at temperature $O^0 c$ and initial temperature is $u(x,0) = x(10-x)$. Find the temperature in the rod at any instant.

Solution : we are given that

$$u_t = u_{xx}, \qquad 0 < x < \ell, t > 0, \qquad \text{... (1.18)}$$

satisfying $$u(0,t) = u(\ell,t) = 0, \qquad \text{... (1.19)}$$

and $$u(x,0) = x(\ell - x) = 0, \qquad 0 \le x \le \ell. \qquad \text{... (1.20)}$$

We assume $$u(x,t) = X(x)T(t) \qquad \text{... (1.21)}$$

$$\Rightarrow u_t = X(x)\mathrm{T}'(t) \text{ and } u_{xx} = X''(x)\mathrm{T}(t)$$

Hence equation (1.18) becomes

$$X(x)\mathrm{T}'(t) = X''(x)\mathrm{T}(t)$$

or $$\frac{X''}{X} = \frac{\mathrm{T}'}{\mathrm{T}} = \lambda(say).$$

We have if λ is zero or positive, the equation (1.18) has trivial solution. Therefore, we assume λ is negative. We choose

$$\lambda = -\alpha^2$$

$$\Rightarrow X'' + \alpha^2 X = 0, \qquad \text{... (1.22)}$$

and $$\mathrm{T}' + \alpha^2 \mathrm{T} = 0 \cdot \qquad \text{... (1.23)}$$

The solutions of (1.22)and (1.24) are respectively given by

$$X(x) = A\cos\alpha x + \beta\sin\alpha x, \qquad \text{... (1.24)}$$

and $$\mathrm{T}(t) = C\exp(-\alpha^2 t) \cdot \qquad \text{... (1.25)}$$

The boundary conditions (1.19) give

$$X(0) = 0 = X(\ell).$$

From equation (1.24) we have for $X(0) = 0$

$$\Rightarrow A = 0,$$

and $$X(\ell) = 0 \Rightarrow 0 = B\sin\alpha\ell.$$

If $B = 0$ we have only trivial solution of (1.18). Hence we assume $B \ne 0$. In this case, we have

$$\sin(\alpha\ell) = 0,$$

$$\Rightarrow \alpha\ell = n\pi \qquad \text{for } n = 1, 2, 3...$$

Let $$\alpha_n = \frac{n\pi}{\ell}, n = 1, 2, 3... \qquad \text{... (1.26)}$$

Hence the corresponding solutions of (1.22) and (1.23) are

$$X_n(x) = B_n \sin\left(\frac{n\pi}{\ell}x\right), \quad \text{and} \qquad \text{... (1.27)}$$

$$\mathrm{T}_n(t) = C_n \exp\left(-\frac{n^2\pi^2}{\ell^2}t\right). \qquad \text{... (1.28)}$$

These are called the corresponding eigen functions. Thus the solution of (1.18) can be written as

$$u_n(x,t) = B_n C_n \exp\left(-\frac{n^2\pi^2}{\ell^2}t\right)\sin\left(\frac{n\pi}{\ell}x\right)$$

or

$$u_n(x,t) = a_n \exp\left(-\frac{n^2\pi^2 t}{\ell^2}\right)\sin\left(\frac{n\pi}{\ell}x\right). \qquad \text{... (1.29)}$$

The by the principle of superposition, the solution of equation (1.18) is given by

$$u(x,t) = \sum_{n=1}^{\infty} u_n(x,t).$$

That is

$$u(x,t) = \sum_{n=1}^{\infty} a_n \exp\left(\frac{-n^2a^2}{\ell^2}t\right)\sin\left(\frac{n\pi}{\ell}x\right) \qquad \text{... (1.30)}$$

if it converges. However, it is given that the initial temperature of the rod is

$$u(x,o) = x(\ell - x), \qquad 0 \le x \le \ell$$

$$\Rightarrow x(\ell - x) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi}{\ell}x\right), \qquad 0 \le x \le \ell \qquad \text{... (1.31)}$$

which is the Fourier sine series of $f(x) = x(\ell - x)$. Hence the Fourier constant a_n is given by

$$a_n = \frac{2}{\ell}\int_0^{\ell} x(\ell - x)\sin\left(\frac{n\pi}{\ell}x\right)dx,$$

$$= \frac{2}{\ell}\left[\ell\int_0^{\ell} x\sin\left(\frac{n\pi}{\ell}x\right)dx - \int_0^{\ell} x^2\sin\left(\frac{n\pi}{\ell}x\right)dx\right],$$

$$a_n = \frac{2}{\ell}\left[\ell\left\{-x\frac{\ell}{n\pi}\cos\left(\frac{n\pi}{\ell}x\right)\right\}_0^\ell + \ell\int_0^\ell \frac{\ell}{n\pi}\cos\left(\frac{n\pi}{\ell}x\right)dx\right] -$$

$$-\frac{2}{\ell}\left[-x^2\frac{\ell}{n\pi}\cos\left(\frac{n\pi}{\ell}x\right)\Bigg|_0^\ell + 2\int_0^\ell \frac{\ell}{n\pi}x\cos\left(\frac{n\pi}{\ell}x\right)dx\right]$$

$$a_n = \frac{2}{\ell}\left[-\frac{\ell^3}{n\pi}(-1)^n + \frac{\ell^3}{n\pi}(-1)^n - 2\frac{\ell^3}{(n\pi)^3}\cos\left(\frac{n\pi}{\ell}x\right)_0^\ell\right]$$

$$a_n = \frac{-4\ell^2}{(n\pi)^3}\left[(-1)^n - 1\right] \qquad \text{... (1.32)}$$

$$\Rightarrow a_n = \frac{8\ell^2}{n^3\pi^3}, \qquad \text{for n is odd,}$$

$$a_n = 0, \qquad \text{for n is even.} \qquad \text{... (1.33)}$$

Hence equation (1.30) is the required solution of equation of (1.18) with

$$a_n = \frac{8\ell^2}{n^3\pi^3}.$$

Uniqueness of the Solution :

Theorem : Show that the solution $u(x,t)$ of the differential equation

$$u_t - ku_{xx} = F(x,t), \qquad 0 < x < \ell,\ t > 0, \qquad \text{... (1.34)}$$

satisfying the initial condition

$$u(x,0) = f(x), \qquad 0 \le x \le \ell, \qquad \text{... (1.35)}$$

and the boundary conditions

$$u(0,t) = u(\ell,t) = 0, \qquad t \ge 0 \qquad \text{... (1.36)}$$

is unique.

Proof. Let $u_1(x,t)$ and $u_2(x,t)$ be two solutions of the equation (1.34) subject to the conditions (1.35) and (1.36).

$$\Rightarrow \frac{\partial u_1}{\partial t} - k\frac{\partial^2 u_1}{\partial x^2} = F(x,t), \qquad 0 < x < \ell \quad t > 0, \qquad \text{... (1.37)}$$

and $$\frac{\partial u_2}{\partial t} - k\frac{\partial^2 u_2}{\partial x^2} = F(x,t), \qquad 0 < x < \ell, \quad t > 0, \qquad \text{... (1.38)}$$

satisfying the conditions

$$u_1(x,0) = f(x), \qquad 0 \le x \le \ell \qquad \text{... (1.39)}$$

$$u_2(x,0) = f(x), \qquad \text{... (1.40)}$$

and $$u_1(0,t) = u,(\ell,t) = 0, \qquad \text{... (1.41)}$$

$$u_2(0,t) = u_2(\ell,t) = 0. \qquad \text{... (1.42)}$$

Subtracting (1.38) from (1.37) we get

$$\frac{\partial}{\partial t}(u_1 - u_2) - k\frac{\partial^2 (u_1 - u_2)}{\partial x^2} = 0,$$

satisfying $$u_1(x,0) - u_2(x,0) = 0,$$

and $$u_1(0,t) - u_2(0,t) = 0 = u_1(\ell,t) - u_2(\ell,t).$$

These equations show that, $v = u_1 - u_2$ satisfies the corresponding equation

$$\frac{\partial v}{\partial t} - k\frac{d^2 v}{\partial x^2} = 0, \qquad 0 \le x < \ell, \; t > 0 \qquad \text{... (1.43)}$$

$$v(0,t) = v(\ell,t) = 0 \quad , t > 0 \qquad \text{... (1.44)}$$

$$v(x,0) = 0, \qquad 0 \le x \le \ell. \qquad \text{... (1.45)}$$

Claim : We prove that $u_1(x,t) = u_2(x,t)$

Let us define a function E (t), such that

$$E(t) = \frac{1}{2k}\int_0^\ell v^2(x,t)dx. \qquad \text{... (1.46)}$$

Since the integrand is positive definite $\Rightarrow E \ge 0$ ... (1.47)

Differentiating (1.46) w.r.t. t we get

$$\frac{dE}{dt} = \frac{1}{k}\int_0^\ell 2v\frac{\partial v}{\partial t}dx,$$

$$= \frac{1}{k}\int_0^\ell 2vk\, u_{xx}\, dx \qquad \text{by equations (1.43)}$$

$$\frac{dE}{dt} = 2\int_0^{\ell} v\, v_{xx}\, dx .$$

Integrating the r.h.s. by parts we get

$$\frac{dE}{dt} = 2\left[v\, v_x\Big|_0^{\ell} - \int_0^{\ell} v_x^2\, dx \right]$$

Therefore , by boundary conditions (1.44) and (1.45) we have

$$v(0,t) = v(\ell,t) = 0 .$$

$$\Rightarrow \frac{dE}{dt} = -2\int_0^{\ell} v_x^2 dx \le 0 \quad \Rightarrow \frac{dE}{dt} \le 0 . \qquad \text{... (1.48)}$$

This shows that E (t) is decreasing function of t. From the condition $v(x,0) = 0$ we have from (1.46) E(0) = 0. Therefore we have

$$E(t) \le 0, \qquad \forall t > 0. \qquad \text{... (1.49)}$$

But, by definition (1.46) $E(t)$ is non-negative.

$$\Rightarrow E(t) = 0, \qquad \forall t > 0$$

$$\Rightarrow v(x,t) \equiv 0 \qquad \text{on } 0 \le x \le \ell, \quad t \ge 0$$

$$\Rightarrow u_1(x,t) = u_2(x,t) .$$

Hence the solution is unique.

Example 2 : The temperature $u(x,t)$ in a rod of length ℓ in governed by the p.d.e.

$$u_t = c^2 u_{xx} .$$

The initial temperature is $u(x,0) = f(x)$. The rod and its ends are perfectly insulated

$$u_x(0,t) = 0 \qquad \text{and} \qquad u_x(\ell,t) = 0 .$$

Find the temperature distribution in the rod.

Solution : Let $u(x,t)$ be the temperature in a rod of length ℓ . We know it is governed by the p.d.e.

$$u_t = c^2 u_{xx} \qquad 0 < x < \ell, t > 0 . \qquad \text{... (1.50)}$$

Given that initial temperature is

$$u(x,0) = f(x) , \qquad \text{... (1.51)}$$

and

$$u_x(0,t) = u_x(\ell,t) = 0 . \qquad \text{... (1.52)}$$

We assume

$$u(x,t) = X(x)T(t) \qquad \text{... (1.53)}$$

be the solution of equation (1.50)

$$\Rightarrow u_x = X'(x)T(t) \Rightarrow u_{xx} = X''(x)T(t),$$

and $$u_t = X(x)T'(t)$$

Hence equation (1.50) becomes

$$X(x)T'(t) = c^2 X''(x)T(t)$$

$$\Rightarrow \frac{X''}{X} = \frac{T'(t)}{c^2 T} = \lambda(say).$$

$$\Rightarrow X'' - \lambda X = 0, \qquad \text{... (1.54)}$$

and $$T' - \lambda c^2 T = 0, \qquad \text{... (1.55)}$$

where λ is either zero or positive or negative. If $\lambda = 0$ and $\lambda > 0$ we know that it has trivial solution. Therefore, we choose $\lambda < 0$ Let $\lambda = -\alpha^2$. Hence equations (1.54) and (1.55) become

$$X''(x) + \alpha^2 X = 0, \qquad \text{... (1.56)}$$

and $$T' + \alpha^2 c^2 T = 0. \qquad \text{... (1.57)}$$

Solving equations (1.56) and (1.57) we get

$$X = A\cos\alpha x + B\sin\alpha x, \qquad \text{... (1.58)}$$

and $$T = C\exp(-\alpha^2 c^2 t), \; C \neq 0. \qquad \text{... (1.59)}$$

Thus the temperature distribution in the rod is given by

$$u(x,t) = (A\cos\alpha x + B\sin\alpha x).C\exp(-\alpha^2 c^2 t). \qquad \text{... (1.60)}$$

To find the constants, we use the given conditions (1.52). From equation (1.60) we find

$$u_x(x,t) = -a(A\sin\alpha x - B\cos\alpha x)C.\exp(-\alpha^2 e^2 t)$$

Thus $$u_x(0,t) = 0 \Rightarrow 0 = B \quad \text{for } C \neq 0.$$

Hence equation (1.60) reduces to

$$u(x,t) = A\cos\alpha x.C\exp(-\alpha^2 c^2 t). \qquad \text{... (1.61)}$$

Also $$u_x(\ell,t) = 0 \Rightarrow 0 = A.\sin(\alpha\ell).C\exp(-\alpha^2 c^2 t).$$

$$\Rightarrow A = 0 \quad \text{for } C \neq 0,$$

or $$\sin(\alpha\ell) = 0 \; \text{for } C \neq 0.$$

If $A = 0$ then we have trivial solution. Therefore, we assume $A \neq 0$ for $C \neq 0$.

$$\Rightarrow \sin(\alpha\ell) = 0\,,$$

$$\Rightarrow \alpha\ell = n\pi, \quad \text{for} \quad n = 1, 2, 3, ...$$

$$\Rightarrow \alpha_n = \frac{n\pi}{\ell}, \ n = 1, 2, 3, ... \qquad ... (1.62)$$

These are called the eigen values. Substituting this in equation (1.61) we get

$$u(x,t) = AC \cos\left(\frac{n\pi}{\ell}x\right) \exp\left(\frac{-n^2\pi^2c^2}{\ell}t\right).$$

or

$$u_n(x,t) = a_n \cos\left(\frac{n\pi}{\ell}x\right) \exp\left(\frac{-n^2\pi^2c^2}{\ell}t\right). \qquad ... (1.63)$$

By the superposition principle, the most general solution of equation (1.50) is given by

$$u(x,t) = \sum_{n=1}^{\infty} u_n(x,t) = \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{\ell}x\right) \exp\left(\frac{-n^2\pi^2c^2}{\ell}t\right). \qquad ... (1.64)$$

Given that the initial temperature in the rod is

$$u(x,0) = f(x)\,.$$

Therefore, from (1.64) we have

$$u(x,0) = f(x) = \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{\ell}x\right). \qquad ... (1.65)$$

This represents the expansion of f (x) in the Fourier cosine series. Consequently , the Fourier constant a_n is given by

$$a_n = \frac{2}{\ell}\int_0^{\ell} f(x) \cos\left(\frac{n\pi}{\ell}x\right) dx\,. \qquad ... (1.66)$$

$$\Rightarrow u(x,t) = \frac{2}{\ell}\sum_{n=1}^{\infty}\int_0^{\ell} f(x) \cos\left(\frac{n\pi}{\ell}x\right) dx \cdot \cos\left(\frac{n\pi}{\ell}x\right) \cdot \exp\left(\frac{-n^2\pi^2c^2}{\ell}t\right). \qquad ... (1.67)$$

2. Heat conduction - Infinite Rod.

Result : Find the temperature distribution in a rod of infinite length satisfying the initial conditions

$$u(x,0) = f(x)\,, \qquad -\infty < x < \infty$$

Solutions : Consider a homogeneous sufficiently thin rod of infinite length such that its surface is insulated. If $u(x,t)$ is the temperature in the rod, then the temperature distribution in the rod is governed by the second order partial differential equation

$$u_t = ku_{xx} \quad , \qquad -\infty < x < \infty, t > 0, \qquad \text{... (2.1)}$$

satisfying

$$u(x,0) = f(x), \qquad -\infty < x < \infty. \qquad \text{... (2.2)}$$

We use the Fourier transform method to solve the equation. Therefore let the Fourier transform of $u(x,t)$ be $U(\alpha,t)$

i.e. $\mathscr{F}\left(u(x,+1)\right) = U(\alpha,t)$

Thus by definition of Fourier transform, we have

$$\mathscr{F}\left(u(x,+1)\right) = U(\alpha,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u(x,t) e^{i\alpha t} dt. \qquad \text{... (2.3)}$$

Also we know the formula for Fourier transform of derivative as

$$\mathscr{F}\left(f^n(x)\right) = (-i\alpha)^n \mathscr{F}\left(f(x)\right). \qquad \text{... (2.4)}$$

Hence taking of the Fourier transform of equation (2.1) and using the formula (2.4) we get

$$U_t = k(-i\alpha)^2 \mathscr{F}(u(x,t)),$$

$$U_t = -k\alpha^2 U(\alpha,t),$$

$$\Rightarrow U_t + k\alpha^2 U = 0. \qquad \text{... (2.5)}$$

This is the first order differential equation, whose solution is obtain by integrating equation (2.5)

$$\Rightarrow U(\alpha,t) = A(\alpha) e^{-\alpha^2 kt}, \qquad \text{... (2.6)}$$

where $A(\alpha)$ is an arbitrary function to be determined from the initial conditions.

From the definition (2.3) we obtain

$$\mathscr{F}\left(u(x,0)\right) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u(x,0) e^{i\alpha x} dx$$

$$\Rightarrow U(\alpha,0) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{i\alpha x} dx \qquad \text{by equation (2.2).}$$

But from equation (2.6), we have

$$U(\alpha,0) = A(\alpha)$$

$$\Rightarrow A(\alpha) = \mathscr{F}\left(f(x)\right) = \mathscr{F}(\alpha).$$

Hence equation (2.6) becomes

$$U(\alpha,t) = \mathcal{F}(\alpha)e^{-\alpha^2 kt}. \qquad \text{... (2.7)}$$

Taking the inverse Fourier transform of equation (2.7) we get

$$\mathcal{F}^{-1}\left[\mathcal{F}\left(u(x+1)\right)\right] = \mathcal{F}^{-1}\left[\mathcal{F}\left(f(x)\right)\mathcal{F}(g)\right] \quad \text{.... for } \mathcal{F}(g) = e^{-\alpha^2 kt}$$

$$\Rightarrow u(x,t) = \mathcal{F}^{-1}\left[\mathcal{F}\left(f * g\right)\right],$$

$$u(x,t) = f * g, \qquad \text{... (2.8)}$$

where $f * g$ is the convolution of $f(x)$ and $g(x)$ over the interval $(-\infty, \infty)$ and is defined by

$$\left(f * g\right)(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x-\xi)g(\xi)d\xi.$$

Thus we have

$$u(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(\xi)g(x-\xi)d\xi, \qquad \text{... (2.9)}$$

where $g = \mathcal{F}^{-1}(e^{-\alpha^2 kt})$

$$\Rightarrow g(x) = \frac{1}{2\sqrt{\pi kt}} \exp\left(-\frac{x^2}{4kt}\right)$$

$$\Rightarrow g(x-\xi) = \frac{1}{2\sqrt{\pi kt}} \exp\left(-\frac{(x-\xi)^2}{4kt}\right).$$

Hence equation (2.9) becomes

$$u(x,t) = \frac{1}{\sqrt{2\pi}} \frac{1}{2\sqrt{\pi \mathrm{kt}}} \int_{-\infty}^{\infty} f(\xi) \exp\left[\frac{\left(x-\xi\right)^2}{4kt}\right] d\xi. \qquad \text{... (2.10)}$$

If $k = 1$ and

$$f(x) = \begin{cases} 0 \text{ when } x<0 \\ a \text{ When } x>0, \end{cases}$$

then we have

$$u(x,t) = \frac{a}{\pi\sqrt{2t}} \int_0^8 \exp\left[-\frac{(x-\xi)^2}{4t}\right] d\xi .$$

Put $\quad \dfrac{x-\xi}{2\sqrt{t}} = \eta ,$

$$\Rightarrow d\xi = 2\sqrt{t}\, d\eta$$

when $\quad \xi = 0, \quad \eta = -\dfrac{x}{2\sqrt{t}}$

and as $\quad \xi \to \infty, \quad \eta \to \infty .$

Thus we have

$$u(x,t) = \frac{a}{\pi\sqrt{2t}} \int_{-x/2\sqrt{t}}^{\infty} e^{-\eta^2} \cdot 2\sqrt{t}\, d\eta$$

$$u(x,t) = \frac{a}{\pi}\sqrt{2} \int_{-x/2\sqrt{t}}^{\infty} e^{-\eta^2}\, d\eta$$

We write this as

$$u(x,t) = \frac{a}{\pi}\sqrt{2}\left[\int_{-x/2\sqrt{t}}^{0} e^{-\eta^2}\, d\eta + \int_0^{\infty} e^{-\eta^2}\, d\eta\right]. \qquad \text{... (2.11)}$$

Now consider the integral

$$\int_{-x/2\sqrt{t}}^{0} e^{-\eta^2}\, d\eta$$

Put $\quad \eta = -y \quad \Rightarrow d\eta = -dy$

When $\quad \eta = \dfrac{-x}{2\sqrt{t}} \quad \Rightarrow y = \dfrac{x}{2\sqrt{t}}$

and $\quad \eta = 0 \quad \Rightarrow y = 0$

Thus we have

$$\int_{\frac{-x}{2\sqrt{t}}}^{0} e^{-\eta^2}\, d\eta = \int_0^{x/2\sqrt{t}} e^{-\eta^2}\, d\eta \qquad \text{... (2.12)}$$

Also consider $\int_0^\infty e^{-\eta^2} d\eta$.

Put $\eta^2 = t \Rightarrow 2\eta d\eta = dt$,

$$\Rightarrow d\eta = \frac{1}{2\sqrt{t}} dt .$$

When $\eta = 0 \Rightarrow t = 0$,

and $\eta \to \infty \Rightarrow t \to \infty$.

Thus $$\int_0^\infty e^{-\eta^2} d\eta = \int_0^\infty e^{-t} \frac{1}{2} t^{\frac{-1}{2}} dt = \frac{1}{2}\int_0^\infty e^{-t} t^{\frac{1}{2}-1} dt$$

$$\Rightarrow \int_0^\infty e^{-\eta^2} d\eta = \frac{1}{2}\sqrt{\frac{1}{2}} = \frac{\sqrt{\pi}}{2}. \qquad \text{... (2.13)}$$

Using (2.12) and (2.13) in equation (2.11) we get

$$u(x,t) = \frac{a}{\pi}\sqrt{2}\left[\int_0^{\frac{x}{2\sqrt{t}}} e^{-\eta^2} d\eta + \frac{\sqrt{\pi}}{2}\right]$$

$$= \frac{a}{\sqrt{2\pi}}\left[1 + \frac{2}{\sqrt{\pi}}\int_0^{\frac{x}{2\sqrt{t}}} e^{-\eta^2} d\eta\right]$$

$$\Rightarrow u(x,t)\frac{a}{\sqrt{2\pi}}\left[1 + erf\left(\frac{x}{2\sqrt{t}}\right)\right], \qquad \text{... (2.14)}$$

where

$erf(x) = \frac{2}{\sqrt{\pi}}\int_0^\infty e^{-\eta^2} d\eta$ is the error function.

