SHIVAJI UNIVERSITY, KOLHAPUR

CENTRE FOR DISTANCE EDUCATION

Real Analysis

(Mathematics)

For

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Paper (MT 301)

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UNIT - I

OPEN SETS, CLOSED SETS AND BOREL SETS

Let $\mathbb{R}$ be the set of real numbers, $\mathbb{Z}$ be the set of integers and $\mathbb{Q}$ denotes the set of rational numbers.

We introduce the concepts of open sets, closed sets and Borel sets in $\mathbb{R}$.

1.1 Open Sets and Closed Sets

1. Definition : Open Set

A set O of real numbers is called open if for every $x \in O$, there exists a real number $r > 0$ such that the internal $(x-r, x+r) \subseteq O$.

2. Note :

(1) For $a < b$, the open interval (a, b) is an open set. Because for any $x \in (a,b)$ choose $r = \min\{b-x, x-a\}$. Then the interval $(x-r, x+r) \subseteq (a,b)$. Also the open interval (a, b) is a bounded open interval.

$$r = \min\{b-x, x-a\} = x-a \text{ and } (x-r, x+r) \subseteq (a,b)$$

(2) For any $a, b \in \mathbb{R}$ we have,

$(a, \infty) = \{x \in \mathbb{R} \mid a < x\}$

$(-\infty, b) = \{x \in \mathbb{R} \mid x < b\}$

$(-\infty, \infty) = \mathbb{R}$

Note that all these sets are open intervals but unbounded. And any unbounded open interval is of the above form.

(3) $\mathbb{R}$ and the empty set f are open.

3. Proposition :

The intersection of any finite collection of open sets is open and the union of any collection of open sets is open .

Proof : Let $\{O_k\}_{k\in I}$ be the collection of open sets where I is an index set. Then for any $x\in\bigcup_{k\in I}O_k$, there exists at least one k for which $x\in O_k$. Since O_k is an open set there exist a real number $r>0$ such that,

$x\in(x-r,x+r)\subseteq O_k\subseteq\bigcup_{k\in I}O_k$. Hence $\bigcup_{k\in I}O_k$ is open.

Next let $\{O_k\}_{k=1}^{n}$ be any finite collection of open sets. If $\bigcap_{k=1}^{n}O_k$ is empty then clearly it is open.

If $\bigcap_{k=1}^{n}O_k$ is non-empty then for any $x\in\bigcap_{k=1}^{n}O_k\Rightarrow x\in O_k$ for $1\le k\le n$

$\Rightarrow$ there exists $r_k>0$ such that $(x-r_k,x+r_k)\subseteq O_K$, $1\le k\le n$

Let $r=\min\{r_1,r_2,.....,r_n\}$. Then $r>0$ and $(x-r,x+r)\subseteq O_k$ for all k, $1\le k\le n$.

Hence $(x-r,x+r)\subseteq\bigcap_{k=1}^{n}O_k$. Therefore $\bigcap_{k=1}^{n}O_k$ is open.

4. Note :

Intersection of any collection of open sets need not be open. For, let $O_n=\left(-\frac{1}{n},\frac{1}{n}\right)n\in\mathbb{N}$ be the open intervals. Then $\bigcap_{n=1}^{\infty}O_n=\{0\}$ which is not open.

5. Proposition :

Every non-empty open set is the disjoint union of a countable collection of open intervals.

Proof : Let O be the non-empty open subset of $\mathbb{R}$. Let $x\in O$ be arbitary. Then there exists $r>0$ such that $(x-r,x+r)\subseteq O$.

Therefore there exists $y>x$ for which $(x,y)\subseteq O$ and $z<x$ such that $(z,x)\subseteq O$. Define the extended real numbers a_x and b_x by

$$a_x=\inf\{z\,|\,(z,x)\subseteq O\},\quad b_x=\sup\{y\,|\,(x,y)\subseteq O\}$$

Then $I_x=(a_x,b_x)$ is an open interval containing x. Further if $a_x<x<w<b_x$ then there exist y such that $x<w<y$ and $(x,y)\subseteq O$, which implies $w\in(x,y)\subseteq O$. Therefore $w\in O$.

Thus $w \in I_x \Rightarrow w \in O$. Hence $I_x \subseteq O$

Next if $b_x \in O$ then there is a real number $r > 0$ such that $(b_x - r, b_x + r) \subseteq O$ and hence $(x, b_x + r) \subseteq O$ which contradicts to the fact that b_x is the supremum of all the elements y such that $(x, y) \subseteq O$. Hence $b_x \notin O$. Similarly $a_x \notin O$.

Next, consider a collection of open intervals $\{I_x\}, x \in O$. For any $x \in O \Rightarrow x \in I_x \subseteq \bigcup_{x \in O} I_x$.

Therefore $O \subseteq \bigcup_{x \in O} I_x$. On the other hand for each $x \in O$, $I_x \subseteq O$ and hence $\bigcup_{x \in O} I_x \subseteq O$. Therefore $O = \bigcup_{x \in O} I_x$. Further for any $x, y \in O$, if $I_x \cap I_y \neq f$ then there is at least one element say $z \in I_x \cap I_y$.

$\Rightarrow a_x < z < b_x$ and $a_y < z < b_y$

$\Rightarrow$ either $a_x \leq a_y < z < b_x \leq b_y$ or $a_x \leq a_y < z < b_y \leq b_x$

$\Rightarrow a_x = a_y$ and $b_x = b_y$ (by the definitions of a_x and b_x)

$\Rightarrow I_x = I_y$

Hence any two sets in $\{I_x\}_{x \in O}$ are either disjoint or equal. Thus $\{I_x\}_{x \in O}$ is a disjoint family of open intervals such that $O = \bigcup_{x \in O} I_x$.

Finally we show that the collection of open intervals $\{I_x\}_{x \in O}$ is countable.

Since each interval of $\mathbb{R}$ contains countably infinite rational numbers, and rational numbers are countably infinite, we conclude that the union $\bigcup_{x \in O} I_x$ is a countable union. (If this union is not countable then we have uncountable union of countable sets of rational numbers which is uncountable set, but set of rational numbers is countable).

Therefore O is the union of countable, disjoint collection of open intervals.

6. Definition :

Let E be any set of real numbers. A real number x is called closure point of E. If every open interval containing x contains a point in E. The collection of all closure points of E is called a closure of E and it is denoted by $\overline{E}$.

For example, if $E = (0,1)$ then $\overline{E} = [0,1]$. Clearly $E \subseteq \overline{E}$ for any set E.

7. Definition :

A set E of real numbers is called a closed set if $E = \overline{E}$.

8. Proposition :

For any set E of real numbers, its closure $\overline{E}$ is closed. Moreover, $\overline{E}$ is the smallest closed set that contains E.

Proof : Let E be any set of real numbers and let $\overline{E}$ be its closure. We prove that $\overline{\overline{E}} = \overline{E}$.

Let x be a closure point of $\overline{E}$. Consider an open interval I_x which contains x. Then I_x contains a point of $\overline{E}$. Let x' be the point such that $x' \in I_x \cap \overline{E}$. Further $x' \in I_x$ and $x' \in \overline{E}$ i.e. x' is a closure point of E and I_x is an open interval containing x'. Therefore there exist a point $x'' \in E \cap I_x$ which shows that every open interval I_x containing the point x also contains a point of E. Hence $x \in \overline{E}$.

Therefore $\overline{E}$ contains all its closure points and hence $\overline{E}$ is closed. i.e. $\overline{\overline{E}} = \overline{E}$

Next if F is any closed set containing E then

$$E \subseteq F \Rightarrow \overline{E} \subseteq \overline{F} = F \Rightarrow \overline{E} \subseteq F$$

Which shows that $\overline{E}$ is the smallest closed set containing E.

9. Proposition

Any set of real numbers is open if and only if its complement in $\mathbb{R}$ is closed.

Proof : Let E be any open subset of $\mathbb{R}$. We show that its complement $\mathbb{R} - E$ is closed.

Consider a closure point x of $\mathbb{R} - E$. Then every open interval containing x also contains a point of $\mathbb{R} - E$. Now if $x \in E$, E is an open set, then there exists an open interval say I_x which contains x and $x \in I_x \subseteq E$. But then I_x is an open interval containing x and contains no point of $\mathbb{R} - E$. Which is a contradiction. Hence $x \notin E$ i.e. $x \in \mathbb{R} - E$.

Thus $\mathbb{R} - E$ contains all its closure points and hence $\mathbb{R} - E$ is closed.

Conversely suppose $\mathbb{R} - E$ is closed. Let $x \in E$ be any point. If every open interval containing x contains a point of $\mathbb{R} - E$, then x is a closure point of $\mathbb{R} - E$. And since $\mathbb{R} - E$ is closed we have $x \in \mathbb{R} - E$. i.e. $x \notin E$ which is contradiction. Hence there exists an open I_x interval containing x which is disjoint from $\mathbb{R} - E$ i.e. $I_x \cap (\mathbb{R} - E) = f$. Hence $I_x \subseteq E$. Thus for any $x \in E$ there exists an open interval I_x such that $x \in I_x \subseteq E$. Which shows that E is open.

10. Note :

(1) Since $E = (E^c)^c$ then the above proposition also states that - A set is closed if and only if its complement is open.

(2) Since $\mathbb{R}^c = f$ and $f^c = R$, and we know that both f and $\mathbb{R}$ are open, the above proposition indicates that both f and $\mathbb{R}$ are also closed.

(3) The union of finite collection of closed sets is closed and the intersection of any collection of closed sets is closed.

1.2 Heine Borel Theorem

1. Definition :

A collection $\{E_i\}_{i\in I}$ is said to be cover of a set E if $E \subseteq \bigcup_{i\in I} E_i$. A sub-collection of the cover that itself also is a cover of E is called a subcover of E. If each set E_i in a cover is open we say that $\{E_i\}_{i\in I}$ is an open cover of E. If the cover $\{E_i\}_{i\in I}$ contains finite number of sets then we call it as a finite cover.

2. Heine-Borel Theorem :

Let F be a closed and bounded set of real numbers. Then every open cover of F has a finite subcover.

Proof : First we consider the case that F is closed and bounded interval i.e. $F = [a, b]$, $a < b$. Let $\mathcal{F}$ be an open cover of [a, b]. Let E be the set defined by

$$E = \{x \in [a,b] \mid [a,x] \text{ can be covered by finite number of sets in } \mathcal{F}\}$$

Then clearly $a \in E$, since $[a,a] = \{a\}$ is covered by finite number of sets in $\mathcal{F}$ (i.e. only one set in $\mathcal{F}$ containing a). Thus $E \neq f$. Since $E \subseteq [a,b]$ it is bounded above by b. Therefore E has a supremum or least upper bound. Let $c = \sup E$. Now $c \in E$ and $\mathcal{F}$ is an open cover of E, there exist an open set $O \in \mathcal{F}$ such that $c \in O$. Therefore there exists $\in > 0$ such that the interval $(c-\in, c+\in) \subseteq O$.

Now $c-\in$ is not supremum of E. Therefore there exist $x > c-\in$ such that $x \in E$. By definition of E, the interval $[a,x]$ is covered by finite number of sets $\{O_1, O_2, ..., O_k\}$ in $\mathcal{F}$. Hence the finite collection $\{O_1, O_2, ..., O_k, O\}$ in $\mathcal{F}$ covers the interval $[a, c+\in)$ i.e. there exist y such that $c < y < c+\in$ and the interval $[a, y]$ is covered by finite number of sets in $\mathcal{F}$, which is a contradiction. since c is the supremum of E such that $[a,c]$ is covered by finite number of sets in $\mathcal{F}$. Thus $c = b$ and $[a,b]$ is covered by finite number of sets from $\mathcal{F}$.

Now, if F is any closed and bounded set and $\mathcal{F}$ is an oepn cover of F, then F contained in some closed and bounded interval $[a,b]$.

Now F is closed set, therefore its complement $R \sim F$ is an open set. Let $O = R \sim F$. Let $\mathcal{F}^*$ be a collection of open sets obtained by adding O to $\mathcal{F}$. i.e. $\mathcal{F}^* = \mathcal{F} \cup \{O\}$. Since $\mathcal{F}$ covers F and O covers complement of F, $\mathcal{F} \cup \{O\}$ covers $[a,b]$. i.e. $\mathcal{F}^*$ is an open cover of $[a,b]$. And by above case $\mathcal{F}^*$ has a finite subcollection of sets which also covers $[a,b]$. If O belongs to this finite subcover of $[a,b]$, then by removing O we get a finite subcover of F which is a subcollection of sets in $\mathcal{F}$. Thus if F is closed and bounded set then there is a finite subcover of set in $\mathcal{F}$.

3. Definition :

A countable collection of sets $\{E_n\}_{n=1}^{\infty}$ is descending or nested provided $E_{n+1} \subseteq E_n$ for all $n \in N$. The collection of sets $\{E_n\}_{n=1}^{\infty}$ is said to be ascending if $E_n \subseteq E_{n+1}$ for all $n \in N$.

4. The Nested Set Theorem :

Let $\{F_n\}_{n=1}^{\infty}$ be a descending countable collection of nonempty closed sets of real numbers for which F_1 is bounded. Then $\bigcap_{n=1}^{\infty} F_n \neq f$.

Proof : Wr prove this theorem by contradiction. Suppose that $\bigcap_{n=1}^{\infty} F_n = f$. Then for any real number x, if $x \in F_n$ for all $n \in \mathbb{N}$ then $x \in \bigcap_{n=1}^{\infty} F_n$ which is not true. Hence there exist a natural number n such that $x \notin F_n$ i.e. $x \in \mathbb{R} - F_n$. Let $\mathbb{R} - F_n = O_n$. Since F_n is closed, O_n is open. Thus for every $x \in \mathbb{R}$ there exist an open set O_n such that $x \in O_n$. Therefore $\mathbb{R} = \bigcup_n O_n$. Further each $F_n \subseteq \mathbb{R}$ for all $n \in \mathbb{N}$ and hence $F_1 \subseteq \mathbb{R}$. Therefore $\{O_n\}_{n=1}^{\infty}$ is an open cover of F_1. The Heine-Borel theorem tells us that there is a natural number k for which $F \subseteq \bigcup_{n=1}^{k} O_n$.

Next $\{F_n\}_{n=1}^{\infty}$ is descending, the sequence of open sets $\{O_n\}_{n=1}^{\infty}$ is ascending, because $O_n = \mathbb{R} - F_n$, $n \in \mathbb{N}$. Hence $\bigcup_{n=1}^{k} O_n = O_k = \mathbb{R} - F_k$. Now $F_k \subseteq F_1$ and $F_1 \subseteq \bigcup_{n=1}^{k} O_n = \mathbb{R} - F_k$

$\Rightarrow F_k = \phi$. Which is a contradiction since F_n 's are nonempty closed sets. Hence $\bigcap_{n=1}^{\infty} F_n \neq \phi$.

1.3 The σ - algebra

1. Definition :

Let X be any set. A collection of $\mathcal{A}$ of subsets of X is called a σ -algebra of subsets of X if

(i) $\phi, X \in \mathcal{A}$

(ii) $A \in \mathcal{A} \Rightarrow X - A \in \mathcal{A}$

(iii) The union of countable collection of sets in $\mathcal{A}$ also belongs to $\mathcal{A}$.

2. Note :

(1) De Morgans Laws implies that the σ -algebra $\mathcal{A}$ is also closed under countable intersection.

(2) The σ -algebra $\mathcal{A}$ is closed w.r.t. the relative complement i.e. $A_1, A_2 \in \mathcal{A} \Rightarrow A_1 - A_2 \in \mathcal{A}$.

3. Examples :

(1) For any set X, $(X \neq \phi)$ the collection $\{\phi, X\}$ is a σ -algebra and it is contained in every σ -algebra of subsets of X.

(2) For any non-empty set X the collection all subsets of X, called as power set of X, is a σ -algebra which contains every σ -algebra of subsets of X. It is denoted by 2^X (or $\mathcal{P}(x)$).

4. Proposition :

Let $\mathcal{F}$ be a collection of subsets of a set X. Then the intersection $\mathcal{A}$ of all σ -algebras of subsets of X that contains $\mathcal{F}$ is a σ -algebra containing $\mathcal{F}$. Moreover it is the smallest σ -algebra of subsets of X containing $\mathcal{F}$ in the sense that any σ -algebra that contains $\mathcal{F}$ also contains $\mathcal{A}$.

Proof :

Let $\{\mathcal{B}_i\}_{i \in I}$ be a collection of σ -algebras of subsets of X such that $\mathcal{F} \subseteq \mathcal{B}_i$, $\forall i \in I$.

Let $\mathcal{A} = \bigcap_i \mathcal{B}_i$. Since $f, X \in \mathcal{B}_i \forall i$, $f, X \in \bigcap_i \mathcal{B}_i \Rightarrow f, X \in \mathcal{A}$

Next, $A \in \mathcal{A} \Rightarrow A \in \bigcap_i \mathcal{B}_i$

$\Rightarrow A \in \mathcal{B}_i$ for all $i \in I$

$\Rightarrow X - A \in \mathcal{B}_i$ for all $i \in I$. Since $\mathcal{B}_i$'s are s -algebras for all *i*.

$\Rightarrow X - A \in \bigcap_i \mathcal{B}_i$

$\Rightarrow X - A \in \mathcal{A}$

Finally if $\{A_k\}$ is a countable collection of sets in $\mathcal{A}$ then, $\{A_k\} \subseteq \mathcal{A} \Rightarrow \{A_k\} \subseteq \bigcap_i \mathcal{B}_i$.

$\Rightarrow \{A_k\} \subseteq \mathcal{B}_i, \forall i \in I \Rightarrow \bigcup_k A_k \in \mathcal{B}_i, \forall i \in I$, since $\mathcal{B}_i$ is a s -algebra.

Hence $\bigcup_k A_k \in \bigcap_i \mathcal{B}_i = \mathcal{A}$. Which proves that $\mathcal{A}$ is a s -algebra.

Also, $\mathcal{F} \subseteq \mathcal{B}_i \quad \forall i \in I \quad \Rightarrow \mathcal{F} \subseteq \bigcap_i \mathcal{B}_i = \mathcal{A}$

Hence, $\mathcal{A}$ is a s -algebra containing $\mathcal{F}$. Now if $\mathcal{C}$ is any s -algebra containing $\mathcal{F}$ then $\mathcal{C} \in \{\mathcal{B}_i\}$. Therefore $\bigcap_i \mathcal{B}_i \subseteq \mathcal{C}$ i.e. $\mathcal{A} \subseteq \mathcal{C}$. This shows that $\mathcal{A}$ is the smallest s -algebra containing $\mathcal{F}$.

5. Definition :

The collection $\mathcal{B}$ of Borel sets of real numbers is the smallest s -algebra of sets of real numbers which contains all of the open sets of real numbers.

6. Note :

Every open set is contained in $\mathcal{B}$. Since $\mathcal{B}$ is closed under complement, and complement of an open set is closed set we infer that all closed sets are Borel sets. Each singleton set is closed and hence it is a Borel set. Since $\mathcal{B}$ is closed under countable union, every countable set is a Borel set.

7. Definition :

A countable intersection of open sets is called G_d set and a countable union of closed set is called F_s set.

G_d and F_s sets are Borel sets.

8. Note :

If $A \in G_d$ then $A = \bigcap_i G_i$, G_i 's are open sets, $\forall i$.

Similalry $B \in F_s$ then $B = \bigcup_i F_i$, F_i 's are closed sets $\forall i$.

Similarly we can construct the families $G_{d6}, G_{d6d}, \ldots$ and $F_{sd}, F_{sds}, \ldots$. All members of these families are Borel sets.

Thus we have following examples of F_s sets.

1. Every closed set is F_s set.
2. Countable sets are F_s sets. (Since these are countable union of singletons which are closed sets.)
3. Open intervals are F_s sets.
4. Countable union of F_s sets is F_s set.

Following are some of the examples of G_d sets.

1. Every open set is G_d set.
2. Every closed interval is G_d set.
3. Countable intersection of G_d sets is G_d set.

Complement of F_s set is G_d set and conversely.

9. Note :

Countable union of closed sets need not be closed and countable intersection of open sets need not be open for,

$$\bigcup_{n=1}^{\infty}\left[a+\frac{1}{n}, b-\frac{1}{n}\right]=(a,b) \qquad \text{and} \qquad \bigcap_{n=1}^{\infty}\left(a-\frac{1}{n}, b+\frac{1}{n}\right)=[a,b]$$

UNIT - II

LEBESGUE MEASURE

Introduction :

Measure theory is the study of special type of set functions initiated by a French Mathematician Henri Lebesgue (1875-1941). It helps in studying problems in Probability theory, Partial differential equations, Hydrodynamics and Quantum Mechanics.

The concept of length of an interval is generalized to define measure of a set of real numbers. Length of a finite interval I is defined as ℓ (I) = b – a where a and b are end points of the interval (a < b), irrespective of whether I is closed, open, open-closed or closed-open. Thus length is a set function defined on a set of intervals. We want to extend the notion of length to any set of real numbers. Therefore we would like to construct a set function m which assigns to each set E a nonnegative extended real number m(E) called a measure of E. Such a set function m should have the following properties.

1. m(E) is defined for any subset of $\mathbb{R}$ i.e. $E \in \mathcal{P}(\mathbb{R})$
2. For an internal I, m(E) = ℓ (I)
3. For a disjoint sequence $\{E_n\}$ of subsets of $\mathbb{R}$ $m\left(\bigcup_n E_n\right) = \sum_n m(E_n)$
4. m is translation invariant i.e. m (E + y) = m(E)

Unfortunately such a set function m satisfying (1) to (4) doesn't exist. Hence we restrict the set P(IR) to a σ-algebra of measurable sets. We first introduce outer measure of a set.

2.1 Lebesgue Outer Measure :

1. Definition : For any set A of real numbers, consider a sequence of non empty open bounded intervals $\{I_k\}_{k=1}^{\infty}$ such that $A \subseteq \bigcup_{k=1}^{\infty} I_k$. We define Lebesgue outer measure of A by,

$$m^*(A) = \inf\left\{\sum_{k=1}^{\infty} \ell(I_k) \mid A \subseteq \bigcup_{k=1}^{\infty} I_k\right\}$$

where $\ell(I_k)$ is the length of the open interval I_k.

2. Properties of m* :

1. $m^* : \mathcal{P}(\mathbb{R}) \to \mathbb{R}^+ U \{\infty\}$

Thus m* is a set function from $\mathcal{P}$ ($\mathbb{R}$) to a nonnegative extended system of real numbers.

2. $m^*(A) \geq 0$ for any $A \in \mathcal{P}(\mathbb{R})$

3. $m^*(\phi) = 0$. For, $\phi \subseteq \left(-\frac{1}{n}, \frac{1}{n}\right)$ for all $n \in \mathbb{N}$ implies,

$$m^*(\phi) = \inf\left\{\ell\left(-\frac{1}{n}, \frac{1}{n}\right) \mid \phi \subseteq \left(-\frac{1}{n}, \frac{1}{n}\right), n \in \mathbb{N}\right\}$$

$$= \inf\left\{\frac{2}{n} \mid n \in \mathbb{N}\right\}$$

$$= 0$$

4. If A is Singleton set then m*(A) = 0

Proof : Let A = {x} then $A \subseteq \left(x - \frac{1}{n}, x + \frac{1}{n}\right)$, $n \in \mathbb{N}$

Therefore, $$m^*(A) = \inf\left\{\ell\left(x - \frac{1}{n}, x + \frac{1}{n}\right) \mid n \in \mathbb{N}\right\}$$

$$= \inf\left\{\frac{2}{n} \mid x \in \mathbb{N}\right\}$$

$$= 0$$

5. m* is monotone i.e. $A \subseteq B \Rightarrow m^*(A) \leq m^*(B)$

Proof : $A \subseteq B$ Then for any sequence $\{J_n\}$ of open intervals such that $B \subseteq UJ_n$ implies $A \subseteq UJ_n$.

Hence,

$$\left\{\sum_n \ell(J_n) \mid B \subseteq \bigcup_n J_n\right\} \subseteq \left\{\sum_n \ell(I_n) \mid A \subseteq \bigcup_n I_n\right\}$$

Taking infimum of both sides,

$$\inf\left\{\sum_n \ell(J_n) \mid B \subseteq \bigcup_n J_n\right\} \geq \inf\left\{\sum_n \ell(I_n) \mid A \subseteq \bigcup_n I_n\right\}$$

$$\Rightarrow \quad m^*(B) \geq m^*(A) \quad \text{or} \quad m^*(A) \leq m^*(B)$$

3. **Proposition :** The outer measure of an interval is its length.

Proof : Let I be any interval.

Case I : I is closed and finite interval.

Let I = [a, b], a < b

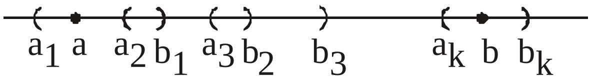
Then for given $\in > 0$, $[a,b] \subseteq (a-\in, b+\in)$

Hence, $m^*[a,b] \le \ell(a-\in, b+\in) = (b+\in)-(a-\in) = b-a+2\in$

Since $\in > 0$ is arbitrary, we have, $m^*[a,b] \le b-a$ ... (i)

Next, consider a countable collection of open intervals such that $[a,b] \subseteq \bigcup_n I_n$. Since [a, b] is closed and bounded set, by Heine Borel theorem, there exist a finite subcover of [a, b].

Now $[a,b] \subseteq \bigcup_n I_n$ there exist an interval $I_1 = (a_1, b_1)$ such that $a \in I_1$, and $a_1 < a < b_1$. If $b_1 < b$ then there exist an interval $I_2 = (a_2, b_2)$ such that $a_2 < b_1 < b_2$.



Continuing in this way we obtain a sequence of open intervals, (a_1, b_1), (a_2, b_2), (a_3, b_3), ... (a_k, b_k) from $\{I_n\}$ such that, $a_i < b_{i-1} < b_i$ $\forall$ i. Since [a, b] is covered by finite number of open intervals, this process must terminates finitely with some interval (a_k, b_k) with $a_k < b < b_k$. Therefore, we get,

$$\sum_n \ell(I_n) \ge \sum_{i=1}^{k} \ell(a_i, b_i)$$

$$= \ell(a_1,b_1) + \ell(a_2,b_2) + \ldots + \ell(a_k,b_k)$$

$$= (b_1 - a_1) + (b_2 - a_2) + \ldots + (b_k - a_k)$$

$$= -a_1 + (b_1 - a_2) + (b_2 - a_3) + \ldots + (b_{k-1} - a_k) + b_k$$

But $a_2 < b_1 < b_2 \Rightarrow b_1 - a_2 > 0$

$a_3 < b_2 < b_3 \Rightarrow b_2 - a_3 > 0$, ..., $b_{k-1} - a_k > 0$

Hence, removing these positive terms from the r.h.s., we get,

$$\sum_n \ell(I_n) \ge -a_1 + b_k$$

Further, $a \in (a_1, b_1) \Rightarrow a_1 < a \Rightarrow -a < -a_1$

Similarly $b \in (a_k, b_k) \Rightarrow b < b_k$. Hence $b - a < b_k - a_1$

Thus we get,

$$\sum_n \ell(I_n) > b - a$$

Taking infimum over all such open covers $\{I_n\}$ of [a, b] we get,

$$\inf\left\{\sum_n \ell(I_n) \mid [a,b] \subseteq \bigcup_n I_n\right\} \geq b - a$$

$$\Rightarrow m^*([a,b]) \geq b - a \qquad \text{... (ii)}$$

From (i) and (ii) we get, $m^*[a, b] = b - a$

Thus for any closed, finite interval I, $m^*(I) = \ell(I)$

Case II : Let I be any finite interval. Then for given $\in > 0$ there exist a closed interval J such that $J \subseteq I$ and $\ell(J) > \ell(I) - \in$.

Therefore we get,

$$\ell(I) - \in < \ell(J) = m^*(J) \leq m^*(I) \qquad \text{... (iii)}$$

Further if $\overline{I}$ is closure of I then $I \subseteq \overline{I}$ clearly $\overline{I}$ is a closed set.

Therefore we get,

$$m^*(I) \leq m^*(\overline{I}) = \ell(\overline{I}) = \ell(I) \qquad \text{... (iv)}$$

From (iii) and (iv) we get,

$$\ell(I) - \in \leq m^*(I) \leq \ell(I)$$

Since, $\in > 0$ is small arbitrary we get,

$m^*(I) = \ell(I)$, where I is any finite interval.

Case III : I is any infinite interval. Since I is infinite interval, for any natural number n, there is a closed interval J such that $J \subseteq I$ and $\ell(J) = n$.

Then,

$$J \subseteq I \Rightarrow m^*(J) \leq m^*(I)$$

$$\Rightarrow \ell(J) \leq m^*(I)$$

$$\Rightarrow n \leq m^*(I) \text{ or } m^*(I) \geq n$$

Thus for any natural number n, $m^*(I) \geq n$.

Hence $m^*(I) = \infty = \ell(I)$

Therefore for any interval I, $m^*(I) = \ell(I)$

Note : The above proposition also asserts that the outer measure m* is a generalization of the length function defined on set of intervals.

4. Proposition : Outer measure is translation invariant. i.e. for any set A and for any real number y,

$$m^*(A+y) = m^*(A).$$

Proof : Let A be any subset of $\mathbb{R}$. If there is a countable collection of open intervals $\{I_k\}_{k=1}^{\infty}$ such that

$A \subseteq \bigcup_{k=1}^{\infty} I_k$, then

$$A \subseteq \bigcup_{k=1}^{\infty} I_k \Leftrightarrow A + y \subseteq \left(\bigcup_{k=1}^{\infty} I_k\right) + y$$

$$\Leftrightarrow A + y \subseteq \bigcup_{k=1}^{\infty} (I_k + y)$$

Also $\ell(I_k) = \ell(I_k + y)$ for all k = 1, 2, 3, ...

Therefore, $$m^*(A) = \inf\left\{\sum_{k=1}^{\infty} \ell(I_k) \mid A \subseteq \bigcup_{k=1}^{\infty} I_k\right\}$$

$$= \inf\left\{\sum_{k=1}^{\infty} \ell(I_k) \mid A + y \subseteq \bigcup_{k=1}^{\infty} (I_k + y)\right\}$$

$$= \inf\left\{\sum_{k=1}^{\infty} \ell(I_k + y) \mid A + y \subseteq \bigcup_{k=1}^{\infty} (I_k + y)\right\}$$

$$= \inf\left\{\sum_{k=1}^{\infty} \ell(J_k) \mid A + y \subseteq \bigcup_{k=1}^{\infty} J_k\right\}$$

$$= m^*(A+y)$$

5. Proposition : Let $\{E_k\}$ be a countable collection of sets of real numbers. (not necessarily disjoint), then

$$m^*\left(\bigcup_k E_k\right) \leq \sum_k m^*(E_k)$$

Proof : If $m^*(E_k) = \infty$ for some k then the inequality holds trivially. Therefore we assume that

If $m^*(E_k) < \infty$ for all k. Then for given $\in > 0$ there exist a countable collection of open intervals $\{I_{k,i}\}_{i=1}^{\infty}$ such that $E_k \subseteq \bigcup_i I_{k,i}$ and,

$$m^*(E_k) + \frac{\in}{2^k} > \sum_{i=1}^{\infty} \ell(I_{k,i}), \qquad \forall \ k = 1,2,...... \qquad \text{... (i)}$$

Since countable union of countable sets is again countable, $\{I_{k,i}\}_{i=1,k=1}^{\infty \ \ \infty}$ is also a countable collection of open intervals such that,

$$\bigcup_{k=1}^{\infty} E_k \subseteq \bigcup_{k=1}^{\infty} \bigcup_{i=1}^{\infty} I_{k,i}$$

i.e. $$\bigcup_{k=1}^{\infty} E_k \subseteq \bigcup_{k=1,i=1}^{\infty} I_{k,i}$$

Thus $\{I_{k,i}\}_{i=1,k=1}^{\infty \ \ \infty}$ is an open cover of $\bigcup_{k=1}^{\infty} E_k$ and hence,

$$m^*\left(\bigcup_{k=1}^{\infty} E_k\right) \le \sum_{i=1,k=1}^{\infty} \ell(I_{k,i})$$

$$= \sum_{k=1}^{\infty} \sum_{i=1}^{\infty} \ell(I_{k,i})$$

$$\le \sum_{k=1}^{\infty} \left(m^*(E_k) + \frac{\in}{2^k} \right) \qquad \text{... (from (i))}$$

$$= \sum_{k=1}^{\infty} m^*(E_k) + \in \sum_{k=1}^{\infty} \frac{1}{2^k}$$

$$= \sum_{k=1}^{\infty} m^*(E_k) + \in$$

Thus, $$m^*\left(\bigcup_{k=1}^{\infty} E_k\right) \le \sum_{k=1}^{\infty} m^*(E_k) + \in$$

Since $\in > 0$ is arbitrary, we get,

$$m^*\left(\bigcup_{k=1}^{\infty} E_k\right) \le \sum_{k=1}^{\infty} m^*(E_k)$$

6. **Note :** The above proposition says that the outer measure m* is countably subadditive.

7. **Corollary :** If A is countable set then m*(A) =0.

Proof : A is countable.

$\Rightarrow$ A = {a_1, a_2, a_3, ... }

$$= \bigcup_{i=1}^{\infty} \{a_i\} = \bigcup_{i=1}^{\infty} A_i \qquad \text{where } A_i = \{a_i\}$$

Therefore, $$m^*(A) = m^*\left(\bigcup_{i=1}^{\infty} A_i\right) \le \sum_{i=1}^{\infty} m^*(A_i) = 0$$

Since A_i's are Singleton sets. $m^*(A_i) = 0$ for all i = 1, 2, 3, ...

Hence, m*(A) = 0

8. **Note :** The set of natural numbers $\mathbb{N}$, the set of integers $\mathbb{Z}$, the set of rational numbers $\mathbb{Q}$ are all countable sets. Hence $m^*(\mathbb{N}) = 0$, $m^*(\mathbb{Z}) = 0$, $m^*(\mathbb{Q}) = 0$.

Any finite set is a countable set hence its outer measure is zero.

9. ***Example :*** Prove that an interval [0, 1] is not countable.

Solution : $m^*([0,1]) = 1 \ne 0$ hence [0, 1] is not countable i.e. [0, 1] is an uncountable set.

Any interval is not countable, since it's outermeasure is not zero.

10. **Example :** Let A be a set of irrational numbers in the interval [0, 1]. Prove that $m^*(A) = 1$.

Solution : Let B be the set of rational numbers in the interval [0, 1]. Then $A \cup B = [0,1]$. Therefore by subadditive property of m^*,

$$m^*[0,1] = m^*(A \cup B) \le m^*(A) + m^*(B)$$

Since B is countable, $m^*(B) = 0$. Also $m^*[0,1] = 1$. Therefore $1 \le m^*(A)$. Also $A \subseteq [0,1]$ implies $m^*(A) \le m^*[0,1] = 1$. Hence $m^*(A) = 1$.

11. **Note :** Outer measure of a countable set is zero. But the converse need not be true i.e. m*(A) = 0 does not imply A is countable. We have the following example.

12. ***Example :*** Cantor's set C is an uncountable set with outer measure zero.

Consider a unit interval [0, 1]

Step 1 : Remove the middle $\left(\frac{1}{3}\right)$ rd part $\left(\frac{1}{3},\frac{2}{3}\right)$

Length of the removed part = $\frac{1}{3}$

Number of intervals remained = 2, $\left[0,\frac{1}{3}\right]$, $\left[\frac{2}{3},1\right]$

Length of each interval present = $\frac{1}{3}$

Step 2 : Remove the middle $\left(\frac{1}{3}\right)$rd of the intervals present in the step 1

Length of the removed part = $\frac{2}{9}=\frac{2}{3^2}$

Number of intervals remained = 4 $= 2^2$

Length of each interval present = $\frac{1}{9}=\frac{1}{3^2}$

At the step n we have,

Length of the removed part = $\frac{2^{n-1}}{3^n}$

Number of intervals remained = 2^n

Length of each interval present = $\frac{1}{3^n}$

Let C_n denotes the union of intervals left at the n^{th} step. Then $C_n=\bigcup_{k=1}^{2^n} I_k$ and $\ell(I_k)=\frac{1}{3^n}$.

The Cantor set C is defined as $C=\bigcap_n C_n$.

Therefore $C \subseteq C_n$ for all $n \in \mathbb{N}$

Hence, $m^*(C) \le m^*(C_n)$

$$= m^*\left(\bigcup_{k=1}^{2^n} I_k\right)$$

$$\le \sum_{k=1}^{2^n} m^*(I_k)$$

$$= \sum_{k=1}^{2^n} \ell(I_k)$$

$$m^*(C) \le \sum_{k=1}^{2^n} \frac{1}{3^n}$$

$$= \frac{1}{3^n}\sum_{k=1}^{2^n} = \frac{2^n}{3^n} = \left(\frac{2}{3}\right)^n$$

i.e. $\quad m^*(C) \le \left(\frac{2}{3}\right)^n \quad$ for all $n \in \mathbb{N}$

But as $n \to \infty$, $\left(\frac{2}{3}\right)^n \to 0$ Hence we must have,

m*(C) = 0

But Cantor's set is uncountable and we have proved that its outer measure is zero.

13. ***Example :*** If m*(A) = 0 then $m^*(A \cup B) = m^*(B)$

Solution : $m^*(A \cup B) \le m^*(A) + m^*(B)$ (Countable sub additive property)

$$\Rightarrow m^*(A \cup B) \le m^*(B) \qquad (m^*(A) = 0)$$

Also $\quad B \subseteq A \cup B \Rightarrow m^*(B) \le m^*(A \cup B)$

Hence, $m^*(A \cup B) = m^*(B)$

14. **Proposition :** Given any set A and any $\in > 0$, there is an open set O such that $A \subset O$ and $m^*(O) \le m^*(A) + \in$. Also there is a set $G \in G_\delta$ such that $A \subseteq G$ and m*(A) = m*(G).