3. Families of Equipotential surfaces :

Definition :

Let $f(x,y,z)=c$ be a one-parameter family of surfaces. We say that this family of surfaces is equipotential if these exists a potential function ψ (which is a solution of Laplace equation $\nabla^2\psi=0$) such that ψ is constant whenever $f(x,y,z)$ is constant.

Note : Not every one parameter family of surfaces $f(x,y,z)=c$ is a family of equipotential surfaces.

Result : Find the condition that a one parameter family of surfaces form a family of equipotential surfaces.

Proof : Let $f(x,y,z)=c$ be a one parameter family of surfaces. By definition, equation (3.1) will be a family of equipotential surfaces if the potential function ψ (Which is a solution of the Laplace equation $\nabla^2\psi=0$) is constant whenever $f(x,y,z)$ is constant.

This means that there must exist a functional relation of the type

$$\psi=F\{f(x,y,z)\} \qquad \text{... (3.2)}$$

between the function ψ and F such that ψ =constant whenever $f(x,y,z)$ =constant.

Differentiating (3.2) partially w.r.t. x we obtain

$$\frac{\partial\psi}{\partial x}=\frac{dF}{df}.\frac{\partial f}{\partial x},$$

... (3.3)

and
$$\frac{\partial^2\psi}{\partial x^2}=\frac{d^2F}{df^2}.\left(\frac{\partial f}{\partial x}\right)^2+\frac{dF}{df}.\frac{\partial^2 f}{\partial x^2}. \qquad \text{... (3.4)}$$

Similarly,
$$\frac{\partial^2\psi}{\partial y^2}=\frac{d^2F}{df^2}\left(\frac{\partial f}{\partial y}\right)^2+\frac{dF}{df}.\frac{\partial^2 f}{\partial y^2}, \qquad \text{... (3.5)}$$

and
$$\frac{\partial^2\psi}{\partial z^2}=\frac{d^2F}{df^2}\left(\frac{\partial f}{\partial z}\right)^2+\frac{dF}{df}.\frac{\partial^2 f}{\partial z^2}. \qquad \text{... (3.6)}$$

Therefore, consider

$$\nabla^2\psi=\frac{\partial^2\psi}{\partial y^2}+\frac{\partial^2\psi}{\partial y^2}+\frac{\partial^2\psi}{\partial z^2},$$

$$\nabla^2\psi = \frac{d^2F}{df^2}\left[\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 + \left(\frac{\partial f}{\partial z}\right)^2\right] + \frac{dF}{df}\left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}\right)$$

$$\nabla^2\psi = F''(f)\ (grad f)^2 + F'(f)\nabla^2 f. \qquad \text{... (3.7)}$$

Since ψ satisfies the Laplace equation in free space

$$\Rightarrow \nabla^2\psi = 0$$

$$\Rightarrow \frac{F''(f)}{F'(f)} = -\frac{\nabla^2 f}{(grad f)^2}. \qquad \text{... (3.8)}$$

This shows that, the condition that the surfaces (3.1) form a family of equipotential surfaces is that the quantity $\frac{\nabla^2 f}{|grad f|^2}$ is a function of f alone. We denote this function by $\chi(f)$. Hence equation (3.8) can be written as

$$\frac{F''(f)}{F'(f)} = -\chi(f),$$

$$\Rightarrow \frac{d^2F}{df^2} + \chi(f)\frac{df}{df} = 0. \qquad \text{... (3.9)}$$

Integrating we get

$$\frac{dF}{df} = A.e^{-\int \chi(f)df} \qquad \text{... (3.10)}$$

where A is a constant. Integrating (3.10) w.r.t. f we get

$$\psi = F(f) = A\int e^{-\int \chi(f)df} df + B \qquad \text{... (3.11)}$$

where B is a constant. This is the general form of the corresponding potential function.

This is the necessary condition that the one-parameter family of surfaces $f(x, y, z) = c$ is a family of equipotential surfaces.

Example 1 : Show the surfaces

$$x^2 + y^2 + z^2 = r^2, \quad r > 0$$

form a family of equipotential surfaces and find the general form a the corresponding potential function.

Solution : Let

$$f(x,y,z) = x^2 + y^2 + z^2 = r^2 \quad \text{... (3.12)}$$

be the one parameter family of surfaces. To show that this family forms a family of equipotential surfaces, we find the potential ψ s.t.

$$\text{grad } f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{dy}, \frac{\partial f}{\partial z}\right) = 2(x,y,z), \quad \text{... (3.13)}$$

$$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = 2+2+2 = 6 \quad \text{... (3.14)}$$

$$\Rightarrow \nabla^2 f = 6 .$$

$$\Rightarrow |\nabla f|^2 = 4(x^2 + y^2 + z^2) \quad \text{... (3.15)}$$

$$\Rightarrow |\nabla f|^2 = 4f .$$

Therefore, the equation

$$\frac{\nabla^2 f}{|\nabla f|^2} = \frac{3}{2f} = \chi(f) \quad \text{... (3.16)}$$

Therefore, the equation $\dfrac{F''(f)}{F'(f)} = -\chi(f)$

has solution $\quad \psi = A\int e^{-\int \chi(f)df} df + B$

$$\Rightarrow \psi = A\int e^{-\frac{3}{2}\int \frac{1}{f}df} .df + B \Rightarrow \psi = A\int e^{\log(f)^{\frac{-3}{2}}} df + B.$$

$$\psi = A\int f^{-\frac{3}{2}} df + B ,$$

This gives

$$\psi = \frac{-2A}{f^{\frac{1}{2}}} + B .$$

$$\Rightarrow \psi = -\frac{2A}{r} + B .$$

Example 2 : Show that the surfaces

$$x^2 + y^2 + z^2 = cx^{\frac{2}{3}}$$

can form an equipotential of surfaces, and find the general form of the potential function.

Solution : One parameter family of surfaces is given by

$$x^2 + y^2 + z^2 = c\, x^{\frac{2}{3}}$$

i.e. $$x^{\frac{-2}{3}}\left(x^2 + y^2 + z^2\right) = c \ .$$

Let $$f\left(x, y, x\right) = x^{\frac{-2}{3}}\left(x^2 + y^2 + z^2\right) = c \qquad \text{... (3.17)}$$

To show this family forms a family of equipotential surfaces we find

$$\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{dy}, \frac{\partial f}{dz}\right)$$

$$\nabla f = \left(\frac{4}{3}x^{1/3} - \frac{2}{3}x^{-5/3}\left(y^2 + z^2\right), 2yx^{-2/3}, 2zx^{-2/3}\right)$$

$$\nabla f = \frac{2}{3}x^{-5/3}\left(2x^2 - y^2 - z^2, 3xy, 3xz\right). \qquad \text{... (3.18)}$$

Now $$\frac{\partial^2 f}{\partial x^2} = \frac{4}{9}x^{-2/3} + \frac{10}{9}x^{-8/3}\left(y^2 + z^2\right),$$

$$\frac{\partial^2 f}{\partial y^2} = 2x^{-2/3}, \quad \frac{\partial^2 f}{\partial z^2} = 2x^{-2/3}.$$

Hence $$\nabla^2 f = \left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}\right) \text{ becomes}$$

$$\nabla^2 f = \frac{4}{9}x^{-2/3} + \frac{10}{9}x^{-8/3}\left(y^2 + z^2\right) + 2x^{-2/3} + 2xx^{-2/3},$$

$$= \frac{40}{9}x^{-2/3} + \frac{10}{9}x^{-8/3}\left(y^2 + z^2\right)$$

Thus, $$\nabla^2 f = \frac{10}{9}x^{-8/3}\left(4x^2 + y^2 + z^2\right).$$

Now $$(\nabla f)^2 = \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 + \left(\frac{\partial f}{\partial z}\right)^2 \qquad \text{... (3.19)}$$

becomes

$$|\text{grand}\, f|^2 = \frac{4}{9}x^{-10/3}\left[\left(2x^2 - y^2 - z^2\right)^2 + 9x^2y^2 + 9x^2z^2\right],$$

$$= \frac{4}{9}x^{-10/3}\left[4x^4 + \left(y^2+z^2\right)^2 - 4x^2\left(y^2+z^2\right) + 9x^2\left(y^2+z^2\right)\right],$$

$$= \frac{4}{9}x^{-10/3}\left[4x^4 + \left(y^2+z^2\right)^2 + 5x^2\left(y^2+z^2\right)\right],$$

$$= \frac{4}{9}x^{-10/3}\left[4x^4 + 4x^2\left(y^2+z^2\right) + \left(y^2+z^2\right)^2 + x^2\left(y^2+z^2\right)\right],$$

$$= \frac{4}{9}x^{-10/3}\left[4x^2\left(x^2+y^2+z^2\right) + \left(y^2+z^2\right)\left(y^2+z^2+x^2\right)\right],$$

$$|\nabla f|^2 = \frac{4}{9}x^{-10/3}\left[\left(4x^2+y^2+z^2\right) + \left(x^2+y^2+z^2\right)\right]. \qquad \text{... (3.20)}$$

So that, $$\frac{\nabla^2 f}{|\nabla f|^2} = \frac{\frac{10}{9}x^{-8/3}\left(4x^2+y^2+z^2\right)}{\frac{4}{9}x^{-10/3}\left(4x^2+y^2+z^2\right)\left(x^2+y^2+z^2\right)},$$

$$= \frac{5}{2}\frac{x^{2/3}}{\left(x^2+y^2+z^2\right)}$$

Therefore, $$\frac{\nabla^2 f}{|\nabla f|^2} = \frac{\frac{5}{2}}{x^{-2/3}\left(x^2+y^2+z^2\right)} \Rightarrow \frac{\nabla^2 f}{|\nabla f|^2} = \frac{5}{2f} = \chi(t) \qquad \text{... (3.21)}$$

This shows that the given set of surfaces forms a family of equipotential surfaces.

Now to find the general form of the corresponding potential function, we know it is given by

$$\psi = A\int e^{-\int \chi(f)df}\, df + B,$$

$$\psi = A\int e^{-\frac{5}{2}\int \frac{1}{f}df}\, df + B,$$

$$\psi = A\int e^{\log(f)^{-5/2}}\, df + B,$$

$$\psi = A\int f^{-5/2} df + B,$$

$$\psi = -\frac{2}{3} A f^{-3/2} + B,$$

or $$\psi = -\frac{2}{3} A\left[x^{-2/3}\left(x^2+y^2+z^2\right)\right]^{-3/2} + B,$$

$$\psi = -\frac{2}{3} A\left[x\left(x^2+y^2+z^2\right)^{-3/2}\right] + B.$$

This is the required potential function.

Example 3 : Show that the family of right circular cones $x^2 + y^2 = cz^2$, where c is a parameter, forms a set of equipotential surfaces and show that the corresponding potential function is of the form $A \log \tan\frac{\theta}{2} + B$, where A and B are constants and θ is the usual polar angle.

Solution : The family of right circular cones is given by

$$x^2 + y^2 = cz^2$$

i.e. $$z^{-2}\left(x^2+y^2\right) = c$$

Let $$f(x,y,z) = z^{-2}\left(x^2+y^2\right) = c. \qquad \text{... (3.22)}$$

To show that, this surfaces form an equipotential surfaces, we find

$$\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right) = \left(2xz^{-2}, 2yz^{-2}, -2z^{-3}\left(x^2+y^2\right)\right)$$

$$\nabla f = 2\left(xz^{-2}, yz^{-2}, -z^{-3}\left(x^2+y^2\right)\right). \qquad \text{... (3.23)}$$

Therefore, $$|\nabla f|^2 = 4\left(x^2z^{-4} + y^2z^{-4} + z^{-6}\left(x^2+y^2\right)^2\right),$$

$$= 4z^{-6}\left(x^2z^2 + y^2z^2 + \left(x^2+y^2\right)^2\right),$$

$$|\nabla f|^2 = 4z^{-6}\left(x^2+y^2\right)\left(x^2+y^2+z^2\right). \qquad \text{... (3.24)}$$

Next $$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} \text{ gives}$$

$$\nabla^2 f = 2z^{-2} + 2z^{-2} + 6z^{-4}\left(x^2+y^2\right),$$

$$\nabla^2 f = 4z^{-2} + 6z^{-4}\left(x^2+y^2\right),$$

$$\nabla^2 f = 2z^{-4}\left(3x^2+3y^2+2z^2\right),$$

$$\nabla^2 f = 2z^{-4}\left(2\left(x^2+y^2+z^2\right)+\left(x^2+y^2\right)\right). \qquad \text{... (3.25)}$$

Therefore,

$$\frac{\nabla^2 f}{|\nabla f|^2} = \frac{2z^{-4}\left[2\left(x^2+y^2+z^2\right)+\left(x^2+y^2\right)\right]}{4z^{-6}\left(x^2+y^2\right)\left(x^2+y^2+z^2\right)}$$

$$= \frac{1}{2}z^2\left[\frac{2}{x^2+y^2}+\frac{1}{x^2+y^2+z^2}\right]$$

$$= \frac{1}{z^{-2}\left(x^2+y^2\right)}+\frac{1}{2z^{-2}\left(x^2+y^2\right)+2}$$

or

$$\frac{\nabla^2 f}{|\nabla f|^2} = \frac{1}{f}+\frac{1}{2(f+1)} = \chi(f) \qquad \text{.... (3.26)}$$

This shows that the one parameter family of surfaces (3.22) forms an equipotential surfaces.

To find the corresponding potential function, we know it is given by

$$\psi = A\int e^{-\int \chi(f)df}\,df + B,$$

$$\psi = A\int e^{-\int\left(\frac{1}{f}+\frac{1}{2(f+1)}\right)df}\,df + B,$$

$$\psi = A\int e^{-\int\left(\log f+\log(f+1)^{1/2}\right)df}\,df + B,$$

$$\psi = A\int\left[\frac{1}{f}+(f+1)^{-1/2}\right]df + B,$$

$$\psi = A\left[\log f + 2\sqrt{f+1}\right] + B\ . \qquad \text{... (3.27)}$$

This is the required potential function. Now we show that potential function is given by

$$\psi = A\tan\frac{\theta}{2} + B.$$

We consider the transformation

$$x = r\sin\theta\cos\phi,\ y = r\sin\theta\sin\phi,\ r = r\cos\theta$$

$$\Rightarrow x^2 + y^2 = r^2 \sin^2 \theta$$

Hence $\quad \Rightarrow f = z^{-2}\left(x^2 + y^2\right) = \tan^2 \theta \qquad \Rightarrow df = 2\tan\theta \sec^2 \theta d\theta$

$$\Rightarrow \sqrt{f+1} = \sqrt{\tan^2 \theta + 1} = \sec\theta \cdot$$

$$\Rightarrow \chi(f) = \frac{1}{f} + \frac{1}{2(f+1)}$$

becomes $\quad \chi(f) = \frac{1}{\tan^2 \theta} + \frac{1}{2\sec^2 \theta}.$

Integrating we get

$$\int \chi(f)\,df = \int \left[\frac{1}{\tan^2 \theta} + \frac{1}{2\sec^2 \theta}\right] 2\tan\theta \sec^2 \theta d\theta,$$

$$= \int \frac{2\sec^2 \theta}{\tan\theta} d\theta + \int \tan\theta d\theta,$$

$$\int \chi(f)\,df = 2\log\tan\theta - \log\cos\theta,$$

$$\int \chi(f)\,df = \log \frac{\tan^2 \theta}{\cos\theta} = \log\left(\frac{\sin^2 \theta}{\cos^3 \theta}\right).$$

Therefore $\quad \psi = A\int e^{-\int \chi(f)df}\, df + B$

becomes $\quad \psi = A\int e^{\log\left(\frac{\cos^3 \theta}{\sin^2 \theta}\right)} \cdot 2\tan\theta \sec^2 \theta d\theta + B,$

$$\psi = 2A\int \frac{\cos^3 \theta}{\sin^2 \theta} \cdot \tan\theta \sec^2 \theta d\theta + B,$$

$$\psi = 2A\int \operatorname{cosec}\theta d\theta + B,$$

$$\psi = A\log\left(\tan\frac{\theta}{2}\right) + B.$$

This is the required family of equipotential surfaces.

Example 4 : Show that surfaces

$$\left(x^2 + y^2\right)^2 - 2a^2\left(x^2 + y^2\right) + a^4 = c$$

can form a family of equipotential surfaces and find the general form of the corresponding potential function.

Solution : The one-parameter family of surfaces is given by

$$f(x,y,z)=\left(x^2+y^2\right)^2-2a^2\left(x^2+y^2\right)+a^4=c\cdot \qquad \text{... (3.28)}$$

To show this surfaces form an equipotential surfaces, we find

$$\nabla f=\left(\frac{\partial f}{\partial x},\frac{\partial f}{\partial y},\frac{\partial f}{\partial z}\right)=\left(4x\left(x^2+y^2\right)-4a^2x,4y\left(x^2+y^2\right)+4a^2y,0\right),$$

$$\Rightarrow \nabla f=\left(4x\left(x^2+y^2\right)-4a^2x,4y\left(x^2+y^2\right)+4a^2y,0\right). \qquad \text{... (3.29)}$$

Therefore $\qquad |\nabla f|^2=\left(\frac{\partial f}{\partial x}\right)^2+\left(\frac{\partial f}{\partial y}\right)^2+\left(\frac{\partial f}{\partial z}\right)^2,$

becomes $\qquad |\nabla f|^2=16x^2\left[\left(x^2+y^2-a^2\right)^2\right]+16y^2\left[\left(x^2+y^2+a^2\right)^2\right],$

$$=16\left[x^2\left\{\left(x^2+y^2\right)^2+a^4-2a^2\left(x^2+y^2\right)\right\}+\right.$$

$$\left.+y^2\left\{\left(x^2+y^2\right)^2+a^4+2a^2\left(x^2+y^2\right)\right\}\right],$$

$$=16\left[\left(x^2+y^2\right)^3+a^4\left(x^2+y^2\right)-2a^2\left(x^2+y^2\right)\left(x^2-y^2\right)\right],$$

$$|\nabla f|^2=16\left(x^2+y^2\right)\left[\left(x^2+y^2\right)^2+a^4-2a^2\left(x^2+y^2\right)\right]. \qquad \text{... (3.30)}$$

Now $\qquad \nabla^2 f=\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}+\frac{\partial^2 f}{\partial z^2}$

becomes $\qquad \nabla^2 f=16\left(x^2+y^2\right) \qquad \text{... (3.31)}$

So that $\qquad \frac{\nabla^2 f}{|\nabla f|^2}=\frac{16\left(x^2+y^2\right)}{16\left(x^2+y^2\right)f}=\frac{1}{f}$

Hence $\qquad \frac{\nabla^2 f}{|\nabla f|^2}=\frac{1}{f}=\chi(f). \qquad \text{... (3.32)}$

This shows that the given set of surfaces (3.28) forms a family of equipotential surfaces.

Now to find the general form of the potential function, we know

$$\psi=A\int e^{-\int\chi(f)df}df+B,$$

$$\psi = A\int e^{-\int \frac{1}{f}df} \cdot df + B,$$

$$\psi = A\int e^{\log f^{-1}} df + B,$$

$$\psi = A\int \frac{1}{f} df + B,$$

$$\psi = A \log f + B,$$

or
$$\psi = A \log\left[\left(x^2+y^2\right)^2 - 2a^2\left(x^2+y^2\right) + a^4\right] + B, \qquad \text{... (3.33)}$$

where A, B are constants. This is a required equipotential function.

□□□

UNIT - VII

LAPLACE EQUATION

1. Introduction :

Various physical phenomena are goverened by the Laplace equation. In this unit we derive the Laplace equation and discuss the method of its solution. Various boundary value problems for the Laplace equation viz., the Dirichlet problem and Neumann problem for certain specified regions are the subject matter of this unit.

Result : Derive Laplace equation.

Proof : Consider two particles m and m_1 at Q and P respectively separated by a distance r. Then by Newton's law of gravitation, the magnitude of the force is directly proportional to the product of the masses and inversely proportional to the square of the distance between them.

$$\Rightarrow \bar{F} = -\frac{Gmm_1}{r^2}, \qquad \text{... (1.1)}$$

where the negative sign indicates the force is attractive. Here G is the gravitational constant. Assuming the unit mass at Q and G = 1, the force at Q due to the mass m_1 at P is given by

$$\bar{F} = -\frac{m_1}{r^2},$$

$$\Rightarrow \bar{F} = \frac{\partial}{\partial r}\left(\frac{m_1}{r}\right). \qquad \text{... (1.2)}$$

Let the particle of unit mass move under the attraction of the particle of mass m_1 at P from infinity upto Q, then the work done by the force $\bar{F}$ is given by

$$\int_\infty^r \bar{F}dr = \int_\infty^r \frac{\partial}{\partial r}\left(\frac{m_1}{r}\right) dr,$$

$$= \int_\infty^r d\left(\frac{m_1}{r}\right),$$

$$\Rightarrow \int_\infty^r \bar{F}dr = \frac{m_1}{r}. \qquad \text{... (1.3)}$$

The gravitational potential is defined to be the amount of work which must be done against gravitational force. Hence the potential V at Q due to a particle at P is given by

$$V = -\frac{m_1}{r}. \quad \text{... (1.4)}$$

From equations (1.2) the intensity of the force at P is given by

$$\bar{F} = -\nabla V \cdot \quad \text{... (1.5)}$$

Now if we consider a system of particles of masses $m_1, m_2, ..., m_n$ which are at distance $r_1, r_2, ..., r_n$ respectively, then the force of attraction at Q due to the system of particles is given by

$$\bar{F} = \sum_{i=1}^{n} \nabla\left(\frac{m_i}{r_i}\right),$$

$$\Rightarrow \bar{F} = \nabla\left(\sum_{i=1}^{n} \frac{m_i}{r_i}\right). \quad \text{... (1.6)}$$

The work done by the force acting on the particle is

$$\int_{\infty}^{r} \bar{F} d\bar{r} = \sum_{i=1}^{n} \frac{m_i}{r_i} = -V, \quad \text{... (1.7)}$$

$$\Rightarrow \nabla^2 V = -\nabla^2\left(\sum_{i=1}^{n} \frac{m_i}{r_i}\right), \qquad , r_i \neq 0$$

$$\Rightarrow \nabla^2 V = -\sum_{i=1}^{n} \nabla^2\left(\frac{m_i}{r_i}\right),$$

where $\bar{r_i} = x_i i + y_i j + z_i k$,

$$\Rightarrow |\bar{r_i}| = r_i = (x_i + y_i + z_i)^{1/2}$$

Thus $\nabla^2\left(\frac{m_i}{r_i}\right) = m_i\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\left(x_i^2 + y_i^2 + z_i^2\right)^{-1/2}$,

where $\frac{\partial}{\partial x}\left(x_i^2 + y_i^2 + z_i^2\right)^{-1/2} = \frac{-x_i}{\left(x_i^2 + y_i^2 + z_i^2\right)^{3/2}}$,

$$\Rightarrow \frac{\partial^2}{\partial x^2}\left(x_i^2 + y_i^2 + z_i^2\right)^{-1/2} = \frac{\left(2x_i^2 - y_i^2 - z_i^2\right)}{\left(x_i^2 + y_i^2 + z_i^2\right)^{5/2}}.$$

Similarly, $\frac{\partial^2}{\partial y^2}\left(x_i^2 + y_i^2 + z_i^2\right)^{-1/2} = \frac{\left(2y_i^2 - z_i^2 - x_i^2\right)}{\left(x_i^2 + y_i^2 + z_i^2\right)^{5/2}}$,

and
$$\frac{\partial^2}{\partial z^2}\left(x_i^2+y_i^2+z_i^2\right)^{-\frac{1}{2}}=\frac{\left(2z_i^2-x_i^2-y_i^2\right)}{\left(x_i^2+y_i^2+z_i^2\right)^{\frac{5}{2}}},$$

$$\Rightarrow\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)\left(\frac{m_i}{r_i}\right)=0,$$

$$\Rightarrow\nabla^2\left(\frac{m_i}{r_i}\right)=0,$$

$$\Rightarrow\nabla^2 V=0\cdot \qquad \text{... (1.8)}$$

This is called the Laplace equation.

Note : In 2-dimensions, the Laplace's equation is given by

$$\nabla^2 u=\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0. \qquad \text{... (1.9)}$$

A solution u (x, y) of equation (1.9) is called 2-dimensional harmonic function.

Solution of Laplace Equation :

Example 1 : Obtain the solution of the two-dimensional Laplace equations $\nabla^2 u=0$ by the method of separation of variables.

Solution : Consider the two-dimensional Laplace equation

$$\nabla^2 u=0,$$

i.e.
$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0. \qquad \text{... (1.10)}$$

To find the solution of (1.10) we assume

$$u(x,y)=X(x)Y(y). \qquad \text{... (1.11)}$$

$$\Rightarrow\frac{\partial^2 u}{\partial x^2}=X''(x)Y(y) \text{ and } \frac{\partial^2 u}{\partial y^2}=X(x)Y''(y)$$

Therefore, equation (1.10) becomes

$$X''Y+XY''(y)=0,$$

or
$$\frac{X''}{X}=-\frac{Y''}{Y}=k \text{ (say)}, \qquad \text{... (1.12)}$$

where k is called the separation constant, and k may be positive, zero or negative.

Case (i) k > 0. Take $k=\alpha^2$, α is real.

Therefore, we get from equations (1.12)

$$X''-\alpha^2 x=0 \text{ and } Y''+\alpha^2 y=0 \qquad \text{... (1.13)}$$

Solutions of these equations are respectively given by

$$X=C_1e^{\alpha x}+C_2e^{-\alpha x} \text{ and } Y=C_3\cos(\alpha y)+C_4\sin(\alpha y). \qquad \text{... (1.14)}$$

Hence the solution of equation (1.10) becomes

$$u(x,y)=\left(C_1e^{\alpha x}+C_2e^{-\alpha x}\right)\left(C_3\cos\alpha y+C_4\sin\alpha y\right). \qquad \text{... (1.15)}$$

Case (ii) If k = 0, then from equations (1.12) we have

$$X''=0 \text{ and } Y''=0.$$

Which provide us

$$X=C_5x+C_6 \text{ and } Y=C_7y+C_8.$$

Hence the solution of (1.10) becomes

$$u(x,y)=\left(C_5x+C_6\right)\left(C_7y+C_8\right). \qquad \text{... (1.16)}$$

Case (iii) Let k < 0. Take $k=-\alpha^2$

Hence the equations (1.12) become

$$X''+\alpha^2x=0 \text{ and } Y''-\alpha^2y=0,$$

which have solutions

$$X=\left(C_9\cos\alpha x+C_{10}\sin\alpha x\right) \text{ and } Y=C_{11}e^{\alpha y}+C_{12}e^{-\alpha y}.$$

Hence the general solution of (1.12) is given by

$$u(x,y)=\left(C_9\cos\alpha x+C_{10}\sin\alpha x\right)\left(C_{11}e^{\alpha y}+C_{12}e^{-\alpha y}\right). \qquad \text{... (1.17)}$$

In all these solutions C_i (i = 1, 2, ...,12) are constants of integration and are to be calculated by using the boundary conditions.

Laplace Equation in Polar Form :

Result : Show that in polar-coordinates r, θ, the two-dimensional Laplace equation $u_{xx}+u_{yy}=0$ takes the form

$$u_{rr}+\frac{1}{r}u_r+\frac{1}{r^2}u_{\theta\theta}=0$$

Proof : In Cartesian co-ordinates the two-dimensional Laplace equation is given by

$$\nabla^2 u=0,$$

$$\Rightarrow u_{xx} + u_{yy} = 0. \qquad \text{... (1.18)}$$

We know the relations between the Cartesian co-ordinates and polar co-ordinates are given by

$$x = r\cos\theta,\ \ y = r\sin\theta,$$

$$\Rightarrow r^2 = x^2 + y^2 \ \text{and}\ \ \theta = \tan^{-1}\left(\frac{y}{x}\right),$$

where $$\frac{\partial r}{\partial x} = \frac{x}{r} = \cos\theta,$$

$$\frac{\partial r}{\partial y} = \frac{y}{r} = \sin\theta,$$

and $$\frac{\partial \theta}{\partial x} = \frac{-y}{x^2 + y^2} = \frac{-\sin\theta}{r},$$

$$\frac{\partial \theta}{\partial y} = \frac{x}{r^2} = \frac{\cos\theta}{r}.$$

We have by Chain rule of partial differentiation

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial x} + \frac{\partial u}{\partial \theta}\cdot\frac{\partial \theta}{\partial x} = u_r\frac{x}{r} + u_\theta\frac{1}{1+\frac{y^2}{x^2}}\left(-\frac{y}{x^2}\right),$$

$$\Rightarrow u_x = u_r\cos\theta + u_\theta\left(\frac{-y}{x^2+y^2}\right)$$

$$\Rightarrow u_x = u_r\cos\theta - u_\theta\left(\frac{\sin\theta}{r}\right) \qquad \text{... (1.19)}$$

and $$u_y = u_r\frac{y}{r} + u_\theta\left(\frac{x^2}{x^2+y^2}\right)\left(\frac{1}{x}\right),$$

$$u_y = u_r\sin\theta + u_\theta\left(\frac{\cos\theta}{r}\right). \qquad \text{... (1.20)}$$

Similarly, we find $$u_{xx} = (u_x)_x = (u_x)_r\frac{\partial r}{\partial x} + (u_x)_\theta\frac{\partial \theta}{\partial x},$$

$$u_{xx} = \left[u_r\cos\theta - u_\theta\left(\frac{\sin\theta}{r}\right)\right]_r\cos\theta + \left[u_r\cos\theta - u_\theta\left(\frac{\sin\theta}{r}\right)\right]_\theta\left(-\frac{\sin\theta}{r}\right),$$

$$u_{xx}=\left[u_{rr}\cos\theta-u_{\theta r}\left(\frac{\sin\theta}{r}\right)+u_\theta\frac{\sin\theta}{r^2}\right]\cos\theta+$$

$$+\left[u_{r\theta}\cos\theta-u_r\sin\theta-u_{\theta\theta}\left(\frac{\sin\theta}{r}\right)-u_\theta\frac{\cos\theta}{r}\right]\left(\frac{-\sin\theta}{r}\right),$$

$$u_{xx}=u_{rr}\cos^2\theta-2u_{r\theta}\frac{\sin\theta\cos\theta}{r}+u_{\theta\theta}\frac{\sin^2\theta}{r^2}+u_r\frac{\sin^2\theta}{r}+2u_\theta\frac{\sin\theta\cos\theta}{r^2}. \quad \text{... (1.21)}$$

Similarly, we find

$$u_{yy}=\left(u_y\right)_y=\left(u_y\right)_r\frac{\partial r}{\partial y}+\left(u_y\right)_\theta\frac{\partial\theta}{\partial y},$$

$$=\left[u_r\sin\theta+u_\theta\frac{\cos\theta}{r}\right]_r\sin\theta+\left[u_r\sin\theta+u_\theta\frac{\cos\theta}{r}\right]_\theta\left(\frac{\cos\theta}{r}\right)$$

$$=\left(u_{rr}\sin\theta+u_{\theta r}\frac{\cos\theta}{r}-u_\theta\frac{\cos\theta}{r^2}\right)\sin\theta+$$

$$+\left(u_{r\theta}\sin\theta+u_{\theta\theta}\frac{\cos\theta}{r}-u_\theta\frac{\sin\theta}{r}+u_r\cos\theta\right)\frac{\cos\theta}{r},$$

$$u_{yy}=u_{rr}\sin^2\theta+2u_{\theta r}\frac{\sin\theta\cos\theta}{r}+u_{\theta\theta}\frac{\cos^2\theta}{r^2}-2u_\theta\frac{\sin\theta\cos\theta}{r^2}+u_r\frac{\cos^2\theta}{r}. \quad \text{... (1.22)}$$

Adding equations (1.21) and (1.22) we get

$$u_{xx}+u_{yy}=u_{rr}+\frac{1}{r^2}u_{\theta\theta}+\frac{1}{r}u_r.$$

Thus $\qquad u_{xx}+u_{yy}=0$

$$\Rightarrow u_{rr}+\frac{1}{r}u_r+\frac{1}{r^2}u_{\theta\theta}=0. \quad \text{... (1.23)}$$

This is the polar form of the 2-dimensional Laplace equation.

Example 2 : Show that the two-dimensional Laplace equation $\nabla^2 u=0$ in polar co-ordinates r, θ has the solution of the form $\sum_n\left(Ar^n+Br^{-n}\right)e^{\pm in\theta}$, where A and B are constants.

Solution : The two-dimensional Laplace equation in plane polar co-ordinates is given by

$$\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}+\frac{1}{r^2}\frac{\partial^2 u}{\partial\theta^2}=0. \quad \text{... (1.24)}$$

Let $\qquad u=R(r)\Theta(\theta) \quad$ ... (1.25)

be the solution of equation (1.24)

$$\Rightarrow u_{rr} = R''(r)\Theta(\theta) \text{ and } u_r = R'(r)\Theta,$$

$$u_{\theta\theta} = R(r)\Theta''(\theta)$$

Hence equation (1.25) becomes

$$R''(r)\Theta(\theta) + \frac{1}{r}R'(r)\Theta(\theta) + \frac{1}{r^2}R(r)\Theta''(\theta) = 0.$$

Dividing this equation by $R(r)\Theta(\theta)$ we get

$$\frac{R''}{R} + \frac{1}{r}\frac{R'(r)}{R} = \frac{1}{r^2}\frac{\Theta''}{\Theta} = 0,$$

or $$\frac{1}{R}\left(r^2R'' + rR'\right) = -\frac{\Theta''}{\Theta} = n^2. \quad \text{(say)} \qquad \ldots (1.26)$$

$$\Rightarrow \Theta'' + n^2\Theta = 0,$$

i.e. $$\frac{d^2\Theta}{d\theta^2} + n^2\Theta = 0, \qquad \ldots (1.27)$$

and $$r^2\frac{d^2R}{dr^2} + r\frac{dR}{dr} - n^2R = 0. \qquad \ldots (1.28)$$

Equation (1.27) provides

$$\Theta = e^{\pm in\theta},$$

or $$\Theta = C\cos n\theta + D\sin n\theta. \qquad \ldots (1.29)$$

Let $R = r^m$ be the solution of equation (1.28). Hence the equation (1.28) becomes

$$r^2\left[m(m-1)r^{m-2}\right] + rmr^{m-1} - n^2r^m = 0,$$

$$\Rightarrow \left(m^2 - n^2\right)r^m = 0,$$

$$\Rightarrow m = \pm n.$$

Hence the solution of (1.28) is given by

$$R = Ar^n + Br^{-n}. \qquad \ldots (1.29)$$

Therefore the solution of equation (1.24) becomes

$$u(r,\theta) = \sum_{n=}\left(A_nr^n + B_nr^{-n}\right)e^{\pm in\theta}, \qquad \ldots (1.30)$$

or $$u(r,\theta) = \sum_{n}\left(A_nr^n + B_nr^{-n}\right)\left(C_n\cos(n\theta) + D_n\sin(n\theta)\right). \qquad \ldots (1.31)$$

Which is the required result.

Laplace Equation in Spherical Polar Co-ordinates :

Result : Show that in spherical polar co-ordinates r, θ, ϕ the Laplace equation $\nabla^2 u = 0$ takes the form

$$\frac{\partial}{\partial r}\left(r^2 \frac{\partial u}{\partial r}\right) + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial u}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2 u}{\partial\phi^2} = 0 .$$

Proof : In Cartersian co-ordinates the Laplace equation is given by

$$\nabla^2 u = u_{xx} + u_{yy} + u_{zz} = 0 . \qquad \text{... (1.32)}$$

To transform equation (1.32) in to spherical polar co-ordinates, we have

$$x = r\sin\theta\cos\phi, y = r\sin\theta\sin\phi, z = r\cos\theta \qquad \text{... (1.33)}$$

$$\Rightarrow r^2 = x^2 + y^2 + z^2, \quad \theta = \tan^{-1}\frac{\sqrt{x^2+y^2}}{z}, \quad \phi = \tan^{-1}\left(\frac{y}{x}\right) \qquad \text{... (1.34)}$$

where $$r_x = \frac{x}{y} = \sin\theta\cos\phi, \quad r_y = \frac{y}{r} = \sin\theta\sin\phi, \quad r_z = \frac{z}{r} = \cos\theta , \qquad \text{... (1.35)}$$

and $$\theta_x = \frac{xz}{r^2\sqrt{x^2+y^2}} \Rightarrow \theta_x = \frac{\cos\theta\cos\phi}{r} ,$$

$$\theta_y = \frac{\cos\theta\sin\phi}{r}, \quad \theta_z = \frac{\sin\theta}{r} \qquad \text{... (1.36)}$$

Similarly, we find

$$\phi_x = -\frac{\sin\phi}{r\sin\theta}, \quad \phi_y = \frac{\cos\phi}{r\sin\theta}, \quad \phi_z = 0 \qquad \text{... (1.37)}$$

Now by using the chain rule of partial differentiation we write

$$u_x = u_r r_x + u_\theta \theta_x + u_\phi \phi_x .$$

Using equations (1.35), (1.36) and (1.37) we get

$$u_x = \sin\theta\cos\phi u_r + \frac{\cos\theta\cos\phi}{r}u_\theta - \frac{\sin\phi}{r\sin\theta}u_\phi . \qquad \text{... (1.38)}$$

In the same way, we find

$$u_y = \sin\theta\sin\phi u_r + \frac{\cos\theta\sin\phi}{r}u_\theta + \frac{\cos\phi}{r\sin\theta}u_\phi , \qquad \text{... (1.39)}$$

and $$u_z = \cos\theta u_r - \frac{\sin\theta}{r}u_\theta . \qquad \text{... (1.40)}$$

Now to find the second order derivative, we again use the chain rule and write

$$u_{xx}=\left(u_x\right)_r r_x+\left(u_x\right)_\theta \theta_x+\left(u_x\right)_\phi \phi_x$$

$$=\left(\sin\theta\cos\phi u_r+\frac{\cos\theta\cos\phi}{r}u_\theta-\frac{\sin\phi}{r\sin\theta}u_\phi\right)_r\cdot\left(\sin\theta\cos\phi\right)+$$

$$+\left(\sin\theta\cos\phi u_r+\frac{\cos\theta\cos\phi}{r}u_\theta-\frac{\sin\phi}{r\sin\theta}u_\phi\right)_\theta\cdot\left(\frac{\cos\theta\cos\phi}{r}\right)+$$

$$+\left(\sin\theta\cos\phi u_r+\frac{\cos\theta\cos\phi}{r}u_\theta-\frac{\sin\phi}{r\sin\theta}u_\phi\right)_\phi\cdot\left(-\frac{\sin\phi}{r\sin\theta}\right),$$

$$\Rightarrow u_{xx}=u_{rr}\sin^2\theta\cos^2\phi+u_{\theta\theta}\frac{\cos^2\theta\cos^2\phi}{r^2}+u_{\phi\theta}\frac{\sin^2\phi}{r^2\sin^2\theta}+$$

$$+u_{r\theta}\left(\frac{2\sin\theta\cos\theta\cos^2\phi}{r}\right)+u_{r\phi}\left(-\frac{2\sin\phi\cos\phi}{r}\right)+$$

$$+u_{\theta\phi}\left(-\frac{2\cos\theta\cos\phi\sin\phi}{r^2\sin\theta}\right)+u_r\left(\frac{\cos^2\theta\cos^2\phi}{r}+\frac{\sin^2\phi}{r}\right)+$$

$$+u_\phi\left(\frac{\sin\phi\cos\phi}{r^2}+\frac{\cos^2\theta\cos\phi\sin\phi}{r^2\sin^2\theta}+\frac{\sin\phi\cos\phi}{r^2\sin^2\theta}\right)+$$

$$+u_\theta\left(\frac{\cos\theta\sin^2\phi}{r^2\sin\theta}-\frac{2\cos\theta\sin\theta\cos^2\phi}{r^2}\right). \qquad \text{... (1.41)}$$

Similarly, the second order derivative

$$u_{yy}=\left(u_y\right)_r r_y+\left(u_y\right)_\theta \theta_y+\left(u_y\right)_\phi \phi_y \text{ gives}$$

$$u_{yy}=u_{rr}\sin^2\theta\sin^2\phi+u_{\theta\theta}\frac{\cos^2\theta\sin^2\phi}{r^2}+u_{\phi\phi}\frac{\cos^2\phi}{r^2\sin^2\theta}+$$

$$+u_{r\theta}\left(\frac{2\sin\theta\cos\theta\sin^2\phi}{r}\right)+u_{r\phi}\left(\frac{2\cos\phi\sin\phi}{r}\right)+u_{\theta\phi}\left(\frac{2\cos\theta\cos\phi\sin\phi}{r^2\sin\theta}\right)+$$

$$+u_r\left(\frac{\cos^2\theta\sin^2\phi}{r}+\frac{\cos^2\phi}{r}\right)+u_\theta\left(-\frac{2\sin\theta\cos\theta\sin^2\phi}{r^2}+\frac{\cos\theta\cos^2\phi}{r^2\sin\theta}\right)+$$

$$+u_\phi\left(-\frac{\sin\phi\cos\phi}{r^2}-\frac{\sin\phi\cos\phi}{r^2\sin^2\theta}-\frac{\cos^2\theta\sin\phi\cos\phi}{r^2\sin^2\theta}\right). \qquad \text{... (1.42)}$$

and $$u_{zz} = (u_z)_r r_z + (u_z)_\theta \theta_z + (u_z)_\phi \phi_z ,$$

$$\Rightarrow u_{zz} = \left(u_r \cos\theta - u_\theta \frac{\sin\theta}{r} \right)_r \cos\theta + \left(u_r \cos\theta - u_\theta \frac{\sin\theta}{r} \right)_\theta \frac{\sin\theta}{r} ,$$

$$\Rightarrow u_{zz} = u_{rr} \cos^2\theta - u_{r\theta} \frac{2\sin\theta\cos\theta}{r} + u_{\theta\theta} \frac{\sin^2\theta}{r^2} + u_r \frac{\sin^2\theta}{r} + u_\theta \frac{\cos\theta\sin\theta}{r^2} . \quad \text{... (1.43)}$$

Adding equations (1.41), (1.42) and (1.43) we obtain

$$\nabla^2 u = 0$$

$$\Rightarrow u_{rr} + \frac{1}{r^2} u_{\theta\theta} + \frac{1}{r^2\sin^2\theta} u_{\phi\phi} + \frac{2}{r} u_r + \frac{\cos\theta}{r^2\sin\theta} u_\theta = 0 . \quad \text{... (1.44)}$$

This can also be written as

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial u}{\partial r} \right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial u}{\partial\theta} \right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 u}{\partial\phi^2} = 0 ,$$

$$\Rightarrow \frac{\partial}{\partial r}\left(r^2 \frac{\partial u}{\partial r} \right) + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial u}{\partial\theta} \right) + \frac{1}{\sin^2\theta}\frac{\partial^2 u}{\partial\phi^2} = 0 . \quad \text{... (1.45)}$$

This is the required Laplace equation in spherical polar co-ordinates.

Example 3 : Show by using the method of separation of variables that the general solution of Laplace's equation in (r, θ, ϕ) co-ordinates is

$$u(r,\theta,\phi) = \sum_{n=0}^{\infty} \left(\alpha_n r^n + \beta_n \frac{1}{r^{n+1}} \right) S_n(\theta,\phi) ,$$

where $$S_n(\theta,\phi) = \sum_{m-0}^{n} P_n^m(\mu)\left(A_{nm} \cos m\phi + B_{nm} \sin m\phi \right) ,$$

$\mu = \cos\theta$, and $P_n^m(\mu)$ is the associated Legendre function and α_n , β_n , A_{nm} and B_{nm} are constants.