Proof : Let $\in > 0$. Then there exist a sequence $\{I_n\}$ of open intervals such that

$$A \subset \bigcup_n I_n \quad \text{and} \quad \sum_n \ell(I_n) < m^*(A) + \in \qquad \text{... (i)}$$

Take $O = \bigcup_n I_n$. Then O is an open set such that $A \subset O$. And

$$m^*(O) = m^*\left(\bigcup_n I_n\right) \leq \sum_n m^*(I_n) = \sum_n \ell(I_n)$$

$$\Rightarrow m^*(O) \leq m^*(A) + \in \qquad \text{(By (i))}$$

Next for $\in = \frac{1}{2^n}$, there is an open set O_n such that $A \subset O_n$ and

$$m^*(O_n) \leq m^*(A) + \frac{1}{2^n} \qquad n = 1, 2, 3, \ldots$$

Take $G = \bigcap_n O_n$, Then $G \in G_\delta$ and $G \subseteq O_n \quad \forall n$

And $A \subseteq O_n, \quad \forall n \Rightarrow A \subseteq \bigcap_n O_n$

$$\Rightarrow A \subseteq G$$

Therefore,

$$m^*(A) \leq m^*(G) \leq m^*(O_n) \leq m^*(A) + \frac{1}{2^n} \qquad \forall n \in \mathbb{N}$$

$$\Rightarrow m^*(A) \leq m^*(G) \leq m^*(A)$$

$$\Rightarrow m^*(A) = m^*(G)$$

2.2 Lebesgue Measurable Sets :

Outer measure has the advantage that it is defined for all subsets of $\mathbb{R}$. But it is not countably additive. It becomes countably additive if we restrict the domain of m* to a 6-algebra of all measurable subsets of $\mathbb{R}$.

We use the following definition due to Caratheodory.

1. Definition : A set E is said to be Lebesgue measurable if for any set A we have,

$$m^*(A) = m^*(A \cap E) + m^*(A \cap E^c)$$

2. Note : For any set A, we can write,

$$A = A \cap \mathbb{R} = A \cap (E \cup E^c) = (A \cap E) \cup (A \cap E^c)$$

Hence, $$m^*(A) = m^*((A \cap E) \cup (A \cap E^c))$$

$$m^*(A) \leq m^*(A \cap E) + m^*(A \cap E^c)$$

Thus, the set E is measurable if for any set A we have.

$$m^*(A) \geq m^*(A \cap E) + m^*(A \cap E^c)$$

3. **Lemma :** If E is measurable then E^c is also measurable.

Proof : E is measurable.

$\Rightarrow$ For any set A, $m^*(A) = m^*(A \cap E) + m^*(A \cap E^c)$

$$= m^*(A \cap E^c) + m^*(A \cap E)$$

$$= m^*(A \cap E^c) + m^*\left(A \cap (E^c)^c\right)$$

which shows that E^c is measurable.

4. ***Example :*** Show that the empty set ϕ and $\mathbb{R}$ are measurable.

Solution : For any set A,

$$A \cap \mathbb{R} = A \quad \text{and} \quad A \cap \mathbb{R}^c = A \cap \phi = \phi$$

Hence, $m^*(A \cap R) + m^*(A \cap \mathbb{R}^c) = m^*(A) + m^*(\phi)$

But $m^*(\phi) = 0$ therefore we get,

$$m^*(A \cap \mathbb{R}) + m^*(A \cap \mathbb{R}^c) = m^*(A)$$

Hence $\mathbb{R}$ is measurable. Since $\mathbb{R}^c = \phi$, ϕ is also measurable.

5. **Preposition :** If m*(E) = 0 then E is measurable.

Proof : Let A be any set. Then

$$m^*(A \cap E) \le m^*(E) = 0 \Rightarrow m^*(A \cap E) = 0$$

Now $A \cap E^c \subseteq A$

$$\Rightarrow m^*(A \cap E^c) \le m^*(A)$$

$$\Rightarrow m^*(A \cap E^c) + m^*(A \cap E) \le m^*(A)$$

Or $m^*(A) \ge m^*(A \cap E) + m^*(A \cap E^c)$

Hence E is measurable.

6. **Note :** Empty set ϕ, any finite set and any countably infinite subsets of $\mathbb{R}$ are measurable. The Cantor's set C is also measurable because its outer measure is zero.

7. **Proposition :** The union of finite collection of measurable sets is measurable.

Proof : First we show that the unionof two measurable sets E_1 and E_2 is measurable. E_1 is measurable. Therefore for any set A we have,

$$m^*(A) = m^*(A \cap E_1) + m^*(A \cap E_1^c) \qquad \text{...... (1)}$$

E_2 is measurable. Therefore for a set $A \cap E_1^c$ we get

$$m^*\left(A \cap E_1^c\right) = m^*\left(A \cap E_1^c \cap E_2\right) + m^*\left(A \cap E_1^c \cap E_2^c\right)$$

$$= m^*\left(A \cap E_2 \cap E_1^c\right) + m^*\left(A \cap \left(E_1 \cup E_2\right)^c\right) \quad \text{...... (2)}$$

Using (2) in (1) we get

$$m^*(A) = m^*(A \cap E_1) + m^*\left(A \cap E_2 \cap E_1^c\right) + m^*\left(A \cap (E_1 \cup E_2)^c\right) \quad \text{....... (3)}$$

Now, $\quad A \cap (E_1 \cup E_2) = (A \cap E_1) \cup (A \cap E_2)$

$$= (A \cap E_1) \cup \left(A \cap E_2 \cap E_1^c\right)$$

$$\Rightarrow m^*\left(A \cap (E_1 \cup E_2)\right) = m^*\left[(A \cap E_1) \cup \left(A \cap E_2 \cap E_1^c\right)\right]$$

$$\leq m^*(A \cap E_1) + m^*\left(A \cap E_2 \cap E_1^c\right) \quad \text{....... (4)}$$

Using (4) and (3) we get

$$m^*(A) \geq m^*\left(A \cap (E_1 \cup E_2)\right) + m^*\left(A \cap (E_1 \cup E_2)^c\right)$$

Thus $E_1 \cup E_2$ is measurable.

Now if $\{E_k\}_{k=1}^n$ is any finite collection of measurable sets then we prove that $\bigcup_{k=1}^n E_k$ is measurable by induction on n. For $n = 1$. E_1 is measurable. Suppose measurability holds for $n - 1$ then $\bigcup_{k=1}^{n-1} E_k$ is measurable and

$$\bigcup_{k=1}^n E_k = \left(\bigcup_{k=1}^{n-1} E_k\right) \cup E_n$$

Hence measurability holds for n and hence for all $n \in \mathbb{N}$. Thus union of finite collection of measurable sets is measurable.

8. **Definition :** A collection $\mathcal{A}$ of subsets of $\mathbb{R}$ is called an algebra of sets if $\mathcal{A}$ is closed under complement and union.

It follows from the DeMorgan's laws that the algebra $\mathcal{A}$ is closed under intersection also.

An algebra $\mathcal{A}$ is called σ-algebra if it is closed under countable union. (σ-algebra or Borel field).

Let $\mathcal{M}$ be the collection of measurable subsets. Since complement of two measurable sets is measurable, $\mathcal{M}$ is an algebra of measurable sets. We further show that $\mathcal{M}$ is σ-algebra.

9. **Lemma :** Let A be any set and E_1, E_2, E_3, ... E_n be a finite sequence of disjoint measurable sets. Then,

$$m^*\left(A\cap\left[\bigcup_{k=1}^{n}E_k\right]\right)=\sum_{k=1}^{n}m^*(A\cap E_k)$$

Proof : We prove the lemma by induction on n.

For n = 1, $m^*(A\cap E_1)=m^*(A\cap E_1)$ which is true trivially.

Let the result be true for n – 1

i.e. $$m^*\left(A\cap\bigcup_{k=1}^{n-1}E_i\right)=\sum_{k=1}^{n-1}m^*(A\cap E_k)$$ holds

Consider,

$$\left[A\cap\left(\bigcup_{k=1}^{n}E_k\right)\right]\cap E_n=A\cap E_n$$

$$\left[A\cap\left(\bigcup_{k=1}^{n}E_k\right)\right]\cap\tilde{E}_n=A\cap\left(\bigcup_{k=1}^{n-1}E_k\right)$$

Since E_n is measurable set we get,

$$m^*\left[A\cap\left(\bigcup_{k=1}^{n}E_k\right)\right]=m^*\left[A\cap\left(\bigcup_{k=1}^{n}E_k\right)\cap E_n\right]+m^*\left[A\cap\left(\bigcup_{k=1}^{n}E_k\right)\cap\tilde{E}_n\right]$$

$$=m^*(A\cap E_n)+m^*\left(A\cap\bigcup_{k=1}^{n-1}E_k\right)$$

$$=m^*(A\cap E_n)+\sum_{k=1}^{n-1}(A\cap E_k)$$ (By induction hypothesis)

$$m^*\left(A\cap\bigcup_{k=1}^{n}E_k\right)=\sum_{i=1}^{n}(A\cap E_k)$$

Thus the result is true for n. Hence by induction, the result is true for all $n\in\mathbb{N}$.

i.e. $$m^*\left(A\cup\left[\bigcup_{k=1}^{n}E_k\right]\right)=\sum_{k=1}^{n}m^*(A\cap E_k)$$ for all $n\in\mathbb{N}$.

10. **Note :** In the above lemma if A = $\mathbb{R}$ then we get,

$$m^*\left(\mathbb{R}\cap\left[\bigcup_{i=1}^{n}E_i\right]\right)=\sum_{i=1}^{n}m^*(\mathbb{R}\cap E_i)$$

$$\Rightarrow m^*\left(\bigcup_{i=1}^{n}E_i\right)=\sum_{i=1}^{n}m^*(E_i)$$

This shows that m* is finitely additive on a disjoint sequence of measurable sets i.e. m* is finitely additive on a class of measurable sets. In the following theorem we prove that $\mathcal{M}$ is closed under countable union.

11. **Theorem :** The collection $\mathcal{M}$ of all measurable sets is σ-algebra.

Proof : Since finite union of measurable sets is measurable and complement of measurable sets is measurable, the collection $\mathcal{M}$ of all measurable sets is an algebra. To prove that $\mathcal{M}$ is σ-algebra we show that $\mathcal{M}$ is closed under countable union. Let E be the countable union of measurable sets. Then there exist a countable collection of pairwise disjoint measurable sets $\{E_k\}$ such that $\bigcup_k E_k = E$.

Let A be any set and let $F_n=\bigcup_{k=1}^{n}E_k$. Then each F_n is a measurable set.

By measurability of F_n we have,

$$m^*(A)=m^*(A\cap F_n)+m^*\left(A\cap F_n^c\right) \quad \text{... (i)}$$

Now $$A\cap F_n = A\cap\left(\bigcup_{k=1}^{n}E_k\right)$$

Hence $$m^*(A\cap F_n)=m^*\left(A\cap\left(\bigcup_{k=1}^{n}E_k\right)\right)$$

$$=\sum_{k=1}^{n}m^*(A\cap E_k) \quad \text{... (ii)}$$

Next $$F_n=\bigcup_{k=1}^{n}E_k\subseteq\bigcup_{k=1}^{\infty}E_k=E$$

$$\Rightarrow F_n\subseteq E\Rightarrow E^c\subseteq F_n^c\Rightarrow A\cap E^c\subseteq A\cap F_n^c$$

$$\Rightarrow m^*\left(A\cap E^c\right)\le m^*\left(A\cap F_n^c\right) \quad \text{... (iii)}$$

Using (ii) and (iii) in (i) we get,

$$m^*(A) \geq \sum_{k=1}^{n} m^*(A \cap E_k) + m^*(A \cap E^c) \quad \text{... (iv)}$$

The l.h.s. is independent of n. Hence letting $n \to \infty$ we get,

$$m^*(A) \geq \sum_{k=1}^{\infty} m^*(A \cap E_k) + m^*(A \cap E^c)$$

But $$m^*(A \cap E) = m^*\left(A \cap \bigcup_{k=1}^{\infty} E_k\right) = m^*\left(\bigcup_{k=1}^{\infty} (A \cap E_k)\right)$$

$$\Rightarrow m^*(A \cap E) \leq \sum_{k=1}^{\infty} m^*(A \cap E_k) \quad \text{... (v)}$$

Using (v) in (iv) we get,

$$m^*(A) \geq m^*(A \cap E) + m^*(A \cap E^c)$$

Which shows that E is measurable.

Thus countable union of measurable sets is measurable which implies that collection of measurable sets is a σ-algebra.

12. Proposition : The interval (a, ∞) is measurable. Also every interval (finite or infinite) is measurable.

Proof : Let A be any set. Let $A \cap (a, \infty) = A_1$ and $A \cap (a, \infty)^c = A \cap (-\infty, a) = A_2$.

We prove that $m^*(A) \geq m^*(A_1) + m^*(A_2)$. If m*(A) = ∞ then the above inequality holds trivially.

If m*(A) < ∞ then for given $\in > 0$ there exist a countable collection of open intervals $\{I_n\}$.

Such that $A \subseteq \bigcup_n I_n$ and $\sum \ell(I_n) < m^*(A) + \in$ ... (i)

Now $A \subseteq UI_n$

$$\Rightarrow A \cap (a, \infty) \subseteq \left(\bigcup_n I_n\right) \cap (a, \infty)$$

$$\Rightarrow A_1 \subseteq \bigcup_n [I_n \cap (a, \infty)]$$

Let $I_n' = I_n \cap (a, \infty)$

Then $\quad A_1 \subseteq \bigcup_n I_n^{'}$ ... (ii)

Similarly $\quad A_2 = A \cap (-\infty, a] \subseteq \left(\bigcup_n I_n\right) \cap (-\infty, a]$

$$\Rightarrow A_2 \subseteq \bigcup_n \left(I_n \cap (-\infty, a]\right)$$

Let $\quad I_n^{''} = I_n \cap (-\infty, a]$

Therefore $\quad A_2 \subseteq \bigcup_n I_n^{''}$ ... (iii)

Further $\quad I_n = I_n \cap \mathbb{R} = I_n \cap \left[(-\infty, a] \cup (a, \infty)\right]$

$$= \left(I_n \cap (-\infty, a]\right) \cup \left(I_n \cap (a, \infty)\right)$$

$I_n = I_n^{'} \cup I_n^{''}$ which is a disjoint union.

$$\Rightarrow \ell(I_n) = \ell\left(I_n^{'}\right) + \ell\left(I_n^{''}\right)$$

$$= m^*\left(I_n^{'}\right) + m^*\left(I_n^{''}\right) \qquad \text{... (iv)}$$

Now from (ii),

$$A_1 \subseteq \bigcup_n I_n^{'}$$

$$\Rightarrow m^*(A_1) \leq m^*\left(\bigcup_n I_n^{'}\right) \leq \sum_n m^*\left(I_n^{'}\right)$$

Similarly from (iii)

$$A_2 \subseteq I_n^{''}$$

$$\Rightarrow m^*(A_2) \leq m^*\left(\bigcup_n I_n^{''}\right) \leq \sum_n m^*\left(I_n^{''}\right)$$

Therefore,

$$m^*(A_1) + m^*(A_2) \leq \sum_n m^*\left(I_n^{'}\right) + \sum_n m^*\left(I_n^{''}\right)$$

$$= \sum_n \left[m^*\left(I_n^{'}\right) + m^*\left(I_n^{''}\right)\right]$$

$$= \sum_n \ell(I_n) < m^*(A) + \in \qquad \text{(from (i))}$$

Thus

$$m^*(A_1)+m^*(A_2)<m^*(A)+\in$$

Since $\in>0$ is arbitrary we have,

$$m^*(A)\geq m^*(A_1)+m^*(A_2)$$

This shows that the interval (a,∞) is measurable.

13. Definition : The smallest σ-algebra containing all open sets is called a family of Borel sets. It is also the smallest σ-algebra containing all closed sets and also all open intervals.

14. Theorem : The collection $\mathcal{M}$ of measurable sets is a s-algebra that contains the s-algebra $\mathcal{B}$ of Borel sets. Each interval, each open set, each closed set, each $\mathcal{G}_d$ set and each $\mathcal{F}_s$ set is measurable.

Proof : We know that $(a,\infty)\in\mathcal{M}$ therefore $\sim(a,\infty)\in\mathcal{M}$ i.e. $(-\infty,a]\in\mathcal{M}$

Also $\quad (-\infty,b)=\bigcup_{n=1}^{\infty}\left(-\infty,b-\frac{1}{n}\right]$

Since countable union of measurable sets is measurable, $(-\infty,b)\in\mathcal{M}$

Next, $\quad (a,b)=(-\infty,b)\cap(a,\infty)$

Intersection of measurable intervals is measurable.

Hence, $(a,b)\in\mathcal{M}$

Thus every open interval is measurable. Each open set is countable union of open intervals. Hence each open set is measurable. Complement of open set is closed set. Hence each closed set is measurable. But the class of Borel sets is the smallest σ-algebra containing all open sets, all closed sets and all open intervals.

Hence the family $\mathcal{B}$ of Borel sets is subset of $\mathcal{M}$ i.e. $\mathcal{B}\subseteq\mathcal{M}$

This shows that every Borel set is measurable. Also each $\mathcal{G}_d$ set is the intersection of countable collection of open sets. Since open sets are measurable and countable intersection of measurable sets is measurable, each $\mathcal{G}_d$ is measurable. Similarly each $\mathcal{F}_s$ set is the countable union of closed sets which are measurable. Hence each $\mathcal{F}_s$ set is also measurable.

15. Proposition : The translate of a measurable set is measurable.

Proof : We know that outer measure is translation invariant. i.e. $m^*(A+x)=m^*(A)$ for any set A. Now if E is a measurable set then for any set A, $A-y$ is also a set for some $y\in\mathbb{R}$. Therefore

$$m^*(A-y)=m^*((A-y)\cap E)+m^*((A-y)\cap E^c)$$

But $x \in (A-y) \cap E \Leftrightarrow x \in A-y$ and $x \in E$

$$\Leftrightarrow x+y \in A \text{ and } x+y \in E+y$$

$$\Leftrightarrow (x+y) \in A \cap (E+y)$$

$$\Leftrightarrow x \in \left[\left(A \cap (E+y)\right) - y\right]$$

Thus $(A-y) \cap E = \left[A \cap (E+y)\right] - y$

Similarly $(A-y) \cap E^c = \left[A \cap (E+y)^c - y\right]$

Therefore we get,

$$m^*(A-y) = m^*\left[A \cap (E+y) - y\right] + m^*\left[A \cap (E+y)^c - y\right]$$

But m^* is translation invariant. Hence we get

$$m^*(A) = m^*\left(A \cap (E+y)\right) + m^*\left(A \cap (E+y)^c\right)$$

Therefore $E+y$ is measurable. Thus translation of measurable sets is also measurable.

2.3 Outer and Inner Approximation of Lebesgue Measurable Sets :

1. Excision Property : If A is a measurable set of finite outermeasure which is contained in B then

$$m^*(B-A) = m^*(B) - m^*(A)$$

Proof : By measurability of A we have

$$m^*(B) = m^*(B \cap A) + m^*(B \cap A^c)$$

$$= m^*(A) + m^*(B-A)$$

$$\Rightarrow m^*(B) - m^*(A) = m^*(B-A) \qquad (\because m^*(A) < \infty)$$

2. Theorem : Let E be any set. Then the following five statements are equivalent.

1) E is measurable.

2) Given $\in > 0$, there is an open set $O \supset E$ with $m^*(O-E) < \in$

3) Given $\in > 0$, there is a closed set $F \subset E$ with $m^*(E-F) < \in$

4) There is a set G in G_δ with $E \subset G$, and $m^*(G-E) = o$

5) There is a set $F \in F_\sigma$ with $F \subset E$, $m^*(E-F) = 0$

Proof : (1) $\Rightarrow$ (2)

Let E be a measurable set. First assume that $m(E) < \infty$. Then by proposition 2.2 (14) for given $\in > 0$ there is an open set $O \supset E$ such that,

$$m^*(O) < m^*(E) + \in \qquad \text{... (i)}$$

Now both E and O are measurable and $O = E \bigcup (O - E)$ which is disjoint union of measurable sets, hence we get,

$$m^*(O) = m(E) + m(O - E)$$

$$\Rightarrow m(O - E) = m(O) - m(E) \qquad (\because m(E) < \infty)$$

$$\Rightarrow m^*(O - E) = m^*(O) - m^*(E) \qquad \text{(m = m* on measurable sets)}$$

$$\Rightarrow m^*(O - E) < \in \qquad \text{(By (i))}$$

Now let $m(E) = \infty$. $\mathbb{R}$ can be expressed as a countable disjoint union of finite intervals.

Let, $$\mathbb{R} = \bigcup_{n=1}^{\infty} I_n$$

Then, $$E = E \bigcap \mathbb{R} = E \bigcap \bigcup_n I_n = \bigcup_n E \bigcap I_n$$

Take $E_n = E \bigcap I_n$. Therefore $E = \bigcup_n E_n$ and each E_n is measurable with $m(E_n) < \infty$. Therefore, there exists an open set $O_n \supset E_n$. such that,

$$m^*(O_n - E_n) < \frac{\in}{2^n}$$

Take $$O = \bigcup_n O_n$$

Then $$\bigcup_n O_n \supseteq \bigcup_n E_n \Rightarrow O \supseteq E \qquad \text{and}$$

$$O - E = \bigcup_n O_n - \bigcup_n E_n \subseteq \bigcup_n (O_n - E_n)$$

Hence, $$m^*(O - E) \le m^*\left(\bigcup_n (O_n - E_n)\right) \le \sum_n m^*(O_n - E_n)$$

$$\le \sum_n \frac{\in}{2^n} = \in \sum_n \frac{1}{2^n} = \in$$

$$\Rightarrow m^*(O - E) < \in.$$

(2) $\Rightarrow$ (4)

Given $\in = \frac{1}{n}$, there is an open set $O_n \supseteq E$ with $m^*(O_n - E) < \frac{1}{n}$. Take $G = \bigcap_n O_n$. Then $G \in G_\delta$ and $G \supseteq E$ and, $G \subset O_n \Rightarrow G - E \subseteq O_n - E$ for all n,

$$\Rightarrow m^*(G-E) \le m^*(O_n - E) < \frac{1}{n} \forall n \in \mathbb{N}$$

Since l.h.s. is independent of n, we get m*(G – E) = 0 (Taking $n \to \infty$) where $G \in G_\delta$.

(4) $\Rightarrow$ (1)

Since m*(G – E) = 0, $G \in G_\delta$ the set G – E is measurable. Also G is measurable.

And E = G – (G – E). Hence E is measurable.

(1) $\Rightarrow$ (3)

E is measurable $\Rightarrow \tilde{E}$ is measurable.

Therefore for given $\in > 0$ there is an open set $O \supset \tilde{E}$ such that,

$$m^*(O - \tilde{E}) < \in \qquad \text{(By (2))}$$

Now, $\quad O - \tilde{E} = O \cap \tilde{\tilde{E}} = E \cap (\tilde{\tilde{O}}) = E - \tilde{O}$

Take $\tilde{O} = F$. Then F is closed set. Also $O \supseteq \tilde{E} \Rightarrow \tilde{O} \subseteq E \Rightarrow F \subseteq E$

Thus there is a closed set $F \subseteq E$ such that, $m^*(E - F) < \in$

(3) $\Rightarrow$ (5)

Given $\in = \frac{1}{n}$ there is a closed set $F_n \subset E$ with $m^*(E - F_n) < \frac{1}{n}$. Take $F = \bigcup_n F_n$.

Then $F \in F_\sigma$ and $F \subset E$. And,

$$m^*(E-F) \le m^*(E - F_n) < \frac{1}{n}, \quad \forall n \in \mathbb{N}$$

Taking $n \to \infty$ we get, m* (E – F) = 0 where $F \in F_\sigma$

(5) $\Rightarrow$ (1)

Since m*(E – F) = 0, E – F is measurable. Also $F \in F_\sigma$. Hence F is measurable.

And E = $E = F \bigcup (E - F)$. Therefore E is measurable.

3. **Theorem :** Let E be a measurable set of finite outer measure. Then for each $\in > 0$, there is a finite collection of open intervals $\{I_k\}_{k=1}^{n}$ for which $O = \bigcup_{k=1}^{n} I_k$, such that,

$$m^*(E - O) + m^*(O - E) < \in$$

Proof : E is measurable set. Therefore by theorem for given $\in > 0$ there exists an open set $\mathcal{U}$ such that $E \subseteq \mathcal{U}$ and $m^*(\mathcal{U} - E) < \in / 2$.

Since E is measurable, by excision property,

$$m^*(\mathcal{U}) - m^*(E) < \in / 2 \;\Rightarrow m^*(\mathcal{U}) < \in / 2 + m^*(E)$$

But $m^*(E) < \infty$. Hence $m^*(\mathcal{U})$ is also finite. Next every open set is the union of disjoint collection of open intervals. Therefore there exists a disjoint collection $\{I_k\}_{k=1}^{\infty}$ of open intervals such that $\mathcal{U} = \bigcup_{k=1}^{\infty} I_k$.

Therefore for each natural number n we have,

$$\sum_{k=1}^{n} \ell(I_k) = \sum_{k=1}^{n} m^*(I_k) \qquad (\because \text{Outer measure of an interval is its length})$$

$$= m^*\left(\bigcup_{k=1}^{n} I_k\right) \quad (\because \text{Outer measure is finitely additive on disjoint measurable sets})$$

$$\leq m^*\left(\bigcup_{k=1}^{\infty} I_k\right) \qquad (m^* \text{ is monotone})$$

$$= m^*(u)$$

Since r.h.s. is independent of n we have

$$\sum_{k=1}^{\infty} \ell(I_k) \leq m^*(u) < \infty$$

$$\Rightarrow \sum_{k=1}^{\infty} \ell(I_k) < \infty$$

This shows that the infinite series $\sum_{k=1}^{\infty} \ell(I_k)$ of positive terms is convergent. Hence for given $\in > 0$ there is an integer n such that, $\sum_{k=n+1}^{\infty} \ell(I_k) < \in / 2$

Take $O = \bigcup_{k=1}^{n} I_k$. Then O is an open set and

$$O \subseteq u \Rightarrow O - E \subseteq u - E$$

$$\Rightarrow m^*(O - E) \leq m^*(u - E) < \in /2$$

$$\Rightarrow m^*(O - E) < \in /2$$

Also, $E \subseteq u \Rightarrow E - O \subseteq u - O = \bigcup_{k=1}^{\infty} I_k - \bigcup_{k=1}^{n} I_k$

$$\Rightarrow E - O \subseteq \bigcup_{k=n+1}^{\infty} I_k$$

Thus $\bigcup_{k=n+1}^{\infty} I_k$ is an open cover of $E - O$.

Hence, $m^*(E - O) = \inf\left\{\sum_k \ell(I_k) \mid E - O \subseteq \bigcup I_k\right\}$

$$\leq \sum_{k=n+1}^{\infty} \ell(I_k) < \in /2$$

Thus we have $m^*(E - O) < \in /2$ and $m^*(O - E) < \in /2$.

Adding these in equations, $m^*(O - E) + m^*(E - O) < \in$

4. ***Example :*** Let E be a measurable set. Prove that there exist a Borel set B_1 and B_2 such that, $B_1 \subseteq E \subseteq B_2$ and $m(B_1) = m(E) = m(B_2)$,

Solution : By proposition, there is a set $G \in G_\delta$ and $F \in F_6$ such that $F \subset E \subset G$ and m* (E – F) = 0, m*(G – E) = 0.

Now, $E = F \bigcup (E - F)$, $G = E \bigcup (G - E)$

$\Rightarrow m(E) = m(F) + m(E - F)$, m(G) = m(E) + m (G – E)

But, m* (E – F) = m(E – F) = 0, m*(G – E) = m(G – E) = 0, (m = m* on measurable sets)

Hence wet get, m(E) = m(F), m(G) = m(E)

Take B_1 = F, B_2 = G Then $B_1 \subseteq E \subseteq B_2$,

B_1 and B_2 are Borel sets and m(B_1) = m(E) = m(B_2).

5. ***Example :*** If E_1 and E_2 are measurable, show that,

$$m(E_1 \cup E_2) + m(E_1 \cap E_2) = m(E_1) + m(E_2)$$

Solution : If either $m(E_1) = \infty$ or $m(E_2) = \infty$ then $m(E_1 \cup E_2) = \infty$ and the equality holds trivially.

If $m(E_1) < \infty$, $m(E_2) < \infty$ then since $E_1 \cup E_2$, $E_1 \cap E_2$ are measurable sets such that $(E_1 \cup E_2) = E_1 \cup (E_2 - E_1)$ and $E_2 = (E_1 \cap E_2) \cup (E_2 - E_1)$.

Since these unions are disjoint we get,

$$m(E_1 \cup E_2) = m(E_1) + m(E_2 - E_1)$$

$$m(E_2) = m(E_1 \cap E_2) + m(E_2 - E_1)$$

$$\Rightarrow m(E_1 \cup E_2) = m(E_1) + m(E_2) - m(E_1 \cap E_2)$$

$$\Rightarrow m(E_1 \cup E_2) + m(E_1 \cap E_2) = m(E_1) + m(E_2)$$

Exercises I :

1. If E_1 and E_2 are measurable sets with finite measure, prove that following are equivalent.

   (a) $m(E_1 \Delta E_2) = 0$ (b) $m(E_1 - E_2) = m(E_2 - E_1)$

   (c) $m(E_1) = m(E_1 \cap E_2) = m(E_2)$

2. If $\{E_i\}$ is a sequence of sets with $m^*(E_i) = 0$ for all $i \in \mathbb{N}$ then prove that $\bigcup_{i=1}^{\infty} E_i$ is a measurable set and has measure zero.

3. If E_1 is a measurable set and $m^*(E_1 \Delta E_2) = 0$ then show that E_2 is measurable.

2.4 Lebesgue Measure :

1. Definition : Lebesgue Measure

A function $m : \mathcal{M} \to \mathbb{R}^+ \cup \{\infty\}$ defined by m(E) = m*(E) is called Lebesgue measure of E. **Where** $\mathcal{M}$ is a s -algebra of Lebesgue measurable sets.

Thus m is a set function obtained by restriction of m* to the family $\mathcal{M}$ of measurable sets. Also for an interval I, m(I) = m*(I) = ℓ (I)

2. **Proposition :** Let $\{E_k\}_{k=1}^{\infty}$ be a sequence of measurable sets then $m\left(\bigcup_k E_k\right) \le \sum_k m(E_k)$

If the sets E_k's are pairwise disjoint then

$$m\left(\bigcup_k E_k\right) = \sum_k m(E_k)$$

Proof : $\{E_k\}$ is a sequence of measurable sets. Therefore $\bigcup_k E_k$ is also measurable and

$$m\left(\bigcup_k E_k\right) = m^*\left(\bigcup_k E_k\right) \le \sum_k m^*(E_k) = \sum_k m(E_k)$$

$$\Rightarrow m\left(\bigcup_k E_k\right) \le \sum_k m(E_k)$$

Now for a finite sequence $\{E_k\}_{k=1}^{n}$ of disjoint measurable sets, we have,

$$m\left(\bigcup_{k=1}^{n} E_k\right) = m^*\left(\bigcup_{k=1}^{n} E_k\right) = \sum_{k=1}^{n} m^*(E_k)$$

$$\Rightarrow m\left(\bigcup_{k=1}^{n} E_k\right) = \sum_{k=1}^{n} m(E_k)$$

Hence m is finitely additive.

Next, for an infinite sequence of disjoint measurable sets we have,

$$\bigcup_{k=1}^{n} E_k \subseteq \bigcup_{k=1}^{\infty} E_k$$

$$\Rightarrow m\left(\bigcup_{k=1}^{n} E_k\right) \le m\left(\bigcup_{k=1}^{\infty} E_k\right)$$

$$\Rightarrow m\left(\bigcup_{k=1}^{\infty} E_k\right) \ge m\left(\bigcup_{k=1}^{n} E_k\right) = \sum_{k=1}^{n} m(E_k)$$

$$\Rightarrow m\left(\bigcup_{k=1}^{\infty} E_k\right) \ge \sum_{k=1}^{n} m(E_k)$$

The l.h.s. is independent of n. Hence as $n \to \infty$, we get,

$$m\left(\bigcup_{k=1}^{\infty} E_k\right) \geq \sum_{k=1}^{\infty} m(E_k)$$

Also by countable sub additivity of m* we get,

$$m\left(\bigcup_{k=1}^{\infty} E_k\right) = m^*\left(\bigcup_{k=1}^{\infty} E_k\right) \leq \sum_{k=1}^{\infty} m^*(E_k) = \sum_{k=1}^{\infty} m(E_k)$$

Hence, $$m\left(\bigcup_{k=1}^{\infty} E_k\right) = \sum_{k=1}^{\infty} m(E_k)$$

3. **Note :** The above proposition says that Lebesgue measure is countably additive.

4. ***Example :*** Prove that countable subsets of $\mathbb{R}$ are measurable.

Solution : If A is countable set then m*(A) = 0. Hence A is measurable.

5. **Definition :** A countable collection of sets $\{E_k\}_{k=1}^{\infty}$ is said to be ascending if $E_k \subseteq E_{k+1}$, $\forall k$. The sequence $\{E_k\}_{k=1}^{\infty}$ is said to be descending if $E_{k+1} \subseteq E_k$, $\forall k$.

6. **Proposition :**

(i) If $\{A_k\}_{k=1}^{\infty}$ is ascending sequence of measurable sets then $m\left(\bigcup_{k=1}^{\infty} A_k\right) = \lim_{k\to\infty} m(A_k)$.

(ii) If $\{B_k\}_{k=1}^{\infty}$ is a descending sequence of measurable sets and $m(B_1) < \infty$, then

$$m\left(\bigcup_{k=1}^{\infty} B_k\right) = \lim_{k\to\infty} m(B_k).$$

Proof :

(i) If $m(A_{k_0}) = \infty$ for some k_0, then

$$A_{k_0} \subseteq \bigcup_{k=i}^{\infty} A_k \Rightarrow m(A_{k_0}) \leq m\left(\bigcup_{k=i}^{\infty} A_k\right)$$

$$\Rightarrow m\left(\bigcup_{k=i}^{\infty} A_k\right) = \infty$$

And $\quad A_{k0} \subseteq A_{k0+n} \qquad \forall n = 1,2,3,.....$

$$\Rightarrow m\left(A_{k0}\right) \le m\left(A_{k0+n}\right) \qquad \forall n = 1,2,....$$

$$\Rightarrow m\left(A_k\right) = \infty \qquad \forall k \ge k_0$$

$$\Rightarrow \lim_{k\to\infty} m\left(A_k\right) = 0$$

Hence we have $m\left(\bigcup_{k=1}^{\infty} A_k\right) = \lim_{k\to\infty} m\left(A_k\right)$

Now if $m\left(A_k\right) < \infty$ for all k = 1, 2, 3,.......

Then define $C_k = A_k - A_{k-1}$, k = 1, 2, 3, ... $\quad \left(A_0 = f\right)$

Then $\{C_k\}_{k=1}^{\infty}$ is a disjoint sequence of measurable sets such that $\bigcup_{k=1}^{\infty} A_k = \bigcup_{k=1}^{\infty} C_k$

$$\Rightarrow m\left(\bigcup_{k=1}^{\infty} A_k\right) = m\left(\bigcup_{k=1}^{\infty} C_k\right)$$

$$= \sum_{k=1}^{\infty} m\left(C_k\right)$$

$$= \sum_{k=1}^{\infty} \left[m\left(A_k\right) - m\left(A_{k-1}\right)\right] \qquad \text{(By excision property)}$$

$$= \lim_{n\to\infty} \sum_{k=1}^{\infty} \left[m\left(A_k\right) - m\left(A_{k-1}\right)\right]$$

$$= \lim_{n\to\infty} \left[m\left(A_k\right) - m\left(A_0\right) + m\left(A_2\right) - m\left(A_1\right) + ... + m\left(A_n\right) - m\left(A_{n-1}\right)\right]$$

$$= \lim_{n\to\infty} m\left(A_n\right) - m\left(A_0\right) \qquad \left(\text{But } A_0 = f \Rightarrow m\left(A_0\right) = 0\right)$$

$$= \lim_{n\to\infty} m\left(A_n\right) = \lim_{k\to\infty} m\left(A_k\right)$$

Thus, $\quad m\left(\bigcup_{k=1}^{\infty} A_k\right) = \lim_{k\to\infty} m\left(A_k\right)$

(ii) Let $\{B_k\}_{k=1}^{\infty}$ be the descending sequence of measurable sets with $m(B_1) < \infty$

Define $D_k = B_1 - B_k$, $k = 1, 2, 3, ...$ where $D_1 = f$

Since $\{B_k\}_{k=1}^{\infty}$ is descending, the sequence $\{D_k\}_{k=1}^{\infty}$ is ascending. Therefore by above result (i) we have

$$m\left(\bigcup_{k=1}^{\infty} D_k\right) = \lim_{k\to\infty} m(D_k)$$

But $\quad \bigcup_{k=1}^{\infty} D_k = \bigcup_{k=1}^{\infty}(B_1 - B_k)$

$$= \bigcup_{k=1}^{\infty}\left(B_1 \cap B_k^c\right)$$

$$= B_1 \cap \bigcup_{k=1}^{\infty} B_k^c \qquad \text{(By distributive law)}$$

$$= B_1 \cap \left(\bigcap_{k=1}^{\infty} B_k\right)^c \qquad \text{(By De Morgam laws)}$$

$$= B_1 - \bigcap_{k=1}^{\infty} B_k$$

Therefore, $\quad m\left(\bigcup_{k=1}^{\infty} D_k\right) = m\left(B_1 - \bigcap_{k=1}^{\infty} B_k\right)$

$$= m(B_1) - m\left(\bigcap_{k=1}^{\infty} B_k\right) \qquad \text{(By excision property)}$$

On the other hand, for all $k = 1, 2, 3, ...$

$$m(D_k) = m(B_1 - B_k)$$

$$= m(B_1) - m(B_k)$$

Therefore,

$$m\left(\bigcup_{k=1}^{\infty} D_k\right) = \lim_{k\to\infty} m(D_k)$$

$$\Rightarrow m(B_1) - m\left(\bigcap_{k=1}^{\infty} B_k\right) = \lim_{k\to\infty} m(B_1) - m(B_k)$$

$$\Rightarrow m(B_1) - m\left(\bigcap_{k=1}^{\infty} B_k\right) = m(B_1) - \lim_{k\to\infty} m(B_k)$$

$$\Rightarrow m\left(\bigcap_{k=1}^{\infty} B_k\right) = \lim_{k\to\infty} m(B_k) \qquad (\because m(B_1) < \infty)$$

7. **Note :** The condition $m(B_1) < \infty$ is essential in the above proposition. We have the following counter example.