Solution : We know the Laplace equation in spherical polar form is given by

$$\frac{\partial}{\partial r}\left(r^2 \frac{\partial u}{\partial r} \right) + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial u}{\partial\theta} \right) + \frac{1}{\sin^2\theta}\frac{\partial^2 u}{\partial\phi^2} = 0 \quad \text{... (1.46)}$$

This can be written as

$$r^2 \frac{\partial^2 u}{\partial r^2} + 2r \frac{\partial u}{\partial r} + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial u}{\partial\theta} \right) + \frac{1}{\sin^2\theta}\frac{\partial^2 u}{\partial\phi^2} = 0 \quad \text{... (1.47)}$$

Since u is a function of r, θ, ϕ we assume

$$u(r,\theta,\phi)=R(r)\Theta(\theta)\Phi(\phi) \qquad \text{... (1.48)}$$

is the solution of equation (1.47). Therefore, we find from (1.48)

$$u_{rr}=R''\Theta\Phi,\quad \psi_r=R'\Theta\Phi,\quad \psi_\theta=R\Theta'\Phi \text{ and } u_\phi=R\Theta\Phi''. \qquad \text{... (1.49)}$$

Substituting these in equation (1.47) we get

$$r^2R''\Theta\Phi+2rR'\Theta\Phi+\frac{1}{\sin\theta}\left[\sin\theta R\Theta''\Phi+\cos\theta R\Theta'\Phi\right]+\frac{1}{\sin^2\theta}R\Theta\Phi''=0.$$

Dividing throughout by $R\Theta\Phi$ we get

$$r^2\frac{R''}{R}+2r\frac{R'}{R}+\frac{\Theta''}{\Theta}+\cot\theta\frac{\Theta'}{\Theta}+\frac{1}{\sin^2\theta}\frac{\Phi''}{\Phi}=0,$$

i.e. $$\frac{r^2}{R}\frac{d^2R}{dr^2}+\frac{2r}{R}\frac{dR}{dr}+\frac{1}{\Theta}\frac{d^2\Theta}{d\theta^2}+\frac{\cot\theta}{\Theta}\frac{d\Theta}{d\theta}+\frac{1}{\Phi\sin^2\theta}\frac{d^2\Phi}{d\phi^2}=0,$$

$$\frac{r^2}{R}\frac{d^2R}{dr^2}+\frac{2r}{R}\frac{dR}{dr}+\frac{1}{\Theta\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{\partial\Theta}{\partial\theta}\right)+\frac{1}{\Phi\sin^2\theta}\frac{d^2\Phi}{d\phi^2}=0,$$

or $$\left[\frac{r^2}{R}\frac{d^2R}{dr^2}+\frac{2r}{R}\frac{dR}{dr}+\frac{1}{\Theta\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{\partial\Theta}{\partial\theta}\right)\right]\sin^2\theta=-\frac{1}{\Phi}\frac{d^2\Phi}{d\phi^2}=m^2 \text{ .(say)} \qquad \text{... (1.50)}$$

Now consider the r.h.s. equations of (1.50)

$$\frac{d^2\Phi}{d\phi^2}+m^2\Phi=0, \qquad \text{... (1.51)}$$

which has solution

$$\Phi(\phi)=Ce^{\pm im\phi}. \qquad \text{... (1.52)}$$

Now the l.h.s. equation of (1.50) becomes

$$\frac{r^2}{R}\frac{d^2R}{dr^2}+\frac{2r}{R}\frac{dR}{dr}+\frac{1}{\Theta\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right)=\frac{m^2}{\sin^2\theta} \qquad \text{... (1.53)}$$

This can also be written as

$$\frac{r^2}{R}\frac{d^2R}{dr^2}+\frac{2r}{R}\frac{dR}{dr}=-\frac{1}{\Theta\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right)+\frac{m^2}{\sin^2\theta}=k \text{ (say)} \qquad \text{... (1.54)}$$

Consider

$$\frac{r^2}{R}\frac{d^2R}{dr^2}+\frac{2r}{R}\frac{dR}{dr}=k\,,$$

$$\Rightarrow r^2\frac{d^2R}{dr^2}+2r\frac{dR}{dr}=kR\,.$$

Take for convenience $k=n(n+1)$

Therefore $$r^2\frac{d^2R}{dr^2}+2r\frac{dR}{dr}-n(n+1)R=0\,, \qquad \text{... (1.55)}$$

and $$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right)-\frac{m^2\Theta}{\sin^2\theta}+n(n+1)\Theta=0$$

$$\Rightarrow\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right)+\left[n(n+1)-\frac{m^2}{\sin^2\theta}\right]\Theta=0 \qquad \text{... (1.56)}$$

Equation (1.55) is a homogeneous linear equation of second order. We put r = e^z (changing the independent variable r to z), hence we find

$$\frac{dr}{dz}=r\Rightarrow\frac{dz}{dr}=\frac{1}{r}$$

$$\Rightarrow\frac{dR}{dr}=\frac{dR}{dz}\cdot\frac{dz}{dr}=\frac{1}{r}\frac{dR}{dz}\Rightarrow r\frac{dR}{dr}=\frac{dR}{dz}$$

Now $$\frac{d^2R}{dr^2}=\frac{d}{dr}\left(\frac{1}{r}\frac{dR}{dz}\right)=\frac{1}{r}\frac{d^2R}{dz^2}\cdot\frac{dz}{dr}-\frac{1}{r^2}\frac{dR}{dz}$$

$$=\frac{1}{r^2}\left(\frac{d^2R}{dz^2}-\frac{dR}{dz}\right),$$

$$\Rightarrow r^2\frac{d^2R}{dr^2}=\frac{d^2R}{dz^2}-\frac{dR}{dz}\,.$$

If $\theta=\frac{d}{dz}$ then we have

$$r\frac{d}{dr}=\theta\,,$$

and $$r^2\frac{d^2}{dr^2}=\theta(\theta-1)\,.$$

Hence equation (1.55) becomes

$$\left(\theta(\theta-1)-2\theta-n(n+1)\right)R=0,$$

$$\Rightarrow(\theta-n)(\theta+n+1)R=0. \qquad ...(1.57)$$

This is a differential equation with constant coefficients whose auxiliary equation is

$$(\theta-n)(\theta+n+1)=0,$$

which has roots $\theta=n,\quad \theta=-(n+1)$.

Hence solution of (1.57) is given by

$$R=C_1e^{nz}+C_2e^{-(n+1)z}. \qquad ...(1.58)$$

Consequently, the solution of equation (1.55) is

$$R=C_1r^n+C_2r^{-(n+1)}. \qquad ...(1.59)$$

Now to find the solution of equation (1.56) we put

$$\mu=\cos\theta,\ \Rightarrow\frac{d\mu}{d\theta}=-\sin\theta,$$

We write

$$\Rightarrow\frac{d\Theta}{d\theta}=\frac{d\Theta}{d\mu}\cdot\frac{d\mu}{d\theta}\Rightarrow\frac{d\Theta}{d\theta}=-\sin\theta\frac{d\Theta}{d\mu},$$

$$\frac{d^2\Theta}{d\theta^2}=-\cos\theta\frac{d\Theta}{d\mu}-\sin\theta\frac{d^2\Theta}{d\mu^2}\cdot\frac{d\mu}{d\theta},$$

and

$$\Rightarrow\frac{d^2\Theta}{d\theta^2}=-\cos\theta\frac{d\Theta}{d\mu}+\sin^2\theta\frac{d^2\Theta}{d\mu^2}.$$

From equation (1.56) we have

$$\frac{1}{\sin\theta}\left[\sin\theta\frac{d^2\Theta}{d\theta^2}+\cos\theta\frac{d\Theta}{d\theta}\right]+\left[n(n+1)-\frac{m^2}{\sin^2\theta}\right]\Theta=0 \qquad ...(1.60)$$

Using the above expressions in equation (1.60) we get

$$-\cos\theta\frac{d\Theta}{d\mu}+\sin^2\theta\frac{d^2\Theta}{d\mu^2}+\frac{\cos\theta}{\sin\theta}(-\sin\theta)\frac{d\Theta}{d\mu}+\left[n(n+1)-\frac{m^2}{\sin^2\theta}\right]\Theta=0,$$

i.e.

$$\left(1-\mu^2\right)\frac{d^2\Theta}{d\mu^2}-2\mu\frac{d\Theta}{d\mu}+\left[n(n+1)-\frac{m^2}{\left(1-\mu^2\right)}\right]\Theta=0. \qquad ...(1.61)$$

This is called as associated Legendre's equation whose solution is given by

$$\Theta=AP_n^m(\mu)=AP_n^m(\cos\theta). \qquad ...(1.62)$$

Using equations (1.52), (1.59) and (1.62) in equation (1.48) we obtain the general solution of given equation (1.46) in the form

$$u(r,\theta,\phi) = \left(A_1 r^n + A_2 r^{-(n+1)}\right) P_n^m(\cos\theta)\cdot e^{\pm im\phi}.$$

By supperposition principle, the general solution can also be written as

$$u(r,\theta,\phi) = \sum_{n=0}^{\infty}\left(\alpha_n r^n + \frac{\beta_n}{r^{n+1}}\right) S_n(\theta,\phi),$$

where $$S_n(\theta,\phi) = \sum_{m=0}^{n} P_n^m(\cos\theta)\left(A_{nm}\cos m\phi + B_{nm}\sin m\phi\right).$$

Kelvin's Inversion Theorem :

Theorem : If $u = u(r,\theta,\phi)$ is a harmonic function, where (r,θ,ϕ) are the spherical polar co-ordinates, then show that

$$\bar{u} = \frac{a^2}{r} u\left(\frac{a^2}{r},\theta,\phi\right)$$

is also a harmonic function, where 'a' is a constant.

Proof : Given that $u(r,\theta,\phi)$ is a harmonic function. $\Rightarrow$ it satisfies Laplace equation.

$$\Rightarrow \frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right) + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial u}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2 u}{\partial\phi^2} = 0. \quad \text{... (1.63)}$$

Claim : We prove that

$$\bar{u} = \frac{a^2}{r} u\left(\frac{a^2}{r},\theta,\phi\right) \text{ is also a harmonic function.}$$

Let $$R = \frac{a^2}{r},$$

$$\Rightarrow \bar{u} = Ru(R,\theta,\phi).$$

Since $u(R,\theta,\phi)$ satisfies the equation (1.63) as it is harmonic.

$$\Rightarrow \frac{\partial}{\partial R}\left(R^2\frac{\partial u}{\partial R}\right) + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial u}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2 u}{\partial\phi^2} = 0. \quad \text{... (1.64)}$$

We claim that

$$\frac{\partial}{\partial r}\left(r^2\frac{\partial \overline{u}}{\partial r}\right)+\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial \overline{u}}{\partial\theta}\right)+\frac{1}{\sin^2\theta}\frac{\partial^2 \overline{u}}{\partial\phi^2}=0\,. \quad \text{... (1.65)}$$

Therefore, consider

$$r^2\frac{\partial \overline{u}}{\partial r}=r^2\frac{\partial}{\partial r}\big(Ru(R,\theta,\phi)\big),$$

$$=r^2R\frac{\partial u}{\partial R}\cdot\frac{\partial R}{\partial r}+r^2\frac{\partial R}{\partial r}\cdot u(R,\theta,\phi),$$

$$\Rightarrow r^2\frac{\partial \overline{u}}{\partial r}=r^2\left(\frac{a^2}{r}\right)\left(-\frac{a^2}{r^2}\right)\frac{\partial u}{\partial R}+r^2\left(-\frac{a^2}{r^2}\right)u(R,\theta,\phi),$$

$$\Rightarrow r^2\frac{\partial \overline{u}}{\partial r}=-\frac{a^4}{r}\frac{\partial u}{\partial R}-a^2u(R,\theta,\phi)\,. \quad \text{... (1.66)}$$

Differentiating this with respect to r we get

$$\frac{\partial}{\partial r}\left(r^2\frac{\partial \overline{u}}{\partial r}\right)=\frac{a^4}{r^2}\frac{\partial u}{\partial R}-\frac{a^4}{r}\frac{\partial^2 u}{\partial R^2}\frac{\partial R}{\partial r}-a^2\frac{\partial u}{\partial R}\frac{\partial R}{\partial r},$$

$$=\frac{2a^4}{r^2}\frac{\partial u}{\partial R}+\frac{a^6}{r^3}\frac{\partial^2 u}{\partial R^2},$$

$$\Rightarrow\frac{\partial}{\partial r}\left(r^2\frac{\partial \overline{u}}{\partial r}\right)=2R^2\frac{\partial u}{\partial R}+R^3\frac{\partial^2 u}{\partial R^2},$$

or

$$\frac{\partial}{\partial r}\left(r^2\frac{\partial \overline{u}}{\partial r}\right)=R\frac{\partial}{\partial R}\left(R^2\frac{\partial u}{\partial R}\right). \quad \text{... (1.67)}$$

Similarly, consider

$$\sin\theta\frac{\partial \overline{u}}{\partial\theta}=\sin\theta\frac{\partial}{\partial\theta}\big(Ru(R,\theta,\phi)\big),$$

$$\sin\theta\frac{\partial \overline{u}}{\partial\theta}=R\sin\theta\frac{\partial u}{\partial\theta}. \qquad \text{as } R\neq R(\theta)$$

Differentiating this with respect to θ we obtain

$$\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial \overline{u}}{\partial\theta}\right)=\frac{\partial}{\partial\theta}\left(R\sin\frac{\partial u}{\partial\theta}\right),$$

$$=R\cos\theta\frac{\partial u}{\partial\theta}+R\sin\theta\frac{\partial^2 u}{\partial\theta^2},$$

$$\Rightarrow \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\overline{u}}{\partial\theta}\right)=\frac{R}{\sin\theta}\left(\cos\theta\frac{\partial u}{\partial\theta}+\sin\theta\frac{\partial^2 u}{\partial\theta^2}\right)$$

$$\Rightarrow \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\overline{u}}{\partial\theta}\right)=\frac{R}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial u}{\partial\theta}\right). \qquad \text{... (1.68)}$$

Next consider

$$\frac{1}{\sin^2\theta}\frac{\partial^2\overline{u}}{\partial\phi^2}=\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\left[Ru(R,\theta,\phi)\right],$$

$$\frac{1}{\sin^2\theta}\frac{\partial^2\overline{u}}{\partial\phi^2}=\frac{R}{\sin^2\theta}\frac{\partial^2 u}{\partial\phi^2}. \qquad \text{... (1.69)}$$

Adding equations (1.67), (1.68) and (1.69) we get

$$\frac{\partial}{\partial r}\left(r^2\frac{\partial\overline{u}}{\partial r}\right)+\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\overline{u}}{\partial\theta}\right)+\frac{1}{\sin^2\theta}\frac{\partial^2\overline{u}}{\partial\phi^2}=$$

$$=R\frac{\partial}{\partial R}\left(R^2\frac{\partial u}{\partial R}\right)+\frac{R}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial u}{\partial\theta}\right)+\frac{R}{\sin^2\theta}\frac{\partial^2 u}{\partial\phi^2},$$

$$=R\left[\frac{\partial}{\partial R}\left(R^2\frac{\partial u}{\partial R}\right)+\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial u}{\partial\theta}\right)+\frac{1}{\sin^2\theta}\frac{\partial^2 u}{\partial\phi^2}\right],$$

$$=0 \qquad \text{by virtue of equation (1.64)}$$

$$\Rightarrow \frac{\partial}{\partial r}\left(r^2\frac{\partial\overline{u}}{\partial r}\right)+\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\overline{u}}{\partial\theta}\right)+\frac{1}{\sin^2\theta}\frac{\partial^2\overline{u}}{\partial\phi^2}=0.$$

This proves that $\overline{u}=\frac{a^2}{r}u\left(\frac{a^2}{r},\theta,\phi\right)$ is also harmonic.

2. Boundary Value Problems :

Any problem of determining a function u (x, y) satisfying Laplace's equation within certain region D and satisfying certain conditions on the boundary B of the region D is called boundary value problem, for the Laplace equation.

There are mainly three types of boundary value problems for Laplace equation viz.

1. The first boundary value problem, called The Dirichlet problem.
2. The second boundary value problem called The Neumann problem.
3. The third boundary value problem called the Mixed Boundary Value problem.

Dirichlet Problem :

There are two types of Dirichlet peoblems -

(i) Interior Dirichlet Problem and

(ii) Exterior Dirichlet Problem.

Interior Dirichlet Problem :

If f is a continuous function on the boundary B of some finite region D, then the problem of determining a function u (x, y) such that

(i) $\nabla^2 u(x, y) = 0$ with D (i.e. u (x, y) is harmonic inside D) and

(ii) *u (x, y) = f* on B (i.e. u coincides with *f* on the boundary B)

is called Interior Dirichlet Problem.

Exterior Dirichlet Problem :

If *f* is a continuous function prescribed on the boundary B of a finite simply connected region D, then the problem of determining a function u (x, y) such that

(i) $\nabla^2 u(x, y) = 0$ outside D and

(ii) *u (x, u) = f* on the boundary B

is called Exterior Dirichlet Problem.

The Neumann Problem :

Interior Neumann Problem :

If *f* is a continuous function defined uniquely at each point of the boundary B of a finite region D, then the problem of determining a fuunction *u (x, y)* such that

(i) $\nabla^2 u(x, y) = 0$ in D (i.e. u is harmonic inside D) and

(ii) Satisfies $\frac{\partial u}{\partial n} = f(s)$ on the boundary B, where $\frac{\partial}{\partial n}$ is the directional derivative along the outward normal (i.e. normal derivative $\frac{\partial u}{\partial n}$ coincides with *f* at every points of B)

is called the interior Neumann problem.

Exterior Neumann Problem :

If *f* is a continuous function prescribed at each point of the smooth boundary B of a bounded simply connected region D. Then finding a function *u (x, y)* satisfying

(i) $\nabla^2 u(x, y) = 0$ outside D and

(ii) $\frac{\partial u}{\partial n} = f$ on the boundary B

is called an exterior Neiumann Problem.

Note : If ψ is the temperature, $\frac{\partial \psi}{\partial n}$ is the heat flux representing the amount of heat crossing per unit volume per unit time along the normal direction.

The Third Voundary Value Problem :

The problem of finding a function *u (x, y)* which is harmonic in D and satisfies the condition

$\frac{\partial u}{\partial n} + h(s)u(s) = 0$ on B where $h(s) \geq 0$ and $h(s) \neq 0$.

The Fourth Boundary Value Problems (Mixed Boundary Value Problem) :

The Robin Problem :

The problem of finding a function *u (x, y)* which is harmonic in D and satisfies different boundary conditions on different portions of the boundary B, such as $u = f_1(s)$ as B_1 $\frac{\partial u}{\partial n} = f_2(s)$ on B_2, where $B_1 \cup B_2 = B$, is called Robin Problem.

Maximum and Minimum Principle :

Theorem : Let D be a region bounded by a simple, closed, piecewise smooth curve B. Let *u (x, y)* be a function which is continuous in a closed region $\bar{D} = D \cup B$ and satisfy the Laplace equation $\nabla^2 u = 0$ (i.e. harmonic in D) in the interior of D. If u is not constant everywhere on $\bar{D}$, then the maximum and minimum values of *u (x, y)* must occur only on the boundary B of D.

Proof : Let D be a region bounded by B inside which the function *u (x, y)* is harmonic.

i.e. $\nabla^2 u = 0$ in D

i.e.
$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \text{ in D.} \qquad \text{... (2.1)}$$

Let the maximum value of *u (x, y)* on B be M. Let the theorem be not true. Therefore we assume that the function *u (x,y)* attains its maximum at some interior point (x_0, y_0) in D and not at any point on the boundary B of D.

If $M_0 = u(x_0, y_0)$ then $M_0 > M$.

Say
$$M_0 = u(x_0, y_0) = M + \in. \qquad \text{... (2.2)}$$

Let us construct an auxiliary function

$$v(x, y) = u(x, y) + \frac{M_0 - M}{4R^2}\left[(x - x_0)^2 + (y - y_0)^2\right], \qquad \text{... (2.3)}$$

where $(x, y) \in D$ and R is the radius of the circle with centre (x_0, y_0) containing D. Since D is bounded as R exists. We observe from equation (2.3) that

$$v(x_0, y_0) = u(x_0, y_0) = M_0. \qquad \text{... (2.4)}$$

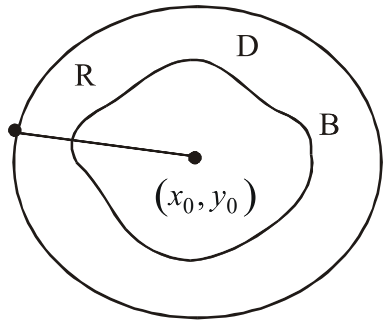
We show that v (x, y) like u (x, y) attains its maximum at a point (x_0, y_0) in D. However, on B we have

$$v(x,y) \le M + \frac{M_0 - M}{4}, \qquad \left(\frac{(x-x_0)^2 - (y-y_0)^2}{R^2} \le 1 \right)$$

$$\Rightarrow v(x,y) \le M + \frac{M_0 - M}{4} < M_0.$$

$\Rightarrow$ the function v (x, y), like u (x, y) must attain its maximum at a point (x_0, y_0) in D.

$\Rightarrow v_{xx} \le 0, v_{yy} \le 0$ at some point in D.



$\Rightarrow v_{xx} + v_{yy} \le 0$ at some point in D.

However, in D we have from equation (2.3)

$$v_{xx} + v_{yy} = u_{xx} + u_{yy} + \frac{M_0 - M}{4R^2}(2+2),$$

$$v_{xx} + v_{yy} = u_{xx} + u_{yy} + \frac{M_0 - M}{R^2},$$

Since u is harmonic in D

$$\Rightarrow u_{xx} + u_{yy} = 0.$$

$$\Rightarrow v_{xx} + v_{yy} = \frac{M_0 - M}{R^2} > 0, \qquad \text{Since } M_0 > M$$

$$\Rightarrow v_{xx} + v_{yy} > 0.$$

This is a contracdiction.

$\Rightarrow$ the maximum of u must be attained on the boundary B.

Green Identity :

Let B be a closed surface in the space and D denote the bounded region enclosed by B. Let $\bar{F}$ be a vector $\in C^1$ in D and continuous on D. Then we know the Gauss divergence theorem is given by

$$\iint_B \bar{F}\hat{n} ds = \iiint_D \nabla \cdot \bar{F} \cdot dV, \qquad \text{... (2.5)}$$

where dV is an element of volume, ds is an element of surface area and $\hat{n}$ is the outward normal. Green's identity is obtained from (2.5).

Let $\overline{F} = \overline{f}\phi$, where $\overline{f}$ is a vector function and ϕ is a scalar function. Then from equation (2.5), we have

$$\iiint_D \nabla\cdot\left(\overline{f}\phi\right)dV = \iint_B \hat{n}\cdot\overline{f}\phi ds$$

we know $\qquad \nabla\cdot\left(\overline{f}\phi\right) = \overline{f}\cdot\nabla\phi + \phi\nabla\cdot\overline{f}$

$$\Rightarrow \iiint_D \left(\overline{f}\cdot\nabla\phi + \phi\nabla\cdot\overline{f}\right)dV = \iint_B \hat{n}\cdot\overline{f}\phi ds,$$

$$\Rightarrow \iiint_D \overline{f}\cdot\nabla\phi dV = \iint_B \hat{n}\cdot\overline{f}\phi ds - \iiint_D \phi\nabla\cdot\overline{f}dV.$$

We choose the vector function

$$\overline{f} = \nabla\psi.$$

Therefore, the above equation yields

$$\iiint_D \nabla\phi\cdot\nabla\phi dV = \iint_B \phi\hat{n}\cdot\nabla\psi ds - \iiint_D \phi\nabla^2\psi dV. \qquad \text{... (2.6)}$$

Since $\hat{n}\cdot\nabla\psi$ is the derivative of ψ in the direction of $\hat{n}$. We denote this directional derivative by

$$\hat{n}\cdot\nabla\psi = \frac{\partial\psi}{\partial n}.$$

Therefore, equation (2.6) reduces to

$$\iiint_D \nabla\phi\cdot\nabla\psi dV = \iint_B \phi\frac{\partial\psi}{\partial n}ds - \iiint_D \phi\nabla^2\psi dV. \qquad \text{... (2.7)}$$

This equation is known as Green's first identity.

Now interchanging the role of ϕ and ψ, we obtain from (2.7) the equation

$$\iiint_D \nabla\psi\cdot\nabla\phi dV = \iint_B \phi\frac{\partial\phi}{\partial n}ds - \iiint_D \psi\nabla^2\phi dV \qquad \text{... (2.8)}$$

Now substracting (2.8) from (2.7) we get

$$\iiint_D \left(\phi\nabla^2\psi - \psi\nabla^2\phi\right)dV = \iint_B \left(\phi\frac{\partial\psi}{\partial n} - \psi\frac{\partial\phi}{\partial n}\right)ds \qquad \text{... (2.9)}$$

This is known as Green's Second identity.

If in particular, $\phi = \psi$ in equation (2.7) then we have

$$\iiint_D (\nabla\phi)^2\, dV = \iint_B \phi \frac{\partial\phi}{\partial n} ds - \iiint_D \phi\nabla^2\phi dV . \qquad \text{... (2.10)}$$

Which is a special case of Green's first identity.

Properties of Harmonic Functions :

Solutions of Laplace equation are called harmonic functions. These functions possess a number of interesting properties.

Theorem 1 : If a harmonic function vanishes everywhere on the boundary then it is identically zero everywhere.

Proof : Let ϕ be a harmonic function in D.

$$\Rightarrow \nabla^2\phi = 0 \text{ in D,} \qquad \text{... (2.11)}$$

and also $\phi = 0$ on B. ... (2.12)

We shall show that $\phi = 0$ in $\bar{D} = D \cup B$.

We know the Green's indentity is given by

$$\iiint_D (\nabla\phi)^2\, dV = \iint_B \phi \frac{\partial\phi}{\partial n} ds - \iiint_D \phi\nabla^2\phi dV . \qquad \text{... (2.13)}$$

Using (2.11) and (2.12) we have from equation (2.13)

$$\iiint_D (\nabla\phi)^2\, dV = 0 . \qquad \text{... (2.14)}$$

Since $(\nabla\phi)^2$ is positive. It follows that the integral (2.14) will be satisfied only if $\nabla\phi = 0$.

$\Rightarrow \phi$ = constant in D. But ϕ is continuous in $\bar{D} = D \cup B$ and $\phi = 0$ on B, it follows from the maximum and minimum principle that $\phi = 0$ in D.

Theorem : If ϕ is a harmonic function in D and $\frac{\partial\phi}{\partial n} = 0$ on B, then ϕ is a constant in $\bar{D}$.

Proof : Let ϕ be a harmonic in D.

$$\Rightarrow \nabla^2\phi = 0 \text{ in D.} \qquad \text{... (2.15)}$$

Also $\frac{\partial\phi}{\partial n} = 0$ on B. ... (2.16)

Then we prove that ϕ = constant in $\bar{D} = D \cup B$.

Now by Green's identity we have

$$\iiint_D (\nabla\phi)^2\, dV = \iint_B \phi \frac{\partial \phi}{\partial n} ds - \iiint_D \phi\nabla^2\phi dV \qquad \text{... (2.17)}$$

Using equations (2.15) and (2.16) we have

$$\iiint_D (\nabla\phi)^2\, dV = 0.$$

Since $(\nabla\phi)^2$ is positive, it follows that the integral will be satisfied only if $\nabla\phi = 0$.

$\Rightarrow \phi$ = constant in D.

Since the value of ϕ is not known on the boundary B, but

$$\frac{\partial \phi}{\partial n} = 0 \text{ on B.}$$

$\Rightarrow \phi$ = constant on B and hence by the maximum and minimum principle it is constant on D.

This proves the theorem.

Uniqueness Theorem :

Theorem : Prove that the solution of the Dirichlet problem, if it exists, is unique.

Proof : Let us suppose that u_1 and u_2 are two solutions of the Dirichlet problem.

$$\Rightarrow \nabla^2 u_1(x, y) = 0 \text{ in D and}$$

$$u_1(x, y) = f(s) \text{ on the boundary B,} \qquad \text{... (2.18)}$$

where f is a continuous function defined on the boundary B. Similarly, we have

$$\nabla^2 u_2(x, y) = 0 \text{ in D and}$$

$$u_2(x, y) = f(s) \text{ on B.} \qquad \text{... (2.19)}$$

Since u_1 and u_2 are harmonic in D, therefore $u_1 - u_2$ is also harmonic in D.

$$\Rightarrow \nabla^2(u_1 - u_2) = 0 \text{ in D.}$$

However, from equations (2.18) and (2.19) we have

$$u_1 - u_2 = 0 \text{ on B} \qquad \text{... (2.20)}$$

By the maximum and minimum principle,

$$u_1 - u_2 \equiv 0 \text{ in D,}$$

$$\Rightarrow u_1 = u_2.$$

(Because if a harmonic function vanishes everywhere on the boundary, then it is identically zero everywhere). This proves the uniqueness.

Other forms of Green's Identity :

By Green's theorem, we know, if u (x, y) and v (x, y) are differentiable functions in D and continuous on the boundary B of D then

$$\int_D \left(\frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} \right) dS = \int_B (Udy - Vdx) \qquad \text{... (2.21)}$$

Let $U = \psi \frac{\partial \phi}{\partial x}$ and $V = \psi \frac{\partial \phi}{\partial y}$

$$\Rightarrow U_x = \psi_x \phi_x + \psi \phi_{xx}, \ V_y = \psi_y \phi_y + \psi \phi_{yy}.$$

Therefore, equation (2.21) becomes

$$\int_D \left(\psi_x \phi_x + \psi \phi_{xx} + \psi_y \phi_y + \psi \phi_{yy} \right) dS = \int_B \psi \left(\phi_x dy - \phi_y dx \right).$$

We use $\phi_x dy - \phi_y dx = \frac{\partial \phi}{\partial n} dS$

Hence, $$\int_D \left(\psi_x \phi_x + \psi \phi_{xx} + \psi_y \phi_y + \psi \phi_{yy} \right) dS = \int_B \psi \frac{\partial \phi}{\partial n} dS. \qquad \text{... (2.22)}$$

On interchanging ϕ and ψ in (2.22) we get

$$\int_D \left(\phi_x \psi_x + \phi \psi_{xx} + \phi_y \psi_y + \phi \psi_{yy} \right) dS = \int_B \phi \frac{\partial \psi}{\partial n} dS. \qquad \text{... (2.23)}$$

Substracting (2.23) from (2.22) we get

$$\int_D \left(\psi \nabla^2 \phi - \phi \nabla^2 \psi \right) dS = \int_B \left(\psi \frac{\partial \phi}{\partial n} - \phi \frac{\partial \psi}{\partial n} \right) dS \qquad \text{... (2.24)}$$

The identities (2.22) and (2.24) are called Green's identities.

Theorem : Show the necessary condition for the existence of the solution of the Neumann problem is that the integral of *f* over the boundary B should vanish.

Proof : Let u (x, y) be the solution of the Neumann interior problem.

$$\Rightarrow \nabla^2 u = 0 \text{ in D}, \qquad \text{... (2.25)}$$

and $$\frac{\partial u}{\partial n} = f(s) \text{ on B}. \qquad \text{... (2.26)}$$

Then we claim that $\int_B f(s)ds = 0$.

We know the Green's identity is given by

$$\int_D \left(\psi\nabla^2\phi - \phi\nabla^2\psi\right) dS = \int_B \left(\psi\frac{\partial\phi}{\partial n} - \phi\frac{\partial\psi}{\partial n}\right) dS . \qquad \text{... (2.27)}$$

Put $\psi = 1$ and $\phi = u$ in (2.27) we get

$$\int_D \nabla^2 u ds = \int_B \frac{\partial u}{\partial n} ds$$

Using equations (2.25) and (2.26) we get

$$\int_B f(s)ds = 0 . \qquad \text{... (2.28)}$$

This proves the result.

Theorem : Show that the solution of the Neumann problem is either unique or it differs from one another by a constant only (i.e. solution is unique up to the addition of a constant).

Proof : Let $u_1(x, y)$ and $u_2(x, y)$ be two solutions of the interior Neumann problem.

$\Rightarrow u_1$ and u_2 are harmonics in D.

i.e. $\quad \nabla^2 u_1 = 0$, in D and $\nabla^2 u_2 = 0$ in D, ... (2.29)

and $\quad \frac{\partial u_1}{\partial n} = f$ on B and $\frac{\partial u_2}{\partial n} = f$ on B. ... (2.30)

Then we claim that $u_1 - u_2 =$ constant.

Consider $\quad v = u_1 - u_2$

Then $\quad \nabla^2 v = \nabla^2 (u_1 - u_2) = \nabla^2 u_1 - \nabla^2 u_2$

$$= 0 \text{ in D.}$$

$$\Rightarrow \nabla^2 v = 0 \text{ in D,} \qquad \text{... (2.31)}$$

and $\quad \frac{\partial v}{\partial n} = \frac{\partial}{\partial n}(u_1 - u_2) = \frac{\partial u_1}{\partial n} - \frac{\partial u_2}{\partial n}$

$$= f - f$$

$$\Rightarrow \frac{\partial v}{\partial n} = 0 \text{ on B.} \qquad \text{... (2.32)}$$

We know the Green's identity

$$\int_D \left(\psi_x\phi_x + \psi\phi_{xx} + \psi_y\phi_y + \psi\phi_{yy}\right) ds = \int_B \psi \frac{\partial \phi}{\partial n} ds . \qquad \text{... (2.33)}$$

Put $\phi = \psi = v$ in (2.23) we get

$$\int_D \left[\left(v_x\right)^2 + vv_{xx} + \left(v_y\right)^2 + yy_{yy}\right] ds = \int_B v \frac{\partial v}{\partial n} ds ,$$

$$\Rightarrow \int_D \left[v_x^2 + v_y^2 + v\nabla^2 v\right] ds = \int_B v \frac{\partial v}{\partial n} ds ,$$

$$\Rightarrow \int_D (\nabla v)^2 ds + \int_D v\left(\nabla^2 v\right) ds = \int_B v \frac{\partial v}{\partial n} ds . \qquad \text{... (2.34)}$$

Using equations (2.31) and (2.32) we obtain from equation (2.34)

$$\int_D (\nabla v)^2 ds = 0 . \qquad \text{... (2.35)}$$

Since $(\nabla v)^2$ is positive. It follows that the integral (2.35) will be satisfied only if $\nabla v = 0$ in D.

$\Rightarrow v$ is constant in D.

$\Rightarrow u_1 - u_2 =$ constant.

This proves that the solution of the Neumann problem differs from one another by a constant.

If constant is zero $\Rightarrow$ the solution is unique.

Stability : A solution is said to be stable if it depends continuously on the initial and/or boundary data.

Stability Theorem : Show that the solution of the Dirichlet problem is stable.

i.e. Show that the solution of the Dirichlet problem depends continuously on the boundary data.

Proof : Let u_1 and u_2 be two solutions of the Dirichlet problem in D and f_1, f_2 be given continuous functions on the boundary B of the region D.

$$\Rightarrow \nabla^2 u_1 = 0 \text{ in D and } u_1 = f_1 \text{ on B.}$$

Similarly $\quad \nabla^2 u_2 = 0$ in D, $u_2 = f_2$ on B.

Let $\quad v = u_1 - u_2$.

$$\Rightarrow \nabla^2 v = \nabla^2\left(u_1 - u_2\right) = \nabla^2 u_1 - \nabla^2 u_2 = 0 \text{ in D}$$

$$\Rightarrow \nabla^2 v = 0 \text{ in D}$$

and $\quad v = f_1 - f_2$ on B.

$\Rightarrow v$ is a solution of the Dirichlet problem with boundary condition $v = f_1 - f_2$ on B.

Therefore, by the maximum and minimum principle, the harmonic function v attains the maximum and minimum values on B.

Equivalently, $f_1 - f_2$ has maximum and minimum value on the boundary B. (i.e. $f_1 - f_2$ must be bounded)

Thus if $|f_1 - f_2| < \in$ on B.

Therefore at any interior point in D, we have, for given $\in > 0$

$$- \in \leq v_{\min} \leq v_{\max} \leq \in$$

$$\Rightarrow |v| < \in \text{ in D}$$

$$\Rightarrow |u_1 - u_2| < \in \text{ in D.}$$

Hence if $|f_1 - f_2| < \in$ on B then $|u_1 - u_2| < \in$ on D.

Thus, small changes in the initial data bring about an arbitrary small change in the solution.

This shows that the solution depends continuously on the boundary data.

i.e. the solution of the Dirichlet problem is stable.

3. Interior Dirichlet Problem for a Circle :

The Dirichlet problem for a circle is defined as follows.

Result : Show that the solution for the Dirichlet Problem for a circle of radius a is given by the Poisson integral formula.

Example 1 : Find the value of $u(r,\theta)$ at any point in the interior of the circle (r = a) D in terms of its values on the boundary B such that u is single valued and continuous within and on a circular region and satisfies the equation

$$\nabla^2 u = 0,\ 0 \leq r \leq a,\ 0 \leq \theta \leq 2\pi$$

subject to $u(a,\theta) = f(\theta)$, $0 \leq \theta \leq 2\pi$, where $f(\theta)$ is continuous function on B.

Solution : Our problem is to solve for $u(r,\theta)$ satisfying the equation

$$\nabla^2 u = 0,\ 0 \leq r \leq a,\ 0 \leq \theta \leq 2\pi, \qquad \text{... (3.1)}$$

subject to the boundary condition

$$u(a,\theta) = f(\theta),\ o \leq \theta \leq 2\pi, \qquad \text{... (3.2)}$$

where $f(\theta)$ is continuous function on the boundary of circle.

We know the polar form of Laplace equation (3.1) is given by

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0 \qquad \text{... (3.3)}$$

We know the solution of the equation (3.3) is given by (Refer equation 1.31)

$$u(r,\theta)=\sum_{n=0}^{\infty}\left(C_n r^n + D_n r^{-n}\right)\left(A_n \cos n\theta + B_n \sin n\theta\right). \qquad \text{... (3.4)}$$

At r = 0, $u(r,\theta)$ must be finite. Hence

$$r^{-n}\to\infty \text{ as } r\to 0 \Rightarrow D_n = 0,$$

$$\Rightarrow u(r,\theta)=\sum_{n=0}^{\infty} r^n\left(A_n \cos n\theta + B_n \sin n\theta\right), \qquad \text{... (3.5)}$$

$$\Rightarrow u(r,\theta)=\frac{a_0}{2}+\sum_{n=1}^{\infty} r^n\left(a_n \cos n\theta + b_n \sin n\theta\right). \quad \text{for } A_0=\frac{a_0}{2} \qquad \text{... (3.6)}$$

Now using the boundary condition

$$u(a,\theta)=f(\theta),$$

we have

$$f(\theta)=\frac{a_0}{2}+\sum_{n=1}^{\infty} a^n\left(a_n \cos n\theta + b_n \sin n\theta\right).$$

This is the Fourier series expansion of $f(\theta)$, hence Fourier constants are given by

$$\Rightarrow a_0=\frac{1}{\pi}\int_0^{2\pi} f(\theta)\,d\theta,$$

$$a_n=\frac{1}{\pi a_n}\int_0^{2\pi} f(\theta)\cos n\theta\,d\theta,$$

and $$b_n=\frac{1}{\pi a_n}\int_0^{2\pi} f(\theta)\sin n\theta\,d\theta, \text{ n = 1, 2, 3, ...}$$

Substituting these values in the solution (3.6) we get

$$u(r,\theta)=\frac{1}{2\pi}\int_0^{2\pi} f(\phi)\,d\phi+\sum_{n=1}^{\infty}\frac{r^n}{a^n}\left[\frac{\cos n\theta}{\pi}\int_0^{2\pi} f(\phi)\cos n\phi\,d\phi+\frac{\sin n\theta}{\pi}\int_0^{2\pi} f(\phi)\cos n\phi\,d\phi\right].$$

Interchanging the order of summation and integrating we get

$$u(r,\theta)=\frac{1}{2\pi}\int_0^{2\pi} f(\phi)\,d\phi+\frac{1}{\pi}\int_0^{2\pi} f(\phi)\sum_{n=1}^{\infty}\left(\frac{r}{a}\right)^n\left[\cos n\phi\cos n\theta+\sin n\phi\sin n\theta\right]d\phi,$$

$$u(r,\theta)=\frac{1}{\pi}\int_0^{2\pi} f(\phi)\left[\frac{1}{2}+\sum_{n=1}^{\infty}\left(\frac{r}{a}\right)^n \cos n(\phi-\theta)\right]d\phi. \qquad \text{... (3.7)}$$

Consider

$$\sum_{n=1}^{\infty}\left(\frac{r}{a}\right)^n\left[\cos n(\phi-\theta)+i\sin\left[n(\phi-\theta)\right]\right]=\sum_{n=1}^{\infty}\left(\frac{r}{a}\right)^n e^{in(\phi-\theta)},$$

$$\sum_{n=1}^{\infty}\left(\frac{r}{a}\right)^n\left[\cos n(\phi-\theta)+i\sin\left[n(\phi-\theta)\right]\right]=\sum_{n=1}^{\infty}\left[\frac{r}{a}e^{i(\phi-\theta)}\right]^n. \qquad \text{... (3.8)}$$

Since $r<a \Rightarrow \frac{r}{a}<1$ and $\left|e^{i(\phi-\theta)}\right|\le 1$.

The expression on the right hand side of the equation (3.8) is a geometric series.

Therefore, $$\sum_{n=1}^{\infty}\left[\frac{r}{a}e^{i(\phi-\theta)}\right]^n=\frac{\left(\frac{r}{a}\right)e^{i(\phi-\theta)}}{1-\left(\frac{r}{a}\right)e^{i(\phi-\theta)}}.$$

Equating the real part on both sides we get

$$\sum_{n=1}^{\infty}\left(\frac{r}{a}\right)^n\cos n(\phi-\theta)=\operatorname{Re}\left[\frac{\left(\frac{r}{a}\right)e^{i(\phi-\theta)}}{1-\frac{r}{a}e^{i(\phi-\theta)}}\right],$$

$$=\operatorname{Re}\left[\frac{\left(\frac{r}{a}\right)e^{i(\phi-\theta)}\left[1-\frac{r}{a}e^{-i(\phi-\theta)}\right]}{\left(1-\frac{r}{a}e^{i(\phi-\theta)}\right)\left(1-\frac{r}{a}e^{-i(\phi-\theta)}\right)}\right],$$

$$=\operatorname{Re}\left[\frac{\left(\frac{r}{a}\right)\left[e^{i(\phi-\theta)}-\frac{r}{a}\right]}{1-\frac{r}{a}\left(e^{i(\phi-\theta)}+e^{-i(\phi-\theta)}\right)+\frac{r^2}{a^2}}\right],$$

$$=\operatorname{Re}\left[\frac{\frac{r}{a}\left[\cos(\phi-\theta)+i\sin(\phi-\theta)-\frac{r}{a}\right]}{\left(1-\frac{r}{a}2\cos(\phi-\theta)-\left(\frac{r}{a}\right)^2\right)}\right],$$

$$\Rightarrow \sum_{n=1}^{\infty}\left(\frac{r}{a}\right)^n \cos n(\phi-\theta)=\frac{\left(\frac{r}{a}\right)\cos(\phi-\theta)-\left(\frac{r}{a}\right)^2}{1-\left(2\frac{r}{a}\right)\cos(\phi-\theta)+\left(\frac{r}{a}\right)^2}.$$

Substituting this on the r.h.s. of (3.7) we get

$$u(r,\theta)=\frac{1}{\pi}\int_0^{2\pi} f(\phi)\left[\frac{1}{2}+\frac{\frac{r}{a}\cos(\phi-\theta)-\frac{r^2}{a^2}}{1-\frac{2r}{a}\cos(\phi-\theta)+\frac{r^2}{a^2}}\right]d\phi,$$

$$u(r,\theta)=\frac{1}{\pi}\int_0^{2\pi} f(\phi)\frac{(a^2-r^2)}{2\left(a^2-2ar\cos(\phi-\theta)+r^2\right)}d\phi. \qquad \text{... (3.9)}$$

This is known as Poisson integral formula for a circle, which gives the unique solution for the Dirchlet problem.