8. **Example :** Let $\{E_n\}_{n=1}^{\infty}$ be a sequence of sets where $E_n = (n, \infty)$. Then $\{E_n\}_{n=1}^{\infty}$ is a decreasing sequence of measurable sets and $\bigcap_{n=1}^{\infty} E_n = \phi \Rightarrow m\left(\bigcap_{n=1}^{\infty} E_n\right) = 0$.

But $m(E_1) = m(1, \infty) = \infty$ which is not finite.

And $\lim_{n\to\infty} m(E_n) = \lim_{n\to\infty} m(n, \infty) = \infty$

Thus $\lim_{n\to\infty} m(E_n) \neq m\left(\bigcap_{n=1}^{\infty} E_n\right)$

The conclusion of the above proposition does not hold since $m(E_1)$ is not finite.

9. **Definition :** For a mesurable set E, a property holds almost everywhere on E if there is a subset E_0 of E such that the property holds for all $x \in E - E_0$ and $m(E_0) = 0$.

10. **The Borel-Cantelli Lemma**

Let $\{E_k\}_{k=1}^{\infty}$ be a countable collection of measurable sets for which $\sum_{k=1}^{\infty} m(E_k) < \infty$. Then almost all $x \in \mathbb{R}$ belongs to at most finitely many of the E_k's.

Proof : For each $n \in \mathbb{N}$ we have,

$$m\left(\bigcup_{k=n}^{\infty} E_k\right) \leq \sum_{k=n}^{\infty} m(E_k) \leq \sum_{k=n}^{\infty} m(E_k) < \infty$$

Let $F_n = \bigcup_{k=n}^{\infty} E_k$. Then $\{F_n\}_{n=1}^{\infty}$ is a decreasing seuence of measurable sets with $m(F_1) = m\left(\bigcup_{k=1}^{\infty} E_k\right) \le \sum_{k=1}^{\infty} m(E_k)$ i.e. $m(F_1) < \infty$. Therefore by continuity of measure we have

$$m\left(\bigcap_{n=1}^{\infty} F_n\right) = \lim_{n\to\infty} m(F_n)$$

$$\Rightarrow m\left(\bigcap_{n=1}^{\infty}\left[\bigcup_{k=n}^{\infty} E_k\right]\right) = \lim_{n\to\infty} m\left(\bigcup_{k=n}^{\infty} E_k\right) \le \lim_{n\to\infty} \sum_{k=n}^{\infty} m(E_k)$$

But if $S_n = \sum_{k=n}^{\infty} m(E_k)$ then $\{S_n\}$ is a decreasing sequence of non-negative real numbers which converges to zero i.e.

$$\lim_{n\to\infty} S_n = 0 \Rightarrow \lim_{n\to\infty} \sum_{k=n}^{\infty} m(E_k) = 0$$

Thus, $m\left(\bigcap_{n=1}^{\infty}\left[\bigcup_{k=n}^{\infty} E_k\right]\right) = 0$

i.e. $m\left\{x \in \mathbb{R} \mid x \in \bigcap_{n=1}^{\infty}\bigcup_{k=n}^{\infty} E_k\right\} = 0$

Thus almost all $x \in R$ does not belong to $\bigcap_{n=1}^{\infty}\bigcup_{k=n}^{\infty} E_k$.

But $x \notin \bigcap_{n=1}^{\infty}\bigcup_{k=n}^{\infty} K_n \Rightarrow x \notin \bigcup_{k=n}^{\infty} E_k$ for all n

$\Rightarrow x \notin E_k$ for all $k \ge n$ and for all n

$\Rightarrow x \notin E_k$ for all k

$\Rightarrow x \in E_k$ for atmost finitely many E_k's.

i.e. almost all $x \in \mathbb{R}$ belongs to atmost finitely many E_k's.

11. **Note :** Some of the properties of Lebesgue measure are named as follows :

1. **Finite additivity :** For any finite disjoint collection $\{E_k\}_{k=1}^{n}$ of measurable sets

$$m\left(\bigcup_{k=1}^{n} E_k\right) = \sum_{k=1}^{n} m(E_k)$$

2. **Monotonicity :** If A and B are measurable sets such that $A \subseteq B$ then $m(A) \leq m(B)$.

3. **Excision :** If A and B are measurable sets with $A \subseteq B$ and $m(A) < \infty$, then

$$m(B - A) = m(B) - m(A)$$

Ans hence if $m(A) = 0$, then $m(B - A) = m(B)$.

4. **Countable monotonicity :** For any collection $\{E_k\}_{k=1}^{\infty}$ of measurable sets which covers a measurable set E.

i.e. $$E \subseteq \bigcup_{k=1}^{\infty} E_k \Rightarrow m(E) \leq \sum_{k=1}^{\infty} m(E_k).$$

5. **Countable additivity :** For any countable collection of disjoint mesurable sets $\{E_k\}_{k=1}^{\infty}$.

$$m\left(\bigcup_{k=1}^{\infty} E_k\right) = \sum_{k=1}^{\infty} m(E_k)$$

2.5 Nonmeasurable Sets

We have defined measurable sets and studied their properties. We have given many examples of measurable sets. Hence it is natural to ask whether there exists any set which is not measurable. The answer is yes but construction of nonmeasurable set is not simple.

We know that $m^*(E) = 0$ if then E is measurable and hence every subset of E is also measurable. Hence nonmeasurable sets have positive outer measure. We show that if E is any set of positive outer measure then there are subsets of E which are not measurable.

We first prove the following result.

1. **Lemma :** Let E be a bounded measurable set of real number. Let $\wedge$ be a bounded, countably infinite set of real numbers for which the collection $\{l + E\}_{l \in \wedge}$ of translations of E is disjoint. Then $m(E) = 0$.

Proof : Since translate of a measurable set is measurable each set $l + E$ is measurable $\forall l \in \wedge$. Hence the collection $\{l + E\}_{l \in \wedge}$ is a countable disjoint collection of measurable sets. Hence by countable additivity of Lebesgue measure we have

$$m\left(\bigcup_{l\in\wedge}(l+E)\right)=\sum_{l\in\wedge}m(l+E)$$

Now E and $\wedge$ are bounded sets. Therefore there exists real numbers L and M such that

$$|x|<L, \quad \forall x\in E$$

$$|l|<M, \quad \forall l\in\wedge$$

We prove that $\bigcup_{l}(l+E)$ is also a bounded set.

Let $y\in\bigcup_{l}(l+E)$ be arbitrary. Then

$$y\in\bigcup_{l}(l+E)\Rightarrow y\in l+E \text{ for some } l\in\wedge$$

$$\Rightarrow y=l+x \text{ for some } l\in A \text{ and for some } l\in E$$

$$\Rightarrow |y|=|l+x|\le|l|+|x|<L+M$$

Since $y\in\bigcup_{l}(l+E)$ is arbitrary we have

$$|y|<L+M \quad \text{for all } y\in\bigcup_{l}(l+E)$$

Hence $\bigcup_{l}(l+E)$ is bounded set and therefore $m\left(\bigcup_{l}(l+E)\right)$ is finite.

Now if $m(E)>0$ then

$$\sum_{l\in\wedge}m(l+E)=\sum_{l}m(E)=m(E)\sum_{l\in\wedge}1=\infty$$

Since $\wedge$ is a countably infinite.

Therefore, $m\left(\bigcup_{l\in\wedge}(l+E)\right)=\sum_{l\in\wedge}m(l+E)$ holds only if $m(E)=0$.

2. Definition : For any set E of real numbers, any two points in E are said to be rationally equivalent if their difference belongs to the set of rational number $\mathbb{Q}$.

i.e. for any $x,y\in E$, $x\sim y$ iff $x-y\in\mathbb{Q}$.

This relation of 'rational equivalence' is an equivalence relation on the set E. For,

(1) $x-x=0,\forall x\in E\Rightarrow x\sim x,\forall x\in E$

(2) $x\sim y\Rightarrow x-y\in\mathbb{Q}\Rightarrow y-x\in\mathbb{Q}\Rightarrow y\sim x$

(3) If $x \sim y$ and $y \sim z \Rightarrow x-y, y-z \in \mathbb{Q} \Rightarrow x-y+y-z \in \mathbb{Q} \Rightarrow x-z \in \mathbb{Q} \Rightarrow x \sim z$.

The relation of 'rational equivalence' is an equivalence relation on the set E and hence partitions E into disjoint equivalence classes.

3. Definition : For the rational equivalence relation on E we form a choice set C_E by taking exactly one member from each equivalene class. By Axiom of choise such a set C_E can be formed.

4. Note : If C_E is the choice set corresponding to the rational equivalence relation on E, then

(i) The difference of two points in C_E is not rational.

(ii) For each point x in E there is a point $c \in C_E$ such that $x \sim c$ i.e. $x - c = q \in \mathbb{Q}$ i.e. $x = c + q$ for some $q \in \mathbb{Q}$.

(iii) For any set $\wedge \subseteq \mathbb{Q}$ the collection $\{ l + C_E \}_{l \in \wedge}$ is disjoint. For if $x \in (l_1 + C_E) \cap (l_2 + C_E)$

$$\Rightarrow x \in l_1 + C_E \text{ and } x \in l_2 + C_E$$

$$\Rightarrow x = l_1 + c_1 \text{ and } x = l_2 + c_2 \text{ for some } c_1, c_2 \in C_E$$

$$\Rightarrow l_1 + c_1 = l_2 + c_2$$

$$\Rightarrow c_1 - c_2 = l_2 - l_1 \in \mathbb{Q}$$

Which is a contradiction since difference of any two points in C_E is not rational. Hence the collection $\{ l + C_E \}_{l \in \wedge}$.

5. Theorem : (Vitali) Any set E of real numbers with positive outer measure contains a subset which is not measurable.

Proof : Since any set of real numbers contains a bounded subset of real numbers, we assume that E is a bounded subset of real numbers with $m^*(E) > 0$.

Let C_E be a choice set for the rational equivalence relation on E. We show that C_E is not measurable.

On the contrary assume that C_E is measurable.

Let $\wedge_0$ be any bounded countably infinite set of rational numbers. Since C_E is measurable, the collection of translates $\{ l + C_E \}_{l \in \wedge_0}$ is disjoint and measurable.

Hence by lemma we get $m(C_E) = 0$.

Since measure is translation invariant we get

$$m(C_E) = m(l + C_E) \qquad \forall\, l \in \wedge_0$$

$$\Rightarrow m(l + C_E) = 0 \qquad \forall\, l \in \wedge_0$$

Also the collection $\{ l + C_E \}_{l \in \wedge_0}$ is disjoint. Hence

$$m\left(\bigcup_{l \in \wedge_0} (l + C_E) \right) = \sum_{l \in \wedge_0} m (l + C_E) = 0$$

$$\Rightarrow m\left(\bigcup_{l \in \wedge_0} (l + C_E) \right) = 0$$

Next E is bounded set. Therefore there exist a real number b such that $|x| < b, \ \forall x \in E$

i.e. $E \subseteq [-b, b]$

Choose the index set $\wedge_0 = [-2b, 2b] \cap \mathbb{Q}$ i.e. $\wedge_0$ contains all rational numbers in [–2b, 2b]

Then $\wedge_0$ is bounded and countably infinite set.

Now if $x \in E$ then by partition of E w.r.t. the equivalence relation, there exists $c \in C_E$ such that $x \sim c \Rightarrow x - c = q$ for some rational number q. But $C_E \subseteq E \subseteq [-b, b]$

$\Rightarrow x, c \in [-b, b]$ $\qquad (\because x \in E$ and $c \in C_E)$

$\Rightarrow -b < x < b, \ -b < c < b$

$\Rightarrow -2b < x - c < 2b$

$\Rightarrow -2b < q < 2b$

$\Rightarrow q \in [-2b, 2b]$

$\Rightarrow q \in \wedge_0$ $\qquad (\because \wedge_0$ contains all rational numbers in [–2b, 2b])

But $x - c = q \Rightarrow x = c + q \in q + C_E, \ q \in \wedge_0$. Hence $x \in \bigcup_{l \in \wedge_0} (l + C_E)$

Since $x \in E$ is arbitrary we get $E \subseteq \bigcup_{l \in \wedge_0} (l + C_E)$.

By monotonicity of outer measure

$$m^*(E) \le m^*\left(\bigcup_{l \in \wedge_0} (l + C_E) \right) \le \sum_{l \in \wedge_0} m^*(l + C_E)$$

$$= \sum_{l \in \wedge_0} m^*(C_E) = \sum m(E)$$

$$= \sum_{l \in \wedge_0} m(C_E) = 0$$

$$\Rightarrow m^*(E) = 0$$

Which is a contradiction because $m*(E)>0$. Hence C_E is not mesurable. But $C_E \subseteq E$.

Therefore E contains a subset that is not measurable.

6. **Theorem :** Outer measure is not additive.

i.e. There are disjoint sets A and B of real numbers for which $m*(A \cup B) < m*(A) + m*(B)$.

Proof : We prove this by contradiction. Suppose $m*(A \cup B) = m*(A) + m*(B)$ holds for every pair of disjoint sets A and B.

Then for any sets E and A of real numbers, $A \cap E$ and $A \cap E^c$ are disjoint sets and $(A \cap E) \cup (A \cap E^c) = A$. Therefore,

$$m*(A) = m*\left[(A \cap E) \cup (A \cap E^c)\right]$$

$$= m*(A \cap E) + m*(A \cap E^c) \quad \text{(By assumption)}$$

This shows that E is measurable. Thus any set of real numbers is measurable which is a contradiction since there exists nonmeasurable sets of real numbers. Hence there must exists a pair of disjoint sets A and B such that,

$$m*(A \cup E) < m*(A) + m*(B).$$

UNIT - III

LEBESGUE MEASURABLE FUNCTIONS

3.1 Measurable Functions :

We first establish the equivalence between the various sets that arise from a function f.

1. Proposition : Let f be an extended real valued function whose domain is measurable. Then the following statements are equivalent.

(i) For each real number c the set $\{x \mid f(x) > c\}$ is measurable.

(ii) For each real number c the set $\{x \mid f(x) \geq c\}$ is measurable.

(iii) For each real number c the set $\{x \mid f(x) < c\}$ is measurable.

(iv) For each real number c the set $\{x \mid f(x) \leq c\}$ is measurable.

Proof : Let D be the domain of f i.e. $f : D \to \mathbb{R}$.

Now $\{x \mid f(x) > c\}^c = \{x \mid f(x) \leq c\}$

Hence $\{x \mid f(x) > c\}$ is measurable iff $\{x \mid f(x) \leq c\}$ is measurable.

Which implies $(i) \Leftrightarrow (iv)$

Similarly $\{x \mid f(x) < c\}^c = \{x \mid f(x) \geq c\}$ implies (ii) $\Leftrightarrow$ (iii)

Next $$\{x \mid f(x) \geq c\} = \bigcap_{n=1}^{\infty} \left\{x \mid f(x) > c - \frac{1}{n}\right\}$$

Therefore if $\{x \mid f(x) > c\}$ is measurable then $\left\{x \mid f(x) > c - \frac{1}{n}\right\}$ is measurable for all n. And countable intersection of measurable sets is measurable. Hence $\{x \mid f(x) \geq c\}$ is measurable. Thus (i) $\Rightarrow$ (ii)

Also $$\{x \mid f(x) > c\} = \bigcup_{n=1}^{\infty} \left\{x \mid f(x) \geq c + \frac{1}{n}\right\}$$

Therefore if (ii) is true then countable union of measurable sets is measurable. Hence $\{x \mid f(x) > c\}$ is measurable. Thus (ii) $\Rightarrow$ (i)

Thus we have, (iv) $\Leftrightarrow$ (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii)

Which shows that all the four statements are equivalent.

2. **Definition :** An extended real valued function *f* is said to be Lebesgue measurable if its domain is measurable and if it satisfies one of four statements of the above proposition.

3. **Proposition :** If a function f is measurable then the set $\{x \mid f(x) = c\}$ is measurable for all $c \in \mathbb{R}$

Proof : Case (i) : $c \in \mathbb{R}$, $c < \infty$

For any finite real number *c*,

$$\{x \mid f(x) = c\} = \{x \mid f(x) \geq c\} \cap \{x \mid f(x) \leq c\}$$

Since f is measurable, the sets $\{x \mid f(x) \geq c\}$ and $\{x \mid f(x) \leq c\}$ are measurable. Hence $\{x \mid f(x) = c\}$ is measurable for all *c*.

Case (ii) $c = +\infty$ or $-\infty$

If $c = +\infty$ then $f(x) = +\infty$ implies $f(x) \geq k$, $\forall k \in \mathbb{N}$.

Hence, $$\{x \mid f(x) = +\infty\} = \bigcap_{k=1}^{\infty} \{x \mid f(x) \geq k\}$$

And if $c = -\infty$ then we can write,

$$\{x \mid f(x) = -\infty\} = \bigcap_{k=1}^{\infty} \{x \mid f(x) \leq -k\}$$

Since *f* is measurable, the sets $\{x \mid f(x) \geq k\}$ and $\{x \mid f(x) \leq -k\}$ are measurable. Countable intersection of measurable sets is measurable. Hence the sets, $\{x \mid f(x) = +\infty\}$ and $\{x \mid f(x) = -\infty\}$ are measurable. Thus $\{x \mid f(x) = c\}$ is measurable for any extended real number *c*.

4. ***Example :*** Show that a function defined by,

$$f(x) = \begin{cases} x + 4 & \text{if } x \geq 2 \\ 8 & \text{if } x < 2 \end{cases}$$

is measurable

Solution : Let *c* be any real number. Then,

$$\{x \mid f(x) \geq c\} = \mathbb{R} \qquad \text{if } c \leq 6$$

$$= (-\infty, 2) \cup [c-4, \infty) \quad \text{if } 6 < c \leq 8$$

$$= [c-4, \infty) \quad \text{if } 8 < c$$

Any interval is measurable. Hence $\{x \mid f(x) \geq c\}$ is measurable for all c i.e. f is measurable function.

5. ***Example :*** Discuss the measurability of $f(x) = e^x$, x > 0.

Solution : Let c be any real number. Then,

$$\{x \mid f(x) \geq c\} = \{x \mid e^x \geq c\} = (0, \infty) \qquad \text{if } c \leq 1$$

$$= (\log_e c, \infty) \qquad \text{if } c > 1$$

The intervals $(0, \infty)$ and $(\log_e c, \infty)$ are measurable for all c, Hence $f(x) = e^x$ is measurable function.

6. **Proposition :** Let f be a function defined on a measurable set E. Then f is measurable if and only if for each open set O, the inverse image of O under f, $f^{-1}(O)$ is measurable.

Proof : If O is any open subset of $\mathbb{R}$ then

$$f^{-1}(O) = \{x \in E \mid f(x) \in O\}$$

First assume that inverse image of an open set is measurable. Then $(c, \infty), c \in \mathbb{R}$ is an open set and hence $f^{-1}(c, \infty)$ is measurable for all $c \in \mathbb{R}$.

But $$f^{-1}(c, \infty) = \{x \in E \mid f(x) \in (c, \infty)\}$$

$$= \{x \in E \mid f(x) > c\}.$$

Thus for all $c \in \mathbb{R}$ the set, $\{x \in E \mid f(x) > c\}$ is measurable. Hence f is measurable function. Conversely suppose f is measurable function. Let O be any open subset of $\mathbb{R}$. Then there is a countable collection $\{I_k\}_{k=1}^{\infty}$ of open, bounded intervals such that $O = \bigcup_{k=1}^{\infty} I_k$. Let $I_k = (a_k, b_k)$, k = 1, 2, 3, ...

Then $I_k = (-\infty, b_k) \cap (a_k, \infty)$.

Let $A_k = (a_k, \infty)$ and $B_k = (-\infty, b_k)$

Therefore $I_k = A_k \cap B_k$, k = 1, 2, 3, ...

Now $$f^{-1}(A_k) = \{x \in E \mid f(x) \in A_k\}$$

$$= \{x \in E \mid f(x) \in (a_k, \infty)\}$$

$$= \{x \in E \mid f(x) > a_k\}$$

Similarly $$f^{-1}(B_k) = \{x \in E \mid f(x) > b_k\}$$

Since f is measurable, $f^{-1}(A_k)$ and $f^{-1}(B_k)$ are measurable for all k = 1, 2, 3, ...

Therefore $$f^{-1}(O) = f^{-1}\left(\bigcup_{k=1}^{\infty} I_k\right) = f^{-1}\left(\bigcup_{k=1}^{\infty}(A_k \cap B_k)\right)$$

$$= f^{-1}\left(\bigcup_{k=1}^{\infty} A_k \cap \bigcup_{k=1}^{\infty} B_k\right)$$

$$= f^{-1}\left(\bigcup_{k=1}^{\infty} A_k\right) \cap f^{-1}\left(\bigcup_{k=1}^{\infty} B_k\right)$$

$$= \left[\bigcup_{k=1}^{\infty} f^{-1}(A_k)\right] \cap \left[\bigcup_{k=1}^{\infty} f^{-1}(B_k)\right]$$

But collection of measurable sets is a s -algebra which is closed under countable union and intersection. Hence $f^{-1}(O)$ is measurable.

Thus *f* is measurable iff inverse image of an open set is measurable.

7. Proposition : A continuous, real valued function defined on measurable domain is measurable.

Proof : Let *f* be a continuous function defined on a measurable set E. i.e. $f : E \to \mathbb{R}$ be continuous function where E is measurable.

Let O be any open subset of $\mathbb{R}$. Since f is continuous there exists an open set $\mathcal{U}$ such that $f^{-1}(O) = E \cap \mathcal{U}$.

Since $\mathcal{U}$ is open it is measurable. Therefore $E \cap \mathcal{U}$ is measurable i.e $f^{-1}(O)$ is measurable. Thus, inverse image of an open set is measurable, hence *f* is measurable function.

8. Proposition : Let *f* be an extended real valued function on E. Then,

(i) If *f* is measurable on E and $f = g$ a.e on E, then g is measurable.

(ii) For a measurable subset D of E, f is measurable on E if and only if the restrictions of *f* to D and E – D are measurable.

Proof :

(i) First assume that *f* is measurable on E. Let $A = \{x \in E \mid f(x) \neq g(x)\}$. Then for any $c \in \mathbb{R}$,

$$\{x \in E \mid g(x) > c\} = \{x \in A \mid g(x) > c\} \cup \{x \in E \mid f(x) > c\} \cap (E - A)$$

Now $\quad f = g \;\; a \cdot e \Rightarrow m(A) = 0$

$\Rightarrow$ A is measurable and every subset of A is measurable.

Therefore $\{x \in A \mid g(x) > c\}$ is measurable. Since f is measurable, $\{x \in E \mid f(x) > c\}$ is measurable. Since E and A are measurable, E – A is also mesurable. Further union and intersection of measurable sets is measurable.

Hence $\{x \in E \mid f(x) > c\}$ is measurable for any $c \in \mathbb{R}$. i.e. g is measurable function.

(ii) For any $c \in \mathbb{R}$ we have,

$$\{x \in E \mid f(x) > c\} = \{x \in D \mid f(x) > c\} \cup \{x \in E - D \mid f(x) > c\}$$

Where D is measurable subset of E.

Thus if f is measurable then its restriction to D and E – D are measurable and conversely if its restriction to D and E – D are measurable then r.h.s. is union of measurable sets which is measurable i.e. f is measurable on E. Thus f is measurable on E iff its restrictions to D and E – D are measurable.

9. Theorem : Let f and g be measurable functions on E that are finite a.e on E. Then for any a and b, $af + bg$ is mesurable and $f \bullet g$ is measurable on E.

Proof : Since f and g are finite $a.e$ we may assume that both f and g are finite on E.

If $a = 0$ then clearly $af = 0$ and hence af is measurable.

If $a \neq 0$ then for any real number c.

$$\{x \in E \mid af(x) > c\} = \left\{x \in E \mid f(x) > \frac{c}{a}\right\} \quad \text{if } a > 0$$

and $$\{x \in E \mid af(x) > c\} = \left\{x \in E \mid f(x) < \frac{c}{a}\right\} \quad \text{if } a < 0$$

Since f is measurable the sets to the r.h.s. are measurable. Thus for all c, $\{x \in E \mid af(x) > c\}$ is measurable. Hence af is measurable function.

Next if $f(x) + g(x) < c$ for some real number c then $f(x) < c - g(x)$. Therefore there exist a rational number q such that $f(x) < q < c - g(x)$.

(Since between any two distinct real numbers there exists countably infinite rational numbers.)

Hence, $$\{x \in E + f(x) + g(x) < c\} = \bigcup_{q \in \mathbb{Q}} \left[\{x \in E \mid f(x) < q\} \cap \{x \in E \mid q < c - g(c)\}\right]$$

$$= \bigcup_{q \in \mathbb{Q}} \left[\{x \in E \mid f(x) < q\} \cap \{x \in E \mid g(x) < c - q\}\right]$$

Since f and g are measurable, $\{x \in E \mid f(x) < q\}$ and $\{x \in E \mid g(x) < c - q\}$ are measurable and countable union of measurable sets is measurable. Hence $f + g$ is measurable function.

Thus if f and g are measurable functions then $af + bg$ is measurable for all a, b.

Now if f is measurable function then for any real number $c \geq 0$,

$$\{x \in E \mid f^2(x) > c\} = \{x \in E \mid f(x) > \sqrt{c}\} \cup \{x \in E \mid f(x) < -\sqrt{c}\}$$

and for $c < 0$, $\{x \in E \mid f^2(x) > c\} = E$.

Hence f^2 is measurable.

Finally for any measurable functions f and g

$$f \cdot g = \frac{1}{2}\{(f+g)^2 - f^2 - g^2\}$$

Since $(f+g)^2$, f^2, g^2 are measurable. The sum and difference of measurable functions is measurable. Hence $f \cdot g$ is measurable.

10. **Note :** The composition of two measurable real valued functions defined on $\mathbb{R}$ need not be measurable.

11. **Proposition :** Let g be a measurable real-valued function defined on E and let f be a continuous real valued function defined on $\mathbb{R}$. Then the composition $f \bullet g$ is a measurable function on E.

Proof : We know that a function is measurable if and only if the inverse image of an open set is measurable.

Consider an open set $O \subseteq \mathbb{R}$. Then,

$$\begin{aligned}(fog)^{-1}(O) &= \left(g^{-1}of^{-1}\right)(O)\\ &= g^{-1}\left(f^{-1}(O)\right)\end{aligned}$$

Since f is continuous $f^{-1}(O)$ is an open set. And since g is measurable, $g^{-1}\left(f^{-1}(O)\right)$ is measurable. Thus the inverse image $(fog)^{-1}(O)$ is measurable. Therefore the composite function fog is measurable.

12. **Note :** If we define a modulus function m by $m: \mathbb{R} \to \mathbb{R}$, $m(x) = |x|$ then m is a continuous function on $\mathbb{R}$ and for any measurable function f,

$$(mof)(x) = m(f(x)) = |f(x)| = |f|(x)$$

i.e. $mof = |f|$. Hence by above result $|f|$ is measurable function.

Further $|f|^p$ is measurable with the same domain E.

13. **Definition :** For a finite family $\{f_k\}_{k=1}^{n}$ of functions defined on a domain E we define,

$$\max\{f_1, f_2, ..., f_n\}(x) = \max\{f_1(x), f_2(x), ..., f_n(x)\} \qquad \forall x \in E$$

and

$$\min\{f_1, f_2, ..., f_n\}(x) = \min\{f_1(x), f_2(x), ..., f_n(x)\} \qquad \forall x \in E$$

14. **Proposition :** For a finite family $\{f_k\}_{k=1}^{n}$ of measurable functions with common domain E, the functions $\max\{f_1, f_2, ..., f_n\}$ and $\min\{f_1, f_2, ..., f_n\}$ are measurable functions.

Proof : For any real number c,

$$\max\{f_1, f_2, f_3, ..., f_n\}(x) > c \Rightarrow \max\{f_1(x), f_2(x), ..., f_n(x)\} > c$$

$$\Rightarrow f_k(x) > c \text{ for some k}$$

Therefore, $$\{x \in E \mid \max\{f_1, f_2, ..., f_n\}(x) > c\} = \bigcup_{k=1}^{n}\{x \in E \mid f_x(x) > c\}$$

Since f_k's are measurable functions, $\{x \mid f_k(x) > a\}$ is measurable for all k = 1, 2, 3, ...n. Also finite union of measurable sets is measurable. Hence $\max\{f_1, f_2, ..., f_n\}$ is measurable.

Similarly for any real number c,

$$\min\{f_1, f_2, ..., f_n\}(x) > c \Rightarrow \min\{f_1(x), f_2(x), ..., f_n(x)\} > c$$

$$\Rightarrow f_k(x) > c \text{ for all k = 1, 2, ...}$$

Therefore, $$\{x \in E \mid \min\{f_1, f_2, ..., f_n\}(x) > c\} = \bigcap_{k=1}^{n}\{x \in E \mid f_k(x) > c\}$$

Since f_k's are measurable functions and intersection of finite collection of measurable sets is measurable sets is measurable, the set $\{x \in E \mid \min\{f_1, f_2, ... f_n\}(x) > c\}$ is also mesurable for all c. Hence the function $\min\{f_1, f_2, ... f_n\}$ is measurable.

15. **Note :** For a function f defined on a set E we define,

$$|f|(x) = \max\{f(x), -f(x)\},\ f^+(x) = \max\{f(x), 0\}$$

and $$f^-(x) = \max\{-f(x), 0\}$$

Therefore if f is measurable on E, by above proposition $|f|$, f^+ and f^- are measurable on E.

3.2 Sequencial Pointwise Limits and Simple Approximation

1. **Definition :** For a sequence $\{f_n\}$ of functions with common domain E, a function f on E and a subset $A \subseteq E$ we have.

(i) The sequence $\{f_n\}$ converges to f pointwise on A if

$$\lim_{n \to \infty} f_n(x) = f(x),\ \forall x \in A$$

(ii) The sequence $\{f_n\}$ converges to f pointwise on a•e on A if it converges to f pointwise on A– B where m (B) = 0.

(iii) The sequence $\{f_n\}$ converges to f uniformly on A provided for each $\in > 0$, there is a positive integer N.

$|f(x) - f_n(x)| < \in$, $\forall x \in A$ and for all $n \geq N$.

2. Proposition : Let $\{f_n\}$ be a sequence of measurable functions on E which converges pointwise a•e on E to a function f. Then f is measurable.

Proof : Since $f_n \to f a \cdot e$ on E, there is a measurable subset E_0 of E such that $m(E_0) = 0$ and $f_n \to f$ on $E - E_0$ pointwise. But f is measurable on E if and only if its restriction to E – E_0 is meeasurable where $m(E_0) = 0$. Therefore without loss of generality we assume that $f_n \to f$ on E pointwise. i.e. $f(x) = \lim_{n\to\infty} f_n(x)$, $\forall x \in E$

Let c be a fixed real number. Then,

$$f(x) < c \text{ for some } x \in E.$$

$$\Rightarrow \exists \text{ in integer } n \text{ such that } f(x) < c - \frac{1}{n} < c.$$

$$\Rightarrow \lim_{n\to\infty} f_n(x) < c - \frac{1}{n} \text{ for some integer } n.$$

$$\Rightarrow \exists \text{ an integer k such that } f_j(x) < c - \frac{1}{n},\ \forall j \geq k$$

Conversely,

$$f_j(x) < c - \frac{1}{n},\ \forall j \geq k \text{ and for some integer } n.$$

$$\Rightarrow \lim_{j\to\infty} f_j(x) < c - \frac{1}{n}, \text{ for some integer } n.$$

$$\Rightarrow f(x) < c - \frac{1}{n}, \text{ for some integer } n$$

$$\Rightarrow f(x) < c$$

Thus $f(x) < c$ if and only if $f_j(x) < c - \frac{1}{n}$, $\forall j \geq k$ and for some integer n.

Since f_j is meeasurable, $\left\{x \in E \mid f_j(x) < c - \frac{1}{n}\right\}$ is measurable.

And hence $\bigcap_{j=k}^{\infty}\left\{x \in E \mid f_j(x) < c - \frac{1}{n}\right\}$ is measurable.

Also union of countable collection of measurable sets is measurable. Therefore,

$$\bigcup_{k=1}^{\infty}\left[\bigcap_{j=k}^{\infty}\left\{x \in E \mid f_j(x) < c - \frac{1}{n}\right\}\right] = \{x \in E \mid f(x) < c\}$$

is measurable. Therefore f is measurable.

Step Functions :

3. Definition : A function $\psi : [a,b] \to \mathbb{R}$ is called a step function if there is a partition, $\{a = x_o, x_1, x_2, ..., x_n = b\}$ of the interval [a, b] such that in every interval (x_{k-1}, x_k), the function ψ is constant. Thus,

$$\psi(x) = c_k \ \forall \ x \in (x_{k-1}, x_k), \text{ k = 1, 2, 3, ..., n}$$

4. Note :

(1) A step function is defined on a closed interval and assumes only finite number of values.

(2) At the endpoint of the interval the values assumed by the step function are arbitrary or may not be assigned. These end points forms a finite set of discontinuities. Hence the set of discontinuities of step function is a set of measure zero.

Following are some of the examples of step function.

(1) $f : [a,b] \to \mathbb{R}$ defined by

$$f(x) = \alpha \quad \text{if} \quad a \le x < c$$

$$f(x) = \beta \quad \text{if} \quad c \le x \le b$$

α, β are constants and a < c < b

(2) The Signum function S defined by

$$S(x) = \begin{cases} 1 & \text{if x > 0} \\ 0 & \text{if x = 0} \\ -1 & \text{if x < 0} \end{cases}$$

is a step function

(3) The greatest integer function $f:(a,b)\to\mathbb{R}$ defined by $f(x)=[x]$ is a step function.

Note that every step function is measurable, since it is defined on closed interval which is measurable and $\{x \mid f(x)>\alpha\}$ are sub intervals of [a, b] which are also measurable for all $\alpha\in\mathbb{R}$.

Characteristic Functions :

5. Definition : Let E be any subset of $\mathbb{R}$. The function $\chi_E:\mathbb{R}\to\{0,1\}$ defined by,

$$\chi_E(x)=\begin{cases}1 & \text{if } x\in E\\ 0 & \text{if } x\notin E\end{cases}$$

is called the characteristic function of E.

Following are some of the properties of a characteristic function.

(1) $\chi_\phi=0$ and $\chi_{\mathbb{R}}=1$

(2) $A\subseteq B\Rightarrow\chi_A\le\chi_B$

(3) $\chi_{A\cup B}=\chi_A+\chi_B-\chi_{A\cap B}$

If A and B are disjoint then we get, $\chi_{A\cup B}=\chi_A+\chi_B$

(4) $\chi_{A\cap B}=\chi_A\cdot\chi_B$

(5) $\chi_{\tilde{A}}=1-\chi_A$

(6) For a disjoint sequence { A_n } of sets we have, $\chi_{UA_n}=\sum_n\chi_{A_n}$

6. ***Example :*** Let A be any set. Prove that the characteristic function χ_A of A is measurable if an only if A is measurable.

Solution : For any $\alpha\in\mathbb{R}$ consider the set,

$$\begin{aligned}\{x \mid \chi_A(x)>\alpha\} &= \mathbb{R} && \text{if } \alpha<0\\ &= A && \text{if } 0\le\alpha<1\\ &= \phi && \text{if } \alpha\ge 1\end{aligned}$$

Therefore, χ_A is measurable

$\Leftrightarrow\{x \mid \chi_A(x)>\alpha\}$ is measurable $\forall\alpha\in\mathbb{R}$

$\Leftrightarrow \mathbb{R}, A, \phi$ are measurable

$\Leftrightarrow$ A is measurable (Since $\mathbb{R}$ and ϕ are always measurable)

Thus χ_A is measurable iff A is measurable.

7. Note :

(1) Existence of non-measurable set implies the existence of non-measurable function. For, if P is a non-measurable set, then χ_P is a non-measurable function.

(2) Sum of two measurable functions is measurable but sum of two non-measurable functions need not be non-measurable.

For, if P is non-measurable set then $\tilde{P}$ is also non-measurable. Hence χ_P and $\chi_{\tilde{P}}$ are non-measurable functions. But $\chi_P + \chi_{\tilde{P}} = \chi_{P \cup \tilde{P}} = \chi_{\mathbb{R}}$ which is a measurable function.