4. The Dirichlet Exterior Problem for a Cicle :

Result : Show that the solution for the exterior Dirichlet problem for a circle of radius a is given by (Poisson integral formula)

$$u(r,\theta)=\frac{1}{2\pi}\int_0^{2\pi}\frac{(r^2-a^2)f(\phi)}{\left[r^2-2ar\cos(\phi-\theta)+a^2\right]}d\phi.$$

Solution : Exterior Dirichlet problem is decribed by

$$\nabla^2 u=0 \text{ for } 0\le\theta\le 2\pi,\ r\ge a, \qquad \text{... (4.1)}$$

and

$$u(a,\theta)=f(\theta),\ 0\le\theta\le 2\pi,\ r=a, \qquad \text{... (4.2)}$$

where $f(\theta)$ is a continuous function of θ on the surface r = a, and $u(r,\theta)$ must be bounded as $r\to\infty$. We know by the method of separation of variables, the general solution of (4.1) in polar co-ordinates is given by

$$u(r,\theta)=\sum_{n=0}^{\infty}\left(C_n r^n+D_n r^{-n}\right)\left(A_n\cos n\theta+B_n\sin n\theta\right) \qquad \text{... (4.3)}$$

Now as $r\to\infty$ we require that $u(r,\theta)$ to be bounded

$$\Rightarrow C_n=0 \qquad \left(\text{As } r^n\to\infty \text{ as } r\to\infty\right)$$

Hence the general solution (4.3) reduces to

$$u(r,\theta)=\sum_{n=0}^{\infty} r^{-n}\left(A_n \cos n\theta + B_n \sin n\theta\right). \qquad \text{... (4.4)}$$

It can also be written as

$$u(r,\theta)=\frac{a_0}{2}+\sum_{n=0}^{\infty} r^{-n}\left(a_n \cos n\theta + b_n \sin n\theta\right). \qquad \text{... (4.5)}$$

Now using the boundary condition

$$u(a,\theta)=f(\theta),$$

$$\Rightarrow f(\theta)=\frac{a_0}{2}+\sum_{n=1}^{\infty} a^{-n}\left(a_n \cos n\theta + b_n \sin n\theta\right).$$

Hence the Fourier constants are given by

$$a_0=\frac{1}{\pi}\int_0^{2\pi} f(\theta)\,d\theta,$$

$$a_n=\frac{a^n}{\pi}\int_0^{2\pi} f(\theta)\cos n\theta\, d\theta,$$

$$b_n=\frac{a^n}{\pi}\int_0^{2\pi} f(\theta)\sin n\theta\, d\theta. \qquad n = 1, 2, 3, ...$$

Substituting these values in equation (4.5) we get the solution as

$$u(r,\theta)=\frac{1}{2\pi}\int_0^{2\pi} f(\phi)\,d\phi+\sum_{n=1}^{\infty} r^{-n}\frac{a^n}{\pi}\left[\cos n\theta\int_0^{2\pi}\cos n\phi f(\phi)\,d\phi+\sin n\theta\int_0^{2\pi}\sin n\phi f(\phi)\,d\phi\right].$$

Interchanging the order of summation and integration we get

$$u(r,\theta)=\frac{1}{2\pi}\int_0^{2\pi} f(\phi)\,d\phi+\frac{1}{\pi}\int_0^{2\pi} f(\phi)\sum_{n=1}^{\infty}\left(\frac{a}{r}\right)^n\left[\cos n\phi\cos n\theta+\sin n\phi\sin n\theta\right]d\phi.$$

$$u(r,\theta)=\frac{1}{\pi}\int_0^{2\pi} f(\phi)\left[\frac{1}{2}+\sum_{n=1}^{\infty}\left(\frac{a}{r}\right)^n\cos n(\phi-\theta)\right]d\phi. \qquad \text{... (4.6)}$$

Consider,

$$\sum_{n=1}^{\infty}\left(\frac{a}{r}\right)^n\left[\cos n(\phi-\theta)+i\sin(\phi-\theta)\right]=\sum_{n=1}^{\infty}\left(\frac{a}{r}\right)^n e^{in(\phi-\theta)}.$$

i.e. $$\sum_{n=1}^{\infty}\left(\frac{a}{r}\right)^n\left[\cos n(\phi-\theta)+i\sin(\phi-\theta)\right]=\sum_{n=1}^{\infty}\left[\left(\frac{a}{r}\right)^n e^{i(\phi-\theta)}\right]^n \qquad \dots (4.7)$$

Since r > a

$$\Rightarrow \frac{a}{r}<1 \text{ and } \left|e^{i(\phi-\theta)}\right|\le 1$$

The expression on the right hand side of the equation (4.7) is a geometric series. Therefore, we have

$$\sum_{n=1}^{\infty}\left[\left(\frac{a}{r}\right)e^{i(\phi-\theta)}\right]^n=\frac{\frac{a}{r}e^{i(\phi-\theta)}}{\left[1-\frac{a}{r}e^{i(\phi-\theta)}\right]},$$

$$=\frac{\frac{a}{r}e^{i(\phi-\theta)}\left[1-\frac{a}{r}e^{i(\phi-\theta)}\right]}{\left[1-\frac{a}{r}e^{i(\phi-\theta)}\right]\left[1-\frac{a}{r}e^{-i(\phi-\theta)}\right]},$$

$$=\frac{\frac{a}{r}\left[e^{i(\phi-\theta)}-\frac{a}{r}\right]}{\left[1-\frac{a}{r}\left(e^{i(\phi-\theta)}+e^{-i(\phi-\theta)}\right)+\frac{a^2}{r^2}\right]},$$

$$\sum_{n=1}^{\infty}\left[\left(\frac{a}{r}\right)e^{i(\phi-\theta)}\right]^n=\frac{\frac{a}{r}\left[\cos(\phi-\theta)+i\sin(\phi-\theta)-\frac{a}{r}\right]}{\left[1-2\frac{a}{r}\cos(\phi-\theta)+\frac{a^2}{r^2}\right]}.$$

Equating the real part on both sides we get

$$\sum_{n=1}^{\infty}\left(\frac{a}{r}\right)^n\cos n(\phi-\theta)=\frac{\frac{a}{r}\left[\cos(\phi-\theta)-\frac{a}{r}\right]}{\left[1-2\frac{a}{r}\cos(\phi-\theta)+\frac{a^2}{r^2}\right]}.$$

Substituting this in equation (4.6) we get

$$u(r,\theta)=\frac{1}{\pi}\int_0^{2\pi}f(\phi)\left[\frac{1}{2}+\frac{\frac{a}{r}\cos(\phi-\theta)-\frac{a^2}{r^2}}{1-\frac{2a}{r}\cos(\phi-\theta)+\frac{a^2}{r^2}}\right]d\phi.$$

$$\Rightarrow u(r,\theta)=\frac{1}{\pi}\int_0^{2\pi} f(\phi)\frac{r^2-a^2}{2\left(r^2-2ar\cos(\phi-\theta)+a^2\right)}d\phi,$$

or
$$u(r,\theta)=\frac{1}{2\pi}\int_0^{2\pi}\frac{\left(r^2-a^2\right)f(\phi)}{\left(r^2-2ar\cos(\phi-\theta)+a^2\right)}. \qquad \text{... (4.8)}$$

This is the required solution of the exterior Dirichlet Problem.

5. Interior Neumann Problem for a Cicle :

The interior Neumann Problem for a circle is described as follows.

Example 1 : Solve

$$\nabla^2 u=0,\ \mathrm{r}<\mathrm{a}$$

subject to the boundary condition

$$\frac{\partial u}{\partial n}=\frac{\partial u}{\partial r}=f(\theta)\ \text{ on r = a },$$

(Because outward normal to the Ole is along the radius vector)

where
$$\int_0^{2\pi} f(\theta)\,d\theta=0.$$

Solution : To find the solution, we solve the equation

$$\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}+\frac{1}{r^2}\frac{\partial^2 u}{\partial\theta^2}=0,\qquad \mathrm{r}<\mathrm{a} \qquad \text{... (5.1)}$$

subject to the boundary condition

$$\frac{\partial u}{\partial r}=f(\theta)\ \text{on r = a},\ 0\le\theta\le 2\pi, \qquad \text{... (5.2)}$$

where $f(\theta)$ is a continuous function of θ on the surface r = a.

We know by the method of spearation of variables, the general solution of equation (5.1) is given by

$$u(r,\theta)=\sum_{n=0}^{\infty}\left(C_n r^n+D_n r^{-n}\right)\left(A_n\cos n\theta+B_n\sin n\theta\right). \qquad \text{... (5.3)}$$

Since at r = 0, the solution should be finite hence we must have $D_n = 0$ ($r^{-n}\to\infty$ as $r\to 0$).

Hence the solution becomes

$$u(r,\theta)=\sum_{n=0}^{\infty} r^n\left(A_n\cos n\theta+B_n\sin n\theta\right).$$

This can be written as

$$u(r,\theta)=\frac{a_0}{2}+\sum_{n=1}^{\infty}r^n\left(a_n\cos n\theta+b_n\sin n\theta\right). \qquad \text{... (5.4)}$$

Differentiating (5.4) w.r.t. r we get

$$\frac{\partial u}{\partial r}=\sum_{n=0}^{\infty}nr^{n-1}\left(a_n\cos n\theta+b_n\sin n\theta\right). \qquad \text{... (5.5)}$$

Now using the boundary condition

$$\frac{\partial u}{\partial r}(a,\theta)=f(\theta),$$

we have
$$f(\theta)=\sum_{n=0}^{\infty}nr^{n-1}\left(a_n\cos(n\theta)+b_n\sin(n\theta)\right). \qquad \text{... (5.6)}$$

This is a Fourier series expansion of $f(\theta)$, where the Fourier constants are given by

$$a_n=\frac{1}{n\pi a^{n-1}}\int_0^{2\pi}f(\theta)\cos(n\theta)\,d\theta,$$

$$b_n=\frac{1}{na^{n-1}\pi}\int_0^{2\pi}f(\theta)\sin(n\theta)\,d\theta.$$

Substituting these values in (5.4) we get

$$u(r,\theta)=\frac{a_0}{2}+\sum_{n=1}^{\infty}\frac{r^n}{n\pi a^{n-1}}\int_0^{2\pi}f(\phi)\left[\cos n\phi\cos n\theta+\sin n\phi\sin n\theta\right].$$

Interchanging the order of summation and integration, we get

$$u(r,\theta)=\frac{a_0}{2}+\frac{1}{\pi}\int_0^{2\pi}f(\phi)\sum_{n=1}^{\infty}\left(\frac{r}{a}\right)^n\frac{a}{n}\cos n(\phi-\theta)\,d\phi. \qquad \text{... (5.7)}$$

Now consider

$$\sum_{n=1}^{\infty}\left(\frac{r}{a}\right)^n\frac{1}{n}e^{in(\phi-\theta)}=\sum_{n=1}^{\infty}\left[\left(\frac{r}{a}\right)e^{i(\phi-\theta)}\right]^n\frac{1}{n},$$

$$=\left[\frac{\frac{r}{a}e^{i(\phi-\theta)}}{1}+\frac{\left\{\frac{r}{a}e^{i(\phi-\theta)}\right\}^2}{2}+\frac{\left\{\frac{r}{a}e^{i(\phi-\theta)}\right\}^3}{3}+\dots\right],$$

$$\sum_{n=1}^{\infty}\left(\frac{r}{a}\right)^n \frac{1}{n}e^{in(\phi-\theta)} = -\log\left[1-\frac{r}{a}e^{i(\phi-\theta)}\right], \quad \text{as } \log(1-x) = -\left[x+\frac{x^2}{2}+\frac{x^3}{3}+...\right]$$

$$\sum_{n=1}^{\infty}\left(\frac{r}{a}\right)^n \frac{1}{n}e^{in(\phi-\theta)} = -\log\left[1-\frac{r}{a}\cos(\phi-\theta)-i\frac{r}{a}\sin(\phi-\theta)\right]. \quad ...(5.8)$$

[Now to find the real part of log z, for z = x + iy

let $W = u + iv = \log z$,

$$\Rightarrow z = x + iy = e^{u+iv},$$

$$\Rightarrow x = e^u \cos v, \quad y = e^y \sin v,$$

$$\Rightarrow e^{2u} = x^2 + y^2$$

$$\Rightarrow u = \log\sqrt{x^2+y^2} \quad].$$

Therefore, equating the real part on both sides of (5.8) we get

$$\sum_{n=1}^{\infty}\left(\frac{r}{a}\right)^n \frac{1}{n}\cos n(\phi-\theta) = -\log\sqrt{\left(1-\frac{r}{a}\cos(\phi-\theta)\right)^2+\left(\frac{r}{a}\sin(\phi-\theta)\right)^2},$$

$$= -\log\sqrt{\frac{a^2-2ar\cos(\phi-\theta)+r^2}{a^2}}.$$

Substituting this in equation (5.7) we get

$$u(r,\theta) = \frac{a_0}{2} - \frac{a}{\pi}\int_0^{2\pi}\log\sqrt{\frac{a^2-2ar\cos(\phi-\theta)+r^2}{a^2}}\cdot f(\phi)\,d\phi. \quad ...(5.9)$$

Thus the required solution of interior Neumann Problem for a circle can also be written as

$$u(r,\theta) = \frac{a_0}{2} - \frac{a}{2\pi}\int_0^{2\pi}\log\left[a^2-2ar\cos(\phi-\theta)+r^2\right]\cdot f(\phi)\,d\phi. \quad ...(5.10)$$

The constant factor a^2 in the argument of log was eliminated by virtue of the necessary condition for the Neumann problem.

6. Exterior Neumann Problem for a Circle :

Result : State the exterior Neumann problem and show that its solution for a circle of radius a is given by

$$u(r,\theta) = \frac{a_0}{2} + \frac{a}{2\pi}\int_0^{2\pi}\log\left[a^2-2ar\cos(\phi-\theta)+r^2\right]\cdot f(\phi)\,d\phi.$$

Proof : The exterior Neumann problem for a circle is described by

$$\nabla^2 u = 0, \quad r > a, \; 0 \le \theta \le 2\pi, \qquad \text{... (6.1)}$$

subject to the condition

$$\frac{\partial u}{\partial n} = \frac{\partial u}{\partial r} = f(\theta), \text{ on the boundary } r = a. \qquad \text{... (6.2)}$$

By the method of separation of variables, we know the general solution of (6.1) in polar form is given by

$$u(r,\theta) = \sum_{n=0}^{\infty} \left(C_n r^n + D_n r^{-n}\right)\left(A_n \cos(n\theta) + B_n \sin(n\theta)\right). \qquad \text{... (6.3)}$$

Now as $r \to \infty$ we require that to $u(r,\theta)$ be finite (bounded)

$$\Rightarrow C_n = 0. \qquad (\text{as } r^n \to \infty \text{ as } r \to \infty)$$

Hence the general solution (6.3) reduces to

$$u(r,\theta) = \sum_{n=0}^{\infty} r^{-n}\left(A_n \cos n\theta + B_n \sin n\theta\right). \qquad \text{... (6.4)}$$

Without loss of generality, it can also be written as

$$u(r,\theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} r^{-n}\left(a_n \cos n\theta + b_n \sin n\theta\right). \qquad \text{... (6.5)}$$

Differentiating equation (6.5) w.r.t. r we get

$$\frac{\partial u}{\partial r} = \sum_{n=1}^{\infty} (-n) r^{-n-1}\left(a_n \cos n\theta + b_n \sin n\theta\right). \qquad \text{... (6.6)}$$

Now using the boundary condition

$$\frac{\partial u}{\partial r}(a,\theta) = f(\theta),$$

we get

$$f(\theta) = \sum_{n=1}^{\infty} (-n) a^{-n-1}\left(a_n \cos n\theta + b_n \sin n\theta\right). \qquad \text{... (6.7)}$$

This is the Fourier series expansion of $f(\theta)$, where the Fourier constants are given by

$$a_n (-n) a^{-n-1} = \frac{1}{\pi}\int_0^{2\pi} f(\theta)\cos n\theta \, d\theta,$$

$$\Rightarrow a_n = -\frac{a^{n+1}}{n\pi}\int_0^{2\pi} f(\theta)\cos n\theta \, d\theta,$$

and $$b_n = -\frac{a^{n+1}}{n\pi}\int_0^{2\pi} f(\theta)\sin n\theta d\theta .$$

Substituting these constants in equation (6.5) we get

$$u(r,\theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} -r^{-n}\frac{a^{n+1}}{n\pi}\int_0^{2\pi} f(\phi)\left[\cos n\phi\cos n\theta + \sin n\phi\sin n\theta\right]d\phi .$$

Interchanging the order of summation and integration we get

$$u(r,\theta) = \frac{a_0}{2} - \frac{a}{\pi}\int_0^{2\pi}\sum_{n=1}^{\infty}\left(\frac{a}{r}\right)^n \cdot \frac{1}{n} f(\phi)\cos n(\phi-\theta)d\phi ,$$

$$u(r,\theta) = \frac{a_0}{2} - \frac{a}{\pi}\int_0^{2\pi} f(\phi)\sum_{n=1}^{\infty}\left(\frac{a}{r}\right)^n \frac{1}{n}\cos n(\phi-\theta)d\phi . \qquad \text{... (6.8)}$$

Now consider

$$\sum_{n=1}^{\infty}\left(\frac{a}{r}\right)^n \frac{1}{n} e^{in(\phi-n)} = \sum_{n=1}^{\infty}\left[\left(\frac{a}{r}\right)e^{i(\phi-n)}\right]^n \frac{1}{n}$$

$$= \left[\frac{\left\{\frac{a}{r}e^{i(\phi-n)}\right\}}{1} + \frac{\left\{\left(\frac{a}{r}\right)e^{i(\phi-n)}\right\}^2}{2} + \frac{\left\{\frac{a}{r}e^{i(\phi-n)}\right\}^3}{3} + \ldots\right]$$

$$= -\log\left[1 - \frac{a}{r}e^{i(\phi-n)}\right] \qquad \ldots\left(\log(1-x) = -\left[x + \frac{x^2}{2} + \frac{x^3}{3} + \ldots\right]\right)$$

$$\sum_{n=1}^{\infty}\left(\frac{a}{r}\right)^n \frac{1}{n} e^{in(\phi-\theta)} = -\log\left(1 - \frac{a}{r}\cos(\phi-\theta) - i\frac{a}{r}\sin(\phi-\theta)\right).$$

Equating the real part on both sides we get

$$\sum_{n=1}^{\infty}\left(\frac{a}{r}\right)^n \frac{1}{n}\cos n(\phi-\theta) = -\log\left|1 - \frac{a}{r}\cos(\phi-\theta) - i\frac{a}{r}\sin(\phi-\theta)\right| ,$$

$$\Rightarrow \sum_{n=1}^{\infty}\left(\frac{a}{r}\right)^n \frac{1}{n}\cos n(\phi-\theta) = -\log\sqrt{\left(1 - \frac{a}{r}\cos(\phi-\theta)^2\right) + \left(\frac{a}{r}\sin(\phi-\theta)^2\right)} ,$$

$$\Rightarrow \sum_{n=1}^{\infty}\left(\frac{a}{r}\right)^n \frac{1}{n}\cos n(\phi-\theta) = -\log\sqrt{\frac{r^2 - 2ar\cos(\phi-\theta) + a^2}{r^2}} .$$

Substituting this in equation (6.8) we get

$$u(r,\theta)=\frac{a_0}{2}+\frac{a}{\pi}\int_0^{2\pi} f(\phi)\log\sqrt{\frac{r^2-2ar\cos(\phi-\theta)+a^2}{r^2}}d\phi,$$

or $$u(r,\theta)=\frac{a_0}{2}+\frac{a}{2\pi}\int_0^{2\pi}\log\left(\frac{r^2-2ar\cos(\phi-\theta)+a^2}{r^2}\right)f(\phi)d\phi. \qquad \text{.... (6.9)}$$

This is the required solution.

7. Interior Dirichlet Problem for a Rectangle :

Result : Solve

$$\nabla^2 u=0,\quad 0\le x\le a,\quad 0\le y\le b, \qquad \text{... (7.1)}$$

subject to the boundary conditions

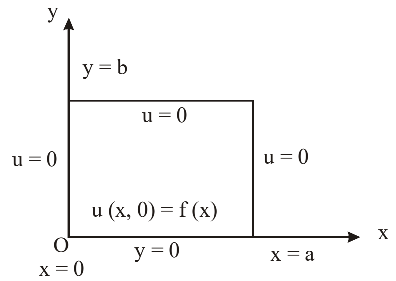
$$u(x,b)=u(a,y)=0, \qquad \text{... (7.2)}$$

$$u(0,y)=0, \qquad \text{... (7.3)}$$

$$u(x,0)=f(x), \qquad 0\le x\le a \qquad \text{... (7.3)}$$

Solution : We assume a variable separable solution of the form

$$u(x,y)=X(x)Y(y). \qquad \text{... (7.5)}$$



Therefore, equation (7.1) becomes

$$X''Y+XY''=0,$$

$$\frac{X''}{X}=-\frac{Y''}{Y}=\lambda \text{ (say)}, \qquad \text{... (7.6)}$$

where λ is a constant, may be positive, zero or negative. For different choices of λ we have three solutions. We have to choose that solution which is consistent with the physical nature of the problem and the boundary conditions.

Case (i) : $\lambda>0$, Take $\lambda=\alpha^2$.

Then we have the equations

$$X''-\alpha^2X=0 \text{ and } Y''+\alpha^2Y=0.$$

Whose solutions are given by

$$X=C_1e^{\alpha x}+C_2e^{-\alpha x},\ Y=C_3\cos(\alpha y)+C_4\sin(\alpha y)$$

Therefore, the general solution of (7.1) is given by

$$u(x,y)=\left(C_1e^{\alpha x}+C_2e^{-\alpha x}\right)\left(C_3\cos(\alpha y)+C_4\sin(\alpha y)\right). \qquad \text{... (7.7)}$$

Now using the boundary condition

$$u(0, y) = 0,$$

we get from equation (7.7)

$$0 = (C_1 + C_2)(C_3 \cos(\alpha y) + C_4 \sin(\alpha y)) = 0,$$

$$\Rightarrow C_1 + C_2 = 0. \qquad \text{.... (7.8)}$$

Again using the boundary condition

$$u(a, y) = 0,$$

we get from equation (7.7)

$$0 = (C_1 e^{\alpha x} + C_2 e^{-\alpha x})(C_3 \cos(\alpha y) + C_4 \sin(\alpha y)),$$

$$\Rightarrow C_1 e^{\alpha a} + C_2 e^{-\alpha a} = 0. \qquad \text{... (7.9)}$$

From equations (7.8) and (7.9) we have $C_1 = 0 = C_2$.

Hence $u(x, y) = 0$ is the only possible trivial solution. Hence we neglect the case $\lambda = 0$.

Case (ii) : If $\lambda = 0$, then from equation (7.6) we have

$$X'' = 0 \text{ and } Y'' = 0.$$

This provides

$$X = (C_5 x + C_6) \text{ and } Y = (C_7 y + C_8).$$

Hence the general solution of (7.1) is given by

$$u(x, y) = (C_5 x + C_6)(C_7 y + C_8). \qquad \text{... (7.10)}$$

Using the boundary conditions

$$u(0, y) = 0 \text{ and } u(a, y) = 0,$$

we get from (7.10)

$$C_6 = 0 = C_5$$

$\Rightarrow u(x, y) = 0$ is a trivial solution. Hence we discard $\lambda = 0$.

Case (iii) : If $\lambda < 0$, Take $\lambda = -\alpha^2$.