Simple Functions :

8. Definition : A function ϕ is called simple function if it is measurable and assumes only a finite number of values.

If $\phi : E \to \mathbb{R}$ is a simple function then there is a finite disjoint sequence $\{E_i\}_{i=1}^n$ of measurable sets such that $E = \bigcup_{i=1}^{n} E_i$ such that $\phi(x) = a_i$, $x \in E_i$, i = 1, 2, ..., n. Thus $\text{Im}\,\phi = \{a_1, a_2, ..., a_n\}$ And the function can be expressed as a linear combination of characteristic functions. Thus,

$$\phi(x) = \sum_{i=1}^{n} a_i \chi_{E_i}(x)$$

This representation of ϕ is not unique. If ϕ is a simple function defined on a measurable set E and $\{a_1, a_2, a_3, ..., a_n\}$ is the set of nonzero values of ϕ then define $A_i = \{x \in E \mid \phi(x) = a_i\}$ and the function ϕ is given by**,**

$$\phi = \sum_{i=1}^{n} a_i \chi_{A_i}$$

This representation of ϕ is unique and called the canonical representation (natural representation). In this representation all a_i's are nonzero and distinct and A_i's are disjoint measurable sets.

Thus a simple function is a finite linear combination of characteristic functions of measurable sets.

9. ***Example :*** Prove that the sum, product and difference of two simple functions are simple.

Solution : Let $\phi = \sum_{i=1}^{m} \alpha_i \chi_{A_i}$ and $\psi = \sum_{i=1}^{n} \beta_i \chi_{B_i}$ be the two simple functions. Then,

$$\phi + \psi = \sum_{i=1}^{m} \alpha_i \chi_{A_i} + \sum_{i=1}^{n} \beta_i \chi_{B_i}$$

$$= \sum_{i=1}^{m+n} \gamma_i \chi_{C_i}$$

where $\gamma_i = \alpha_i$ i = 1, 2, ..., m

$= \beta_{i-m}$ i = m+1, ..., m+n

and $C_i = A_i$ i = 1, 2, ..., m

$= B_{i-m}$ i = m+1, m+2, ..., n

Since A_i, B_i are measurable, each C_i is also measurable and hence $\phi + \psi$ is a simple function. Similarly $\phi - \psi$ is also a simple function.

Now,

$$\phi \cdot \psi = \sum_{i=1}^{m} \alpha_i \chi_{A_i} \cdot \sum_{i=1}^{n} \beta_i \chi_{B_i}$$

$$= \sum_{i,j} \alpha_i \beta_j \chi_{A_i} \cdot \chi_{Bj}$$

$$= \sum_{i,j} \gamma_{ij} \chi_{A_i \cap Bj}$$

Since A_i and B_j are measurable, $A_i \cap B_j$ are also measurable for all i, j. Hence $\phi \cdot \psi$ is a simple function.

10. The Simple Approximation Lemma

Let f be a measurable real valued functions on E. Assume that f is bounded on E and there is an integer $M \geq 0$ such that $|f| \leq M$ on E. Then for each $\in > 0$, there are simple functions $f_\in$ and $y_\in$ defined on E such that $f_\in \leq f \leq y_\in$ and $0 \leq y_\in - f_\in < \in$ on E.

Proof : Since f is bounded on E, $|f(x)| \leq M$ for all $x \in E$. $-M \leq f(x) \leq M$, $\forall x \in E$ i.e. $f(E) \subseteq [-M, M]$. Let (c, d) be an open bounded interval that contains $f(E)$, i.e. $f(E) \subseteq (c, d)$.

Let $c = y_0 < y_2 < y_2 < ...... < y_{n-1} < y_n = d$ be a partition of the closed bounded interval [*c*, *d*] such that the successive elements differ by less than $\in > 0$ (given)

i.e. $y_k - y_{k-1} < \in \qquad \forall$ k = 1, 2, ... n

Define $I_k = [y_{k-1}, y_k)$ and $f^{-1}(I_k) = E_k$, k = 1, 2, ...n

Each interval I_k is open and *f* is measurable. Hence $f^{-1}(I_k) = E_k$ is measurable for all k = 1, 2, 3, ...n.

Define simple functions $f_{\in}$ and $y_{\in}$ on E by

$$f_{\in} = \sum_{k=1}^{n} y_{k-1} \cdot c_{E_k} \text{ and } y_{\in} = \sum_{k=1}^{n} y_k \cdot c_{E_k}$$

Then for any $x \in E$, $f(x) \subseteq (c, d)$

Therefore there exists unique k such that $y_{k-1} \le f(x) < y_k$

Since $f(x) \subseteq I_k$, $x \in f^{-1}(I_k) = E_k$. Therefore $c_{E_k}(x) = 1$

and $\qquad f_{\in}(x) = y_{k-1} \cdot c_{E_k(x)} = y_{k-1}$

$$y_{\in}(x) = y_k \cdot c_{E_k(x)} = y_k$$

Hence, $f_{\in}(x) \le f(x) < y_{\in}(x)$ and $y_{\in}(x) - f_{\in}(x) = y_k - y_{k-1} < \in$

Since $x \in E$ is arbitrary we get,

$$f_{\in} \le f < y_{\in} \text{ and } y_{\in} - f_{\in} < \in \text{ on E.}$$

11. The Simple Approximation Theorem

An extended real valued function *f* on a measurable set E is measurable if and only if there is a sequence $\{f_n\}$ of simple functions on E which converges pointwise on E to *f* and $|f_n| \le |f|$ for all *n*, on E.

If *f* is nonnegative we may choose $\{f_n\}$ to be increasing.

Proof : Since each simple function is measurable, the sequence $\{f_n\}$ is a sequence of measurable functions which converges to *f* pointwise on E. Henece *f* is measurable. Conversely assume that *f* is measurable. Since every function is a difference of nonnegative functions, we further assume that $f \ge 0$ on E.

Let n be a natural number. Define E_n by

$$E_n = \{x \in E \mid f(x) \leq n\}$$

Then E_n is measurable and the restriction of f to E_n is nonnegative bounded measurable function.

i.e. $f : E_n \to \mathbb{R}$, $f(x) \geq 0$, $\forall x \in E_n$ and $f(x) \leq n$, $\forall x \in E_n$ (By definition of E_n)

Therefore by simple Approximation Lemma and by taking $\in = \frac{1}{n}$, there exists simple function f_n and y_n such that $0 \leq f_n \leq f \leq y_n$ on E_n and $0 \leq y_n - f_n < \frac{1}{n}$ on E_n.

$\Rightarrow 0 \leq f_n \leq f$ and $0 \leq f - f_n \leq y_n - f_n < \frac{1}{n}$ on E_n.

Now $f_n : E_n \to \mathbb{R}$ can be extended to E by setting $f_n(x) = n$, $\forall x$ such that $f(x) > n$.

Hence, $o \leq f_n \leq f$ on E.

We show that the sequence $\{f_n\}$ converges to f pointwise on E.

Let $x \in E$ be arbitrary.

Case I : $f(x)$ is finite. Choose a natural number N such that $f(x) < N$. Then for any $n \geq N$, $f(x) < N \Rightarrow f(x) < n$ and hence $0 \leq f(x) - f_n(x) < \frac{1}{n}$, $\forall n \geq N$.

$$\Rightarrow \lim_{n\to\infty} f_n(x) = f(x)$$

Case II : $f(x) = \infty$. Since $f(x) > n$ for all n. $f_n(x) = n$, $\forall n$.

Therefore $\lim_{xn\to\infty} f_n(x) = \infty = f(x)$.

Thus there exists a sequence $\{f_n\}$ of simple functions such that $f_n \to f$ pointwise on E. Now if $y_n = \max\{f_1, f_2, ...f_n\}$. Then $\{y_n\}$ is an increasing sequence of simple functions and $0 \leq f_n \leq f$ for all n.

$\Rightarrow 0 \leq \max\{f_1, f_2, ...f_n\} \leq f$ for all n

$\Rightarrow 0 \leq y_n \leq f$ for all n on E

and $\lim_{n\to\infty} y_n \leq f$

Also $y_n = \max(f_1, f_2, ..., f_n) \geq f_n$ for all n

$\Rightarrow \lim_{n\to\infty} y_n \geq \lim_{n\to\infty} f_n = f$

Hence $\lim_{n\to\infty} y_n = f$ where $\{y_n\}$ is increasing sequence of simple functions.

3.3 Littlewood's Three Principles

There are three principles roughly expressed in the following terms :

1. Every measurable set is nearly a finite union of open intervals.
2. Every measurable function is nearly continuous.
3. Every pointwise convergence sequence of measurable functions is nearly uniformly convergent.

We have already discussed first wto of these principles. One of the versions of the third principle is given by Egoroff's Theorem.

To prove Egoroff's Theorem we require the following Lemma.

1. Lemma : Let E be a measurable set of finite measure. Let $\{f_n\}$ be a sequence of measurable functions on E that converges pointwise on E to a real valued function f. Then for each $h > 0$ and $d > 0$, there is a measurable subset A of E and there is an index N such that $|f_n - f| < h$ on A for all $n \geq N$ and $m(E - A) < d$.

Proof : For each k, $|f - f_k|(x) = |f(x) - f_k(x)|$, $x \in E$.

Since each f_n is measurable and $f_n \to f$ pointwise on E, f is also measurable. Hence $|f - f_k|$ is measurable function for all k. Therefore the set, $\{x \in E \mid |f(x) - f_k(x)| < h\}$ is measurable for all k.

Let $E_n = \{x \in E \mid |f(x) - f_k(x)| < h \text{ for all } k \geq n\}$

Then $E_n = \bigcap_{k=n}^{\infty} \{x \in E \mid |f(x) - f_k(x)| < h\}$

Since intersection of a countable collection of measurable sets is measurable. Therefore E_n is a measurable set for all n. Further $\{E_n\}$ is an ascending collection of measurable sets

$$\left(\because E_n = \bigcap_{k=n}^{\infty} \{x \in E \mid |f(x) - f_k(x)| < h\} \subseteq \bigcap_{k=n+1}^{\infty} \{x \in E \mid |f(x) - f_k(x)| < h\} = E_{n+1} \right)$$

Next $E_n \subseteq E$ for all n $\Rightarrow \bigcup_{n=1}^{\infty} E_n \subseteq E$.

On the other hand if $x \in E$ then $f_n \to f$ pointwise on E implies, $\lim_{n\to\infty} f_n(x) = f(x)$.

Therefore for $h > 0$ there exists an integer N such that $|f_n(x) - f(x)| < h$ for all $n \geq N$.

Hence $x \in E_N$

Thus $x \in E \Rightarrow x \in E_N$ for some N.

Therefore $E \subseteq \bigcup_{n=1}^{\infty} E_n$. Thus $E = \bigcup_{n=1}^{\infty} E_n$.

By continuity of Lebesgue measure we get

$$m(E) = \lim_{n\to\infty} m(E_n)$$

Since $m(E)$ is finite, for given d, there is an index N such that $|m(E) - m(E_n)| < d$ for all $n \geq N$. In particular for N,

$$0 \leq m(E) - m(E_N) < d \qquad (\because E_n \subseteq E \Rightarrow m(E_n) < m(E))$$

Take $A = E_N$. Then A is measurable and

$$m(E - A) = m(E - E_N)$$

$$= m(E) - m(E_n), \qquad \text{(By excision property)}$$

$$< d$$

Thus on $A = E_N$, $|f_n(x) - f(x)| < h$ for all $n \geq N$ and $m(E - A) < d$.

2. Egoroff's Theorem

Let E be a measurable set with finite measure. Let $\{f_n\}$ be a sequence of measurable functions defined on E that converges pointwise on E to a real valued function *f*. Then for each $\in > 0$, There is a closed set F contained in E for which $f_n \to f$ uniformly on F and $m(E - F) < \in$.

Proof : Let $\in > 0$ be arbitrary. For any integer $n \in \mathbb{N}$, and for $h = \frac{1}{n}$ and $d = \frac{\in}{2^{n+1}}$, there exists a mesurable set A_n and an index N (n) such that

$$|f_k(x) - f(x)| < \frac{1}{n} \text{ on } A_n \text{ for all } k \geq N(n) \qquad \text{..... (1)}$$

and $m(E - A_n) < \frac{\in}{2^{n+1}}$ .... (2) (By Lemma)

Take $A=\bigcap_{n=1}^{\infty} A_n$. Therefore A is measurable and ,

$$m(E-A)=m\left(E\cap A^c\right)$$

$$=m\left(E\cap\left(\bigcap_{n=1}^{\infty} A_n\right)^c\right)$$

$$=m\left(E\cap\bigcup_{n=1}^{\infty} A_n{}^c\right)$$

$$=m\left(\bigcup_{n=1}^{\infty} E\cap A_n{}^c\right)$$

$$=m\left(\bigcup_{n=1}^{\infty}(E-A_n)\right)$$

$$\leq\sum_{n=1}^{\infty} m\left(E-A_n\right)$$

$$<\sum_{n=1}^{\infty}\frac{\in}{2^{n+1}}=\frac{\in}{2} \qquad \left(\text{Since } \sum_{n=1}^{\infty}\frac{1}{2^{n+1}}=\frac{1}{2}\right)$$

$$\Rightarrow m(E-A)<\frac{\in}{2} \qquad \text{.... (3)}$$

We calim that $\{f_n\}$ converges to *f* uniformly on A. For given $\in>0$ choose an integer n_0 such that $\frac{1}{n_0}<\in$. Therefore by (1)

$$\left|f_k(x)-f(x)\right|<\frac{1}{n_0} \text{ on } A_{n_0} \text{ for all } k\geq N\left(n_0\right).$$

But $A\subseteq A_{n_0}$ and $\frac{1}{n_0}<\in$. Therefore we get

$$\left|f_k(x)-f(x)\right|<\in \text{ on A for all } k\geq N\left(n_0\right)$$

Which shows that $\{f_n\}$ converges uniformly on A and we have $m(E-A)<\frac{\in}{2}$.

Further A is measurable set. Therefore there exists a closed set $F \subseteq A$ such that $m^*(A-F) < \frac{\in}{2}$. SInce F is closed, it is measurable. Hence A – F is also measurable and $m^*(A-F) = m(A-F) < \frac{\in}{2}$.

Now $E-F = E \cap F^c = E \cap F^c \cap \mathbb{R} = E \cap F^c \cap (A \cup A^c)$

$= (E \cap F^c \cap A) \cup (E \cap F^c \cap A^c) \subseteq (A \cap F^c) \cup (E \cap A^c)$

$\Rightarrow E - F \subseteq (A \cap F^c) \cup (E \cap A^c)$

$\Rightarrow m(E-F) \le m(A-F) + m(E-A) < \frac{\in}{2} + \frac{\in}{2}$

$\Rightarrow m(E-F) < \in$

Thus $\{f_n\}$ converges uniformly on F and $m(E-F) < \in$ where F is closed set contained in E.

3. **Note :** We have proved the following result which is a formulation of Littlewood's first principle.

If E is measurable set of finite measure then for each $\in > 0$, there is a fnite disjoint collection of open intervals whose union is u and

$$m(E-\mathcal{U}) + m(\mathcal{U}-E) < \in$$

i.e. every measurable set of finite measure is nearly equal to finite union of open intervals.

Next we prove a precise version of Littlewood's second principle.

4. **Proposition :** Let f be a simple function defined on a set E. Then for every $\in > 0$, there is a continuous function g on $\mathbb{R}$ and a closed set $F \subseteq E$ such that $f = g$ on F and $m(E-F) < \in$.

Proof : Since f is a simple function, f takes only finite distinct values on E. Let a_1, a_2, ... a_n be the finite number of distinct values of f on E.

Let $E_k = \{x \in E \mid f(x) = a_k\}$, k = 1, 2, 3, ... n

Therefore the collection $\{E_k\}_{k=1}^{n}$ is a disjoint collection of measurable sets whose union is E. Hence by theorem there exists closed sets F_k, k = 1, 2, 3, ...n such that

$F_k \subseteq E_k$ and $m(F_k - E_k) < \frac{\in}{n}$

Take $F=\bigcup_{k=1}^{n}F_k$. Then F is also a closed set and

$$m(E-F)=m\left(E\cap F^c\right)$$

$$=m\left(\bigcup_{k=1}^{n}E_k\cap F^c\right)=m\left(\bigcup_{k=1}^{n}\left(E_k\cap F^c\right)\right)$$

$$=m\left(\bigcup_{k=1}^{n}\left(E_k-F\right)\right)$$

But $\quad E_k-F=E_k\cap\left(\bigcup_{i=1}^{n}F_i\right)^c$

$$=E_k\cap\bigcap_{i=1}^{n}F_i^c$$

$$=\bigcap_{i=1}^{n}E_k\cap F_i^c$$

But $F_k\subseteq E_k$ for all k = 1, 2, 3, ... n and E_k's are disjoint.

Hence $E_k\cap F_i^c=E_k$ if $i\neq k$ and $E_k\cap F_k{}^c=E_k-F_k$. Therefore we get,

$$E_k-F=\bigcap_{i=1}^{n}\left(E_k-F_i\right)=E_k\cap\left(E_k-F_k\right)$$

$$=E_k-F_k$$

Therefore,

$$m(E-F)=m\left(\bigcup_{k=1}^{n}E_k-F_k\right)$$

$$=\sum_{k=1}^{n}m\left(E_k-F_k\right)\leq\sum_{k=1}^{n}\frac{\in}{n}$$

$$\leq\sum_{k=1}^{n}\frac{\in}{n}$$

$$= \frac{\in}{n} \sum_{k=1}^{n} 1 = \frac{\in}{n} \cdot n = \in$$

$$\Rightarrow m(E-F) < \in$$

Now define a function $g : F \to \mathbb{R}$ by $g(x) = a_k$ if $x \in F_k$, k = 1, 2, ... n. Since the collection $\{F_k\}_{k=1}^{n}$ is disjoint, *g* is properly defined. We show that *g* is continuous on F.

Let $x \in F$ be arbitrary. Then $x \in F_k$ for some k. Let $\in > 0$ be arbitrary. Then we can find $d > 0$ such that $(x-d, x+d) \cap F_i = f$ for all $i \neq k$.

And for any $y \in (x-d, x+d) \cap F$

$g(y) = a_k$. Hence $|g(x) - g(y)| = 0 < \in$ for all $y \in F$ such that $|x-y| < d$.

This shows that *g* is continuous at *x*.

Since $x \in F$ is arbitrary, *g* is continuous on F. This function *g* which is continuous on a closed set F can be extended to a continuous function on $\mathbb{R}$. And for this extended continuous function *g* we have

$f(x) = g(x)$ on F and $m(E-F) < \in$.

5. Lusin's Theorem

Let *f* be a real valued measurable function on E. Then for each $\in > 0$ there is a continuous function *g* on $\mathbb{R}$ and a closed set F contained on E for which *f* = *g* on F and $m(E-F) < \in$.

Proof : We prove the theorem for a measurable set E such that $m(E) < \infty$. Since *f* is a measurable function, by Simple Approximation Theorem, there is a sequence $\{f_n\}$ of simple functions defined on E which converges to *f* pointwise on E. Let *n* be a natural number, for each simple function f_n and for any $\in > 0$ there exists a continuous function g_n on $\mathbb{R}$ and a closed set F_n contained in E such that,

$$f_n = g_n \text{ on } f_n \text{ and } m(E-F_n) < \frac{\in}{2^{n+1}}.$$

Also by Egoroff's theorem there is a closed set F_0 such that $F_0 \subseteq E$ and $\{f_n\}$ converges to *f* uniformly on F_0 and $m(E-F_0) < \frac{\in}{2}$.

Define $F = \bigcap_{n=0}^{\infty} F_n$. Then,

$$m(E-F)=m\left(E-\bigcap_{n=0}^{\infty}F_n\right)=m\left(E\cap\left(\bigcap_{n=0}^{\infty}F_n\right)^c\right)$$

$$=m\left(E\cap\bigcup_{n=0}^{\infty}F_n{}^c\right)$$

$$=m\left(\bigcup_{n=0}^{\infty}\left(E\cap F_n{}^c\right)\right)=m\left(\bigcup_{n=0}^{\infty}\left(E-F_n\right)\right)$$

$$=m\left((E-F_0)\cup\bigcup_{n=1}^{\infty}(E-F_n)\right)$$

$$\leq m(E-F_0)+m\left(\bigcup_{n=1}^{\infty}(E-F_n)\right)$$

$$\leq m(E-F_0)+\sum_{n=1}^{\infty}m(E-F_n)$$

$$<\frac{\in}{2}+\sum_{n=1}^{\infty}\frac{\in}{2^{n+1}}=\frac{\in}{2}+\frac{\in}{2}=\in$$

$$\Rightarrow m(E-F)<\in$$

Also F is countable intersection of closed set and hence it is closed. Also $F\subseteq F_n$ for all n and $f_n=g_n$ on F_n. Hence each f_n is continuous on F and $f_n=g_n$ on F. Also $\{f_n\}$ converges uniformly on F to the function f $(\because F\subseteq F_0)$. Here f is also continuous on F. And this function f on F can be extended to a continuous function g defined on $\mathbb{R}$ such that $f=g$ on F where $m(E-F)<\in$.

UNIT - IV

LEBESGUE INTEGRAL

Introduction :

We have studied theory of Riemann integration which is very useful in solving many mathematical problems. But there are some drawbacks. First of all the Riemann integral of a function is defined on a closed interval and cannot be defined on arbitrary set. Some problems in Probability theory, Hydrodynamics Quantum mechanics requires integration of a function over a set which may not be an interval. Further the function *f* must be bounded and continuous almost every where so that its Riemann integral exist. Also, for a sequence { f_n } of functions which converges to *f*, the sequence of integrals, { f_n } need not converge to $\int f$ or even $\int f$ does not exist sometimes.

Henry Lebesgue in his classical work introduced the concept of an integral based on the measures theory which generalizes the Riemann integral. The theory of Lebesgue integral tries to overcome the drawbacks of Riemann integral.

4.1 Riemann Integral :

1. Let *f* be a bounded real valued function defined on the interval [a, b]. Let P be a partition of [a, b] given by,

$$P = \{a = x_o < x_1 < x_2 < ... < x_n = b\}$$

Consider the sums,

$$U(f,P) = \sum_{i=1}^{n} (x_i - x_{i-1}) \cdot M_i \text{ and } L(f,P) = \sum_{i=1}^{n} (x_i - x_{i-1}) \cdot m_i$$

where $M_i = \sup\limits_{x \in (x_{i-1}, x_i)} f(x)$ and $m_i = \inf\limits_{x \in (x_{i-1}, x_i)} f(x)$

where i = 1, 2, 3, ..., n.

The upper Riemann integral of *f* over [a, b] is defined by,

$$(R)\overline{\int_a^b} f(x)dx = \inf U(f,P)$$

and the lower Riemann integral of *f* over [a, b] is defined by

$$(R)\underline{\int_a^b} f(x)dx = \sup L(f,P)$$

where the supremum and infimum are taken over all possible partitions P of [a, b].

If, $$(R)\int_{a}^{\overline{b}} f(x)dx = (R)\int_{\underline{a}}^{b} f(x)dx$$

then we say that Riemann integral of *f* over [a, b] exists and the common value of lower and upper integral is called the Riemann integration of *f* over [a, b]. Thus,

$$(R)\int_{a}^{b} f(x)dx = (R)\int_{\underline{a}}^{b} f(x)dx = (R)\int_{a}^{\overline{b}} f(x)dx$$

Note that in order that the function *f* be Riemann integrable, it is necessary for it to be bounded. We give another definition of Riemann integral of a bounded function using step functions.

Let $\psi : [a,b] \to \mathbb{R}$ be a function defined by

$$\psi(x) = c_i\ ,\ x_{i-1} < x < x_i \quad (\text{i} = 1, 2, 3, ..., \text{n})$$

where $a = x_0 < x_1 < x_2 < ... < x_n = b$ is a partition of [a, b]. ψ is called a step function.

Observe that,

$L(y\ ,\ p) = \sum c_i\,(x_i - x_{i-1}) = U(y\ ,\ p)$ for any partition P of [a, b].

Thus step function y is integrable and $(R)\int_{a}^{b} y = \sum c_i\,(x_i - x_{i-1})$.

2. Definition :

For any function *f* we define lower and upper Riemann integrals as follows :

$$(R)\int_{\underline{a}}^{b} f = \sup\left\{ (R)\int_{a}^{b} f \mid f \text{ is a step function and } f \le f \text{ on}[\text{a,b}] \right\}$$

and $$(R)\int_{a}^{\overline{b}} f = \inf\left\{ (R)\int_{a}^{b} y \mid y \text{ is a step function and } y \ge f \text{ on}[\text{a,b}] \right\}$$

3. ***Example :*** If $f : [0,1] \to \mathbb{R}$ defined by

f(x) = 1 if x is rational

f(x) = 0 if x is irrational

Show that $(R)\int_{\underline{0}}^{1} f(x)dx = 0$, $(R)\int_{0}^{\overline{1}} f(x)dx = 1$

The function *f* is called Dirichlet's function.

Solution : For any partition P of [0, 1]

$M_i = 1$ and $m_i = 0 \qquad \forall\, i$

Hence,

$$U(f,P) = \sum_i (x_i - x_{i-1}) \cdot M_i = \sum_i (x_i - x_{i-1}) = 1$$

$$L(f,P) = \sum_i (x_i - x_{i-1}) \cdot m_i = 0$$

Hence inf $U(f, P) = 1$ and sup $L(f, P) = 0$

$$\Rightarrow (R)\overline{\int_0^1} f(x)dx = 1 \quad \text{and} \quad (R)\underline{\int_0^1} f(x)dx = 0$$

4. Note :

(1) In the above example the given function is not Riemann integrable. In due course we show that its Lebesgue integral exist.

(2) A sequence $\{f_n\}$ of Riemann integrable functions need not converge to a Riemann integrable function.

4.2 Lebesgue Integral of a Bounded Measurable Functions :

1. Definition : Let E be a measurable set. The function χ_E defined by**,**

$$\chi_E(x) = 1 \quad \text{if} \quad x \in E$$

$$\chi_E(x) = 0 \quad \text{if} \quad x \notin E$$

is called the characteristic function of E,

We define Lebesgue integral of χ_E by

$$\int \chi_E = m(E)$$

2. Definition : A measurable real valued function y defined on a set E is said to be simple if it takes only finite number of real values.

If y takes distinct valyes $a_1, a_2, \ldots a_n$ on E, then define $E_i = \{x \in E \mid y(x) = a_i\} = y^{-1}(a_i)$.

Then $$y = \sum_{i=1}^{n} a_i c_{E_i} \text{ on E.}$$

This is called a canonical representation of y. In this representation all E_i**'s** are disjoint and a_i's are distinct.

3. **Definition :** For a simple function y defined on a set of finite measure E, we define integral of y over E by $\int_E y = \sum_{i=1}^{n} a_i m(E_i)$ where $y = \sum_{i=1}^{n} a_i c_{E_i}$ is a canonical representation.

4. ***Example :*** If $\phi = 2\chi_{A_1} + 3\chi_{A_2}$ where $A_1 = [2, 3]$, $A_2 = [4, 7]$ find $\int \phi$

Solution :

$$\int \phi = \sum_i a_i m(A_i)$$

$$= 2 \cdot m(A_1) + 3 \cdot m(A_2)$$

$$= 2\ell(A_1) + 3 \cdot \ell(A_2)$$

$$= 2 \times 1 + 3 \times 3$$

$$= 11$$

5. ***Example :*** If $f : [0,1] \to \mathbb{R}$ defined by

$f(x) = 1$ if x is rational

$f(x) = 0$ if x is irrational

Find $\int_I f$ where I = [0, 1]

Solution : If A is a set of rational numbers in [0, 1] then $f = \chi_A$. Hence

$$\int_I f = \int \chi_A = m(A) = 0 \qquad \text{since A is countable}$$

The following lemma shows that the elementary integral is independent of the choice of the representation of the simple function.

6. **Lemma :** Let $\{E_i\}_{i=1}^{n}$ be a finite disjoint collection of measurable subsets of a set of finite measure E. If $f = \sum_{i=1}^{n} a_i c_{E_i}$, a_i's are real numbers, $1 \le i \le n$ then $\int_E f = \sum_{i=1}^{n} a_i m(E_i)$.

Proof : Given $f = \sum_{i=1}^{n} a_i \cdot c_{E_i}$. Here E_i's are disjoint but the numbers a_i's need not be distinct. Hence the representation of may not be canonical.

Let $\{l_1, l_2, ... l_m\}$ be the distinct values of f.

Define $A_j = \{x \in E \mid f(x) = l_j\}$, $1 \le j \le m$

Then $f = \sum_{j=1}^{m} l_i \cdot c_{A_j}$ is a canonical representation of f and hence $\int_E f = \sum_{j=1}^{m} l_i m(A_j)$.

Now for each j, let I_j be the set of indices i in the set of indices I = {1, 2, ...n} such that $a_i = l_j$.

Then $I = \bigcup_{j=1}^{m} I_j$. Therefore,

$$m(A_j) = m\left(\bigcup_{i \in I_j} E_i\right) \qquad \left(\text{on } \bigcup_{i \in I_j} E_i, \ f(x) = l_j\right)$$

$$= \sum_{i \in I_j} m(E_i) \qquad (\text{Since } E_i\text{'s are disjoint})$$

Therefore,

$$\sum_{j=1}^{m} l_j m(A_j) = \sum_{j=1}^{m} \sum_{i \in I_j} l_j m(E_i)$$

$$= \sum_{j=1}^{m} \sum_{i \in I_j} a_j m(E_i) \qquad (\because a_i = l_j \text{ on } I_j)$$

$$= \sum_{i=n}^{n} a_j m(E_i)$$

Hence $$\int_E f = \sum_{i=1}^{n} a_i m(E_i)$$

7. **Proposition :** Let ϕ and ψ be the simple functions which vanishes outside a set of finite measure E.

Then (1) $\int a\phi + b\psi = a\int \phi + b\int \psi$ (2) $\phi \geq \psi a \cdot e \Rightarrow \int \phi \geq \int \psi$

Proof :

(1) Let $\phi = \sum_{i=1}^{m} \alpha_i \cdot \chi_{A_i}$ and $\psi = \sum_{j=1}^{n} \beta_i \cdot \chi_{B_j}$ be the canonical representation of ϕ and ψ on a set of finite measure E. Let A_o and B_o be the sets where ϕ and ψ are zero respectively.

Then, $$E = \bigcup_{i=0}^{m} A_i = \bigcup_{j=0}^{n} B_j$$

Therefore, $E = E \bigcap E$

$$= \bigcup_{i=0}^{m} A_i \cap \bigcup_{j=0}^{n} B_j$$

$$= \bigcup_{(i,j)} \left(A_i \cap B_j \right)$$

$$= \bigcup_{k=0}^{\ell} E_k \qquad \text{where } E_k = A_i \bigcap B_j$$

Since $\{A_i\}_{i=0}^{m}$ and $\{B_j\}_{j=0}^{n}$ are disjoint collections of measurable sets. Therefore the collection $\{E_k\}_{k=0}^{\ell}$ is also a disjoint collection of measurable sets and

$$\phi = \sum_{k=0}^{\ell} a_k \chi_{E_k}, \quad \psi = \sum_{k=0}^{\ell} b_k \chi_{E_k}$$

Hence,

$$a\phi + b\psi = \sum_{k=0}^{\ell} \left(aa_k + bb_k \right) \chi_{E_k}$$

And,

$$\int a\phi + b\psi = \sum_{k=0}^{\ell} \left(aa_k + bb_k \right) m \left(E_k \right)$$

$$= \sum_{k=0}^{\ell} aa_k m\left(E_k\right) + \sum_{k=0}^{\ell} bb_k m\left(E_k\right)$$

$$= a \sum_{k=0}^{\ell} a_k m\left(E_k\right) + b \sum_{k=0}^{\ell} b_k m\left(E_k\right)$$

$$\int a\phi + b\psi = a\int \phi + b\int \psi$$

(2) For a = 1, b = –1 above result becomes,

$$\int \phi - \psi = \int \phi - \int \psi$$

Now $\phi \geq \psi a \cdot e$

$$\Rightarrow \phi - \psi \geq 0 a \cdot e$$

$$\Rightarrow \int \phi - \psi \geq 0$$

$$\Rightarrow \int \phi - \int \psi \geq 0$$

$$\Rightarrow \int \phi \geq \int \psi$$

8. **Note :** If $\phi = \sum_{i=1}^{n} a_i \cdot \chi_{E_i}$ is any representation for ϕ where a_i's are not necessary distinct and E_i's need not be pairwise disjoint then,

$$\phi = a_1 \cdot \chi_{E_1} + a_2 \cdot \chi_{E_2} + a_3 \cdot \chi_{E_3} + \ldots + a_n \cdot \chi_{E_n}$$

$$\Rightarrow \int \phi = \int \left(a_1 \cdot \chi_{E_1} + a_2 \cdot \chi_{E_2} + \ldots + a_n \cdot \chi_{E_n} \right)$$

$$= a_1 \int \chi_{E_1} + a_2 \int \chi_{E_2} + \ldots + a_n \int \chi_{E_n}$$

$$= a_1 m(E_1) + a_2 m(E_2) + \ldots + a_3 m(E_3)$$

$$= \sum_{i=1}^{n} a_i m(E_i)$$

Thus for any representation of simple function ϕ,

$$\int \phi = \sum_{i=1}^{n} a_i m(E_i)$$

9. **Note :** A step function takes only a finite number of values and each internal is measurable. Hence every step function is a simple function.

10. **Definition :** Let f be a boundd real-valued function defined on a set of finite measure E. We define the lower and Upper Lebesgue integral of f over E by

Lower Lebesgue Integral $= \sup \left\{ \int_E f \mid f \text{ is simple and } f \leq f \text{ on E} \right\}$

Upper Lebesgue Integral $= \inf \left\{ \int_E y \mid y \text{ is simple and } f \leq y \text{ on E} \right\}$

11. **Definition :** A bounded function f on a domain E of finite measure is said to be Lebesgue integrable over E if its Upper and Lower Lebesgue integrals over E are equal. The common value of the Upper and Lower integrals is called the Lebesgue integral or simply the integral of f over E and it is denoted by $\int_E f$.

12. **Theorem :** Let f be a bounded function defined on the closed bounded interval [A, b]. If f is Riemann integrable over [a, b], then it is Lebesgue integrable over [a, b] and the two integrals are equal.

Proof : f is Riemann integrable over [a, b]

$\Rightarrow$ Riemann lower and upper integrals are equal.

$$\Rightarrow \sup\left\{(R)\int_I f \mid f \text{ is a step function, } f \le f\right\}$$

$$= \inf\left\{(R)\int_I y \mid y \text{ is a step function, } f \le y\right\}$$

Now for simple function f and y such that

$$f \le f \le y$$

$$\Rightarrow \int_E f \le \int_E y$$

$$\Rightarrow \sup_{f \le f} \int_E f \le \inf_{f \le y} \int_E y \qquad \text{.... (1)}$$

where f and y are simple functions. But,

$$\inf_{\substack{y \ge f \\ y-simple}} \int_E y \le \inf_{\substack{y \ge f \\ y-step}} \int_E y = \sup_{\substack{f \le f \\ f-step}} \int_E f \le \sup_{\substack{f \le f \\ f-simple}} \int_E f \qquad \text{.... (2)}$$

(Supremum over larger set is larger and infinum over smaller set is larger. And every step function is a simple function.)

$$\Rightarrow \inf_{\substack{y \ge f \\ y-simple}} \int_E y \le \sup_{\substack{f \le f \\ f-simple}} \int_E f \qquad \text{.... (3)}$$

Hence from (1) and (3) we get

$$\sup_{\substack{f \le f \\ f-simple}} \int_E f = \inf_{\substack{f \le y \\ y-simple}} \int_E y$$

Hence *f* is Lebesgue integrable. The inequality (2) implies that all the terms are equal. Hence Lebesgue integral of *f* is equal to Riemann integral of *f*.

13. Example : Let E be the set of rational numbers in [0, 1]. Let *f* be a Dirichlet's function defined on [0, 1] by

$$\left.\begin{aligned} f(x) &= 1 \text{ if } x \in E \\ &= 0 \text{ if } x \notin E \end{aligned}\right\} \forall x \in [0,1]$$

Then $\int_{[0,1]} f = \int_{[0,1]} 1 \cdot c_E = 1 \cdot m(E) = 0$ (Since E is countable)

Earlier we have shown that *f* is not Riemann integrable. Thus *f* is Lebesgue integrable but not Riemann integrable and Lebesgue integral of *f* is $\int_{[0,1]} f = 0$.

14. Theorem : Let f be a bounded measurable function on a set of finite measure E. Then f is integrable over E.

Proof : Let n be a natural number. Take $\in = \frac{1}{n}$. By simple approximation Lemma, there exists two simple functions f_n and y_n on E such that

$$f_n \leq f \leq y_n \text{ and } 0 \leq y_n - f_n \leq \frac{1}{n} \text{ on E.}$$

Applying monotone property and linearity property, we get

$$0 \leq \int_E y_n - f_n = \int_E y_n - \int_E f_n \leq \int_E \frac{1}{n} = \frac{1}{n}\int_E 1 = \frac{1}{n} m(E)$$

$$\Rightarrow 0 \leq \inf\left\{\int_E y \mid y \text{ is simple function, } y \geq f\right\} - \sup\left\{\int_E f \mid f \text{ is simple function, } f \leq f\right\}$$

$$\leq \int_E y_n - \int_E f_n \leq \frac{1}{n} \cdot m(E) \qquad \forall n \in \mathbb{N}$$

$$\Rightarrow \inf\left\{\int_E y \mid y \text{ is simple function, } y \geq f\right\} = \sup\left\{\int_E f \mid f \text{ is simple function, } f \leq f\right\}$$

$\Rightarrow$ f is Lebesgue integrable over E.

15. Proposition : If f and g are bounded measurable function defined on a measurable set of finite measure then (i) $\int_E \alpha f + \beta g = \alpha \int_E f + \beta \int_E g$

(ii) $f = g \; a \cdot e \Rightarrow \int_E f = \int_E g$

(iii) $f \leq g \; a \cdot e \Rightarrow \int_E f \leq \int_E g$, Hence, $\left|\int_E f\right| \leq \left|\int_E g\right|$

(iv) If $A \leq f(x) \leq B$ then $A \cdot m(E) \leq \int_E f \leq B \cdot m(E)$

(v) If A and B are disjoint measurable sets of finite measure then $\int_{A \cup B} f = \int_A f + \int_B f$.

Proof :

(i) First we prove that,

$$\int_E \alpha f = \alpha \int_E f \qquad \alpha \in \mathbb{R}$$

If a = 0 then equality holds trivially.

If a > 0 then,

$$\int_E \alpha f = \inf_{\alpha f \le \psi} \int_E \psi = \inf_{f \le \psi/\alpha} \int_E \psi$$

Let $\frac{\psi}{\alpha} = \phi \Rightarrow \psi = \alpha\,\phi$

Therefore, $$\int_E \alpha f = \inf_{f \le \phi} \int_E \alpha\phi$$

$$= \inf_{f \le \phi} \alpha \int_E \phi$$

$$= \alpha \inf_{f \le \phi} \int_E \phi$$

$$= \alpha \int_E f$$

If a < 0 then

$$\int_E \alpha f = \inf_{\alpha f \le \psi} \int_E \psi$$

$$= \inf_{f \ge \psi/\alpha} \int_E \psi \qquad \text{(since a is negative)}$$

$$= \inf_{f \ge \phi} \int_E \alpha\phi \qquad \frac{\psi}{\alpha} = \phi \Rightarrow \psi = \alpha\,\phi$$

$$= \inf_{f \ge \phi} \alpha \int_E \phi$$

$$= \alpha \sup_{f \ge \phi} \int_E \phi \qquad \text{(since } a \text{ is negative)}$$

$$= \alpha \int_E f$$

Thus for all $\alpha \in \mathbb{R}$

$$\int_E \alpha f = \alpha \int_E f$$

Now we show that,

$$\int_E f + g = \int_E f + \int_E g$$

Let ψ_1 and ψ_2 be the two simple functions such that $f \leq \psi_1$, $g \leq \psi_2$ therefore $f + g \leq \psi_1 + \psi_2$.

Then,

$$\int_E f + g = \inf_{f+g \leq \psi} \int_E \psi \leq \int_E \psi_1 + \psi_2 = \int_E \psi_1 + \int_E \psi_2$$

Taking infimum over all simple functions $\psi_1 \geq f$ and $\psi_2 \geq g$ we get,

$$\int_E f + g \leq \inf_{f \leq \psi_1} \int_E \psi_1 + \inf_{g \leq \psi_2} \int_E \psi_2$$

$$\Rightarrow \int_E f + g \leq \int_E f + \int_E g \qquad \text{... (a)}$$

Similarly if $\phi_1 \leq f$ and $\phi_2 \leq g$ are simple functions such that $\phi_1 + \phi_2 \leq f + g$, then . And,

$$\int_E f + g = \sup_{\phi \leq f+g} \int_E \phi$$

$$\geq \int_E \phi_1 + \phi_2$$

$$= \int_E \phi_1 + \int_E \phi_2$$

Taking supremum over all simple functions $\phi_1 \leq f$ and $\phi_2 \leq g$ we get,

$$\int_E f + g \geq \sup_{\phi_1 \leq f} \int_E \phi_1 + \sup_{\phi_2 \leq g} \int_E \phi_2$$

$$\Rightarrow \int_E f + g \geq \int_E f + \int_E g \qquad \text{... (b)}$$

Hence from (a) and (b) we get,

$$\int_E f + g = \int_E f + \int_E g$$

Therefore, $\int_E \alpha f + \beta g = \int_E \alpha f + \int_E \beta g$

$$= \alpha \int_E f + \beta \int_E g$$

(ii) If $f = g$ a.e. then $f - g = 0$ a.e.

If ψ is a simple function such that $\psi \geq f - g = 0$ a.e.

$$\Rightarrow \psi \geq 0 \ \ a \cdot e \Rightarrow \int_E \psi \geq 0$$

Taking infimum over all simple functions $\psi \geq f - g$ we get,

$$\inf_{\psi \geq f-g} \int_E \psi \geq 0$$

$$\Rightarrow \int_E f - g \geq 0$$

$$\Rightarrow \int_E f - \int_E g \geq 0$$

$$\Rightarrow \int_E f \geq \int_E g$$

Similarly if ϕ is a simple function such that $\phi \leq f - g = 0$ a.e., we can show that,

$$\int_E f - g \leq 0 \Rightarrow \int_E f \leq \int_E g$$

Hence, $\int_E f = \int_E g$

iii) $f \leq g$ a.e.

$\Rightarrow 0 \leq g - f$ a.e.

If ψ is a simple function such that $\psi \geq g - f \geq 0$ then $\psi \geq 0$ a.e.

$$\Rightarrow \int_E \psi \geq 0$$

$$\Rightarrow \inf_{\psi \geq g-f} \int_E \psi \geq 0$$

$$\Rightarrow \int_E g - f \geq 0$$

$$\Rightarrow \int_E g - \int_E f \geq 0$$

$$\Rightarrow \int_E g \geq \int_E f$$

Thus $f \leq g$ a.e. $\Rightarrow \int_E f \leq \int_E g$

Since f is bounded on E,

$$-|f| \leq f \leq |f|$$

$$\Rightarrow \int_E -|f| \leq \int_E f \leq \int_E |f|$$

$$\Rightarrow -\int_E |f| \leq \int_E f \leq \int_E |f|$$

$$\Rightarrow \left|\int_E f\right| \leq \int_E |f|$$

iv) $A \leq f(x) \leq B, \ \forall \ x \in E$

$$\Rightarrow A \cdot 1 \leq f(x) \leq B \cdot 1, \ \forall x \in E$$

$$\Rightarrow A \cdot \chi_E(x) \leq f(x) \leq B\chi_E(x)$$

$$\Rightarrow \int A \cdot \chi_E(x) \leq \int_E f(x) \leq \int B\chi_E(x)$$

$$\Rightarrow A \cdot m(E) \leq \int_E f \leq B \cdot m(E)$$

v) If A and B are disjoint measurable sets then χ_A, χ_B , $\chi_{A \cup B}$ are measurable functions and

$$\chi_{A \cup B} = \chi_A + \chi_B$$

$$\Rightarrow f \cdot \chi_{A \cup B} = f(\chi_A + \chi_B)$$

$$= f \cdot \chi_A + f \cdot \chi_B$$

$$\Rightarrow \int_E f \cdot \chi_{A \cup B} = \int_E f \cdot \chi_A + \int_E f \cdot \chi_B$$

$$\Rightarrow \int_{A \cup B} f = \int_A f + \int_B f$$

16. **Note :** From the above proposition we conclude that,

1) If $f(x) \geq 0$ on E then $\int_E f(x)dx \geq 0$ and if $f(x) \leq 0$ on E, then $\int_E f(x)dx \leq 0$

2) If m(E) = 0 then $\int_E f = 0$

3) If f (x) = K a.e. on E, then $\int_E f = Km(E)$

4) The result (ii) in the above proposition is one of the advantage over the Riemann integral.

Change in the value of function f on a set of measure zero has no effect on the Lebesgue integrability of f or on the value of its integral. On the other hand changing the values of a Riemann integrable function on a set of measure zero may affect the Riemann integrability of the function or the value of its integral.

In the above proposition, converse of (ii) need not be true. We discuss the following example.

17. ***Example :*** Let $f:[-1,1] \to \mathbb{R}$ and $g:[-1,1] \to \mathbb{R}$ be the functions defined by,

$$f(x) = 2, \quad x \leq 0$$
$$= 0, \quad x > 0$$

and $g(x) = 1 \quad \forall x$

Then clearly $f \neq g \;\; a \cdot e$. In fact they are not equal even for a single point in the domain.

Also $f = 2 \cdot \chi_{[-1,0]}$ and $g = \chi_{[-1,1]}$

Therefore, $$\int_{-1}^{1} f = \int_{-1}^{1} 2\chi_{[-1,0]} = 2m[-1,0] = 2$$

and $$\int_{-1}^{1} g = \int \chi_{[-1,1]} = m[-1,1] = 2$$

Thus, $\int_{-1}^{1} f = \int_{-1}^{1} g$ but $f \neq g$

18. **Proposition :** Let $\{f_n\}$ be a sequence of bounded measurable functions on a set of finite measure E. If $\{f_n\} \to f$ uniformly on E, then $\lim_{n\to\infty} \int_E f_n = \int_E f$.

Proof : Since the convergence is uniform and each f_n is bounded, the limit function f is bounded. Also f is a pointwise limit of a sequence of measurable functions. Therefore f is also measurable. Since $f_n \to f$ uniformly, for given $\in > 0$ there is a positive integer N such that

$$|f \to f_n| < \frac{\in}{m(E)} \text{ for all } n \geq N \text{ and for all } x \in E$$

Therefore

$$\left|\int_E f \to \int_E f_n\right| = \left|\int_E f - f_n\right| \leq \int_E |f - f_n| \leq \int_E \frac{\in}{m(E)} = \frac{\in}{m(E)} \cdot m(E) = \in$$

i.e. $\left|\int_E f \to \int_E f_n\right| < \in$ for all $n \geq N$

$\Rightarrow \int_E f_n \to \int_E f$ i.e. $\lim\limits_{n\to\infty} \int_E f_n = \int_E f$

19. Note : If Convergence $f_n \to f$ is not uniform then $\int f_n$ need not converge to $\int f$. **We** have the following :

For each natural number n we define f_n on $\mathbb{R}^+$ by $f_n(x) = n$, if $x \in \left[\frac{1}{n}, \frac{2}{n}\right] = 0$ otherwise

Then $\int f_n = 1$ for all n. Therefore $\lim \int f_n = 1$. But $\lim\limits_{n\to 0} f_n = 0 \Rightarrow \int f = 0$.

Hence $\lim \int f_n \neq \int f$.

20. The Bounded Convergence Theorem

Let $\{f_n\}$ be a sequence of measurable functions on a set of finite measure E. Suppose $\{f_n\}$ is uniformly pointwise bounded on E. i.e. there exists a number $M \geq 0$ such that $|f_n| \leq M$ on E for all n. If $\{f_n\} \to f$ pointwise on E then $\lim\limits_{n\to\infty} \int_E f_n = \int_E f$.

Proof : Since pointwise limit of a sequence of measurable functions is measurable. Therefore f is measurable. Also $|f_n| \leq M$ for all n on E $\Rightarrow |f| \leq M$ on E. Now for any measurable subset of E, $E = A \cup (E - A)$. Therefore for any n,

$$\int_E f_n - \int_E f = \int_E (f_n - f) = \int_{A\cup(E-A)} (f_n - f) = \int_A (f_n - f) + \int_{E-A} (f_n - f)$$

$$= \int_A (f_n - f) + \int_{E-A} f_n + \int_{E-A} (-f)$$

$$\Rightarrow \left| \int_E f_n - \int_E f \right| \le \int_A |f_n - f| + \int_{E-A} |f_n| + \int_{E-A} |-f|$$

$$\le \int_A |f_n - f| + 2M \int_{E-A} 1$$

$$\le \int_A |f_n - f| + 2Mm(E - A)$$

Now for any $\in > 0$, and E is a set of finite measure, by Egoroff's theorem, there exists a measurable set A of E for which $f_n \to f$ uniformly on A and $m(E - A) < \frac{\in}{4M}$. Since $f_n \to f$ uniformly on A, there exists an integer N such that

$$|f_n - f| < \frac{\in}{2m(E)}, \ \forall n \ge N \text{ and } \forall x \in A$$

Therefore for $n \ge N$ we get,

$$\left| \int_E f_n - \int_E f \right| \le \int_A |f_n - f| + 2Mm(E - A)$$

$$\Rightarrow \left| \int_E f_n - \int_E f \right| \le \frac{\in \cdot m(A)}{2m(E)} + 2M \frac{\in}{4M}$$

$$< \frac{\in}{2} + \frac{\in}{2} = \in$$

$$\Rightarrow \left| \int_E f_n - \int_E f \right| < \in \text{ for all } n \ge N.$$

Hence the sequence $\left\{ \int_E f_n \right\}$ converges to $\int_E f$.

i.e. $$\lim_{n\to\infty} \int_E f_n = \int_E f$$

21. **Note :** A measurable function f on E is said to vanish outside a set of finite measure if there is a subset E_0 of E for which $m(E_0) < \infty$ and $f = 0$ on $E - E_0$.

We define a support of a function f as a set, supp $f = \{x \in E \mid f(x) \neq 0\}$

If a function f vanishes outside a set of finite measure then f has a finite support.

If f is bounded measurable function defined on a set E and has a finite support E_0 then

$$\int_E f = \int_{E_0 \cup (E - E_0)} f = \int_{E_0} f + \int_{E - E_0} f = \int_{E_0} f$$

where E_0 is measurable set with finite measure and $f = 0$ on $E - E_0$. This definition also holds for a measurable set E with $m(E) = \infty$.

4.3. Lebesgue Integral of a Non-negative Measurable Function :

1. **Definition :** Let f be a nonnegative measurable function on E. We define integral of f over E by

$$\int_E f = \sup \{ \int_E h \mid h \text{ is bounded measurable function of finite support and } 0 \leq h \leq f \text{ on E}\}$$

i.e. $$\int_E f = \sup_{h \leq f} \int_E h,$$

Where supremum is taken over all bounded measurable functions on E with finite measurable support.

2. **Theorem : Chebychev's Inequality :** Let f be a nonnegative measurable function on E. Then for any $l > 0$,

$$m\{x \in E \mid f(x) \geq l\} \leq \frac{1}{l} \int_E f$$

Proof : For any real $l > 0$, define a set E_l by,

$$E_l = \{x \in E \mid f(x) \geq l\}$$

Then E_l is measurable for all l.

First suppose that $m(E_l) = \infty$.

For any natural number n, define sets $E_{l,n}$ by $E_{l,n} = E_l \cap [-n, n]$.

Then the sequence $\{E_{l,n}\}$ is an increasing sequence of measurable sets such that

$$\bigcup_{n-1}^{\infty} E_{l,n} = E_l \qquad \left(\because E_l = E_l \cap \mathbb{R} = E_l \cap \bigcup_{n=1}^{\infty}[-n,n] = \bigcup_{n=1}^{\infty} E_{l,n}\right)$$

Define a function $y_n,\ n \in \mathbb{N}$ by

$$y_n(x) = l \cdot c_{E_{l,n}}(x),\ \forall x \in E$$

Then $y_n(x) = l$ if $x \in E_{l,n}$ and $y_n(x) = 0$ if $x \notin E_{l,n}$. Hence $y_n(x) \le f(x)$ on E for all n.

$$\Rightarrow \int_E f \ge \int_E y_n \text{ for all } n$$

$$\Rightarrow \int_E f \ge \lim_{n\to\infty} \int_E y_n$$

But $$\int_E y_n = \int_E l \cdot c_{E_{l,n}} = l \cdot m\left(E_{l,n}\right)$$

$$\Rightarrow \lim_{n\to\infty} \int_E y_n = \lim_{n\to\infty} \int_E l \cdot m\left(E_{l,n}\right)$$

$$= l \lim_{n\to\infty} m\left(E_{l,n}\right)$$

$$= l\, m\left(\bigcup_{n=1}^{\infty} E_{l,n}\right) \qquad \text{(By continuity of Lebesgue measure)}$$

$$= l\, m\left(E_l\right)$$

$$= \infty$$

Hence $\Rightarrow \int_E f \ge \lim_{n\to\infty} \int_E y_n = \infty \qquad \Rightarrow \int_E f = \infty$

Therefore, $\dfrac{1}{l}\int_E f = m\left(E_l\right)$

Now consider, $m\left(E_l\right) < \infty$. Define a function h by $h = l \cdot c_{E_l}$. Then $h \le l$ on E. Therefore h is bounded measurable function and support of h is E_l whose measure is finite.

Therefore, $f \ge h \ge 0$ on E_l.

$$\Rightarrow \int_E f \ge \int_E h = \int_{E_l} h = l \int_{E_l} 1 = l \cdot m\left(E_l\right)$$

$$\Rightarrow m(E_l) \le \frac{1}{l}\int_E f$$

$$\Rightarrow m\{x \in E \mid f(x) \ge l\} \le \frac{1}{l}\int_E f$$

3. **Proposition :** Let f be a nonnegative measurable function on E. Then $\int_E f = 0$ if and only if $f = 0$ a.e on E.

Proof : First assume that $\int_E f = 0$. Then by Chebychev's inequality for each natural number n we have

$$\Rightarrow m\left\{x \in E \mid f(x) \ge \frac{1}{n}\right\} \le \frac{1}{n}\int_E f = 0$$

$$\Rightarrow m\left\{x \in E \mid f(x) \ge \frac{1}{n}\right\} = 0$$

Therefore by countable additive property of Lebesgue measure,

$$\{x \in E \mid f(x) > 0\} = \bigcup_{n=1}^{\infty}\left\{x \in E \mid f(x) \ge \frac{1}{n}\right\}$$

$$\Rightarrow m\{x \in E \mid f(x) > 0\} = m\left\{\bigcup_{n=1}^{\infty}\left\{x \in E \mid f(x) \ge \frac{1}{n}\right\}\right\}$$

$$\le \sum_{n=1}^{\infty} m\left\{x \in E \mid f(x) \ge \frac{1}{n}\right\}$$

$$= 0$$

$$\Rightarrow m\{x \in E \mid f(x) > 0\} = 0$$

$\Rightarrow f = 0 \ a \cdot e$ on E.

Conversely suppose that $f = 0$ a.e on E.

Let f be a simple function and h be a bounded measurable function of finite support such that

$$f \ge h \ge f \ge 0.$$

Then $\quad f = 0 \ a \cdot e \Rightarrow f = 0 a \cdot e$

$$\Rightarrow \int_E f = 0$$

But $\int_E h = \sup_{f \le h} \int_E f$, f is a simple function such that $f \le h$. Hence $\int_E h = 0$ since $\int_E f = 0$ for all $f \le h$.

Next $\int_E f = \sup_{n \le f} \int_E h$. Hence $\int_E f = 0$, since $\int_E h = 0$ for all bounded measurable functions with finite support such that $h \le f$.

Thus $f = 0 \; a \cdot e \Leftrightarrow \int_E f = 0$.

4. **Theorem :** If f and g are non negative measurable functions then,

(i) $\int_E \alpha f = \alpha \int f$, $a > 0$

(ii) $\int_E f + g = \int_E f + \int_E g$

(iii) $f \le g$ a.e. then $\int_E f \le \int_E g$

Proof :

(i) Since $f \ge 0$, $a > 0$, $\alpha f \ge 0$

By definition,

$$\int_E \alpha f = \sup_{h \le \alpha f} \int_E h$$

$$= \sup_{\frac{h}{\alpha} \le f} \int_E h \qquad \text{Let } \frac{h}{\alpha} = K \quad \therefore h = \alpha K$$

$$= \sup_{K \le f} \int_E \alpha K$$

$$= \sup_{K \le f} \alpha \int_E K$$

$$= \alpha \sup_{K \le f} \int_E K$$

$$= \alpha \int_E f$$

(ii) Let h and k be the bounded measurable functions such that $h \le f$ and $k \le g$

Then, $h + k \le f + g$

Now, $\int_E f+g = \sup_{\ell \le f+g} \int_E \ell$, ℓ is bounded measurable function.

$$\ge \int_E h+k \qquad \text{since } h+k \le f+g$$

$$= \int_E h + \int_E k$$

Taking supremum over all bounded measurable functions h, k such that $h \le f$ and $k \le g$ we get,

$$\int_E f+g \ge \sup_{h\le f} \int_E h + \sup_{k \le g} \int_E k$$

$$\Rightarrow \int_E f+g \ge \int_E f + \int_E g \qquad \text{... (1)}$$

Next, let ℓ be the bounded measurable function defined on a set of finite measure such that $\ell \le f+g$. Define the functions h and k by,

$$h(x) = \min\{f(x), \ell(x)\} \quad \text{and} \quad k(x) = \ell(x) - h(x)$$

Then h and k are bounded measurable functions. Further if f(x) < ℓ(x) then h(x) = f(x) and $k(x) = \ell(x) - h(x) \le f(x) + g(x) - f(x) = g(x)$.

And if $f(x) \ge \ell(x)$ then $h(x) = \ell(x)$ and $k(x) = \ell(x) - h(x) = \ell(x) - \ell(x) = 0 \le g(x)$

Thus $\forall x \in E$, $h(x) \le f(x)$, $k(x) \le g(x)$

Now $\quad k = \ell - h$

$$\Rightarrow \ell = k+h$$

$$\Rightarrow \int_E \ell = \int_E k+h$$

$$= \int_E k + \int_E h$$

$$\Rightarrow \int_E \ell \le \sup_{k\le g} \int_E k + \sup_{k \le f} \int_E h$$

$$\Rightarrow \int_E \ell \le \int_E g + \int_E f$$

Taking supremum over all bounded measurable functions $\ell \le f+g$, we get,

$$\sup_{\ell \le f+g} \int_E \ell \le \int_E f + \int_E g$$

$$\Rightarrow \int_E f+g \le \int_E f + \int_E g \qquad \text{... (2)}$$

From (1) and (2) we get,

$$\int_E f+g=\int_E f+\int_E g$$

(iii) Let $f \le g$ a.e. If h is bounded measurable function such that $h \le f$ then $h \le g$ a.e.

Therefore, $\{h \mid h\le f\} \subseteq \{h \mid h \le g\}$

$$\Rightarrow \sup_{h\le f}\int_E h \le \sup_{h\le g}\int_E h$$

$$\Rightarrow \int_E f \le \int_E g$$

5. Theorem : (Additivity Over Domains of Integration)

Let *f* be a nonnegative measurable function on E. If A and B are disjoint measurable subsets of E then,

$$\int_{A\cup B} f=\int_A f+\int_B f$$

In particular if E_0 is a subset of E of measure zero then,

$$\int_E f=\int_{E-E_0} f$$

Proof : Since A and B are measurable, $A\cup B$ is also measurable and the functions c_A, c_B and $c_{A\cup B}$ are measurable. Since A and B are disjoint, we have

$$c_{A\cup B}=c_A+c_B$$

Therefore, $$\int_{A\cup B} f=\int_E f\cdot c_{A\cup B}$$

$$=\int_E f\,[c_A+c_B]$$

$$=\int_E [f\cdot c_A+f\cdot c_B]$$

$$=\int_E f\cdot c_A+\int_B f\cdot c_B \qquad \text{(Linearily property)}$$

$$=\int_A f+\int_B f$$

Next if E_0 is a measurable subset of E then

$$E_0 = E_0 \cup (E - E_0)$$

Hence by above property,

$$\int_E f = \int_{E_0 \cup (E-E_0)} f = \int_{E_0} f + \int_{E-E_0} f$$

Now $m(E_0) = 0$. Hence $m\{x \in E_0 \mid f(x) > 0\} = 0$.

i.e. $f = 0 \; a \cdot e$ on E_0. Hence $\int_{E_0} f = 0$.

Therefore we get, $\int_E f = \int_{E-E_0} f$ where $m(E_0) = 0$.

6. Fatou's Lemma

Let $\{f_n\}$ be a sequence of nonnegative measurable functions on E. If $f_n \to f$ pointwise a.e on E, then

$$\int_E f \le \liminf \int_E f_n = \underline{\lim} \int_E f_n$$

Proof : Since Lebesgue measure over a set of measure zero is zero, we assume that $f_n \to f$ pointwise on E. Also $\{f_n\}$ is a sequence of nonnegative, measurable functions, the limit function f is nonnegative and measurable.

Let h be a bounded measurable function of finite support such that $h \le f$ on E.

Let $M \ge 0$ be a real number such that $|h| \le M$.

Let $E_0 = \{x \in E \mid h(x) \ne 0\}$. Then $m(E_0) < \infty$. For each natural number n, define a function h_n on E by

$$h_n(x) = \min\{h(x), f_n(x)\}, \; \forall x \in E$$

Then h_n is measurable for each n. And $0 \le h_n \le M$ on E_0 for all n and $h_n = 0$ on $E - E_0$.

Thus h_n is bounded $\forall n$. Further for each $x \in E$, $h(x) \le f(x)$ and $f_n(x) \to f(x)$ implies $h_n(x) \to h(x)$. Thus $\{h_n\}$ is a sequence of bounded measurable functions which converges pointwise on E to h. Therefore by bounded convergence theorem,

$$\lim_{n\to\infty} \int_E h_n = \lim_{n\to\infty} \int_{E_0} h_n = \int_{E_0} h = \int_E h \qquad \text{(Since } h_n = 0 \text{ and } h = 0 \text{ on } E - E_0 \text{)}$$

But for each n, $h_n \le f_n$. Hence $\int_E h_n \le \int_E f_n$, for all n.

Thus $\int_E h = \lim_{n\to\infty} \int_E h_n = \underline{\lim} \int_E h_n \le \underline{\lim} \int_E f_n$

Taking supremum over all bounded measurable functions $h \le f$ we get,

$$\sup_{n\le f} \int_E h \le \liminf \int_E f_n$$

$$\Rightarrow \int_E f \le \liminf \int_E f_n$$

7. **Note :** The inequality in Fatou's Lemma may be strict. We have the following example :

8. **Example :** Let $E = (0,1]$. For any natural number n, define $f_n = n \cdot c_{\left(0,\frac{1}{n}\right]}$ on E.

Then $\{f_n\}$ is a sequence of nonnegative measurable functions such that $f_n \to f = 0$ on E.

Hence $\int_E f = 0$. But $\int_E f_n = \int_E n \cdot c_{\left(0,\frac{1}{n}\right]} = nm\left(0,\frac{1}{n}\right]$.

$\Rightarrow \int_E f_n = 1 \quad \forall n$. Hence $\lim_{n\to\infty} \int_E f_n = 1$.

Thus $$\int_E f < \lim_{n\to\infty} \int_E f_n = \underline{\lim} \int_E f_n .$$

9. **Monotone Convergence Theorem :**

Let { f_n } be an increasing sequence of nonnegative measurable functions on E, and let $f = \lim f_n$ a.e. pointwise on E. Then,

$$\int_E f = \lim \int_E f_n$$

Proof : Since { f_n } is a sequence of nonnegative measurable functions, by Fatou's lemma

$$\int_E f \le \underline{\lim} \int_E f_n \qquad \text{... (1)}$$

Also { f_n } is an increasing sequence and $f_n \to f$ a.e. Hence $f_n \le f$ for all $n \in \mathbb{N}$ which implies, $\int_E f_n \le \int_E f$ for all $n \in \mathbb{N}$

$$\Rightarrow \sup_{n\geq k} \int_E f_n \leq \int_E f \quad \text{for all } \; k \in \mathbb{N}$$

$$\Rightarrow \inf_k \sup_{n\geq k} \int_E f_n \leq \int_E f$$

$$\Rightarrow \overline{\lim} \int_E f_n \leq \int_E f \qquad \text{... (2)}$$

From (1) and (2) we get

$$\int_E f \leq \underline{\lim} \int_E f_n \leq \overline{\lim} \int_E f_n \leq \int_E f$$

$$\Rightarrow \int_E f = \underline{\lim} \int_E f_n = \overline{\lim} \int_E f_n$$

$$\Rightarrow \int_E f = \lim \int_E f_n$$

10. **Corollary :** Let { u_n } be a sequence of nonnegative measurable functions, and let $f = \sum_{n=1}^{\infty} u_n$. Then

$$\int f = \sum_{n=1}^{\infty} \int u_n$$

Proof : Define a sequence of functions { f_n } by,

$$f_n = \sum_{k=1}^{n} u_k$$

Since, u_k's are nonnegative measurable functions f_n's are nonnegative measurable and { f_n } is an increasing sequence of nonnegative measurable functions and

$$f_n \to f = \sum_{k=1}^{\infty} u_k$$

Therefore by monotonic convergence theorem we get,

$$\int f = \lim \int f_n$$

$$= \lim_{n\to\infty} \int \sum_{k=1}^{n} u_k$$

$$= \lim_{n\to\infty} \int (u_1 + u_2 + \ldots.. + u_n)$$

$$= \lim_{n\to\infty} \int u_1 + \int u_2 + \ldots.. + \int u_n$$

$$= \lim_{n\to\infty} \sum_{k=1}^{n} \int u_k$$

$$= \sum_{k=1}^{\infty} \int u_k$$

Thus, $\int f = \sum_{k=1}^{\infty} \int u_k$

or $\int \sum_{k=1}^{\infty} u_k = \sum_{k=1}^{\infty} \int u_k$

11. Proposition : Let f be a nonnegative measurable function and $\{ E_i \}$ be a disjoint sequence of measurable sets. If $E = UE_i$ then $\int_E f = \sum_i \int_{E_i} f$

Proof : Let $u_i = f \cdot \chi_{E_i}$

Then, $f \cdot \chi_E = f \cdot \chi_{\bigcup_i E_i}$

$$= f\left[\chi_{E_1} + \chi_{E_2} + ...\right]$$

$$= f \cdot \sum_i \chi_{E_i}$$

$$= \sum_i f \cdot \chi_{E_i}$$

$$= \sum_i u_i$$

Hence by corollary 19,

$$\int_E f = \int f \cdot \chi_E = \sum_i \int u_i$$

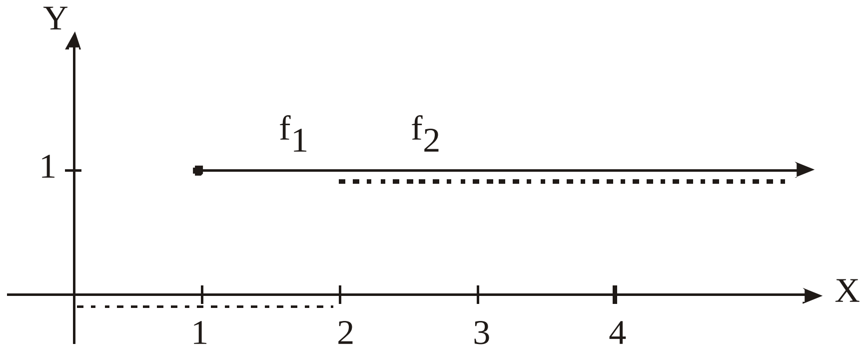
$$= \sum_i \int f \cdot \chi_{E_i}$$

$$\Rightarrow \int_E f = \sum_i \int_{E_i} f$$

12. ***Example :*** Show that Monotone Convergence theorem need not be true for decreasing sequence of functions.

Solution : Let { f_n } be a sequence of functions defined by,

$$f_n(x) = \begin{cases} 0 & x < n \\ 1 & x \geq n \end{cases} \quad \text{for all } x \in \mathbb{R}$$



$f_1 > f_2 > f_3 > \ldots\ldots\ldots$

Then { f_n } is a decreasing sequence of measurable functions and $f_n \to 0 = f$.

Hence $\int_{\mathbb{R}} f = 0$

But $\int_{\mathbb{R}} f_n = \int_{\mathbb{R}} \chi_{[n,\infty)} = m[n,\infty) = \infty$ for all n.

Therefore, $\lim_{n\to\infty} \int_{\mathbb{R}} f_n = \infty$

Thus $\int_{\mathbb{R}} f \neq \lim_{n\to\infty} \int_{\mathbb{R}} f_n$. Which shows that the Monotone Convergence Theorem is not true for decreasing sequence of functions.

13. ***Example :*** Show that we may have strict inequality in Fatou's Lemma.

Solution : Let { f_n } be a sequence of functions defined by,

$$f_n(x) = \begin{cases} 1 & \text{if } n \leq x < n+1 \\ 0 & \text{otherwise} \end{cases}$$

i.e. $f_n(x) = \chi_{[n,n+1)}$

Then { f_n } is a sequence of nonnegative measurable functions and $f_n \to 0 = f$

Hence $\int_{\mathbb{R}} f = 0$. And $\int_{\mathbb{R}} f_n = \int_{\mathbb{R}} \chi_{[n,n+1)} = 1, \forall n$. Therefore $\underline{\lim} \int_{\mathbb{R}} f_n = 1$

And we get, $0 = \int_{\mathbb{R}} f < \underline{\lim} \int_{\mathbb{R}} f_n = 1$

This shows that strict inequality holds in Fatou's Lemma.

14. Definition : A nonnegative measurable function f is said to be integrable over a measurable set E if $\int_E f < \infty$.

15. Proposition : Let f and g be the two nonnegative measurable functions. If f is integrable over E and g(x) < f(x) on E then g is also integrable and

$$\int_E f - g = \int_E f - \int_E g$$

Proof : Since f and g are nonnegative measurable functions and g < f on E, f – g $\geq$ 0 on E.

Therefore, $f = (f - g) + g$

$$\Rightarrow \int_E f = \int_E (f - g) + g$$

$$= \int_E f - g + \int_E g$$

But f is integrable on $E \Rightarrow \int_E f < \infty$

And $g < f \Rightarrow \int_E g < \int_E f < \infty$

$$\Rightarrow \int_E g < \infty$$

$\Rightarrow g$ is integrable over E.

Hence, $\int_E f = \int_E f - g + \int_E g$

$$\Rightarrow \int_E f - g = \int_E f - \int_E g$$

16. Proposition : Let f be a nonnegative function which is integrable over E. Then for given $\in > 0 \exists \, \delta > 0$. Such that for every set $A \subseteq E$ with $m(A) < \delta$ we have,

$$\int_A f < \in$$

Proof : If f is nonnegative and bounded then assume that $\sup|f(x)| < M$ for some finite positive real number M. Then for given $\in > 0$ choose $\delta < \in/M$, such that for any set A with $m(A) < \delta$ we get,

$$\int_A f < \int_A M = M \cdot m(A) < M \cdot \delta < M \cdot \frac{\in}{M} = \in$$

$$\Rightarrow \int_A f < \in$$

Thus the result is true for nonnegative bounded function.

Next if f is not bounded then define a sequence $\{f_n\}$ by,

$$\begin{aligned} f_n(x) &= f(x) && \text{if } f(x) \le n \\ &= n && \text{if } f(x) > n \end{aligned}$$

Clearly $f_n(x) \le n \quad \forall n$ and $\forall x$

And $f_{n+1} \ge f_n$ for all n. Thus $\{f_n\}$ is an increasing sequence of bounded measurable functions, and $f_n \to f$. Also $f_n \ge 0$ for all n.

Hence by Monotone Convergence Theorem,

$$\int_E f = \lim_{n\to\infty} \int_E f_n$$

Therefore for given $\in > 0 \exists$ an integer N such that,

$$\left|\int_E f_n - \int_E f\right| < \frac{\in}{2} \quad \text{for all } n \ge N$$

$$\Rightarrow \left|\int_E f_N - \int_E f\right| < \frac{\in}{2}$$

$$\Rightarrow \left|\int_E (f_N - f)\right| < \frac{\in}{2}$$

$$\Rightarrow \left|\int_E (f - f_N)\right| < \frac{\in}{2}$$

Since, $f \ge f_N$, $f - f_N \ge 0 \Rightarrow \int_E f - f_N \ge 0$

Hence $\left|\int_E f - f_N\right| = \int_E f - f_N$

Thus $\int_E f - f_N < \frac{\in}{2}$

Choose $\delta > 0$ such that $\delta < \frac{\in}{2N}$

Then for any set A with $m(A) < \delta$ we have,

$$\int_A f = \int_A (f - f_N) + f_N$$

$$= \int_A (f - f_N) + \int_A f_N$$

$$\leq \frac{\in}{2} + \int_A N \qquad \text{(since } f_N \leq N\text{)}$$

$$= \frac{\in}{2} + N \cdot m(A)$$

$$< \frac{\in}{2} + N \cdot \delta < \frac{\in}{2} + N \cdot \frac{\in}{2N} = \frac{\in}{2} + \frac{\in}{2} = \in$$

$$\Rightarrow \int_A f < \in$$

17. Proposition : Let f be a nonnegative integrable function over E. Then f is finite a.e on E.

Proof : For any natural number n, we have,

$$\{x \in E \mid f(x) = \infty\} \subseteq \{x \in E \mid f(x) \geq n\}$$

$$\Rightarrow m\{x \in E \mid f(x) = \infty\} \leq m\{x \in E \mid f(x) \geq n\}, \ \forall n$$

By Chebychev's inequality we have,

$$m\{x \in E \mid f(x) \geq n\} \leq \frac{1}{n} \int_E f$$

Therefore, $m\{x \in E \mid f(x) = \infty\} \leq \frac{1}{n} \int_E f \ , \forall n$

$$\Rightarrow m\{x \in E \mid f(x) = \infty\} = 0$$

$\Rightarrow$ f is finite a.e on E.

18. Beppo Levi's Lemma :

Let $\{f_n\}$ be an increasing sequence of nonnegative measurable functions on E. If the sequence of integrals $\left\{\int_E f_n\right\}$ is bounded then $\{f_n\}$ converges pointwise on E to a measurable function f that is finite a.e on E. and $\lim_{n\to\infty}\int_E f_n = \int_E f < \infty$.

Proof : Since $\{f_n\}$ is a monotonic (increasing) sequence of measurable functions on E, $f_n \to f$ on E where f is also an extended real valued function.

Thus $f(x) = \lim_{n\to\infty} f_n(x)$, $\forall x \in E$.

Since $\{f_n\}$ is increasing sequence, by Monotone Convergence Theorem we have

$$\int_E f = \lim_{n\to\infty}\int_E f_n$$

But $\left\{\int_E f_n\right\}$ is bounded. Hence $\int_E f$ is finite i.e. $\int_E f < \infty$. Therefore f is integrable on E and hence by above proposition f is finite a.e on E.

UNIT - V

THE GENERAL LEBESGUE INTEGRAL

Introduction :

We have defined Lebesgue integration of simple functions, bounded measurable functions and nonnegative measurable functions. Now we define Lebesgue integration of any measurable function.