Hence from equation (7.6) we have

$$X'' + \alpha^2 X = 0 \text{ and } Y'' - \alpha^2 Y = 0,$$

which have solutions

$$X = C_9 \cos\alpha x + C_{10} \sin\alpha x \text{ and } Y = C_{11}e^{\alpha y} + C_{12}e^{-\alpha y}.$$

Therefore, the general solution of (7.1) is given by

$$u(x,y) = (C_9 \cos\alpha x + C_{10} \sin\alpha x)\left(C_{11}e^{\alpha y} + C_{12}e^{-\alpha y}\right). \qquad \text{... (7.11)}$$

Now using the boundary condition

$$u(0,u) = 0,$$

we obtain $\quad C_9 = 0.$

Also the boundary condition $u(a,y) = 0$ yields

$$0 = C_{10} \sin(\alpha a)\left(C_{11}e^{\alpha y} + C_{12}e^{-\alpha y}\right).$$

If $C_{10} = 0$, we will have again a trivial solution. Therefore, we assume

$$C_{10} \neq 0$$

$$\Rightarrow \sin\alpha a = 0,$$

$$\Rightarrow \alpha a = n\pi, \qquad \forall n = 1, 2, ...$$

or $\quad \alpha = \dfrac{n\pi}{a},$

Take $\quad \alpha_n = \dfrac{n\pi}{a}. \qquad \forall n = 1, 2, ... \qquad \text{... (7.12)}$

These are called the eigen values. Hence the possible non-trivial solution is given by

$$u_n(x,y) = C_{10} \sin\left(\frac{n\pi}{a}x\right)\left(a_n \exp\left(\frac{n\pi}{a}y\right) + b_n \exp\left(-\frac{n\pi}{a}y\right)\right)$$

By superposition principle, the most general solution of (7.1) is given by

$$u(x,y) = \sum_{n=1}^{\infty} u_n(x,y)$$

Hence

$$u(x,y) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi}{a}x\right)\left[a_n \exp\left(\frac{n\pi}{a}y\right) + b_n \exp\left(-\frac{n\pi}{a}y\right)\right] \qquad \text{... (7.13)}$$

Now using the boundary condition

$$u(x,b) = 0,$$

we have from equation (7.13)

$$0=\sin\left(\frac{n\pi}{a}x\right)\left[a_n\exp\left(\frac{n\pi}{a}y\right)+b_n\exp\left(-\frac{n\pi}{a}y\right)\right],$$

$$\Rightarrow a_n\exp\left(\frac{n\pi b}{a}\right)+b_n\exp\left(-\frac{n\pi b}{a}\right)=0,$$

$$\Rightarrow b_n=-a_n\frac{\exp\left(\frac{n\pi b}{a}\right)}{\exp\left(-\frac{n\pi b}{a}\right)}, \qquad \text{n} = 1, 2, \ldots$$

Substituting this in (3.13) we get

$$u(x,y)=\sum_{n=1}^{\infty}\frac{a_n\sin\left(\frac{n\pi x}{a}\right)}{\exp\left(-\frac{n\pi x}{a}\right)}\left[\exp\left\{\frac{n\pi}{a}(y-b)\right\}-\exp\left\{-\frac{n\pi}{a}(y-b)\right\}\right],$$

$$=\sum_{n=1}^{\infty}\frac{2a_n\sin\left(\frac{n\pi x}{a}\right)}{\exp\left(-\frac{n\pi x}{a}\right)}\cdot\sinh\left\{\frac{n\pi}{a}(y-b)\right\}, \qquad \left(\text{Since } \sinh x=\frac{e^x-e^{-x}}{2}\right)$$

$$\Rightarrow u(x,y)=\sum_{n=1}^{\infty}A_n\sin\left(\frac{n\pi x}{a}\right)\sinh\left\{\frac{n\pi}{a}(y-b)\right\}, \text{ for } A_n=\frac{2a_n}{\exp\left(-\frac{n\pi b}{a}\right)}. \qquad \ldots (7.14)$$

Now using the boundary condition

$$u(x,0)=f(x), \qquad 0\le x\le a,$$

we have from equation (7.14)

$$f(x)=\sum_{n=1}^{\infty}A_n\sin\left(\frac{n\pi x}{a}\right)\sinh\left(-\frac{n\pi b}{a}\right) \qquad \ldots (7.15)$$

This is a Fourier sine series, where the Fourier constant is given by

$$A_n\sinh\left(-\frac{n\pi b}{a}\right)=\frac{2}{a}\int_0^a f(x)\sin\left(\frac{n\pi}{a}x\right)dx,$$

or

$$A_n=\frac{2}{a\sinh\left(-\frac{n\pi b}{a}\right)}\int_0^a f(x)\sin\left(\frac{n\pi}{a}x\right)dx. \qquad \ldots (7.16)$$

Thus the general solution for the Dirichlet problem for a rectangle is given by

$$u(x,y)=\sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi}{a}x\right)\sinh\left\{\frac{n\pi}{a}(y-b)\right\},$$

where $$A_n=\frac{2}{a\sinh\left(-\frac{n\pi b}{a}\right)}\int_0^a f(x)\sin\left(\frac{n\pi}{a}x\right)dx\,. \quad \text{... (7.17)}$$

8. The Neumann Problem for a rectangle

Result : Solve the equation

$$\nabla^2 u=0 \quad 0\le x\le a,\quad 0\le y\le b, \quad \text{... (8.1)}$$

subject to the boundary conditions.

$$u_x(0,y)=u_x(a,y)=0, \quad \text{... (8.2)}$$

$$u_y(x,0)=0, \quad \text{... (8.3)}$$

$$u_y(x,b)=f(x). \quad \text{... (8.4)}$$

Solution : By variable separable method, we have obtained the general solution of equation (8.1) in the form

$$u(x,y)=(C_1\cos\alpha x+C_2\sin\alpha x)\left(C_3e^{\alpha y}+C_4e^{-\alpha y}\right). \quad \text{... (8.5)}$$

Differentiating equation (8.5) w.r.t. x and y we get

$$u_x(x,y)=\alpha\left[C_1\sin\alpha x+C_2\cos\alpha x\right]\left(C_3e^{\alpha y}+C_4e^{-\alpha y}\right). \quad \text{... (8.6)}$$

Now using the boundary condition (8.2) viz.

$u_x(0,y)=0$ gives

$C_2=0$.

Therefore, equation (8.5) becomes

$$u(x,y)=C_1\cos\alpha x\left(C_3e^{\alpha y}+C_4e^{-\alpha y}\right). \quad \text{... (8.7)}$$

Now the boundary condition

$u_x(a,y)=0$ gives

$$0=\alpha C_1\sin\alpha a\left(C_3e^{\alpha y}+C_4e^{-\alpha y}\right).$$

If $C_1=0$ then we get trivial solution of (8.1). Therefore, for non-trivial solution, we assume $C_1\neq 0$.

$\Rightarrow \sin(\alpha a)=0$,

$\Rightarrow \alpha a=n\pi$, n = 1, 2, 3, ...

$$\Rightarrow \alpha = \frac{n\pi}{a}. \qquad n = 1, 2, 3, ...$$

Let for each n,

$$\alpha_n = \frac{n\pi}{a}, \qquad n = 1, 2, ... \qquad ... (8.8)$$

These are called the eigen values. Thus the possible solution is given by putting α_n in equation (8.7).

$$u(x, y) = \cos\left(\frac{n\pi}{a}x\right)\left(Ae^{\frac{n\pi}{a}y} + Be^{-\frac{n\pi}{a}y}\right). \qquad ... (8.9)$$

Differentiating (8.9) w.r.t. y we get

$$u_y(x, y) = \frac{n\pi}{a}\cos\left(\frac{n\pi}{a}x\right)\left[Ae^{\frac{n\pi}{a}y} - Be^{-\frac{n\pi}{a}y}\right].$$

Now using the boundary condition

$$u_y(x, 0) = 0$$

we have
$$0 = \frac{n\pi}{a}\cos\left(\frac{n\pi}{a}x\right)(A - B)$$

$$\Rightarrow A - B = 0 \Rightarrow \mathrm{A = B}.$$

Thus the solution (8.9) becomes

$$u(x, y) = A\cos\left(\frac{n\pi}{a}x\right)\left[\exp\left(\frac{n\pi}{a}y\right) + \exp\left(-\frac{n\pi}{a}y\right)\right],$$

$$u(x, y) = 2A\cos\left(\frac{n\pi}{a}x\right)\cosh\left(\frac{n\pi}{a}y\right).$$

Using the superposition principle and for $2A = A_n$, we write the general solution of (8.1) as

$$u(x, y) = \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi}{a}x\right)\cosh\left(\frac{n\pi}{a}h\right). \qquad ... (8.10)$$

Finally using the boundary condition (8.4)

$$u_y(x, b) = f(x),$$

we have

$$f(x) = \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi}{a}x\right)\cdot\left(\frac{n\pi}{a}\right)\sinh\left(\frac{n\pi}{a}b\right),$$

which is the Fourier cosine series. The corresponding Fourier constant is given by

$$\left(\frac{n\pi}{a}\right)A_n \cdot \sinh\left(\frac{n\pi}{a}b\right) = \frac{2}{a}\int_0^a f(x)\cos\left(\frac{n\pi}{a}x\right)dx$$

$$\Rightarrow A_n = \frac{2}{n\pi \sinh\left(\frac{n\pi}{a}b\right)} \cdot \int_0^a f(x)\cos\left(\frac{n\pi}{a}x\right)dx. \qquad \text{... (8.11)}$$

Hence the required solution of the Neumann problem for a rectangle is given by

$$u(x,y) = \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi}{a}x\right)\sinh\left(\frac{n\pi}{a}y\right),$$

where the constant A_n is given by

$$A_n = \frac{2}{n\pi \sinh\left(\frac{n\pi b}{a}\right)}\int_0^a f(x)\cos\left(\frac{n\pi}{a}x\right)dx. \qquad \text{... (8.12)}$$

9. The Dirichlet Problem for the Upper Half Plane

Result : Find the solution of the problem

$$\nabla^2 u = 0, \ -\infty < x < \infty, y > 0,$$

$$u(x,0) = f(x), \ -\infty < x < \infty,$$

such that u is bounded as $y \to \infty$, u and u_x vanish as $|x| \to \infty$.

Solution : Given that

$$u_{xx} + u_{yy} = 0, \quad -\infty < x < \infty, \ y > 0, \qquad \text{... (9.1)}$$

$$u(x,0) = f(x), \quad -\infty < x < \infty, \qquad \text{... (9.2)}$$

with the conditions that u is bounded as $y \to \infty$ and u and u_x vanish as $|x| \to \infty$.

We use the technique of Fourier transform to solve the problem. Let $U(\alpha, y)$ be the Fourier transform of u (x, y) in the variable x. Therefore, by definition we have

$$U(\alpha, y) = \mathcal{F}\{u(x,y\} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} u(x,y)e^{i\alpha x}dx. \qquad \text{... (9.3)}$$

Now applying the Fourier transform to equation (9.1) we get

$$\mathcal{F}\left\{\frac{\partial^2 u}{\partial x^2}\right\} + \mathcal{F}\left\{\frac{\partial^2 u}{\partial y^2}\right\} = 0.$$

Since Fourier transform for derivative is given by

$$\mathcal{F}\left\{f^{(n)}(n)\right\}=(-i\alpha)^n\,\mathcal{F}\{f(x)\}.$$

$$\Rightarrow(-i\alpha)^2\,\mathcal{F}\{u(x,y)\}+\mathcal{F}\left\{\frac{\partial^2 u}{\partial y^2}\right\}=0.$$

$$\Rightarrow-\alpha^2U(\alpha,y)+U_{yy}=0,$$

i.e. $$U_{yy}-\alpha^2U=0. \qquad \text{... (9.4)}$$

Its solution is given by

$$U(\alpha,y)=A(\alpha)e^{\alpha y}+B(\alpha)e^{-\alpha y}. \qquad \text{... (9.5)}$$

Since we require that the solution $U(\alpha,y)$ be bounded as $y\to\infty$, therefore, for $\alpha>0$, we must have $A(\alpha)=0$, and for $\alpha<0$, $B(\alpha)=0$.

Therefore, we have

$$U(\alpha,y)=U(\alpha,0)e^{-|\alpha|y}, \qquad \text{... (9.6)}$$

where $$U(\alpha,0)=\mathcal{F}\{u(x,0)\}$$

$$=\mathcal{F}\{f(x)\} \qquad \text{by equation (9.2)}$$

$$\Rightarrow U(\alpha,0)=F(\alpha), \qquad \text{by definition of Fourier series.}$$

Hence $$U(\alpha,y)=\mathcal{F}(\alpha)\cdot e^{-|\alpha|y}. \qquad \text{... (9.7)}$$

Also by definition of inverse Fourier transform, we have

$$\mathcal{F}^{-1}\left\{e^{-|\alpha|y}\right\}=\sqrt{\frac{2}{\pi}}\left(\frac{y}{y^2+x^2}\right). \qquad \text{... (9.8)}$$

We write equation (9.7) on using (9.8) as

$$\mathcal{F}\{u(x,y)\}=\mathcal{F}\{f(x)\}\,\mathcal{F}\left\{\sqrt{\frac{2}{\pi}}\cdot\left(\frac{y}{y^2+x^2}\right)\right\},$$

$$\mathcal{F}\{u(x,y)\}=\mathcal{F}\left\{f*\sqrt{\frac{2}{\pi}}\cdot\left(\frac{y}{y^2+x^2}\right)\right\} \qquad \left(\text{Since } \mathcal{F}(f*g)=\mathcal{F}(f)\cdot\mathcal{F}(g)\right)$$

Taking inverse Fourier transform on both sides we get

$$u(x,y) = f * \sqrt{\frac{2}{\pi}}\left(\frac{y}{y^2+x^2}\right),$$

$$u(x,y) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(\xi)\cdot\sqrt{\frac{2}{\pi}}\left(\frac{y}{y^2+(x-\xi)^2}\right)d\xi \qquad \text{(by convolution theorem.)}$$

$$u(x,y) = \frac{y}{\pi}\int_{-\infty}^{\infty}\frac{f(\xi)}{y^2+(x-\xi)^2}d\xi\,. \qquad \text{... (9.9)}$$

This is the required solution of the Dirichlet problem for the upper half plane.

10. The Neumann Problem for the Upper Half Plane :

Result : Find the solution of the problem

$$\nabla^2 u = 0, \qquad -\infty < x < \infty, y > 0,$$

$$u_y(x,0) = g(x), \qquad -\infty < x < \infty,$$

such that u is bounded as $y \to \infty$, u and u_x vanish as $|x| \to \infty$ and $\int_{-\infty}^{\infty} g(x)dx = 0$.

Solution : We reformulate the problem by introducing a new variable v (x, y) as

$$v(x,y) = u_y(x,y). \qquad \text{... (10.1)}$$

Then

$$u = \int_{\alpha}^{u} v(x,\eta)\,d\eta \qquad \text{... (10.2)}$$

Also

$$\nabla^2 v(x,y) = \nabla^2 u_y = \frac{\partial}{\partial y}(\nabla^2 u) = 0,$$

and

$$v(x,0) = u_y(x,0) = g(x).$$

Thus our problem is reformulated in to the new variable v as

$$\nabla^2 v(x,y) = 0, \qquad -\infty < x < \infty, \ y > 0, \qquad \text{... (10.3)}$$

$$v(x,0) = g(x), \qquad -\infty < x < \infty. \qquad \text{... (10.4)}$$

Since u is bounded as $y \to \infty \Rightarrow v$ is also bounded as $y \to \infty$.

Integrating equation (10.1) with respect to y we obtain

$$u(x,y) = \int_{a}^{y} v(x,\eta)d\eta$$

Since, u is bounded as $y \to \infty \Rightarrow \int_a^y v(x,\eta)d\eta$ is bounded as $y \to \infty$.

$\Rightarrow$ the integrand v (x, y) is bounded on $y \to \infty$.

Also from equation (10.1) we find

$$v_x(x,y) = \frac{\partial}{\partial x}u_y = \frac{\partial}{\partial y}u_x$$

$$\Rightarrow \lim_{|x|\to\infty} v_x(x,y) = \frac{\partial}{\partial y}\lim_{|x|\to\infty} u_x(x,y) = 0.$$

$$\Rightarrow v_x(x,y) \to 0 \text{ as } |x| \to \infty.$$

Also from equation (10.2), we have

$$\lim_{|x|\to\infty} u = \int_a^y \lim_{|x|\to\infty} v(x,\eta)d\eta = 0$$

$$\Rightarrow v \to 0 \text{ as } |x| \to \infty.$$

However, we know the solution of the problem is given by

$$v(x,y) = \frac{y}{\pi}\int_{-\infty}^{\infty} \frac{g(\xi)}{(\xi - x)^2 + y^2}d\xi. \qquad \text{... (10.5)}$$

Hence the solution of the original problem becomes

$$u(x,y) = \int_\alpha^y v(x,\eta)d\eta.$$

On using (10.5) we get

$$u(x,y) = \frac{1}{\pi}\int_\alpha^y \eta \int_{-\infty}^{\infty} \frac{g(\xi)}{(\xi - x)^2 + \eta^2}\, d\xi d\eta$$

Consider $$\frac{1}{2}\int_\alpha^y \frac{2\eta}{(\xi - x)^2 + \eta^2}d\eta = \frac{1}{2}\log\left[(\xi - x)^2 + \eta^2\right]_\alpha^y,$$

$$\int_\alpha^y \frac{\eta}{(\xi - x)^2 + \eta^2}d\eta = \frac{1}{2}\log\left[\frac{(\xi - x)^2 + y^2}{(\xi - x)^2 + \alpha^2}\right],$$

$$u(x,y) = \frac{1}{2\pi}\int_\alpha^y g(\xi)\log\left[\frac{(\xi - x)^2 + y^2}{(\xi - x)^2 + \alpha^2}\right]d\xi. \qquad \text{... (10.6)}$$

This determines the solution of the problem.

UNIT - VIII

RIEMANN'S METHOD OF SOLUTION OF LINEAR HYPERBOLIC EQUATIONS

Introduction :

In this unit we shall discuss Riemann's method of solving a linear second order hyperbolic partial differential equations which are in the canonical forms. The method is illustrated in the following theorem.

Theorem : Describe Riemann's method of solving a linear second order hyperbolic equation .

Proof : Let

$$L[u] = u_{xy} + au_x + bu_y + Cu = f(x, y)\,, \qquad \text{... (1.1)}$$

be a linear, second order hyperbolic equation, which is in a canonical form, where a, b, c are functions of x and y only .

Define another operator M such that

$$M[u] = v_{xy} - (av)_x - (bv)_y + cv, \qquad \text{... (1.2)}$$

where *v (x)* is a function having continuous second order partial derivatives. The operator M is called the adjoint operator of L.

Consider

$$vL[u] - uM[v] = v[u_{xy} + au_x + bu_y + cu] - u[v_{xy} - (av)_x - (bv)_y + cv],$$
$$= \left(Nu_{xy} - uv_{xy}\right) + \left(vau_x + u(av)_x\right) + \left(vbu_y + u(bv)_y\right).$$

We write

$$vu_{xy} - uv_{xy} = (vu_x)_y - (uv_y)_x\,,$$

$$vau_x = (avu)_x - u(av)_x\,,$$

$$vbu_y = (bvu)_y - u(bv)_y\,.$$

Therefore

$$vL[u] - uM[v] = \left(vu_x\right)y - \left(uv_y\right)_x + \left(avu\right)_x + \left(bvu\right)_y\,,$$

$$= \left(avu - uv_y\right)_x + \left(bvu + vu_x\right)_y$$

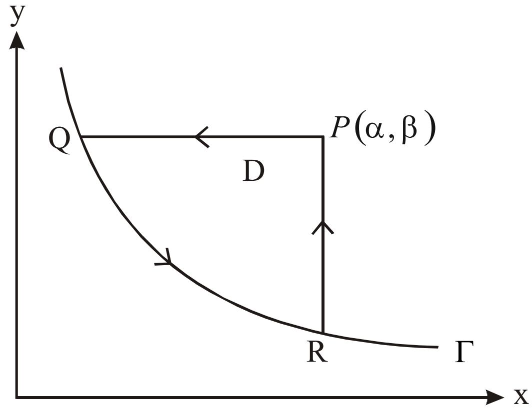
$$\Rightarrow vL[u] - uM[v] = U_x + V_y\,, \qquad \text{... (1.3)}$$

where

$$U = avu - uv_y,$$

$$V = bvu + vu_x . \qquad \text{... (1.4)}$$

Let $P(\alpha, \beta)$ be a point at which the solution is to be found. Let the characteristics through P intersect the initial curve Γ at Q and R. We assume that u, u_x, u_y are prescribed along Γ. Let C be a closed contour PQRP bounding the region D. We now apply Green's theorem to this region.



Now from equation (1.3) we have

$$\iint_D \left(vLu - uMv\right]dxdy = \iint_D \left(U_x + V_y\right)dxdy .$$

Using Green's theorem we write

$$\iint_D \left(vLu - uMv\right]dxdy = \oint_c (Udy - Vdx)$$

$$\iint_D \left(vLu - uMv\right]dxdy = \int_Q^R (Udy - Vdx) + \int_R^P Udy - Vdx) + \int_P^Q (Udy - Vdx) \qquad \text{... (1.5)}$$

Now along PQ, y is constant $\Rightarrow dy = 0$ and along PR , x is constant and hence dx = 0.

Therefore, above equation becomes

$$\iint_D (vLu - uMv) = \int_Q^R (Udy - Vdx) + \int_R^P Udy - \int_P^Q Vdx \qquad \text{... (1.6)}$$

Now consider

$$\int_P^Q Vdx = \int_P^Q bvudx + \int_P^Q vu_x dx \qquad \text{by equation (1.4)}$$

$$= \int_P^Q bvudx + [uv]_P^Q - \int_P^R uv_x dx$$

$$\int_P^Q Vdx = [uv]_P^Q + \int_P^Q u(bv - v_x)dx$$

Substituting this in equation (1.6) we get

$$\iint_D (vLu - uMv)dxdy = \int_Q^R (Udy - Vdx) + \int_R^P Udy - [uv]_Q + [uv]_p - \int_p^Q u(bv - v_x)dx,$$

$$\Rightarrow [uv]_p = [uv]_Q + \int_P^Q u(bv - v_x)dx + \int_P^R u(av - v_y)dy -$$

$$-\int_Q^R (Udy - Vdx) + \iint_D (vLu - uMv)dxdy. \qquad \text{... (1.7)}$$

The function v is quite arbitrary. We choose the function v such that it is the solution of the adjoint equation M[v] = 0 satisfying the conditions

$v_x = bv$ on $y = \beta$, (i.e along PQ)

$v_y = av$ on $x = \alpha$, (i.e along RP)

and $v = 1$ at $x = \alpha$ and $y = \beta$. (i.e at point P) ... (1.8)

Such a function $v(x, y, \alpha, \beta)$, if it exists, is called a Riemann function or Green's function.

Hence equation (1.7) reduces to

$$[u]_p = [uv]_Q - \int_Q^R (Udy - Vdx) + \iint_D vLu \text{ dxdy},$$

$$[u]_p = [uv]_Q - \int_Q^R [(avu - uv_y)dy - (bvu + vu_x)dx] + \iint_D vf \text{ dxdy}$$

$$[u]_p = [uv]_Q - \int_Q^R uv(ady - bdx) + \int_Q^R (uv_y dy + vu_x dx) + \iint_D vf \text{ dxdy} \qquad \text{... (1.9)}$$

Equation (1.9) finds u at p provided u and u_x are prescribed along the curve Γ. However, when u and u_y are prescribed along Γ, then to find u at p we use the identify

$$\int_Q^R d(uv) = \int_Q^R [(uv)_x dx + (uv)_y dy], \qquad \text{by chain rule}$$

$$[uv]_Q = [uv]_R - \int_Q^R [(uv)_x dx + (uv)_y \text{ dy}]. \qquad \text{... (1.10)}$$

Substituting this in (1.9) we get

$$[u]_p = [uv]_R - \int_Q^R uv(ady - bdx) - \int_Q^R \left(uv_x dx + vu_y \text{ dy}\right) + \iint_D vf \text{ dxdy} \qquad \text{... (1.11)}$$

On adding equations (1.9) and (1.11) we get

$$[u]_p = \frac{1}{2}\Big[[uv]_Q + [uv]_R\Big] - \int_Q^R uv(ady - bdx) + \iint_D vfdxdy \; +$$

$$+\frac{1}{2}\int_Q^R \left(uv_y dy + vu_x dx\right) - \frac{1}{2}\int_Q^R \left(uv_x dx + vu_y dy\right). \qquad \text{... (1.12)}$$

By using any of the equations (1.9), (1.11) and (1.12) whichever is suitable, value of u at p can be obtained provided values of u, u_x or u_y are known along the curve Γ .