5.1 General Lebesgue Integral

1. Definition : Let f be any function defined on E. The positive part of f is defined as,

$$f^+(x) = \max\left(f(x), 0\right), \ \forall x \in E$$

Similarly negative part of f is defined as $f^-(x) = \max\left(-f(x), 0\right)$ for all $x \in E$. Note that both f^+ and f^- are nonnegative functions.

2. Note : (1) $f = f^+ - f^-$ on E. For if $f(x) > 0$ then $f^+(x) = f(x)$ and $f^-(x) = 0$. Hence $f^+(x) - f^-(x) = f(x)$. Similarly if $f(x) < 0$ then $f^+(x) = 0$ and $f^-(x) = -f(x)$.

Hence $f^+(x) - f^-(x) = 0 - \left(-f(x)\right) = f(x)$.

(2) Similarly $|f| = f^+ + f^-$ on E.

3. *Example :* f is measurable if and only if f^+ and f^- are measurable.

Solution : Let f be measurable. By definition

$$f^+ = \max\left(f, 0\right) \text{ and } f^- = \max\left(-f, 0\right)$$

Since maximum of measurable functions is measurable, f^+ and f^- are measurable.

Conversely, if both f^+ and f^- are measurable then $f^+ - f^-$ is measurable.

$\Rightarrow f$ is measurable.

5. *Example :* If $f(x) = x$, $x \in [-1,1]$, find f^+ and f^-.

Solution : By definition, $f^+(x) = \max\left(f(x), 0\right)$

$= \max(x, 0) \qquad x \in [-1,1]$

Hence, $\quad f^+(x) = x \qquad$ if $0 \le x \le 1$

$\qquad = 0 \qquad -1 \le x < 0$

Similarly $\qquad f^{-}(x) = \max(-f(x), 0)$

$\qquad\qquad = \max(-x, 0) \qquad x \in [-1,1]$

Hence, $\qquad f^{-}(x) = 0 \qquad$ if $0 < x \leq 1$

$\qquad\qquad = -x \qquad$ if $-1 \leq x \leq 0$

6. Note :

(1) The representation of f as $f = f^{+} - f^{-}$ is not unique. For, $f_1 = f^{+} + C, f_2 = f^{-} + C$ then $f = f_1 - f_2$.

(2) If f is measurable then f^{+} and f^{-} are measurable. Hence $|f| = f^{+} + f^{-}$ is also measurable. Converse need not be true. i.e. $|f|$ is measurable but f need not be measurable.

For example, if P is a nonmeasurable subset of $E = [0,1)$ then define a function $f : E \to \mathbb{R}$

by $\qquad f(x) = 1 \qquad$ if $x \in P$

$\qquad\qquad = -1 \qquad$ if $x \notin P$

Then f is not measurable. But $|f|(x) = 1$ is measurable.

7. Proposition : Let f be a measurable function on E. Then f^{+} and f^{-} are integrable over E if and only if $|f|$ is integrable over E.

Proof : First assume that f^{+} and f^{-} are integrable on E. Since f^{+} and f^{-} are nonnegative measurable functions by linearity property,

$$\int_E |f| = \int_E f^{+} + f^{-} = \int_E f^{+} + \int_E f^{-} < \infty$$

$$\Rightarrow |f| \text{ is integrable over E.}$$

Conversely if $|f|$ is integrable over E. Then we have $0 \leq f^{+} \leq |f|$ and $0 \leq f^{-} \leq |f|$.

And by monotone property we get,

$$\int_E f^{+} \leq \int_E |f| < \infty \text{ and } \int_E f^{-} < \int_E |f| < \infty$$

Hence both f^{+} and f^{-} are integrable over E.

8. Definition : A measurable function f on E is said to be integrable over E if $|f|$ is integrable over E.

i.e. f is integrable over E iff both f^{+} and f^{-} are integrable over E. And we define integral of f by

$$\int_E f = \int_E f^+ - \int_E f^-$$

9. **Note :** For a nonnegative function *f*, $f = f^+$ and $f^- = 0$ on E.

10. **Proposition :** Let f be integrable over E. Then f is finite a.e on E and, $\int_E f = \int_{E-E_0} f$, where $E_0 \subseteq E$ with $m(E_0) = 0$.

Proof : Since *f* is integrable over E, by definition, $|f|$ is also integrable over E. Also $|f|$ is nonnegative measurable function. Hence $|f|$ is finite a.e on E.

But $|f| = f^+ + f^-$. Hence both f^+ and f^- are finite a.e. Hence $f = f^+ - f^-$ is finite a.e on E.

Let E_0 be a measurable subset of E with $m(E_0) = 0$.

Then $\int_E f = \int_E f^+ - f^-$

$$= \int_E f^+ - \int_E f^-$$

$$= \int_{E-E_0} f^+ - \int_{E-E_0} f^- \quad (\because \text{Integral of } f^+ \text{ and } f^- \text{ is zero over a set of measure zero})$$

$$= \int_{E-E_0} f^+ - f^-$$

$$= \int_{E-E_0} f$$

11. **Proposition : (The Integral Comparison Test)**

Let *f* be a measurable function on E. Suppose there is a nonnegative, integrable function *g* over E such that $|f| \le g$ on E. Then *f* is integrable over E and $\left|\int_E f\right| \le \int_E |f|$.

Proof : Since $|f|$ and *g* are nonnegative measurable functions over E and $|f| \le g$ implies

$$\int_E |f| \le \int_E g < \infty, \text{ since } g \text{ is integrable on E.}$$

$\Rightarrow |f|$ is integrable over E.

Hence by definition *f* is integrable over E.

And, $$\int_E |f| = \left|\int_E \left(f^+ - f^-\right)\right|$$

$$= \left|\int_E f^+ - \int_E f^-\right| \le \left|\int_E f^+\right| + \left|\int_E f^-\right|$$

Since $\quad f^+,\ f^- \ge 0,\ \int_E f^+ \ge 0,\ \int_E f^- \ge 0.$

Hence, $$\left|\int_E f\right| \le \int_E f^+ + \int_E f^- = \int_E \left(f^+ + f^-\right) = \int_E |f|$$

$$\Rightarrow \left|\int_E f\right| \le \int_E |f|$$

12. **Note :** Two functions f and g are integrable over E then the sum $f + g$ may not be properly defined at points in E where f and g take infinite values of opposite sign. Hence we define the function $f + g$ on a subset A of E where both f and g are finite and then $m(E - A) = 0$ [E – A is a set where either f is infinite or g is infinite or both f and g are infinite and since f and g are integrable. f and g are finite a.e. Hence $m(E - A) = 0$]

If $f + g$ is integrable over A then we define

$$\int_E (f+g) = \int_{A \cup (E-A)} (f+g) = \int_A f+g + \int_{E-A} f+g = \int_A f+g$$

13. **Proposition :** Let f and g be integrable function over E. Then,

(a) αf is integrable over E, and $\int_E \alpha f = \alpha \int_E f$

(b) $f + g$ is integrable over E, and $\int_E f+g = \int_E f + \int_E g$

(c) If $f \le g$ a•e, then $\int_E f \le \int_E g$

Proof :

(a) If $\alpha > 0$ then,

$$\int_E \alpha f = \int_E \alpha\left(f^+ - f^-\right) = \int_E \left(\alpha f^+ - \alpha f^-\right)$$

$$= \alpha \int_E f^+ - \alpha \int_E f^-$$ (since f^+ and f^- are non-negative measurable)

$$= \alpha\int_E f^+ - \alpha\int_E f^-$$

$$= \alpha\left[\int_E f^+ - \int_E f^-\right]$$

$$\int_E \alpha f = \alpha\int_E f$$

Next if $a = -1$ then $a\ f = (-1)f = -f$

Therefore, $\alpha f = -f = -\left(f^+ - f^-\right) = f^- - f^+$

And $\displaystyle\int_E \alpha f = \int_E f^- - f^+$

$$\int_E -f = \int_E f^- - \int_E f^+$$

$$= -\left[\int_E f^+ - \int_E f^-\right]$$

$$= -\int_E f$$

Finally if $a < 0$ then $a = -k$ where $k > 0$

Therefore, $\displaystyle\int_E \alpha f = \int_E -kf = -\int_E kf = -k\int_E f = \alpha\int_E f$

Hence for all $\alpha \in \mathbb{R}$, we have,

$$\int_E \alpha f = \alpha\int_E f$$

(b) If f_1 and f_2 are nonnegative integrable functions such that $f = f_1 - f_2$. And $f = f^+ - f^-$.

Then $f = f_1 - f_2 = f^+ - f^-$

$$\Rightarrow f_1 + f^- = f^+ + f_2$$

$$\Rightarrow \int_E f_1 + f^- = \int_E f^+ + f_2$$

$$\Rightarrow \int_E f_1 + \int_E f^- = \int_E f^+ + \int f_2$$

$$\Rightarrow \int_E f_1 - \int_E f_2 = \int_E f^+ - \int_E f^-$$ (Since f_1, f_2, f^+ and f^- are integrable)

But $\int_E f = \int_E f^+ - \int_E f^-$

Hence $\int_E f = \int_E f_1 - \int_E f_2$

This shows that $\int_E f$ is independent of the choice of representation for f.

Now if f and g are integrable functions then $f = f^+ - f^-$ and $g = g^+ - g^-$ where f^+, f^-, g^+, g^- are nonnegative integrable functions.

Therefore, $$f + g = \left(f^+ - f^-\right) - \left(g^+ - g^-\right)$$

$$= \left(f^+ + g^+\right) - \left(f^- + g^-\right)$$

$$\Rightarrow \int_E f + g = \int_E \left(f^+ + g^+\right) - \int_E \left(f^- + g^-\right)$$

$$= \int_E f^+ + \int_E g^+ - \int_E f^- - \int_E g^-$$

$$= \int_E f^+ - \int_E f^- + \int_E g^+ - \int_E g^-$$

$$= \int_E f + \int_E g$$

(c) If $f \le g$ a.e. then $0 \le g - f$ a.e.

But $0 \le g - f$ a.e.

$$\Rightarrow 0 \le \int_E g - f$$

$$\Rightarrow 0 \le \int_E g - \int_E f$$

$$\Rightarrow \int_E f \le \int_E g$$

14. Corollary : (Additivity over Domain of Integration)

Let f be integrable over E. Assume that A and B are disjoint mesurable subsets of E. Then

$$\int_{A \cup B} f = \int_A f + \int_B f$$

Proof : We have $|f \cdot c_A| \le |f|$ and $|f \cdot c_B| \le |f|$ on E. By integral comparison test, the measurable function $f \cdot c_A$ and $f \cdot c_B$ are integrable over E ($\because$ f is integrable $|f|$ is integrable). And since A and B are disjoint.

$$f \cdot c_{A \cup B} = f\,[c_A + c_B]$$

$$= f \cdot c_A + f \cdot c_B \quad \text{on E.}$$

Hence, $$\int_{A \cup B} f = \int_E f \cdot c_{A \cup B} = \int_E [f \cdot c_A + f \cdot c_B]$$

$$= \int_E f \cdot c_A + \int_E f \cdot c_B$$

$$= \int_A f + \int_B f$$

15. ***Example :*** If f is integrable function, prove that $|f|$ is also integrable and $\left|\int f\right| \le \int_E |f|$. Does integrability of $|f|$ implies that of f ?

Solution : For any function f, $f = f^+ - f^-$ where $f^+ \ge 0$, $f^- \ge 0$. If f is integrable then f^+ and f^- are integrable. Hence $|f| = f^+ + f^-$ is integrable.

Further $|f| = f^+ + f^- \ge f^+ - f^- = f$

$$\Rightarrow |f| \ge f \quad \text{and} \quad |f| \ge -f$$

$$\Rightarrow \int_E |f| \ge \int_E f \quad \text{and} \quad \int_E |f| \ge -\int_E f$$

$$\Rightarrow \int_E |f| \ge \int_E f \quad \text{and} \quad -\int_E |f| \le \int_E f$$

$$\Rightarrow -\int_E |f| \le \int_E f \le \int_E |f|$$

$$\Rightarrow \left|\int_E f\right| \le \int_E |f|$$

Finally $|f|$ is integrable then f^+ and f^- are integrable. Hence $f = f^+ - f^-$ is integrable.

16. Lebesgue Convergence Theorem :

Statement : Let g be an integrable function over E and let { f_n } be a sequence of measurable functions such that $|f_n| \le g$ on E for all n and $f_n \to f$ a.e. on E, then f is integrable over E and $\int_E f = \lim \int_E f_n$

Proof : $|f_n| \le g$ for all n

$\Rightarrow -g \le f_n \le g$ for all n

$\Rightarrow 0 \le f_n + g$ and $0 \le g - f_n \ \forall n$

Therefore { f_n + g } and { g – f_n } are the sequences of nonnegative measurable functions such that $f_n + g \to f + g$ and $g - f_n \to g - f$.

Therefore by Fatou's lemma,

$$\int_E f + g \le \underline{\lim} \int_E f_n + g \quad \text{and} \quad \int_E g - f \le \underline{\lim} \int_E g - f_n$$

Now, $|f_n| \le g \forall n \Rightarrow |f| \le g$

Since g is integrable, $|f|$ is integrable and hence f is integrable. Also each f_n is integrable.

Hence we get,

$$\int_E f + \int_E g \le \underline{\lim} \left\{ \int_E f_n + \int_E g \right\} \text{ and } \int_E g - \int_E f \le \underline{\lim} \left\{ \int_E g - \int_E f_n \right\}$$

$$\Rightarrow \int_E f + \int_E g \le \underline{\lim} \int_E f_n + \int_E g \text{ and } \int_E g - \int_E f \le \int_E g - \overline{\lim} \int_E f_n$$

Since g is integrable, $\int_E g < \infty$. Hence we get,

$$\int_E f \le \underline{\lim} \int_E f_n \text{ and } -\int_E f \le -\underline{\lim} \int_E f_n$$

$$\Rightarrow \int_E f \le \underline{\lim} \int_E f_n \text{ and } \int_E f \ge \overline{\lim} \int_E f_n$$

$$\Rightarrow \int_E f \le \underline{\lim} \int_E f_n \le \overline{\lim} \int_E f_n \le \int_E f$$

$$\Rightarrow \int_E f = \underline{\lim} \int_E f_n = \overline{\lim} \int_E f_n$$

$$\Rightarrow \int_E f = \lim \int_E f_n$$

17. Theorem : (General Lebesgue Dominated Convergence Theorem)

Let $\{ f_n \}$ be a sequence of measurable functions on E that converges a.e on E to f. Suppose there is a sequence $\{ g_n \}$ of nonnegative functions on E that converges pointwise a.e on E to g and dominates $\{ f_n \}$ on E in the sense that $|f_n| \le g_n$ on E $\forall n$.

If $\lim\limits_{n\to\infty} \int_E g_n = \int_E g < \infty$, then $\lim\limits_{n\to\infty} \int_E f_n = \int_E f$.

Proof : $|f_n| \le g_n$ for all n.

$\Rightarrow -g_n \le f_n \le g_n$ for all n.

$\Rightarrow 0 \le f_n + g_n$ and $0 \le g_n - f_n$ for all n.

Therefore $\{ f_n + g_n \}$ and $\{ g_n - f_n \}$ are the sequences of nonnegative measurable functions such that $f_n + g_n \to f + g$ and $g_n - f_n \to g - f$ a•e.

By Fatou's lemma, we get,

$$\int_E f + g \le \underline{\lim} \int_E f_n + g_n \quad \text{and} \quad \int_E g - f \le \underline{\lim} \int_E g_n - f_n$$

Now $|f_n| \le g_n \forall n \Rightarrow |f| \le g$

Since g and g_n are integrable, f and f_n are integrable. Hence we get,

$$\int_E f + \int_E g \le \underline{\lim} \left\{ \int_E f_n + \int_E g_n \right\} \quad \text{and} \quad \int_E g - \int_E f \le \underline{\lim} \left\{ \int_E g_n - \int_E f_n \right\}$$

$$\Rightarrow \int_E f + \int_E g \le \underline{\lim} \int_E f_n + \underline{\lim} \int_E g_n \quad \text{and} \quad \int_E g - \int_E f \le \underline{\lim} \int_E g_n - \overline{\lim} \int_E f_n$$

But $g_n \to g$ a•e and $\int_E g = \lim \int_E g_n$

Therefore, we get

$$\int_E f + \int_E g \le \underline{\lim} \int_E f_n + \int_E g \quad \text{and} \quad \int_E g - \int_E f \le \int_E g - \overline{\lim} \int_E f_n$$

$$\Rightarrow \int_E f \le \underline{\lim} \int_E f_n \quad \text{and} \quad -\int_E f \le -\overline{\lim} \int_E f_n$$

$$\Rightarrow \int_E f \le \underline{\lim} \int_E f_n \quad \text{and} \quad \int_E f \ge \overline{\lim} \int_E f_n$$

$$\Rightarrow \int_E f \le \underline{\lim} \int_E f_n \le \overline{\lim} \int_E f_n \le \int_E f$$

$$\Rightarrow \int_E f = \underline{\lim} \int_E f_n = \overline{\lim} \int_E f_n$$

$$\Rightarrow \int_E f = \lim \int_E f_n$$

5.2 Characterization of Riemann and Lebesgue Integrability

1. Lemma : Let $\{f_n\}$ and $\{y_n\}$ be sequences of functions which are integrable over E such that $\{f_n\}$ is increasing while $\{y_n\}$ is decreasing on E. Let *f* be a function on E such that $f_n \leq f \leq y_n$ on E for all n.

If $\lim_{n\to\infty} \int_E (y_n - f_n) = 0$, then $\{f_n\} \to f$ pointwise a.e on E, $\{y_n\} \to f$ a.e on E and f is integrable over E and

$$\lim_{n\to\infty} \int_E f_n = \int_E f \, , \; \lim_{n\to\infty} \int_E y_n = \int_E f$$

Proof : For $x \in E$, define $f^*(x) = \lim_{n\to\infty} f_n(x)$ and $y^*(x) = \lim_{n\to\infty} y_n(x)$. The functions f^* and y^* are extended real valued functions. The sequences $\{f_n\}$ and $\{y_n\}$ are monotonic and hence f^* and y^* are properly defined on E. Further f^* and y^* are pointwise limit of sequences of measurable functions and hence f^* and y^* are measurable functions.

Also $f_n \leq f \leq y_n, \quad \forall n \quad \Rightarrow \quad f^* \leq f \leq y^*.$

And since $\{f_n\}$ is increasing and $\{y_n\}$ is decreasing, we have

$$f_n \leq f^* \leq f \leq y^* \leq y_n, \text{ for all n}$$

$$\Rightarrow 0 \leq y^* - f^* \leq y_n - f_n, \text{ for all n}$$

$$\Rightarrow 0 \leq \int_E (y^* - f^*) \leq \int_E (y_n - f_n), \text{ for all n}$$

$$\Rightarrow 0 \leq \int_E (y^* - f^*) \leq \lim_{n\to\infty} \int_E (y_n - f_n) = 0$$

$$\Rightarrow \int_E (y^* - f^*) = 0$$

$$\Rightarrow y^* - f^* = 0 \text{ a.e on E.}$$

$$\Rightarrow y^* = f^* \text{ a.e on E}$$

But $f^* \le f \le y^*$. Hence $f^* = f = y^*$, a.e. on E.

And $f_n \to f^* = f$, $y_n \to y^* = f$, a.e on E.

Since f^* and y^* are measurable, f is also measurable.

Further $f_n \le f \le y_n$ for all n.

$$\Rightarrow f_1 \le f \le y_1$$

$$\Rightarrow \int_E f_1 \le \int_E f \le \int_E y_1$$

Since f_1 and y_1 are integrable, $\int_E f_1 < \infty$, $\int_E y_1 < \infty$.

Hence $\int_E f < \infty$. Therefore f is integrable over E.

Next $\quad f_n \le f \le y_n$ for all n.

$$\Rightarrow 0 \le f - f_n \text{ and } 0 \le y_n - f$$

$$\Rightarrow 0 \le f - f_n \le y_n - f_n \text{ and } \Rightarrow 0 \le y_n - f \le y_n - f_n$$

$$\Rightarrow 0 \le \int_E f - f_n \le \int_E y_n - f_n, \quad 0 \le \int_E y_n - f \le \int_E y_n - f_n$$

$$\Rightarrow 0 \le \int_E f - \int_E f_n \le \int_E y_n - f_n, \quad 0 \le \int_E y_n - \int_E f \le \int_E y_n - f_n$$

Taking limit as $n \to \infty$ and $\lim_{n\to\infty} \int_E y_n - f_n = 0$.

$$\Rightarrow 0 \le \lim_{n\to\infty} \int_E f - \int_E f_n \le 0, \quad 0 \le \lim_{n\to\infty} \int_E y_n - \int_E f \le 0$$

$$\Rightarrow \int_E f = \lim_{n\to\infty} \int_E f_n = \lim_{n\to\infty} \int_E y_n .$$

2. **Theorem :** Let f be a bounded function on a set of finite measure E. Then f is Lebesgue integrable over E if and only if f is measurable.

Proof : We know that a bounded measurable function on a set of finite measure E is Lebesgue integrable. Conversely we show that a bounded, Lebesgue integrable function is measurable.

Let f be integrable function on a set of finite measure E. Also f is bounded. Since f is integrable, lower and upper Lebesgue integrals are equal. i.e.

$$\sup\left\{\int_E f \mid f \text{ is simple}, f \le f \text{ on E}\right\} = \inf\left\{\int_E y \mid y \text{ is simple}, f \le y \text{ on E}\right\}$$

Therefore there exists a sequence of simple functions $\{f_n\}$ and $\{y_n\}$ on E such that,

$$f_n \le f \le y_n \forall n \text{ and } \lim_{n\to\infty}\int_E f_n = \lim_{n\to\infty}\int_E y_n$$

$$\Rightarrow \lim_{n\to\infty}\int_E (y_n - f_n) = 0$$

Since maximum and minimum of simple functions is again simple function we can replace each f_n by $\max_{1\le i\le n} f_i$ and each y_n by $\max_{1\le i\le n} y_i$.

Then the sequence $\{f_n\}$ becomes increasing and $\{y_n\}$ becomes decreasing such that $f_n \le f \le y_n,\ \forall n$. Also we have $\Rightarrow \lim_{n\to\infty}\int_E y_n - f_n = 0$. Hence by above Lemma we get $\{f_n\} \to f, \{y_n\} \to f$ a.e. on E and since the convergence is pointwise, f is measurable.

3. Note : If a bounded function on a closed and bounded interval [a, b] is Riemann integrable over [a, b] then it is Lebesgue integrable over [a, b] and the two integrals are equal. The above theorem suggest that a bounded Lebesgue integrals function is measurable. Hence we have following theorem in which we prove the equivalence of Riemann integrability and measurability (or continuity a.e)

4. Theorem : Let f be a bounded function on the closed bounded interval [a, b]. Then f is Riemann integrable over [a, b] if and only if the set of points in [a, b] at which f fails to be continuous has measure zero i.e. f is continuous on [a, b] a.e.

[Since continuity implies measurability, the above theorem states that Riemann integrability on a closed bounded interval implies measurability]

Proof : First we assume that f is Riemann integrable over [a, b]. Then Riemann upper integral and lower integrals are equal. Therefore there are sequences $\{p_n'\}$ and $\{p_n''\}$ of partitons of [a, b] such

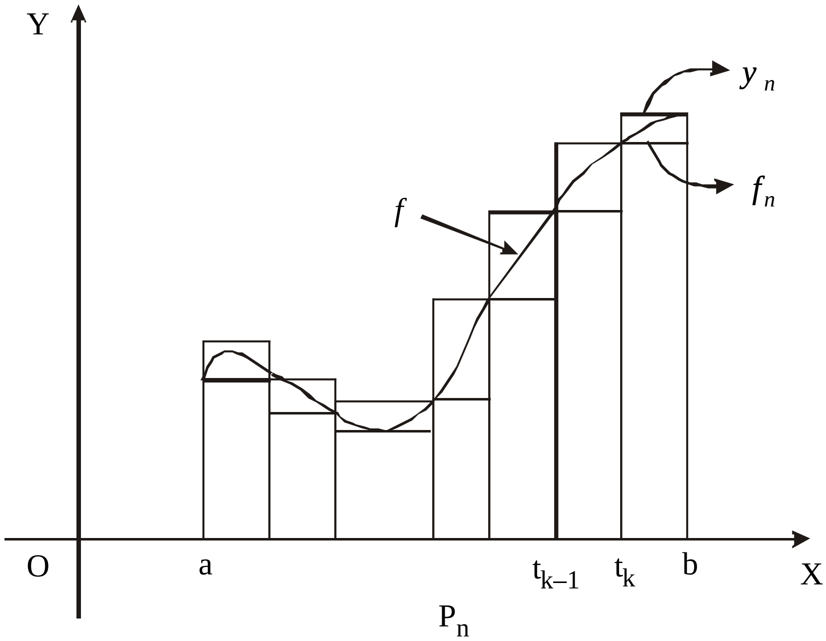
that $$\lim_{n\to\infty} U(f, p_n') = \lim_{n\to\infty} L(f, p_n'')$$

$$\Rightarrow \lim_{n\to\infty}\left[U(f, p_n') - L(f, p_n'')\right] = 0$$

where $U(f, p_n')$ and $L(f, p_n'')$ are upper and lower Darbaux sums.

Under refinement of partition of [a, b] the lower Darbaux sum increases and the upper Darbaux sum decreases. Hence, we form a common refinement p_n of p_n' and p_n'' so that $\{p_n\}$ is refinement of both $\{p_n'\}$ and $\{p_n''\}$.

Also we construct the common refienement sequence $\left\{ p_n \right\}$ such that p_{n+1} is refinement of p_n for all n. Therefore, $\lim_{n\to\infty}\left[U\left(f, p_n \right) - L\left(f, p_n \right) \right] = 0$.



For each integer n we define lower step function f_n associated with f w.r.t. the partition P_n which agrees with f at the partition points of P_n and in each open interval of the partition P_n, f_n assumes constant value equal to the infimum of f on that interval.

Similarly for each integer n we define upper step function y_n which agrees with f at the partition points of P_n and y_n takes constant value equal to the supremum of f on that interval.

Therefore by definition of the Darbaux sums we get,

$$L\left(f, P_n \right) = \int_a^b f_n \text{ and } U\left(f, P_n \right) = \int_a^b y_n \text{, for all n.}$$

Further the sequences $\{f_n\}$ and $\{y_n\}$ are sequences of integrable functions such that $f_n \le f \le y_n$ for all n on [a, b]. Moreover each P_{n+1} is refinement of P_n implies the sequence $\{f_n\}$ is increasing and $\{y_n\}$ is decreasing. Therefore,

$$\lim_{n\to\infty}\int_a^b (y_n - f_n) = \lim_{n\to\infty}\left[U\left(f, P_n \right) - L\left(f, P_n \right) \right] = 0$$

Hence by theorem $\{f_n\} \to f$ and $\{y_n\} \to f$ on [a, b] pointwise a.e.

Let E be the set of points where either $\{y_n(x)\}$ or $\{f_n(x)\}$ fail to converge to $f(x)$. Then $m(E)=0$ (since $y_n \to f$ a.e, $f_n \to f$ a.e). Let E_0 be the union of E and the set of all partition points in the sequence $\{P_n\}$. Then $m(E_0)=0$ since E_0 is the union of countable set and the set E whose measure is zero.

We claim that f is continuous at each point in [a, b] – E_0.

Let $x_0 \in [a,b]-E_0$ be arbitrary and let $\in > 0$.

Since $\{y_n(x_0)\}$ and $\{f_n(x_0)\}$ converges to $f(x_0)$ there exists an integer n_0 such that

$$|y_n(x_0)-f(x_0)| < \in,\ |f_n(x_0)-f(x_0)| < \in \qquad \forall n \ge n_0$$

$$\Rightarrow y_{n_0}(x_0)-f(x_0),\ f(x_0)-f_{n_0}(x_0) < \in$$

$$\Rightarrow f(x_0)-\in < f_{n_0}(x_0) \le f(x_0) \le y_{n_0}(x_0) < f(x_0)+\in$$

Since x_0 is not a partition point there exists $d>0$ such that $(x_0-d, x_0+d) \subseteq I_{n_0}$, where I_{n_0} is some open interval corresponding to the partition P_{n_0}.

Therefore if $(x-x_0)<d$ then

$$y_{n_0}(x_0) = f_{n_0}(x) \le f(x) \le y_{n_0}(x) = y_{n_0}(x_0)$$

$$\Rightarrow f(x_0)-\in < f(x) < f(x_0)+\in$$

$$\Rightarrow |f(x)-f(x_0)| < \in$$

Thus $|x-x_0|<d \Rightarrow |f(x)-f(x_0)|<\in$. Which shows that f is continuous at x_0. Since $x_0 \in [a,b]-E_0$ is arbitray, f is continuous a.e on [a, b].

Next we prove the converse.

Let f be continuous on [a, b] a.e. Let $\{P_n\}$ be a sequence of partitions of [a, b] for which $\|P_n\| \to 0$.

Let $\{f_n\}$ and $\{y_n\}$ be the sequences of lower and upper step functions associated with the function f over the partition P_n. Then $f_n \le f \le y_n$ for all n on [a, b].

Let $x_0 \in [a,b]$ such that f is continuous at x_0 and x_0 is not a point of any partition P_n. Then for given $\in > 0$ there is $d>0$ such that

$$|x-x_0|<d \Rightarrow |f(x)-f(x_0)| < \frac{\in}{2}$$

$$\Rightarrow f(x_0)-\frac{\in}{2}< f(x)< f(x_0)+\frac{\in}{2}$$

Choose an integer N such that $\|P_n\|<d$ for all $n \geq N$. Let I_n be the open interval of the partition P_n such that $x_0 \in I_n$. Then

$$I_n \subseteq (x_0-d, x_0+d)$$

But $\forall x \in I_n,\ f_n(x) \leq f(x) \leq y_n(x)$.

and $f(x_0)-\frac{\in}{2} \leq f_n(x_0) \leq f(x_0) \leq y_n(x_0) \leq f(x_0)+\frac{\in}{2}$

$$\Rightarrow f_n(x_0) \leq f(x_0) \leq f(x_0)+\frac{\in}{2},\ f(x_0) \leq y_n(x_0) \leq f(x_0)+\frac{\in}{2}$$

$$\Rightarrow 0 \leq f(x_0)-f_n(x_0) \leq \frac{\in}{2}<\in,\ 0 \leq y_n(x_0)-f(x_0) \leq \frac{\in}{2}<\in$$

$\Rightarrow 0 \leq f(x_0)-f_n(x_0)<\in,\ \Rightarrow 0 \leq y_n(x_0)-f(x_0)<\in$ for all $n \geq N$.

$$\Rightarrow f_n(x_0) \to f(x_0),\ y_n(x_0) \to f(x_0).$$

$$\Rightarrow \lim_{n\to\infty} f_n(x_0)=f(x_0),\ \Rightarrow \lim_{n\to\infty} y_n(x_0)=f(x_0)$$

Since $x_0 \in [a,b]$ is such that f is continuous at x_0 and x_0 is not the point of any partition P_n, we get $f_n \to f$, $y_n \to f$ a.e on [a, b].

$\Rightarrow \lim_{n\to\infty} f_n = f$, $\lim_{n\to\infty} y_n = f$ a.e on [a, b].

Further since f is bounded on [a, b], the functions f_n and y_n are also bounded on [a, b]. Therefore by bounded convergence theorem,

$$\lim_{n\to\infty}\int_a^b f_n=\int_a^b f,\ \lim_{n\to\infty}\int_a^b y_n=\int_a^b f$$

$$\Rightarrow \lim_{n\to\infty}\int_a^b (y_n-f_n)=0$$

The Riemann integration of a step function is same as its Lebesgue integral, we have

$\int_a^b y_n = U(f,P_n)$ and $\int_a^b f_n = L(f,P_n)$ for all n.

But $$\int_a^{\overline{b}} f = \inf_{P_n} U(f, P_n)$$

$$\Rightarrow \int_a^{\overline{b}} f \le U(f, P_n), \quad \text{for all n}$$

Similarly, $$\int_{\underline{a}}^{b} f = \sup_{P_n} L(f, P_n)$$

$$\Rightarrow \int_{\underline{a}}^{b} f \ge L(f, P_n), \quad \text{for all n.}$$

Thus $0 \le \int_a^{\overline{b}} f - \int_{\underline{a}}^{b} f \le U(f, P_n) - L(f, P_n)$, for all n.

$$\Rightarrow 0 \le \int_a^{\overline{b}} f - \int_{\underline{a}}^{b} f \le \int_a^b y_n - \int_a^b f_n$$

$$\Rightarrow 0 \le \int_a^{\overline{b}} f - \int_{\underline{a}}^{b} f \le \int_a^b (y_n - f_n)$$

Taking limit as $n \to \infty$ we get,

$$\int_a^{\overline{b}} f - \int_{\overline{a}}^{b} f = 0 \quad \text{i.e.} \quad \int_a^{\overline{b}} f = \int_{\overline{a}}^{b} f$$

Hence *f* is Riemann integrable. Thus a bounded function *f* is continuous a.e on [a, b] implies *f* is Riemann integrable on [a, b].

DIFFERENTIABILITY OF MONOTONE FUNCTIONS

6.1 Vitali's Lemma

1. Definition

A closed, bounded interval [c, d] is said to be nondegenrate if c < d.

2. Definition

A collection $\mathcal{F}$ of closed, bounded, nondegenerate intervals is said to be a cover of a set E, in the sense of Vitali, if for every $x \in E$ and $\in > 0$ there is an interval I in $\mathcal{F}$ such that $x \in I$ and $\ell(I) < \in$.

3. The Vitali Covering Lemma

Let E be a set of finite outer measure. Let $\mathcal{F}$ be a collection of closed, bounded intervals that covers E in the sense of Vitali. Then for each $\in > 0$, there is a finite disjoint subcollection $\{I_k\}_{k=1}^{n}$ of $\mathcal{F}$ such that,

$$m^*\left[E - \bigcup_{k=1}^{n} I_k\right] < \in$$

Proof : Since $m^*(E) < \infty$, there is an openset O such that $E \subseteq O$ and $m^*(O) < \infty$.

Since $\mathcal{F}$ is a Vitali covering of E, we can assume that each interval of $\mathcal{F}$ is contained in O.

Let $\{I_k\}_{k=1}^{\infty}$ be a disjoint collection of sets in $\mathcal{F}$.

Then $\bigcup_{k=1}^{\infty} I_k \subseteq O$

$$\Rightarrow m\left(\bigcup_{k=1}^{\infty} I_k\right) \leq m(O)$$

$$\Rightarrow \sum_{k=1}^{\infty} m\left(I_k\right) \leq m(O)$$

$$\Rightarrow \sum_{k=1}^{\infty} \ell\left(I_k\right) \leq m(O) < \infty \qquad .... (1)$$

Now $\mathcal{F}$ is a Vitali covering of E. Therefore $\forall x \in E$ and $\forall \in > 0$, $\exists I \in \mathcal{F}$ such that

$x \in I$ and $\ell(I) < \in$.

$$\Rightarrow E \subseteq \bigcup_{I \in \mathcal{F}} I$$

Let $\mathcal{F}_n = \left\{ I \in \mathcal{F} \mid I \cap \bigcup_{k=1}^{n} I_k = f \right\}$ ..... (2)

Then $E \subseteq \bigcup_{I \in \mathcal{F}} I \Rightarrow E \subseteq \left(\bigcup_{I \in \mathcal{F}} I \right) \cup \left(\bigcup_{k=1}^{n} I_k \right)$

$$\Rightarrow E - \bigcup_{k=1}^{n} I_k \subseteq \bigcup_{I \in \mathcal{F}_n} I \qquad .... (3)$$

Now if $\{I_k\}_{k=1}^{n}$ is a finite disjoint subcollection of intervals in $\mathcal{F}$ such that $E \subseteq \bigcup_{k=1}^{n} I_k$ then the proof is complete

$$\left(\because E - \bigcup_{k=1}^{n} I_k = f \text{ and } m^*\left(E - \bigcup_{k=1}^{n} I_k \right) = m^*(f) = 0 \Rightarrow m^*\left(E - \bigcup_{k=1}^{n} I_k \right) < \in, \forall \in > 0 \right)$$

If E is not covered by $\bigcup_{k=1}^{n} I_k$ then there exist $x \in E - \bigcup_{k=1}^{n} I_k$. Then from (3) we can find an interval $I \in \mathcal{F}_n$ such that $x \in I \in \mathcal{F}_n$.

Since $I \subseteq O$, $\forall I \in \mathcal{F}$ and $\mathcal{F}_n \subseteq \mathcal{F}$, $I \subseteq O$, $\forall I \in \mathcal{F}_n$,

$\Rightarrow \ell(I) \leq m(O) < \infty$.

Hence $\ell(I)$ is finite for all $I \in \mathcal{F}_n$.