Example (1) : Show that for the linear hyperbolic equation

$$u_{xy} + \frac{u}{4} = 0,$$

the Riemann function is

$$v(x, y, \alpha, \beta) = J_0\left(\sqrt{(x-\alpha)(y-\beta)}\right),$$

where J_o is the Bessel function of the first kind of order zero.

Solution : The linear hyperbolic equation is given by

$$L[u] = u_{xy} + \frac{u}{4} = 0. \qquad \text{... (1.13)}$$

Comparing this equation with the standard linear hyperbolic equation (1.1) we have

$$a = o, b = o, c = \frac{1}{4} \qquad \text{and } f(x, y) = 0.$$

Hence the adjoint operator M is defined by

$$M[v] = v_{xy} + \frac{v}{4}. \qquad \text{... (1.14)}$$

We see that

$$M = L$$

$\Rightarrow$ L is self adjoint.

Therefore, we have

$$vL[u]-uM[v]=v\left(u_{xy}+\frac{u}{4}\right)-u\left(v_{xy}+\frac{v}{4}\right),$$

$$= vu_{xy}-uv_{xy},$$

$$= (vu_x)_y-(uv_y)_x,$$

$$vL[u]-uM[v]=U_x+V_y. \qquad \text{... (1.15)}$$

We choose the Riemannian function in such a way that

$$M[v]=0.$$

$$v_x=0 \quad \text{on} \quad y=\beta,$$

$$v_y=0 \quad \text{on} \quad x=\alpha,$$

and $\quad v=1 \quad$ at $\quad x=\alpha$ and $y=\beta$.

Let $\quad v=v(\eta)$,

where η is a single valued differentiable function of x and y.

Let $\quad \eta^2=(x-\alpha)(y-\beta)$,

$$\Rightarrow 2\eta\eta_x=(y-\beta),$$

$$\Rightarrow \eta_x=\frac{1}{2\eta}(y-\beta),$$

and $\quad \eta_y=\dfrac{1}{2\eta}(x-\alpha)$.

Hence, we have

$$v_x=\frac{1}{2\eta}v_\eta(y-\beta),$$

$$v_y=\frac{1}{2\eta}v_\eta(x-\alpha).$$

$$\Rightarrow v_x=v_\eta\eta_x,\quad v_y=v_\eta\eta_y,$$

$$\Rightarrow v_{xy}=\frac{1}{2}\left[v_{\eta\eta}\eta_y\frac{1}{\eta}(y-\beta)+v_\eta\frac{1}{\eta}-v_\eta\frac{1}{\eta^2}(y-\beta)\eta_y\right],$$

$$\Rightarrow v_{xy} = \frac{1}{4}\left(v_{\eta\eta} + \frac{1}{\eta} v_{\eta} \right). \qquad\qquad v_{xy} = \frac{v}{4}(0)$$

$\Rightarrow$ the equation is transformed to the ordinary differential equation

$$v_{\eta\eta} + \frac{1}{\eta} v_{\eta} + v = 0, \qquad \text{... (1.18)}$$

where $v(\eta)$ satisfies $v(o) = 1$. Equation (1.18) is the Bessel equation of order zero, whose solution is given by $J_0(\eta)$,

$$\Rightarrow v = J_o\left(\sqrt{(x-\alpha)(y-\beta)}\right).$$

This is the required Riemann function .

Example (2) : Show that the Green's function for the equation $u_{xy} + u = o$ is

$$v(x, y; \alpha, \beta) = J_0\left(2\sqrt{(x-\alpha)(y-\beta)}\right),$$

where J_0 is the Bessel's function of first kind and of order zero .

Solution : Here the linear hyperbolic second order partial differential equation is given by

$$L(u) = u_{xy} + u = 0. \qquad \text{... (1.19)}$$

Comparing this equation with the standard canonical hyperbolic equation we get

$$a = 0, b = 0, c = 1 \qquad \text{and } f(x,y) = 0.$$

The adjoint operator M is given by

$$M[v] = v_{xy} + v = 0. \qquad \text{... (1.20)}$$

We see that M = L, proving L is self adjoint.

Hence $\qquad vL[u] - uM[v] = v\left(u_{xy} + u\right) - u\left(v_{xy} + v\right),$

$$= v\ u_{xy} - u\ v_{xy},$$

$$\Rightarrow vL[u] - uM[v] = (u\ u_y)_x - (u\ v_x)_y,$$

$$= U_x + V_y,$$

where $\qquad U = uu_y, V = uv_x. \qquad$ ... (1.21)

We choose v such that $M[v] = 0$,

$$v_x = 0 \quad \text{on} \quad y = \beta,$$

$$v_y = 0 \quad \text{on} \quad x = \alpha, \qquad \text{... (1.22)}$$

and $\qquad v = 1 \quad$ at $\quad x = \alpha\,,\;\; y = \beta\,.$

Let $\qquad v = v(\eta),$

where η is a single valued differentiable function of x and y.

Let $\qquad \eta^2 = 4(x-\alpha)(y-\beta)$

$$\Rightarrow \eta_x = \frac{2}{\eta}(y-\beta) \quad \text{and} \quad \Rightarrow \eta_y = \frac{2}{\eta}(x-\alpha).$$

Therefore

$$v_x = v_\eta \eta_x = \frac{2}{\eta} v_\eta (y-\beta)$$

and
$$v_y = \frac{2}{\eta} v_\eta (x-\alpha)$$

Thus
$$v_{xy} = \frac{2}{\eta}\eta_y v_{\eta\eta}(y-\beta) - \frac{2}{\eta^2} v_\eta \eta_y (y-\beta) + \frac{2}{\eta} v_\eta$$

$$\Rightarrow v_{xy} = \frac{4}{\eta^2} v_{\eta\eta}(x-\alpha)(y-\beta) - \frac{4}{\eta^3} v_\eta (x-\alpha)(y-\beta) + \frac{2}{\eta} v_\eta\,,$$

$$\Rightarrow v_{xy} = v_{\eta\eta} + \frac{1}{\eta} v_\eta\,. \qquad \text{... (1.23)}$$

Thus the equation

$$v_{xy} + v = 0\,,$$

is transformed to the equation

$$v_{\eta\eta} + \frac{1}{\eta} v_\eta + v = 0\,. \qquad \text{... (1.24)}$$

This is a Bessel equation of order zero, whose solution is given by

$$v(x, y; \alpha, \beta) = J_0(\eta)$$

$$v(x, y; \alpha, \beta) = J_o\left(2\sqrt{(x-\alpha)(y-\beta)}\right). \qquad \text{... (1.25)}$$

Example 3 : Show that

$$v(x, y; \alpha, \beta) = \frac{(x+y)\left[2xy + (\alpha-\beta)(x-y) + 2\alpha\beta\right]}{(\alpha+\beta)^3}$$

is the Riemann function for the second order p.d.e

$$u_{xy}+\frac{2}{x+y}\left(u_x+u_y\right)=0.$$

Hence obtain the solution of the equation in the form

$$v=\left(2y^3-3y^2x+3yx^2-2x^3\right),$$

subject to $\quad u=0, \quad u_x=3x^2 \quad \text{on} \quad y=x.$

Solution : A linear second order hyperbolic equation is given by

$$L[u]=u_{xy}+\frac{1}{x+y}\left(u_x+u_y\right)=0, \qquad \text{... (1.26)}$$

where $\quad a=\frac{2}{x+y}, \quad b=\frac{2}{x+y}, \quad c=0, \quad f(x,y)=0.$

The operator M is defined by

$$M[v]=v_{xy}-\left(\frac{2}{x+y}v\right)_x-\left(\frac{2}{x+y}v\right)_y, \qquad \text{... (1.27)}$$

such that

i) $\quad M[v]=0,$

ii) $\quad v_x=\frac{2}{x+y}v \quad$ on $\quad y=\beta, \qquad$ ... (1.28)

iii) $\quad v_y=\frac{2}{x+y}v \quad$ on $\quad x=\alpha,$

iv) $\quad$ and $v=1$ at $x=\alpha \quad y=\beta.$

Now to show

$$v(x,y;\alpha,\beta)=\frac{(x+y)\left[2xy+(\alpha-\beta)(x-y)+2\alpha\beta\right]}{(\alpha+\beta)^3} \qquad \text{... (1.29)}$$

in a Riemann's (Green's) function, we simply show that v defined in (1.29) must satisfy the equation (1.27) and the addition (1.28). We find

$$v_x=\frac{(x+y)}{(\alpha+\beta)^3}\left[2y+\alpha-\beta\right]+\frac{1}{(\alpha+\beta)^3}\left[2xy+(\alpha-\beta)(\alpha-y)+2\alpha\beta\right],$$

$$v_x = \frac{1}{(\alpha+\beta)^3}\left[2xy + 2y^2 + 2xy + (\alpha-\beta)(x-y+x+y) + 2\alpha\beta\right],$$

$$v_x = \frac{1}{(\alpha+\beta)^3}\left[4xy + 2y^2 + 2x(\alpha-\beta) + 2\alpha\beta\right]. \qquad \text{... (1.30)}$$

Next $$v_{xy} = \frac{1}{(\alpha+\beta)^3}[4x+4y]$$

$$\Rightarrow v_{xy} = \frac{4(x+y)}{(\alpha+\beta)^3}. \qquad \text{... (1.31)}$$

Also $$v_y = \frac{1}{(\alpha+\beta)^3}\left[4xy + 2x^2 - 2y(\alpha-\beta) + 2\alpha\beta\right] \qquad \text{... (1.32)}$$

Now consider

$$M[v] = v_{xy} - \frac{2}{x+y}(v_x + v_y) + \frac{4}{(x+y)^2}(v) \qquad \text{... (1.33)}$$

On using equations (1.30), (1.31), and (1.32) in (1.33) we get

$$M[v] = \frac{4(x+y)}{(\alpha+\beta)^3} - \frac{2}{(x+y)}\frac{1}{(\alpha+\beta)^3}\left[8xy + 2(x^2+y^2) + 2(x-y)(\alpha-\beta) + 4\alpha\beta\right] +$$

$$+\frac{4}{(x+y)^2(\alpha+\beta)^3}(x+y)\left[2xy + (\alpha-\beta)(x-y) + 2\alpha\beta\right],$$

$$M[v] = \frac{4(x+y)}{(\alpha+\beta)^3} - \frac{4}{(x+y)(\alpha+\beta)^3}(x+y)^2$$

$$\Rightarrow M[v] = 0.$$

$\Rightarrow v$ satisfies the condition (i) in (1.28). Now along $y=\beta$ equation (1.30) becomes

$$(V_x)_{y=\beta} = \frac{1}{(\alpha+\beta)^3}\left[4x\beta + 2\beta^2 + 2x(\alpha-\beta) + 2\alpha\beta\right].$$

$$(V_x)_{y=\beta} = \frac{1}{(\alpha+\beta)^3}\left[2\beta^2 + 2x(\alpha+\beta) + 2\alpha\beta\right] \qquad \text{... (1.34)}$$

Also $$\left(\frac{2}{x+y}v\right)_{y=\beta}=\frac{1}{(\alpha+\beta)^3}\left[4x\beta+2(\alpha-\beta)(x-\beta)+4\alpha\beta\right]$$

$$=\frac{1}{(\alpha+\beta)^3}\left[2x(\alpha+\beta)-2\beta(\alpha-\beta)+4\alpha\beta\right]$$

$$\Rightarrow\left(\frac{2}{x+y}v\right)_{y=\beta}=\frac{1}{(\alpha+\beta)^3}\left[2\beta^2+2x(\alpha+\beta)+2\alpha\beta\right]. \quad ...(1.35)$$

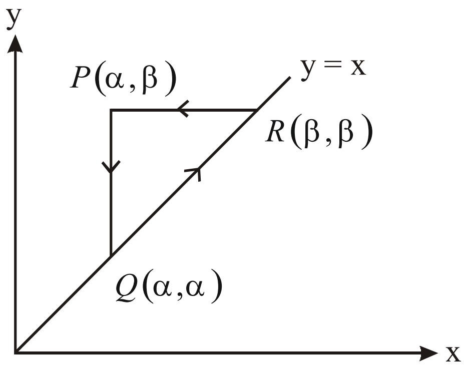
Equations (1.34) and (1.35) show that the condition (ii) of equation (1.28) is satisfied.
Similarly condition (iii) can be verified .

Now consider at $x=\alpha$, $y=\beta$

$$v|_p=\frac{(\alpha+\beta)}{(\alpha+\beta)^3}\left[2\alpha\beta+(\alpha-\beta)(\alpha-\beta)+2\alpha\beta\right],$$

$$v|_p=\frac{1}{(\alpha+\beta)^2}\left[4\alpha\beta+\alpha^2-2\alpha\beta+\beta^2\right]$$

$$v|_p=1$$



This shows that the equations (iv) of (1.28) also verified.
Hence

$$v(x,y;\alpha,\beta)=\frac{x+y}{(\alpha+\beta)^3}\left[2xy+(\alpha-\beta)(x-y)+2\alpha\beta\right]$$

in the Riemann function of the given p.d.e. (1.26). Now to find it solution , we consider

$$vL[u]-uM[v]=U_x+V_y,$$

where $$U=\frac{2}{x+y}vu-uv_y, \quad ...(1.36)$$

and $$V=\frac{2}{x+y}vu+vu_x \quad ...(1.37)$$

It is given that along the curve $\Gamma: y=x$ that is along QR

$$u=o \text{ and } u_x=3x^2. \quad ...(1.38)$$

Also along QR the Riemann's function is given by

$$v = \frac{2x}{(\alpha+\beta)^3}\left[2x^2 + 2\alpha\beta\right]$$

$$\Rightarrow v = \frac{4x\left(x^2 + \alpha\beta\right)}{(\alpha+\beta)^3}. \qquad \text{... (1.39)}$$

We know the value of u at P in given by

$$[u]_P = [uv]_Q - \int_Q^R uv(ady - bdx) + \int_Q^R \left(uv_y dy + vu_x dx\right) + \iint_D vfdxdy,$$

where in this carve $f(x, y) = 0$. Using (1.38) and (1.39) we get

$$[u]_P = \int_Q^R vu_x dx,$$

$$= \int_Q^R \frac{4x\left(x^2 + \alpha\beta\right)}{(\alpha+\beta)^3} 3x^2 dx,$$

$$= \frac{12}{(\alpha+\beta)^3}\int_\alpha^\beta \left(x^5 + x^3\alpha\beta\right)dx,$$

$$= \frac{12}{(\alpha+\beta)^3}\left[\frac{x^6}{6} + \frac{x^4}{4}\alpha\beta\right]_\alpha^\beta,$$

$$= \frac{1}{(\alpha+\beta)^3}\left[2\left(\beta^6 - \alpha^6\right) + 3\alpha\beta\left(\beta^4 - \alpha^4\right)\right],$$

$$= \frac{1}{(\alpha+\beta)^3}\left[2\beta^6 - 2\alpha^6 + 3\alpha\beta^5 - 3\alpha^5\beta\right].$$

$$u(\alpha,\beta) = \frac{1}{(\alpha+\beta)^3}\left[2\beta^6 + 2\alpha\beta^5 + \alpha\beta^5 - 2\alpha^6 - 2\alpha^5\beta - \alpha^5\beta\right],$$

$$= \frac{1}{(\alpha+\beta)^3}\left[2\beta^5(\alpha+\beta) - 2\alpha^5(\alpha+\beta) + \alpha\beta\left(\beta^4 - \alpha^4\right)\right],$$

$$=\frac{1}{(\alpha+\beta)^3}\left[2(\alpha+\beta)\left(\beta^5-\alpha^5\right)+\alpha\beta\left(\beta^4-\alpha^4\right)\right].$$

$$=\frac{(\beta-\alpha)}{(\alpha+\beta)^3}2(\alpha+\beta)\left[2\beta^4 2\beta^3\alpha+2\beta^2\alpha^2+2\beta\alpha^3+2\alpha^4+\alpha\beta^3+\alpha^3\beta\right],$$

$$=\frac{(\beta-\alpha)}{(\alpha+\beta)^2}\left[2\beta^4+3\alpha\beta^3+3\alpha^3\beta+2\alpha^2\beta^2+2\alpha^4\right],$$

$$u(\alpha,\beta)=(\beta-\alpha)\left(2\beta^2-\alpha\beta+2\alpha^2\right),$$

$$u(\alpha,\beta)=2\beta^3-2\alpha\beta^2-\alpha\beta^2+\alpha^2\beta+2\alpha^2\beta-2\alpha^3,$$

$$u(\alpha,\beta)=2\beta^3-3\alpha\beta^2+3\alpha^2\beta-2\alpha^3.$$

Thus u at any point (x,y) is given by

$$u(x,y)=2y^3-3y^2x+3x^2y-2x^3.$$

Note : We see that the solution of the Cauchy problem at a point (α,β) depends only on the Cauchy data on the curve Γ . The knowledge of the Riemann's -Green function therefore enables us to solve the p.d.e with the carve data .

Harnack's Theorem :

Let us prove the following lemma first.

Lemma : Let D be a bounded domain, bounded by a smooth closed curve B. Let $\{u_n(x,y)\}$ be a sequence of functions each of which is continuous on $\overline{D}=DUB$ and harmonic in D. If $\{u_n(x,y)\}$ converges uniformly on B, then u_n converges uniformly on $\overline{D}$.

Proof : Let $\{u_n\}$ be a sequence of functions, converges uniformly on the boundary B. Then by definition, for given $\in>o$ we can always find N such that

$$|u_m(x,y)-u_n(x,y)|<\in \text{ on B} \quad \forall m,n>N.$$

Since each of $u_n(x,y)$ is harmonic in D.

$$\Rightarrow \nabla^2 u_n=0 \text{ in D},$$

and

$$\nabla^2 u_m=0 \text{ in D}.$$

$$\Rightarrow \nabla^2(u_m-u_n)=0 \quad \text{in D},$$

$\Rightarrow u_m - u_n$ is harmonic in D.

By the Maximum and minimum principle, we have

$$|u_m(x,y) - u_n(x,y)| < \in \quad \text{on } \overline{D} \quad \forall m,n > N(\in).$$

Hence the result.

Harnnack ' Theorem :

Let D be a bounded domain, bounded by a closed smooth curve B. Let $u_n(x,y)$ be a sequence of functions, each of which continuous on $\overline{D}$ and harmonic in D. If $u_n(x,y)$ converges uniformly on B, then u_n converges an $\overline{D}$ to a limit function which is continuous on $\overline{D}$ and harmonic in D.

Proof : Let $\{u_n\}$ be a sequence of functions , each of which continuous on $\overline{D} = DUB$ and harmonic in D.

$$\Rightarrow \nabla^2 u_n(x,y) = 0 \text{ in D} \qquad \text{... (1.40)}$$

Let the sequence $\{u_n\}$ converge uniformly on the boundary B.

$\Rightarrow$ for given $\in > 0$, we can find a number N such that

$$|u_m(x,y) - u_n(x,y)| < \in \text{ on B } \forall m,n > N \qquad \text{... (1.41)}$$

Since by (1.40) each of $u_n(x,y)$ is harmonic in D.

$$\Rightarrow (u_m - u_n) \text{ is harmonic in D.}$$

Then by maximum and minimum principle,

we have $$|u_m(x,y) - u_n(x,y)| < \in \text{ on } \overline{D} \ \forall \text{m,n} \in \text{N} \qquad \text{... (1.42)}$$

i.e. the sequence $\{u_n\}$ converges uniformly on $\overline{D}$. We know that "On a closed bounded set, a uniformly convergent sequence of continuous functions converges to a function which is continuous on that set ."

Since $u_n(x,y)$ converges uniformly on $\overline{D}$, let it converges to $u(x,y)$. Then $u(x,y)$ is also continuous on $\overline{D}$.

We now show that $u(x,y)$ is hammonic in D.

Let $(x,y) \in D$. Since D is open, therefore $\exists$ a circle with centre at (x,y) and radius 'a' which is contained in D, whose equations is

$$(X-x)^2+(Y-y)^2=a^2,$$

where , $X = x + a\cos\tau$, $Y = y + a\sin\tau$ is any point on the circle.

Let $$u_n(\tau) = u_n(x + a\cos\tau, y + a\sin\tau).$$

By equation (1.40), $u_n(x,y)$ is harmonic inside the circle and continuous on the circle, then we know $u_n(x,y)$ is given by Possion integral formula

$$u_n(\xi,n)=\frac{1}{2\pi}\int_0^{2\pi}\frac{(1-\rho^2)}{1-2\rho\cos(\theta-\tau)+\rho^2}u_n(\tau)d\tau\,.$$

We have $$\rho=\left(r/a\right),$$

and $$(\xi-x)^2+(n-y)^2=r^2<a^2.$$

Hence

$$u(\xi,n)=\lim_{x\to\infty}u_n(\xi,n) \qquad \text{As } u_n(x,y) \text{ converges to } u(x,y)$$

$$=\lim_{x\to\infty}\frac{1}{2\pi}\int_0^{2\pi}\frac{(1-\rho^2)}{1-2\rho\cos(\theta-\tau)+\rho^2}u_n(\tau)d\tau\,,$$

$$=\frac{1}{2\pi}\int_0^{2\pi}\frac{(1-\rho^2)}{1-2\rho\cos(\theta-\tau)+\rho^2}\lim_{x\to\infty}u_n(\tau)d\tau\,.$$

Since the sequence $u_n(x,y)$ converges uniformly to $u(x,y)$ therefore limit and the integral have been interchanged

$$u(\xi,n)=\frac{1}{2\pi}\int_0^{2\pi}\frac{(1-\rho^2)}{1-2\rho\cos(\theta-\tau)+\rho^2}u(\tau)d\tau$$

Hence u is harmonic in the region $(x-\xi)^2+(y-n)^2<a^2$ for all points (ξ,n) . Since (x,y) is an arbitrary point of D.

$\Rightarrow u$ is harmonic in D.

This proves the theorem.

Exercise:

1. Show that

$$v(x, y; \alpha, \beta) = \frac{x^2 + \alpha^2 - (y-\beta)^2}{2x^2}$$

is the Green's function for the second order partial differential equation

$$u_{xx} - u_{yy} - \frac{2}{x}u_x = 0 .$$

□□□

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