Let $S_n = \sup\{\ell(I) \mid I \in \mathcal{F}_n\}$

Choose a set $I_{n+1} \in \mathcal{F}_n$ such that $\ell(I_{n+1}) > \frac{S_n}{2}$. Then the collection $\{I_1, I_2, ... I_n, I_{n+1}\}$ is a disjoint collection of sets in $\mathcal{F}$. Inductively we can obtain a countable disjoint collection of sets $\{I_k\}_{k=1}^{\infty}$ in $\mathcal{F}$ such that $\ell(I_{n+1}) > \frac{S_n}{2} \geq \frac{\ell(I)}{2}$, for all $I \in \mathcal{F}_n$.

i.e. $\ell\left(I_{n+1}\right) > \dfrac{\ell(I)}{2}$, $\forall I$ with $I \cap \bigcup_{k=1}^{n} I_k = f$. .... (4)

Next, for this countable collection $\{I_k\}_{k=1}^{\infty}$ we have

$I_k \subseteq O$, $\forall k$ = 1, 2, 3,....

$$\Rightarrow \bigcup_{k=1}^{\infty} I_k \subseteq O$$

$$\Rightarrow m\left(\bigcup_{k=1}^{\infty} I_k\right) \leq m(O)$$

$$\Rightarrow \sum_{k=1}^{\infty} m\left(I_k\right) \leq m(O) < \infty$$

$$\Rightarrow \sum_{k=1}^{\infty} \ell\left(I_k\right) < \infty$$

Thus $\sum_{k=1}^{\infty} \ell\left(I_k\right)$ converges. Hence $\lim_{k\to\infty} \ell\left(I_k\right) = 0$

i.e. $\left\{\ell\left(I_k\right)\right\} \to 0$ as $k \to \infty$ .... (5)

Let n be any natural number. If $\bigcup_{k=1}^{n} I_k$ is not a cover of E, then there exists $x \in E - \bigcup_{k=1}^{n} I_k$.

Since $\mathcal{F}$ is Vitali covering of E there exists an interval $I \in \mathcal{F}$ such that $x \in I$ and $I \cap \left(\bigcup_{k=1}^{n} I_k\right) = f$.

Then *I* must have nonempty intersection with some member of $\{I_k\}_{k=1}^{\infty}$. Otherwise if $I_k \cap I = f$, $\forall k$,

then $I \cap \bigcup_{k=1}^{n-1} I_k = f$, $\forall n$ = 1,2 ,3, ...

By (4) $\ell\left(I_n\right) > \dfrac{\ell(I)}{2}$ for all n = 1, 2, ...

Which is a contradiction since $\ell\left(I_k\right) \to 0$.

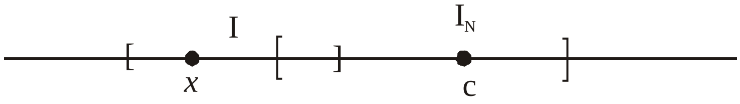
Hence *I* intesects with some member of $\{I_k\}_{k=1}^{\infty}$.

Let N be the least natural number such that $I \cap I_N \neq f$.

Then N > n and $I \cap \bigcup_{k=1}^{N-1} I_k = f$.

And by (4), $\ell(I_N) > \dfrac{\ell(I)}{2}$ i.e. $2\ell(I_N) > \ell(I)$.

Since $x \in I$ and $I \cap I_N \neq f$, the distance of x from the centre of I_N is at the most $\ell(I) + \dfrac{1}{2}\ell(I_N)$.



But $\ell(I) < 2\ell(I_N)$. Hence the distance between x and the centre of I_N is at the most.

$$\ell(I) + \frac{1}{2}\ell(I_N) < 2\,\ell(I_N) + \frac{1}{2}\,\ell(I_N) = \frac{5}{2}\,\ell(I_N)$$

Therefore $x \in 5 * I_N$.

Thus $x \in E - \bigcup_{k=1}^{n} I_k \Rightarrow x \in 5 * I_N$ for some N > n.

Hence $E - \bigcup_{k=1}^{n} I_k \subseteq \bigcup_{k=n+1}^{\infty} 5 * I_k$ .... (6)

Since n is arbitrary. This relation holds for all n = 1, 2, 3, ...

Now for any $\in > 0$, since $\sum_{k=1}^{\infty} \ell(I_k)$ converges, we can find an integer n such that

$$\sum_{k=n+1}^{\infty} \ell(I_k) < \frac{\in}{5} \qquad \text{... (7)}$$

For this n we have

$$E - \bigcup_{k=1}^{n} I_k \subseteq \bigcup_{k=n+1}^{\infty} 5 * I_k$$

$$\Rightarrow m^*\left(E - \bigcup_{k=1}^{n} I_k\right) \leq m^*\left(\bigcup_{k=n+1}^{\infty} 5 * I_k\right)$$

$$\leq \sum_{k=n+1}^{\infty} m^*(5^* I_k)$$

$$= \sum_{k=n+1}^{\infty} 5m^*(I_k)$$

$$= 5\sum_{k=n+1}^{\infty} \ell(I_k)$$

$$< 5 \cdot \frac{\in}{5} \qquad \text{(by (7))}$$

$$\Rightarrow m^*\left(E - \bigcup_{k=1}^{n} I_k\right) < \in.$$

4. Definition

For a real valued function f, let x be an interior point of its domain. We define the Upper derivative of f at x as

$$\overline{D}f(x) = \lim_{h\to 0}\left[\sup_{0<|t|\leq h} \frac{f(x+t) - f(x)}{t}\right]$$

Similarly the lower derivative of f at x is defined as

$$\underline{D}f(x) = \lim_{h\to 0}\left[\inf_{0<|t|\leq h} \frac{f(x+t) - f(x)}{t}\right]$$

Clearly $\underline{D}f(x) \leq \overline{D}f(x)$. If $\underline{D}f(x) = \overline{D}f(x)$ then f is said to be differentiable at x and the common value of the upper and lower derivatives is denoted by $f'(x)$.

5. Note

Let f be a continuous on closed bounded interval [c, d] and differentiable on its interior (c, d) then by Mean value theorem, there exists $z \in (c, d)$ such that

$$f'(z) = \frac{f(d) - f(c)}{d - c}$$

If $f' \geq a$ on (c, d) then $f'(z) \geq a$ and we get

$$a \leq f'(z) = \frac{f(d) - f(c)}{d - c}$$

$\Rightarrow a \cdot (d-c) \leq [f(d) - f(c)]$

The following theorem generalizes this inequality.

6. Theorem

Let f be an increasing function on the closed bounded interval [a, b]. Then for each $a > 0$.

$$m^*\{x \in (a,b) \mid \bar{D}f(x) \geq a\} \leq \frac{1}{a}[f(b) - f(a)]$$

and $m^*\{x \in (a,b) \mid \bar{D}f(x) = \infty\} = 0$

Proof : Let $a > 0$. Define $E_a = \{x \in (a,b) \mid \bar{D}f(x) \geq a\}$ choose $a' \in (0, a)$ i.e. $0 < a' < a$.

Let $\mathcal{F}$ be a collection of closed, bounded intervals [c, d] contained in (a, b) such that

$f(d) - f(c) \geq a'(d-c)$

Since $\bar{D}f(x) \geq a$ on E_a, we have

$$\bar{D}f(x) = \lim_{h \to 0} \sup_{0<|t|\leq h} \frac{f(x+t) - f(x)}{t} \geq a$$

$\Rightarrow \dfrac{f(x+t) - f(x)}{t} \geq a > a' > 0$ (*f* is increasing)

$\Rightarrow f(x+t) - f(x) \geq a'(t)$ $t \to 0$

$\Rightarrow f(x+t) - f(x) \geq a'(x+t-x)$

Thus for every $\in > 0 \exists$ an interval $[x, x+t] \in \mathcal{F}$ such that

$\ell[x, x+t] = t < \in$ $(\because t \to 0)$

Hence $\mathcal{F}$ is a Vitali covering of E_a. Hence by Vitali covering Lemma there is a finite disjoint subcollection $\{[c_k, d_k]\}_{k=1}^{n}$ of intervals in $\mathcal{F}$ such that

$$m^*\left[E_a - \bigcup_{k=1}^{n}[c_k, d_k]\right] < \in$$

Now $E_a \subseteq \left(\bigcup_{k=1}^{n}[c_k, d_k]\right) \cup \left(E_a - \bigcup_{k=1}^{n}[c_k, d_k]\right)$

$$\Rightarrow m^*(E_a) \leq m^*\left(\bigcup_{k=1}^{n}[c_k, d_k]\right) + m^*\left(E_a - \bigcup_{k=1}^{n}[c_k, d_k]\right)$$

$$\leq \sum_{k=1}^{n} m^*[c_k, d_k] + \in$$

$$= \sum_{k=1}^{n} (d_k - c_k) + \in$$

But $\quad [c_k, d_k] \in \mathcal{F} \Rightarrow f(d_k) - f(c_k) \geq a'(d_k - c_k)$

i.e. $\quad (d_k - c_k) \leq \frac{1}{a'}[f(d_k) - f(c_k)]$

Hence, $\quad m^*(E_a) \leq \frac{1}{a'} \sum_{k=1}^{n} [f(d_k) - f(c_k)] + \in$

Now f is increasing on [a, b] and $\{[c_k, d_k]\}_{k=1}^{\infty}$ is a disjoint collection of subintervals in [a, b]. Therefore,

$$\sum_{k=1}^{n} [f(d_k) - f(c_k)] \leq f(b) - f(a)$$

Thus for each $\in > 0$ and $a' \in (0, a)$, $m^*(E_a) \leq \frac{1}{a'}[f(b) - f(a)] + \in$

Since $\in > 0$ is arbitrary we get,

$$m^*(E_a) \leq \frac{1}{a}[f(b) - f(a)]$$

$$\Rightarrow m^*\{x \in (a,b) \mid \overline{D}f(x) \geq a\} \leq \frac{1}{a} \cdot [f(b) - f(a)]$$

Next for each natural number n,

$$\{x \in (a,b) \mid \overline{D}f(x) = \infty\} \subseteq \{x \in (a,b) \mid \overline{D}f(x) \geq n\}$$

$$\Rightarrow \{x \in (a,b) \mid \overline{D}f(x) = \infty\} \subseteq E_n \text{ for all n = 1, 2,3 , ...}$$

Hence by above result,

$$m^*\{x \in (a,b) \mid \overline{D}f(x) = \infty\} \leq m^*(E_n) \leq \frac{1}{n}(f(b) - f(a)) \qquad \forall \text{ n = 1, 2, 3, ...}$$

$$\Rightarrow m^*\{x \in (a,b) \mid \overline{D}f(x) = \infty\} = 0$$

6.2 Lebesgue's Theorem

Lebesgue's theorem is one of the important theorem in mathematical analysis (1904).

1. Lebesgue's Theorem

If the function *f* is monotone on the open interval (a, b) then it is differentiable almost everywhere on (a, b).

Proof : Let *f* be increasing on (a, b). Further assume that (a, b) is bounded [$\because$ a and b are extended real numbers. (a, b) need not be bounded].

For rational numbers a and b define the sets

$$E_{a,b} = \{x \in (a,b) \mid \overline{D}f(x) > a > b > \underline{D}f(x)\}$$

Then, $\{x \in (a,b) \mid \overline{D}f(x) > \underline{D}f(x)\} = \bigcup_{\substack{a,b \\ \text{rationals}}} \mathrm{E}_{a,b}$

$$\Rightarrow m\{x \in (a,b) \mid \overline{D}f(x) > \underline{D}f(x)\} = m\left(\bigcup_{(a,b)} \mathrm{E}_{a,b}\right) \le \sum_{a,b} m\left(\mathrm{E}_{a,b}\right)$$

We prove that $\mathrm{E}_{a,b}$ has measure zero $\forall$ rationals a , b . Let a , b be any two fixed rational numbers with $a > b$. Let $E = \mathrm{E}_{a,b}$.

Let $\in > 0$. Then there exists an open set O for which $E \subseteq O \subseteq [a,b]$ and $m(O) < m^*(E) + \in$.

Let $\mathcal{F}$ be the collection of closed bounded intervals [c, d] contained in O for which

$$f(d) - f(c) < b(d-c)$$

Since $\underline{D}f(x) < b$ on E, $\mathcal{F}$ is a Vitali covering of E. This vitali covering Lemma tells us that there is a finite disjoint subcollection $\{[c_k, d_k]\}_{k=1}^{n}$ of $\mathcal{F}$ for which

$$m^*\left[E - \bigcup_{k=1}^{n} [c_k, d_k]\right] < \in$$

Thus each interval in $\{[c_k, d_k]\}_{k=1}^{n}$ is such that

$$f(d_k) - f(c_k) < b(d_k - c_k) \qquad \forall \text{ k = 1, 2, 3, ... n}$$

and $\quad [c_k, d_k] \subseteq O$

$$\Rightarrow \bigcup_{k=1}^{n} [c_k, d_k] \subseteq O$$

$$\Rightarrow m\left(\bigcup_{k=1}^{n}[c_k, d_k]\right) \le m(O)$$

$$\Rightarrow \sum_{k=1}^{n} m\left([c_k, d_k]\right) \le m(O) \qquad (\because \text{ The collection } \{[c_k, d_k]\} \text{ is disjoint})$$

$$\Rightarrow \sum_{k=1}^{n} \ell\left([c_k, d_k]\right) \le m(O)$$

$$\Rightarrow \sum_{k=1}^{n} (d_k - c_k) \le m(O)$$

Also by property of each interval $[c_k, d_k]$,

$$f(d_k) - f(c_k) < b\,(d_k - c_k)$$

$$\Rightarrow \sum_{k=1}^{n} \left[f(d_k) - f(c_k)\right] \le b \sum_{k=1}^{n} (d_k - c_k)$$

$$\Rightarrow \sum_{k=1}^{n} \left[f(d_k) - f(c_k)\right] \le b\, m(O) < b\,[m^*(E) + \in]$$

Now for $x \in E \cap [c_k, d_k]$ we have $\bar{D}f(x) > a$.

Hence lemma

$$m^*\{E \cap [c_k, d_k]\} \le \frac{1}{a}\left[f(d_k) - f(c_k)\right], \ \forall \text{ k = 1, 2, ... n}$$

But $\quad E = \left(E \cap \bigcup_{k=1}^{n}[c_k, d_k]\right) \cup \left(E \cap \left(\bigcup_{k=1}^{n}[c_k, d_k]\right)^{\sim}\right)$

$$\Rightarrow m^*(E) = m^*\left(\bigcup_{k=1}^{n} E \cap [c_k, d_k]\right) + m^*\left(E - \bigcup_{k=1}^{n}[c_k, d_k]\right)$$

$$\le \sum_{k=1}^{n} m^*\left(E \cap [c_k, d_k]\right) + \in$$

$$\le \frac{1}{a} \sum_{k=1}^{n} \left[f(d_k) - f(c_k)\right] + \in$$

$$\leq \frac{1}{a} b \left[m^*(E) + \in\right] + \in$$

$$= \frac{b}{a} m^*(E) = \frac{1}{a} \cdot \in + \in$$

i.e. $$m^*(E) \leq \frac{b}{a} m^*(E) + \frac{1}{a} \cdot \in + \in$$

Since $\in > 0$ is arbitrary we have

$$m^*(E) \leq \frac{b}{a} m^*(E)$$

But $0 \leq m^*(E) < \infty$ and $b < a$

$\Rightarrow \frac{b}{a} < 1$, Therefore we must have . $m^*(E) = 0$.

Thus $E = E_{a,b} = \left\{x \in (a,b) \mid \overline{D}f(x) > a > b > \underline{D}f(x)\right\}$ has measure zero. Hence $\overline{D}f(x) = \underline{D}f(x)$ a.e on [a, b] i.e. f is differentiable a.e on [a, b].

2. Definition :

Let f be integrable over a closed bounded interval [a, b]. Let f is extended to $(b, b+1]$ by assuming the value $f(b)$ on this interval.

For all h, $0 < h \leq 1$ we define the divided difference function $\text{Diff}_h f$ average value function $\text{Av}_h f$ on [a, b] by

$$Diff_h f(x) = \frac{f(x+h) - f(x)}{h} \quad \text{and} \quad Av_h f(x) = \frac{1}{h} \int_x^{x+h} f$$

where $0 < h \leq 1$ and $x \in [a, b]$.

3. Note : For all $a \leq u < n \leq b$ we have

$$\int_u^n Diff_h f = \int_u^n \frac{f(x+h) - f(x)}{h} = \frac{1}{h}\left[\int_u^n f(x+h) - \int_n^u f(x)\right]$$

$$= \frac{1}{h}\left[\int_{u+h}^{n+h} f(x) - \int_u^n f(x)\right]$$

$$=\frac{1}{h}\left[\int_{u+h}^{n} f(x)+\int_{n}^{v+h} f(x)-\int_{u}^{u+h} f(x)-\int_{u+h}^{n} f(x)\right]$$

$$=\frac{1}{h}\int_{v}^{n+h} f(x)-\frac{1}{h}\int_{u}^{u+h} f(x)$$

$$= Av_h f(n)-Av_h f(u)$$

Corollary : Let f be an increasing function on the closed bounded interval [a, b]. Then f's is integrable over [a, b] and $\int_a^b f' \leq f(b)-f(a)$

Proof : Since f is increasing on [a, b], it is measurable and hence its divided difference functions are also measurable. Since f is increasing (monotonic) on [a, b] it is differentiable a.e (a, b) [by Lebesgue Theorem]. For each positive integer n, define

$$Diff_{\frac{1}{n}} f(x)=\frac{f\left(x+\frac{1}{n}\right)-f(x)}{\frac{1}{n}}, \ x\in[a,b]$$

Then $\left\{Diff_{\frac{1}{n}} f\right\}_{n=1}^{\infty}$ is a sequence of nonnegative mesaurable functions and

$$\lim_{n\to\infty} Dif_{\frac{1}{n}} f(x)=\lim_{n\to\infty}\frac{f\left(x+\frac{1}{n}\right)-f(x)}{\frac{1}{n}}$$

$$= f'(x) \text{ a.e on [a, b]}$$

Thus $\left\{Diff_{\frac{1}{n}} f\right\}$ converges to f 'a.e. on [a, b].

Hence by Fatou’s Lemma,

$$\int_a^b f' \leq \liminf_{n\to\infty}\int_a^b Diff_{\frac{1}{n}} f$$

But $\displaystyle\int_a^b Diff_{\frac{1}{n}} f(x) = \int_a^b \frac{f\left(x+\frac{1}{n}\right) - f(x)}{\frac{1}{n}} dx$

$$= n\left[\int_a^b f\left(x+\frac{1}{n}\right)dx - \int_a^b f(x)dx\right], \text{(substitution for } x+\frac{1}{n}=t)$$

$$= n\left[\int_{a+\frac{1}{n}}^{b+\frac{1}{n}} f(x)dx - \int_a^b f(x)dx\right]$$

$$= n\left[\int_{a+\frac{1}{n}}^{b} f(x)dx + \int_b^{b+\frac{1}{n}} f(x)dx - \int_a^{a+\frac{1}{n}} f(x)dx - \int_{a+\frac{1}{n}}^{b} f(x)dx\right]$$

$$= n\left[\int_b^{b+\frac{1}{n}} f(x)dx - \int_a^{a+\frac{1}{n}} f(x)dx\right]$$

$$= \frac{1}{\left(1/n\right)}\int_b^{b+\frac{1}{n}} f - \frac{1}{\left(1/n\right)}\int_a^{a+\frac{1}{n}} f$$

$$= Av_{1/n} f(b) - Av_{1/n} f(a)$$

$$\leq f(b) - f(a) \qquad \left(\because f(x) = f(b) \text{ on } [b, b\_1]\right.$$

$$\left.\text{and } f(a) \leq f(x) \text{ on } \left[a, a+\frac{1}{n}\right]\right)$$

$$\Rightarrow \int_a^b Diff_{1/n} f(x) \leq f(b) - f(a) \qquad \forall n$$

$$\Rightarrow \limsup_{n\to\infty}\int_a^b Diff_{1/n} f \leq f(b) - f(a)$$

Therefore we get

$$\int_a^b f' \le \liminf_{n\to\infty} \int_a^b Diff_{1/n} f \le \limsup_{n\to\infty} \int_a^b Diff_{1/n} f \le f(b) - f(a)$$

$$\Rightarrow \int_a^b f' \le f(b) - f(a)$$

6.3 Functions of Bounded Variations

1. Definition : Total Variation

Let *f* be a real valued function defined on the closed bounde interval [a, b]. Let $P = \{x_0, x_1 .... x_k\}$ be a partition of [a, b]. The variation of *f* with respect to partition P is defined by

$$V(f, p) = \sum_{i=1}^{K} \left| f(x_i) - f(x_{i-1}) \right|$$

And the total variation of *f* on [a, b] is defined by,

$$TV(f) = \sup\{V(f, P) | \text{P is a partition of } [a, b]\}$$

2. Definition

A real valued function *f* on the closed, bounded interval [a, b] is said to be of bounded variation on [a, b], if

$$TV(f) < \infty$$

3. Example

Let *f* be an increasing function on [a, b] show that *f* is of bounded variations on [a, b].

Solution : Let P be a partition of [a, b] given by

$$P = \{a = x_0 < x_1 < x_2 < ...... < x_K = b\}$$

Then $$V(f, p) = \sum_{i=1}^{k} \left| f(x_i) - f(x_{i-1}) \right|$$

$$= \sum_{i=1}^{k} \left[f(x_i) - f(x_{i-1}) \right] \qquad \text{(Since } f \text{ is increasing)}$$

$$= \left[f(x_i) - f(x_0) \right] + \left[f(x_2) - f(x_1) \right] + ... + \left[f(x_k) - f(x_{k-1}) \right]$$

$$= f(x_k) - f(x_0)$$

$$= f(b) - f(a)$$

Hence $\sup\limits_{P} V(f, p) = f(b) - f(a)$

$$\Rightarrow TV(f) = f(b) - f(a) < \infty$$

Therefore *f* is of bounded variations on [a, b].

4. Definition

A real valued function *f* is said to be Lipschitz function if there exists a real number $c \geq 0$ such that

$$|f(x') - f(x)| \leq c|x' - x| \qquad \forall x, x' \in [a, b]$$

Lipschitz functions are continuous but converse need not be true.

5. Example

Let *f* be a Lipschitz function on [a, b] show that *f* is of bounded variation on [a, b].

Solution : Since *f* is a Lipschitz function on [a, b], there exists $c \geq 0$ such that,

$$|f(u) - f(v)| \leq c|u - v| \qquad \forall u, v \in [a, b]$$

Therefore for any partition $P = \{x_0, x_1, ...x_k\}$ of [a, b]

$$V(f, p) = \sum_{i=1}^{k} |f(x_i) - f(x_{i-1})|$$

$$\leq \sum_{i=1}^{k} c|x_i - x_{i-1}|$$

$$= c\sum_{i=1}^{k} [x_i - x_{i-1}]$$

$$= c\{(x_1 - x_0) + (x_2 - x_1) + ... + (x_k - x_{k-1})\}$$

$$= c(x_k - x_0)$$

$$= c(b - a)$$

$$\Rightarrow V(f, p) \leq c(b - a)$$

Taking supremum over all partitions of [a, b] we get,

$$\sup_{p} V(f, p) \leq c(b - a)$$

$$\Rightarrow TV(f) \leq c(b - a) < \infty$$

Hence *f* is of bounded variations on [a, b].

6. Example

Define a function f on [0, 1] by

$$f(x) = \begin{cases} x\cos\left(\dfrac{p}{2x}\right) & \text{if } 0 < x \le 1 \\ 0 & \text{if } x = 0 \end{cases}$$

Show that f is continuous on [0, 1] but not of bounded variations on [a, b].

Solution : For any natural number n consider the partition P_n of [0, 1] given by

$$P_n = \left\{0, \frac{1}{2n}, \frac{1}{2n-1}, \ldots, \frac{1}{3}, \frac{1}{2}, 1\right\}$$

The $\quad V(f, P_n) = \sum_{i=1}^{k} \left| f(x_i) - f(x_{i-1}) \right|$

$$= \left| f(x_i) - f(x_0) \right| + \left| f(x_2) - f(x_1) \right| + \ldots + \left| f(x_k) - f(x_{k-1}) \right|$$

$$= \left| f\left(\frac{1}{2n}\right) - f(x_0) \right| + \left| f\left(\frac{1}{2n-1}\right) - f\left(\frac{1}{2n}\right) \right| + \ldots + \left| f\left(\frac{1}{3}\right) - f\left(\frac{1}{2}\right) \right|$$

$$= \left| \frac{1}{2n}\cos\left(2n, \frac{p}{2}\right) - 0 \right| + \left| \frac{1}{2n-1}\cos\left((2n-1), \frac{p}{2}\right) - \frac{1}{2n}\cos\left(2n\frac{p}{2}\right) \right|$$

$$+ \ldots + \left| \cos\frac{p}{2} - \frac{1}{2}\cos p \right|$$

But $\cos\left(n \cdot \dfrac{p}{2}\right) = \pm 1$ if n is even and $\cos\left(n \cdot \dfrac{p}{2}\right) = 0$ if n is odd. Hence we get

$$V(f, P_n) = \frac{1}{2n} + \frac{1}{2n} + \frac{1}{2n-2} + \frac{1}{2n-2} + \ldots + \frac{1}{2} + \frac{1}{2}$$

$$= 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n}$$

As $n \to \infty \left(1 + \dfrac{1}{2} + \dfrac{1}{3} + \ldots + \dfrac{1}{n}\right) \to \infty$ since the series $\sum_{n=1}^{\infty} \dfrac{1}{n}$ is divergent.

Hence $V(f, P_n) \to \infty$ as $n \to \infty$

Therefore f is not a function of bounded variations.

7. Note

Let $c \in [a,b]$ be any element, P be a partition of [a, b] and P' be the refinement of P obtained by adding c to the partition P.

Let $P = \{a = x_0 < x_1 < ....x_n = b\}$

Then, $P' = \{a = x_0 < x_1 < ....x_{i-1} < c < x_i < ...x_n = b\}$

Therefore, $$V(f, p) = \sum_{k=1}^{n} \left(f(x_k) - f(x_{k-1}) \right)$$

$$= \sum_{k=1}^{i} \left| f(x_k) - f(x_{k-1}) \right| + \sum_{k=i+1}^{n} \left| f(x_k) - f(x_{k-1}) \right|$$

$$= \sum_{k=1}^{i-1} \left| f(x_k) - f(x_{k-1}) \right| + \left| f(x_i) - f(x_{i-1}) \right|$$

$$+ \sum_{k=i+1}^{n} \left| f(x_i) - f(x_{i-1}) \right|$$

$$\leq \sum_{k=1}^{i-1} \left| f(x_k) - f(x_{k-1}) \right| + \left| f(c) - f(x_{i-1}) \right| + \left| f(x_i) - f(c) \right|$$

$$+ \sum_{k=i+1}^{n} \left| f(x_k) - f(x_{k-1}) \right|$$

$$= V(f, P')$$

Thus $V(f, P) \leq V(F, P')$ $\qquad \forall P \subseteq P'$

This shows that finer the partition, larger is the variation.

8. Lemma

Let *f* be a function of bounded variations on the closed and bounded interval [a, b]. Then *f* can be expressed as the difference of two increasing functions on [a, b] as follows,

$$f(x) = \left[f(x) + TV\left(f_{[a,x]} \right) \right] - TV\left(f_{[a,x]} \right) \qquad \forall x \in [a,b]$$

Proof : Let $c \in [a,b]$ be arbitrary. Let P be a partition of [a, b] containing c. Then P induces the partitions P_1 and P_2 of [a, c] and [c, b] respectively and we have

$$V\left(f_{[a,b]}, P \right) = V\left(f_{[a,c]}, P_1 \right) + V\left(f_{[c,b]}, P_2 \right)$$

Taking supremum over P, P_1 and P_2 we get,

$$TV\left(f_{[a,b]}\right)=TV\left(f_{[a,c]}\right)+TV\left(f_{[c,b]}\right)$$

If f is a function of bounded variations on [a, b] then $TV\left(f_{[a,b]}\right)<\infty$ and hence $TV\left(f_{[a,x]}\right)<\infty$ for all $x\in[a,b]$.

Therefore of $a\le u<v\le b$ then

$$TV\left(f_{[a,v]}\right)=TV\left(f_{[a,u]}\right)+TV\left(f_{[u,v]}\right)$$

$$\Rightarrow TV\left(f_{[a,v]}\right)-TV\left(f_{[a,u]}\right)=TV\left(f_{[u,v]}\right) \qquad \forall a\le u<v\le b$$

Let $T:[a,b]\to\mathbb{R}$ be a function defined by

$$T(x)=TV\left(f_{[a,x]}\right)$$

T is called the total variation function for f and for $a\le u<v\le b$, we have

$$TV\left(f_{[a,v]}\right)-TV\left(f_{[a,u]}\right)=TV\left(f_{[u,v]}\right)\ge 0$$

$$\Rightarrow T(v)-T(u)\ge 0$$

$$\Rightarrow T(v)\ge T(u)$$

Thus $\quad u<v\Rightarrow T(u)\le T(v)$

Hence T is increasing function. i.e. $TV\left(f_{[a,x]}\right)$ is increasing function on [a, b].

Next for $a\le u<v\le b$, let $P=\{u,v\}$ be the partition of [u, v]. Then,

$$f(u)-f(v)\le\left|f(v)-f(u)\right|=V\left(f_{[u,v]},P\right)\le TV\left(f_{[u,v]}\right)$$

And $\quad TV\left(f_{[u,v]}\right)=TV\left(f_{[a,v]}\right)-TV\left(f_{[a,u]}\right)$

Therefore,

$$f(u)-f(v)\le TV\left(f_{[a,v]}\right)-TV\left(f_{[a,u]}\right)$$

$$\Rightarrow f(u)+TV\left(f_{[a,u]}\right)\le f(v)+TV\left(f_{[a,v]}\right)$$

Thus, $\quad u<v\Rightarrow f(u)+TV\left(f_{[a,u]}\right)\le f(v)+TV\left(f_{[a,v]}\right)$

This shows that $f(x)+TV\left(f_{[a,x]}\right)$ is an increasing function on [a, b].

Finally, $f(x)=\left[f(x)+TV\left(f_{[a,x]}\right)\right]-TV\left(f_{[a,x]}\right),\ \forall x\in[a,b]$

i.e. f can be expressed as a differene of two increasing functions on [a, b].

9. Jordan's Theorem

A function f is of bounded variations on the closed bounded interval [a, b] if and only if it is the difference of two increasing functions on [a, b].

Proof : Let f be a function of bounded variations on [a, b]. Then by preceding lemma f can be expressed as the difference of increasing functions.

Conversely let $f = g - h$ on [a, b] where g and h are increasing functions on [a, b].

Let $P = \{x_0, x_1, x_2, ...x_k\}$ be a partition of [a, b]. Then

$$
\begin{aligned}
V(f,P) &= \sum_{i=1}^{k} \left| f(x_i) - f(x_{i-1}) \right| \\
&= \sum_{i=1}^{k} \left| \left(g(x_i) - h(x_i)\right) - \left(g(x_{i-1}) - h(x_{i-1})\right) \right| \\
&= \sum_{i=1}^{k} \left| \left(g(x_i) - g(x_{i-1})\right) + \left(h(x_i) - h(x_{i-1})\right) \right| \\
&\le \sum_{i=1}^{k} \left\{ \left| g(x_i) - g(x_{i-1}) \right| + \left| h(x_i) - h(x_{i-1}) \right| \right\} \\
&= \sum_{i=1}^{k} \left[g(x_i) - g(x_{i-1}) \right] + \sum_{i=1}^{k} \left[h(x_i) - h(x_{i-1}) \right] \\
&= g(b) - g(a) + h(b) - h(a)
\end{aligned}
$$

Thus, $V(f,P) \le g(b) - g(a) + h(b) - h(a)$ holds for any partition P of [a, b].

Taking supremum over all partitions P of [a, b] we get,

$TV(f) \le g(b) - g(a) + h(b) - h(a) < \infty \Rightarrow f$ is of bounded variations on [a, b].

10. Definition

A function f of bounded variations can be expressed as the difference of two monotonic increasing functions. This representation of f is called as Jordan decomposition of f. The above theorem says that Jordan decomposition exists for a function of bounded variations.

11. Corollary

If a function f is of bounded variations on closed bounded interval [a, b] then it is differentiable almost everywhere on the open interval (a, b) and $f\,'$ is integrable over [a, b].

Proof : According to Jordan theroem, f is the difference of two increasing functions on [a, b]. Let $f = g - h$ where g and h are increasing. Hence g' and h' exists a.e on (a, b) and therefore $f\,' = g' - h'$ exists a.e on (a, b). Also by theorem $f\,'$ is integrable over [a, b].

UNIT - VII

CONTINUOUS FUNCTIONS

7.1 Absolutely Continuous Functions

1. Definition

A real valued function *f* on a closed bounded interval [a, b] is said to be absolutely continuous on [a, b] if for every $\in > 0$, there is $d > 0$ such that for every finite disjoint collection $\{(a_k, b_k)\}_{k=1}^{n}$ of open intervals in (a, b) with

$$\sum_{k=1}^{n}(b_k - a_k) < d \quad \text{then} \quad \sum_{k=1}^{n}\left|f(b_k) - f(a_k)\right| < \in$$

2. Note

If *f* is absolutely continuous on [a, b] then for any $c \in [a,b]$ for given $\in > 0$, there is $d > 0$ such that $|x - c| < d \Rightarrow \left|f(x) - f(c)\right| < \in$.

Therefore *f* is uniformly continuous at c. Since $c \in [a,b]$ is arbitrary, *f* is continuous on [a, b].

Thus absolute continuity implies continuity. But the converse need not be true.

3. Proposition

If the function *f* is Lipschitz on a closed bounded interval [a, b] then it is absolutely continuous on [a, b].

Proof : Let *f* be a Lipschitz function. Then there exists a real number c such that

$$\left|f(u) - f(v)\right| \le c|u - v| \text{ for all } u, v \in [a,b]$$

Then for given $\in > 0$ choose $d = \dfrac{\in}{c}$ then

$$|u - v| < d \Rightarrow \left|f(u) - f(v)\right| \le c \cdot |u - v| < c \cdot \frac{\in}{c} = \in$$

Hence *f* is absolutely continuous on [a, b].

4. Note

Absolutely continuous functions need not be Lipschitz for example $f(x)=\sqrt{x}$, $0 \le x \le 1$ is absolutely continuous but not Lipschitz.

5. Theorem

Let *f* be the absolutely continuous on a closed bounded interval [a, b]. Then *f* is the difference of two increasing absolutely continuous functions and hence *f* is a function of bounded variations on [a, b].

Proof : Let *f* be the absolutely continuous function on [a, b]. Therefore for given $\in = 1$ choose $d > 0$ such that for a partition P of [a, b] containing N closed intervals, $\{[c_k, d_k]\}_{k=1}^{N}$, $|d_k - c_k| < d$ for all k = 1, 2, ... N.

Then on any sub-interval $[c_k, d_k]$ of [a, b]

$|c_k - d_k| < d \Rightarrow |f(c_k) - f(d_k)| < 1$ (since *f* is absolutely continuous)

Therefore for any finite collection $\{(x_i, x_i')\}$ of disjoint intervals in $[c_k, d_k]$ we have

$$\sum_i |x_i' - x_i| < d \Rightarrow \sum_{p_k} |f(x_i') - f(x_i)| < 1$$

Taking supremum over all partitions p_k of $[c_k, d_k]$ we get

$$TV\left(f_{[c_k, d_k]}\right) \le 1, \ \forall, 1 \le k \le N$$

But $$TV\left(f_{[a,b]}\right) = \sum_{k=1}^{N} TV\left(f_{[c_k, d_k]}\right)$$

$$\le \sum_{k=1}^{N} 1 = N < \infty$$

Therefore *f* is a function of bounded variations on [a, b].

Now any function *f* of bounded variations on [a, b] can be written as,

$$f(x) = \left[f(x) + TV\left(f_{[a,x]}\right)\right] - TV\left(f_{[a,x]}\right)$$

where $TV\left(f_{[a,x]}\right)$ is a total variation function on [a, b].

Also sum of two absolutely continuous functions is continuous. Hence it is sufficient to prove that the total variation function $TV\left(f_{[a,x]}\right)$ on [a, b] is absolutely continuous.

Let $\in > 0$ be given. Since f is absolutely continous, for a collection $\{(c_k, d_k)\}_{k=1}^{n}$ of disjoint open intervals, there is $d > 0$ such that

$$\sum_{k=1}^{n} |d_k - c_k| < d \Rightarrow \sum_{k=1}^{n} |f(d_k) - f(c_k)| < \frac{\in}{2}$$

Now P_k if is any partition of $[c_k, d_k]$, k = 1, 2, ... n then,

$$V\left(f, \bigcup_{k=1}^{n} P_k\right) < \frac{\in}{2}$$

$$\Rightarrow \sum_{k=1}^{n} V\left(f_{[c_k, d_k]}, P_k\right) < \frac{\in}{2}$$

Taking supremum over all partitions P_k of $[c_k, d_k]$, k = 1, 2, ... n we get,

$$\sum_{k=1}^{n} TV\left(f_{[c_k, d_k]}\right) \leq \frac{\in}{2} < \in$$

But $\quad TV\left(f_{[c_k, d_k]}\right) = TV\left(f_{[a, d_k]}\right) - TV\left(f_{[a, c_k]}\right)$

Hence $\sum \left[TV\left(f_{[a, d_k]}\right) - TV\left(f_{[a, c_k]}\right)\right] < \in$

Since $TV\left(f_{[a,x]}\right)$ is increasing on [a, b] we have,

$$\sum_{k=1}^{n} |d_k - c_k| < d \Rightarrow \sum_{k=1}^{n} \left|TV\left(f_{[a, d_k]}\right) - TV\left(f_{[a, c_k]}\right)\right|$$

This shows that $TV\left(f_{[a,x]}\right)$ is absolutely continuous on [a, b]. Hence f is the difference of two increasing, absolutely continuous functions on [a, b].

6. Note

$f = (T + f) - T$ where T and T + f are increasing. Above proposition says that if f is absolutely continuous then T is also absolutely continuous and also T + f is absolutely continuous. Thus f can be expressed as a difference of increasing, absolutely continuous functions.

7. Definition : Uniformly Integrable Functions

A family $\mathcal{F}$ of measurable functions defined on E is said to be uniformly integrable over E if for each $\in > 0$ there is a $d > 0$ such that for each $f \in \mathcal{F}$ if A is a measurable of E with $m(A) < d$ then $\int_A |f| < \in$.

8. Note : For $0 < h \le 1$ we have

$$\mathrm{Diff_h}\, f(x) = \frac{f(x+h) - f(x)}{h}, \quad \mathrm{Av_h}\, f(x) = \frac{1}{h}\int_x^{x+h} f(t)dt, \quad \forall x \in [a,b]$$

And as $h \to 0$, $\mathrm{Diff_h}\, f(x) \to f'(x)$ and $\mathrm{Av_h}\, f(x) \to f(x)$, $\forall x \in [a,b]$.

9. Theorem

Let f be continuous on the closed bounded interval [a, b]. Then f is absolutely continuous on [a, b] if and only if the family of divided difference functions, $\{\mathrm{Diff_h}\, f\}_{0<h\le 1}$ is uniformly integrable over [a, b].

Proof : First assume that the family $\{\mathrm{Diff_h}\, f\}_h$ is uniformly integrable over [a, b]. Let $\in > 0$ choose $d > 0$ such that

$$m(E) < d \Rightarrow \int_E |\mathrm{Diff_h}\, f| < \frac{\in}{2}, \quad \forall 0 < h \le 1 \qquad \text{.... (1)}$$

To prove that f is absolutely continuous on [a, b] let $\{(c_k, d_k)\}_{k=1}^{n}$ be the disjoint collection of open subintervals of (a, b) for which

$$\sum_{k=1}^{n} [d_k - c_k] < d \qquad [d \text{ is taken from uniform integrability}] \quad \text{.... (2)}$$

Now for all, $0 < h \le 1$ and $1 \le k \le n$ we have

$$\int_{c_k}^{d_k} \mathrm{Diff_h}\, f = \mathrm{Av_h}\, f(c_k) - \mathrm{Av_h}\, f(d_k)$$

Therefore,

$$\sum_{k=1}^{n} |\mathrm{Av_h}\, f(d_k) - \mathrm{Av_h}\, f(c_k)| = \sum_{k=1}^{n} \left| \int_{c_k}^{d_k} \mathrm{Diff_h}\, f \right|$$

$$\leq \sum_{k=1}^{n} \int_{c_k}^{d_k} \left|\text{Diff}_{\text{h}}\ f\right|$$

$$= \int_{\bigcup_{k=1}^{n}(c_k, d_k)} \left|\text{Diff}_{\text{h}}\ f\right|$$

Let $E = \bigcup_{k=1}^{n}(c_k, d_k)$. Hence we can write,

$$\sum_{k=1}^{n}\left|\text{Av}_{\text{h}}\ f\,(d_k) - \text{Av}_{\text{h}}\ f\,(c_k)\right| \leq \int_E \left|\text{Diff}_{\text{n}}\ f\right| \qquad \text{..... (3)}$$

Using (1) in (3) (Since $m(E) < d$) we get

$$\sum_{k=1}^{n}\left|\text{Av}_{\text{h}}\ f\,(d_k) - \text{Av}_{\text{h}}\ f\,(c_k)\right| < \frac{\in}{2},\quad \forall 0 < h \leq 1$$

Since *f* is continous, taking limit as $h \to 0$ we get,

$$\sum_{k=1}^{n}\left|\ f\,(d_k) - f\,(c_k)\right| < \frac{\in}{2} < \in$$

where $\sum_{k=1}^{n}[d_k - c_k] < d$. Hence *f* is absolutely continous.

To prove the converse, suppose that *f* is a absolutely continuous. Since every absolutely continuous function is a difference of two increasing functions. We prove the converse for increasing function *f*. Now *f* is increasing $\Rightarrow$ the divided difference functions are nonnegative. To prove that the family $\{\text{Diff}_{\text{h}}\ f\}_{0<h\leq 1}$ of divided difference functions is uniformly integrable we prove that for given $\in > 0$ there $d > 0$ such that for any measurable subset E of (a, b) $m(E) < d$ implies

$$\int_E \text{Diff}_{\text{h}}\ f < \in,\ \forall\ \ 0 < h \leq 1$$

Now any measurable set E is contained in a G_d set G such that $m(G - E) = 0$. Every G_d set is the intersection of descending sequence of open sets. And every open set is a disjoint union of countable collection of open intervals. Therefore every open set can be expresed as a union of finite disjoint collection of open intervals. Hence we prove that for a collection $\{(c_k, d_k)\}_{k=1}^{n}$ of finite disjoint open sub-intervals of (a, b) if $E = \bigcup_{k=1}^{n}(c_k, d_k)$ then

$$m(E) < d \Rightarrow \int_E \mathrm{Diff_h}\, f < \frac{\in}{2}$$

We show that that such d exists for given $\in > 0$. Now let $\in > 0$ be arbitrary. The function *f* is absolutely continuous on [a, b + 1]. Therefore for given $\in > 0$ there is $d > 0$ such that if $\{(c_k, d_k)\}_{k=1}^{n}$ is a disjoint collection of open sub-intervals of (a, b) with $\sum_{k=1}^{n} [d_k - c_k] < d$.

$$\Rightarrow \sum_{k=1}^{n} \left(f(d_k) - f(c_k) \right) < \frac{\in}{2}$$

Now for $a \le u < v \le b$ we have

$$\int_u^v \mathrm{Diff_h}\, f = \int_u^v \frac{f(t+h) - f(t)}{h} dt$$

$$= \frac{1}{h}\left[\int_u^v f(t+h) dt - \int_u^v f(t) dt \right]$$

$$= \frac{1}{h}\left[\int_{u+h}^{v+h} f(t) dt - \int_u^v f(t) dt \right]$$

$$= \frac{1}{h}\left[\int_{u+h}^{v} f(t) dt + \int_v^{v+h} f(t) dt - \int_u^{u+h} f(t) dt - \int_{u+h}^{v} f(t) dt \right]$$

$$= \frac{1}{h}\left[\int_v^{v+h} f(t) dt - \int_u^{u+h} f(t) dt \right]$$

$$= \frac{1}{h}\left[\int_0^h f(v+t) dt - \int_0^h f(u+t) dt \right]$$

$$= \frac{1}{h} \int_0^h \left[f(v+t) - f(u+t) \right] dt$$

Let $g(t) = f(v+t) - f(u+t)$. Therefore,

$$\int_u^v \mathrm{Diff_h}\, f = \frac{1}{h} \int_0^h g(t) dt \qquad \text{where } 0 \le t \le 1 \text{ and } 0 < h \le 1$$

Now if $E=\bigcup_{k=1}^{n}(c_k,d_k)$ then

$$\int_E \text{Diff}_h\, f = \int_{\bigcup_{k=1}^{n}(c_k,d_k)} \text{Diff}_h\, f = \sum_{(c_k,d_k)}^{n} \text{Diff}_h\, f$$

$$=\sum_{k=1}^{n}\frac{1}{h}\int_0^h g(t)dt \qquad \text{where } 0\le t\le 1$$

and $\quad g(t)=f(d_k+t)-(c_k+t)$

Therefore, $\quad \int_E \text{Diff}_h\, f=\frac{1}{h}\int_0^h\sum_{k=1}^{n}[f(d_k+t)-f(c_k+t)]dt$

Now $\quad \sum_{k=1}^{n}[d_k-c_k]=\sum_{k=1}^{n}[(d_k+t)-(c_k+t)]<d$

$$\Rightarrow \sum_{k=1}^{n}[f(d_k+t)-f(c_k+t)]<\frac{\in}{2}$$

But $\quad \int_E \text{Diff}_h\, f=\frac{1}{h}\int_0^h\sum_{k=1}^{n}\left[f(d_k+t)-f(c_k+t)\right]$

$$\le\frac{1}{h}\int_0^h\frac{\in}{2}dt=\frac{1}{h}\frac{\in}{2}\cdot h=\frac{\in}{2}$$

$$\Rightarrow \int_E \text{Diff}_h\, f\le\frac{\in}{2}<\in,\ 0<h\le 1$$

Which shows that the family $\{\text{Diff}_n\, f\}_h$ is uniformly integrable over [a, b].

10. Note

For a non-degenerate, closed bounded interval [a, b], let $\mathcal{F}_{Lip}$, $\mathcal{F}_{AC}$ and $\mathcal{F}_{BV}$ denote the families of functions on [a, b] which are Lipschtiz, absolutely continous and of bounded variations respectively. Then the following strict inclusion holds.

$$\mathcal{F}_{Lip}\subseteq\mathcal{F}_{AC}\subseteq\mathcal{F}_{BV}$$

Each of these collections are closed w.r.t. linear combination. Also the functions in any of these collections has total variation function in the same collection and hence any function in one of these collections may be expressed as a difference of two increasing functions in the same collection.

7.2 Integrating Derivatives : Differentiating Indefinite Integrals

1. Definition

Let f be a continuous function on the closed bounde interval [a, b]. By taking $u = a$ and $v = b$ we get,

$$\int_a^b \mathrm{Diff}_h\, f = \mathrm{Av}_h\, f(b) - \mathrm{Av}_h\, f(a), \text{ where } 0 < h \le 1$$

This is called a discrete formulation of the fundamental theorem of integral calculus.

2. Note

Since f is continuous $\mathrm{Av}_h\, f(b) \to f(b)$ and $\mathrm{Av}_h\, f(a) \to f(a)$ as $h \to 0^+$.

Further if f is absolutely continous we prove that

$$\int_a^b \mathrm{Diff}_h\, f \to \int_a^b f' \text{ as } h \to 0$$

3. Theorem : (Fundamental theorem of intergal calculus for Lebesgue integral)

Let f be absolutely continuous function on the closed bounded interval [a, b]. Then f is differentiable almost every where on [a, b], its derivative f' is integrable over [a, b] and $\int_a^b f' = f(b) - f(a)$

Proof : By discrete formulation of the fundamental theorem of Integral calculus we have

$$\int_a^b \mathrm{Diff}_h\, f \to \mathrm{Av}_h\, f(b) - \mathrm{Av}_h\, f(a) \qquad \text{... (1)}$$

Taking limit as $h \to 0^+$ we get

$$\lim_{h\to 0^+}\left(\int_a^b \mathrm{Diff}_h f\right) = \lim_{h\to 0^+}\left[\mathrm{Av}_h f(b) - \mathrm{Av}_h f(a)\right]$$

$$= f(b) - f(a)$$

put $h = \frac{1}{n}$. Therefore as $h \to 0^+$, $n \to \infty$. Hence we get,

$$\lim_{n\to\infty}\left(\int_a^b \mathrm{Diff}_{1/n} f\right) = f(b) - f(a) \qquad \text{.... (2)}$$

Now *f* is absolutely continuous function on [a, b]. Hence *f* can be expressed as a difference of two increasing functions. By Lebsegue theorem increasing functions are differentiable a.e on [a, b]. Hence *f* is also differentiable on [a, b] a.e. Therefore the sequence $\left\{\mathrm{Diff}_{\frac{1}{n}} f\right\}_{n=1}^{\infty}$ converges pointwise almost every where on [a, b] to $f\,'$ ...... (3)

Also the sequence $\left\{\mathrm{Diff}_{\frac{1}{n}} f\right\}_{n=1}^{\infty}$ is uniformly integrable over [a, b]. Therefore by Vitali Convergence Theorem we can write

$$\lim_{n\to\infty}\left(\int_a^b \mathrm{Diff}_{\frac{1}{n}} f\right) = \int_a^b \lim_{n\to\infty} \mathrm{Diff}_{\frac{1}{n}} f = \int_a^b f\,' \qquad \text{..... (4)}$$

Therefore from (2) and (4) we get

$$\int_a^b f\,' = f(b) - f(a)$$

4. Definition

A function *f* on a closed bounded interval [a, b] is called the indefinite integral of a function *g* over [a, b] if *g* is Lebesgue integrable over [a, b] and $f(x) = f(a) + \int_a^x g(t)\,dt, \;\; \forall x \in [a,b].$

5. Theorem

A function *f* on a closed bounded interval [a, b] is absolutely continuous on [a, b] if and only if it is an indefinite integral over [a, b].

Proof : First suppose that f is absolutely continuous on [a, b]. For each $x \in [a,b]$, f is absolutely continuous over [a, x]. Hence by above theorem we have

$$\int_a^x f' = f(x) - f(a)$$

$$\Rightarrow f(x) = f(a) + \int_a^x f'$$

Thus f is the indefinite integral of $f\,'$ over [a, b]. Conversely suppose that f is the indefinite integral of g over [a, b]. Let $\{(a_k, b_k)\}_{k=1}^n$ be the disjoint collection of open subintervals of (a, b).

We define $E = \bigcup_{k=1}^n (a_k, b_k)$. Then,

$$\sum_{k=1}^n |f(b_k) - f(a_k)| = \sum_{k=1}^n \left| \int_{b_k}^{a_k} g \right| \qquad \left(\because f(x) = f(a) + \int_a^x g, \forall x \in [a,b] \right)$$

$$\le \sum_{k=1}^n \int_{a_k}^{b_k} |g|$$

$$= \int_{\bigcup_{k=1}^n (a_k, b_k)} |g|$$

$$= \int_E |g|$$

Now $|g|$ is integrable over [a, b]. Therefore for given $\in > 0$ there is $d > 0$ such that for any measurable subset E of [a, b] with $m(E) < d$, $\int_E |g| < \in$.

Therefore for, $E = \bigcup_{k=1}^n (a_k, b_k)$

$$m(E) < d \Rightarrow m\left(\bigcup_{k=1}^n (a_k, b_k) \right) < d$$

$$\Rightarrow \sum_{k=1}^n m(a_k, b_k) < d$$

$$\Rightarrow \sum_{k=1}^{n} [b_k - a_k] < d$$

$$\Rightarrow \sum_{k=1}^{n} \left| f(b_k) - f(a_k) \right| < \in$$

Which shows that *f* is absolutely continuous over [a, b].

6. Corollary

Let *f* be a monotone function on the closed bounded interval [a, b]. Then f is a absolutely continuous on [a, b] if and only if

$$\int_a^b f' = f(b) - f(a)$$

Proof : If *f* is absolutely continous on [a, b] then by above theorem,

$$f(x) = f(a) + \int_a^x f', \ \forall x \in (a, b]$$

For $x = b$, we get $\int_a^b f' = f(b) - f(a)$

Conversely, assume that f is increasing on [a, b] and $\int_a^b f' = f(b) - f(a)$. Then for any $x \in [a, b]$

$$0 = \int_a^b f' - f(b) + f(a) = \int_a^x f' + \int_x^b f' - f(b) + f(x) - f(x) + f(a)$$

$$0 = \left[\int_a^x f' - f(x) + f(a) \right] + \left[\int_x^b f' - f(b) + f(x) \right] \qquad \text{.... (2)}$$

Since *f* is increasing on [a, b],

$$\int_a^x f' \le f(x) - f(a), \quad \int_x^b f' \le f(b) - f(x)$$

$$\Rightarrow \int_a^x f' - f(x) - f(a) \le 0, \quad \int_x^b f' - f(b) + f(x) \le 0$$

Thus sum of two nonpositive terms is zero (from (2)). Hence each of them must be zero,

i.e. $\displaystyle\int_b^x f' - f(x) + f(a) = 0$

$$\Rightarrow f(x) = f(a) + \int_a^x f' \qquad \forall x \in [a,b]$$

Thus *f* is indefinite integral of *f* ' and hence *f* is absolutely continuous on [a, b].

7. Lemma

Let *f* be integral over closed bounded interval [a, b]. Then $f(x) = 0$ for almost all $x \in [a,b]$ if and only if $\displaystyle\int_{x_1}^{x_2} f = 0$ for all $(x_1, x_2) \subseteq [a,b]$.

Proof : Clearly if $f(x) = 0$ for almost all $x \in [a,b]$ then $\displaystyle\int_{x_1}^{x_2} f = 0$ for all $(x_1, x_2) \subseteq [a,b]$.

Conversely suppose that the condition holds

i.e. $\displaystyle\int_{x_1}^{x_2} f = 0 \quad \forall (x_2, x_1) \subseteq [a,b]$ .... (1)

We claim that, $\displaystyle\int_E f = 0$ for all measurable sets $E \subseteq [a,b]$. .... (2)

Since every open set is a countable union of disjoint open intervals, the equation (1) holds for all open sets. The continuity of integration says that every G_d set G satisfies equation (1) since it is the countable intersection of open sets.

Further every measurable set E of [a, b] can be expressed as $E = G \sim E_0$ where G is G_d sets and E_0 is a set of measure zero. Hence equation (1) holds for any measurable subset of [a, b]. Therefore our claim (2) holds.

Next measurability of *f* implies, the sets

$$E^+ = \{x \in [a,b] \mid f(x) \geq 0\} \text{ and } E^- = \{x \in [a,b] \mid f(x) \leq 0\}$$

are measurable. Therefore $f = f^+ - f^-$ and $f = f^+$ on E^+ and $f = -f^-$ on E^-.

Where both f^+ and f^- are nonnegative measurable functions. Hence,

$$\int_a^b f^+ = \int_{E^+} f = 0 \text{ and } \int_a^b \left(-f^-\right) = \int_{E^-} f = 0$$

$$\Rightarrow \int_a^b f^+ = 0 \text{ and } \int -f^- = 0 \text{ or } \int f^- = 0$$

$\Rightarrow f^+ = 0 \; a.e\,,\; f^- = 0 \; a.e$ (Since f^+ and f^- are nonnegative measurable functions)

But $f = f^+ - f^-$. Hence $f = 0 \; a.e$ on $[a, b]$

8. Theorem

Let f be integrable over the closed bounded interval [a, b] then $\frac{d}{dx}\left[\int_a^x f\right] = f(x)$ for almost all $x \in [a,b]$.

Proof : Define a function F on $[a, b]$ by

$$F(x) = \int_a^x f\,,\; \forall x \in [a,b]$$

Then F is an indefinite integral of some integrable function on $[a, b]$ and hence it is absolutely continuous. Therefore F is differentiable almost everywhere on [a, b] and its derivative F' is integrable.

Now if $[x_1, x_2]$ is any closed interval contained in [a, b], then

$$\int_{x_1}^{x_2} [F' - f] = \int_{x_1}^{x_2} F' - \int_{x_1}^{x_2} f = F(x_2) - F(x_1) - \int_{x_1}^{x_2} f$$

$$= \int_a^{x_2} f - \int_a^{x_1} f - \int_{x_1}^{x_2} f = \int_{x_1}^{x_2} f - \int_{x_1}^{x_2} f$$

$$= 0$$

i.e. $= \int_{x_1}^{x_2} (F' - f) = 0 \quad \forall [x_1, x_2] \subseteq [a,b]$ ($\because [x_1, x_2]$ is arbitrary)

$\Rightarrow \int_{[a,b]} [F' - f] = 0$ (Taking $x_1 = a$ and $x_2 = b$)

$\Rightarrow F' - f = 0$ a.e on [a, b]

$$\Rightarrow F' = f \text{ a.e on } [a, b]$$

$$\Rightarrow \frac{d}{dx}\left[\int_a^x f\right] = f \text{ a.e on } [a, b]$$

9. Note

Above theorem shows that the differential operator $\frac{d}{dx}$ and the integral operator $\int_a^x$ are inverses of each other and that differentiation is reverse process of integration and vice-versa a.e.

THE L^P SPACES

8.1 Normed Linear Spaces

1. Definition

Let E be a measurable set of real numbers. Let $\mathcal{F}$ be a collection of all measurable extended real valued functions on E which are finite a.e on E.

Let $f, g \in \mathcal{F}$. We define a relation on $\mathcal{F}$ by $f \cong g$ if $f(x) = g(x)$ a.e on E. Then,

(i) $f \cong f$ for all $f \in \mathcal{F}$ since $f(x) = f(x)$ on E.

(ii) $f \cong g \Rightarrow g \cong f$, $\forall f, g \in \mathcal{F}$.

(iii) $f \cong g$ and $g \cong h \Rightarrow f = g$ a.e on E and $g = h$ a.e on E.

Let $E_1 = \{x \in E \mid f(x) \neq g(x)\}$, $E_2 = \{x \in E \mid g(x) \neq h(x)\}$

Then $m(E_1) = 0 = m(E_2)$

Therefore, $\{x \in E \mid f(x) = h(x)\} = \left[E - (E_1 \cup E_2)\right] \cup E_3$

$$= (E \cup E_3) - \left[(E_1 \cup E_2) - E_3\right]$$

$$= E - \left[(E_1 \cup E_2) - E_3\right]$$

where $E_3 \subseteq E_1 \cap E_2$ such that $E_3 = \{x \in E_1 \cap E_2 \mid f(x) = h(x)\}$

Since $m(E_1) = m(E_2) = 0$, $m(E_3) = 0$. Also $m(E_1 \cup E_2) = 0$.

Therefore $m((E_1 \cup E_2) - E_3) = 0$. This shows that $f(x) = h(x)$ a.e on E i.e. $f \cong h$.

(i), (ii) and (iii) implies that the relation ‘ $\cong$ ’ is an equivalence relation on $\mathcal{F}$. This equivalence relation on $\mathcal{F}$ induces a partition of $\mathcal{F}$ into disjoint collection of equivalence classes denoted by $\mathcal{F} / \cong$.

If $f, g \in \mathcal{F}$ and a, b are real numbers then $a[f] + b[g] \in \mathcal{F} / \equiv$.

i.e. $\mathcal{F} / \equiv$ is a linear space. The zero element of this equivalence class is the set of all functions which vanish a.e on E.

2. Definition

A set of all equivalence classes $[f] \in \mathcal{F}/\cong$ such that $\int_E |f|^P < \infty$, $1 \le P < \infty$ is called an $L^P(E)$ space.

Thus $\quad L^P(E) = \left\{ [f] \mid \int_E |f|^P < \infty \right\}$

3. Note

(1) If $f \cong g$ then $[f] = [g]$.

But $f \cong g \Rightarrow f = g$ a.e on E.

$$\Rightarrow \int_E |f|^P = \int_E |g|^P$$

i.e. any member of the equivalent class gives same value of the integral. Therefore $L^P(E)$ is properly defined $\forall P, 1 \le P < \infty$.

(2) For any two real numbers a and b,

$$|a+b| \le |a| + |b| \le 2\max\{|a|,|b|\}$$

Hence $|a+b|^P \le 2^P \left[\max\{|a|,|b|\}\right]^P = 2^P \max\left\{|a|^P, |b|^P\right\}$

$$\Rightarrow |a+b|^P \le 2^P \left\{|a|^P + |b|^P\right\}$$

(3) If $[f],[g] \in L^P(E)$ and a, b are real numbers than,

$$\int_E |a\,f + b\,g|^P \le \int_E 2^P \left[|a\,f|^P + |b\,g|^P\right]$$

$$= 2^P |a|^P \int_E |f|^P + 2^P |b|^P \int_E |g|^P < \infty,$$

since $\int_E |f|^P < \infty$ and $\int_E |g|^P < \infty$.

Hence $[a\,f + b\,g] \in L^P(E)$ i.e. $L^P(E)$ is a linear space. For $P = 1$, $L^1(E)$ is a space of all equivalent classes of integrable functions.

4. Definition

Let $\mathcal{F}$ be a collection of all measurable extended real valued functions on E which are finite a.e on E. A function $f \in \mathcal{F}$ is called essentially bounded if there is some real number M > 0 such that $|f(x)| \leq M$ for almost all $x \in E$.

If such a real number M exists it is called an essential upper bound for *f*.

5. Definition : $L^{\infty}(E)$

A collection of all equivalent classes [*f*] for which *f* is essentially bounded is called $L^{\infty}(E)$ space. $L^{\infty}(E)$ is also a linear subspace of $\mathcal{F} / \cong$.

6. Note

For simplicity and convenience we say that the equivalence classes [*f*] as functions and denote them by *f* insterad of [*f*]. Thus

$$L^{P}(E) = \left\{ f \mid \int_{E} |f|^{P} < \infty \right\} \text{ where } f = [f].$$

7. Definition : Norm on Linear Spaces

Let X be a linear space. A real valued functional $\|\bullet\|$ on X is called a norm if $(\|\bullet\| : X \to \mathbb{R})$.

(i) $\|f + g\| \leq \|f\| + \|g\|,\ f, g \in X$

(ii) $\|a f\| = |a| \|f\|,\ \forall f \in X$ and $\forall a \in \mathbb{R}$.

(iii) $\|f\| \geq 0$ and $\|f\| = 0$, iff $f = 0$.

A linear space X together with a norm is called a normed linear space.

A function $f \in X$ is called unit function if $\|f\| = 1$. Note that for any $f \in X$, $f \neq 0$ then $\frac{f}{\|f\|}$ is a unit function. $\frac{f}{\|f\|}$ is a normalization of *f*.

8. Example

Show that $L^1(E)$ is a normed linear space.

Solution : We have, $L^{1}(E) = \left\{ f \mid \int_{E} |f| < \infty \right\}$.

i.e. space of all Lebesgue integrable function is the $L^1(E)$. Clearly $L^1(E)$ is linear space (since Lebesgue integration is linear). Define a norm on $L^1(E)$ by

$$\|f\|_1 = \int_E |f|, \ \forall f \in L^1(E)$$

Then,

(i) $f, g \in L^1(E) \Rightarrow f$ and g are finite a.e on E.

And $|f+g| \le |f| + |g|$ a.e. on E.

$$\Rightarrow \int_E |f+g| \le \int_E |f| + |g| = \int_E |f| + \int_E |g| < \infty$$

$$\Rightarrow f + g \in L^1(E) \text{ and } \|f+g\|_1 \le \|f\|_1 + \|g\|_1$$

(ii) $$\|a f\| = \int_E |a f| = \int_E |a| |f| = |a| \int_E |f|$$

(iii) For $f \in L^1(E)$ such that $\|f\|_1 = 0$ then

$$\int_E |f| = 0 \Rightarrow f = 0 \text{ a.e on E.}$$

$\Rightarrow [f]$ is the zero of $L^1(E)$.

$\Rightarrow f = 0$

Also $f = 0$ on $E \Rightarrow \int_E |f| = 0 \Rightarrow \|f\|_1 = 0$.

Hence $\|\bullet\|$ is a norm on $L^1(E)$. Therefore $L^1(E)$ is a normed linear space.

9. Example

Show that $L^\infty(E)$ is a normed linear space.

Solution : We have $L^\infty(E) = \{ f \mid f \text{ is essentially bounded on } E\}$

$$\Rightarrow L^\infty(E) = \{ f \mid |f(x)| \le \text{M a.e on E for some real number } M > 0\}$$

For a function $f \in L^\infty(E)$ we define,

$$\|f\|_\infty = \inf \{M \mid |f(x)| \le \text{M a.e on E}\}$$

= infimum of the essential supremum of f.

Hence $|f| \le \|f\|_\infty$ a.e on E.

First we prove the triangle in equality for the norm. For each natural number n, $\|f\|_\infty + \frac{1}{n}$ is not the infimum of the essential supremum of *f*. Hence,

$$|f(x)| \le \|f\|_\infty + \frac{1}{n} \text{ a.e. on E.}$$

i.e. $\exists$ a set E_n such that $|f| \le \|f\|_\infty + \frac{1}{n}$ on $E - E_n$ and $m(E_n) = 0$.

Let $E_\infty = \bigcup_{n=1}^{\infty} E_n$. Then $m(E_\infty) = m\left(\bigcup_{n=1}^{\infty} E_n\right) \le \sum_{n=1}^{\infty} m(E_n) = 0$

And $|f| \le \|f\|_\infty$ on $E - E_\infty$ and $m(E_\infty) = 0$

i.e. $|f| \le \|f\|_\infty$ a.e on E.

Thus $\|f\|_\infty$ is the smallest essential supremum of *f*. Now if $f, g \in L^\infty(E)$. Then,

$$|f(x) + g(x)| \le |f(x)| + |g(x)| \le \|f\|_\infty + \|g\|_\infty \text{ a.e on E.}$$

But $\|f + g\|_\infty$ is the smallest essential supremum.

Hence $\|f + g\|_\infty = \|f\|_\infty + \|g\|_\infty$.

Next $\|a\, f\|_\infty$ is the smallest essential supremum of $a\, f$.

$$\Rightarrow |a\, f| \le \|a\, f\|_\infty \text{ a.e. on E.}$$

$$\Rightarrow |a||f| \le \|a\, f\|_\infty \text{ a.e. on E.}$$

$$\Rightarrow |f| \le \frac{1}{|a|}\|a\, f\|_\infty \text{ a.e on E, } a \ne 0.$$

$$\Rightarrow \|f\|_\infty \le \frac{1}{|a|}\|a\, f\|_\infty \qquad (\because \|f\|_\infty \text{ is the smallest upper bound of } |f|)$$

$$\Rightarrow |a|\|f\|_\infty \le \|a\, f\|_\infty$$

Also $\quad |f| \le \|f\|_\infty$ a.e. on E.

$$\Rightarrow |a||f| \le |a|\|f\|_\infty \text{ a.e on E}$$

$\Rightarrow |a\,f| \le |a| \|f\|_\infty$ a.e on E

$\Rightarrow \|a\,f\|_\infty \le |a| \|f\|_\infty$

Thus $\|a\,f\|_\infty = |a| \cdot \|f\|_\infty$ $(a \ne 0)$

Clearly $\|f\|_\infty \ge 0$. And if $\|f\|_\infty = 0$ then

$|f(x)| \le \|f\|_\infty = 0$ a.e on E.

$\Rightarrow |f(x)| = 0$ a.e on E.

$\Rightarrow f = 0$ a.e. on E.

$\Rightarrow f \cong 0$

Thus $\|\bullet\|_\infty$ is a norm on $L^\infty(E)$ and therefore $L^\infty(E)$ is a normed linear space.

8.2 The Inequalities

1. Definition

For any measurable set E, $f \in L^P(E)$, $1 < P < \infty$ we define a function $\|\bullet\|$ on $L^P(E)$ by

$$\|f\|_P = \left[\int_E |f|^P\right]^{1/P}$$

We show that $\|\bullet\|$ is a norm on $L^P(E)$.

2. Definition

If $p \in (1, \infty)$ is a real number then its conjugate q is also real number such that $\frac{1}{p} + \frac{1}{q} = 1$.

Note that if $p \in (1, \infty)$, its conjugate $q = \frac{p}{p-1}$ also lies in $(1, \infty)$. Conjugate of 1 is ∞ and vice-versa.

3. Theorem : Young's Inequality

For $1 < p < \infty$, and a conjugate q of p, and for any two positive real numbers a and b,

$$ab < \frac{a^p}{p} + \frac{b^q}{q}$$

Proof : Define a function g by

$$g(x)=\frac{1}{p}x^p+\frac{1}{q}-x,\ \forall x>0$$

Differentiating w.r.t. x we get,

$$g'(x)=x^{p-1}-1$$

Then $g'(x)>0$ if $x>1$ and $g'(x)<0$ if $x<1$ and $g'(x)=0$ at $x=1$. Also $g''(x)=(p-1)x^{p-2}$.

Hence $g''(x)>0,\ \forall x>0$ since $p>1$.

Therefore g is minimum at $x=1$ and $g_{\min}=\frac{1}{p}+\frac{1}{q}-1=0$.

Hence $g(x)\geq 0,\ \forall x\in(0,\infty)$.

$$\Rightarrow \frac{1}{p}x^p+\frac{1}{q}-x\geq 0,\ \forall x\in(0,\infty)$$

$$\Rightarrow \frac{x^p}{p}+\frac{1}{q}\geq x,\ \forall x\in(0,\infty)$$

$$\Rightarrow x\leq\frac{x^p}{p}+\frac{1}{q},\ \forall x\in(0,\infty)$$

Take $x=\frac{a}{b^{q-1}}$. Then we get

$$\left(\frac{a}{b^{q-1}}\right)\leq\frac{1}{p}\left(\frac{a}{b^{q-1}}\right)^p+\frac{1}{q}$$

$$\Rightarrow \frac{a}{b^{q-1}}\leq\frac{1}{p}\frac{a^p}{b^{(q-1)p}}+\frac{1}{q}$$

$$\Rightarrow \frac{a}{b^{q-1}}\leq\frac{1}{p}\frac{a^p}{b^q}+\frac{1}{q} \qquad (\because (q-1)p=q)$$

$$\Rightarrow \frac{a\cdot b^q}{b^{q-1}}\leq\frac{a^p}{p}+\frac{b^q}{q}$$

$$\Rightarrow a \cdot b \le \frac{a^p}{p} + \frac{b^q}{q}$$, where $a > 0$, $b > 0$ and p and q are conjugates of each other.

4. Theorem : Holder's Inequality

Let E be any measurable set and $1 \le p < \infty$ and q the conjugate of p. If $f \in L^P(E)$ and $g \in L^q(E)$ then their product $f \cdot g$ is integrable over E and

$$\int_E |f \cdot g| \le \|f\|_p \cdot \|g\|_q$$

Moreover if $f \ne 0$ then the function f, given by

$$f^* = \|f\|_p^{1-p} \cdot \text{sgn}(f) \cdot |f|^{p-1} \text{ belongs to } L^q(E).$$

and $\int_E f \cdot f^* = \|f\|_p$ and $\|f^*\|_q = 1$

Proof :

Case 1 : p = 1

If p = 1 then $q = \infty$. Therefore if $f \in L^1(E)$ and $g \in L^\infty(E)$.

then $\int_E |f| < \infty$ and $\|f\|_1 = \left(\int |f|^1\right)^{1/1} = \int_E |f|$

and $|g| \le \|g\|_\infty$ a.e on E.

$$\text{Therefore } \int_E |f \cdot g| = \int_E |f| \cdot |g|$$

$$\le \int_E |f| \cdot \|g\|_\infty$$

$$= \|g\|_\infty \int_E |f| = \|g\|_\infty \cdot \|f\|_1$$

$$\text{Thus } \int_E |f \cdot g| \le \|f\|_1 \cdot \|g\|_\infty$$

Next for p = 1, $f^* = \|f\|_1^{1-1} \text{sgn}(f) |f|^{1-1}$

$$= \text{sgn}(f) \in L^\infty(E)$$

since $\text{sgn}(f) = \pm 1 \in L^\infty(E)$

And $\int_E f \cdot f^* = \int_E f \cdot \text{sgn}(f) = \int_E |f| = \|f\|_1$

Also $\|f^*\|_\infty = \|\text{sgn}(f)\|_\infty = 1$ since 1 is the essential supremum of sgn (*f*).

Thus the result holds for p = 1.

Case II : p > 1 let $f \neq 0$, $g \neq 0 \ni f \in L^p(E)$, $g \in L^q(E)$.

If Holder's Inequality is true when *f* is replaced by $\dfrac{f}{\|f\|_p}$ and *g* is replaced by $\dfrac{g}{\|g\|_q}$ then it is true for *f* and *g*. Hence we assume that *f* and *g* are normalized functions. i.e. $\|f\|_p = 1$, $\|g\|_q = 1$.

Then, $\left(\int_E |f|^p\right)^{1/p} = 1$, $\left(\int_E |g|^q\right)^{1/q} = 1$

$$\Rightarrow \int_E |f|^p = 1, \Rightarrow \int_E |g|^q = 1$$

Now $|f|^p$ and $|g|^q$ are integrable over E.

Therefore *f* and *g* are finite a.e on E. Therefore by Young's Inequality, taking $a = |f|$, $b = |g|$,

$$|f| \cdot |g| \leq \frac{|f|^p}{p} + \frac{|g|^p}{q} \text{ a.e on E.}$$

$$\Rightarrow \int_E |f \cdot g| \leq \int_E \frac{|f|^p}{p} + \int \frac{|g|^p}{q}$$

$$= \frac{1}{p}\int_E |f|^p + \frac{1}{q}\int |g|^p$$

$$= \frac{1}{p}(1) + \frac{1}{q}(1) = \frac{1}{p} + \frac{1}{q}$$

$$= 1$$

$\Rightarrow \int_E |f \cdot g| \leq 1$ i.e. is integrable over E.

Hence $f, g \in L^1(E)$

Also for any functions f and g (not normalised) $\dfrac{f}{\|f\|_p}$ and $\dfrac{g}{\|g\|_q}$ are normalised. Hence we get

$$\int_E \left| \frac{f}{\|f\|_p} \cdot \frac{g}{\|g\|_q} \right| \leq 1$$

$$\Rightarrow \int_E |f \cdot g| \leq \|f\|_p \cdot \|g\|_q$$

Finally for any function $f \in L^p(E)$,

$$f \cdot f^* = f \cdot \|f\|_p^{1-p} \operatorname{sgn}(f) \cdot |f|^{p-1}$$

$$= \|f\|_p^{1-p} \cdot |f|^p, \text{ a.e on E} \qquad (\because f \cdot \operatorname{sgn}(f) = |f| \text{ a.e. on E})$$

$$\Rightarrow \int_E f \cdot f^* = \int_E \|f\|_p^{1-p} \cdot |f|^p$$

$$= \|f\|_p^{1-p} \cdot \int_E |f|^p = \|f\|_p^{1-p} \cdot \|f\|_p^p$$

$$= \|f\|_p^{(1-p)+p} = \|f\|_p$$

And $$\|f^*\|_q = \left(\int_E |f^*|^q \right)^{1/q}$$

$$= \left(\int_E \left| \|f\|_p^{1-p} \operatorname{sgn}(f) \cdot |f|^{p-1} \right|^q \right)^{1/q}$$

$$= \|f\|_p^{(1-p)} \left(\int_E |f|^{(p-1)q} \right)^{1/q}$$

$$= \|f\|_p^{1-p} \cdot \left(\int_E |f|^p \right)^{\frac{1}{p} \times \frac{p}{q}}$$

$$= \|f\|_p^{1-p} \cdot \|f\|_p^{\frac{p}{q}} = \|f\|_p^{(1-p)+\frac{p}{q}}$$

$$= \| f \|_p^0 \qquad \left(\because \frac{p}{q} + (1-p) = 0 \right)$$

$$= 1$$

Thus $\int\limits_E f \cdot f^* = \| f \|_p$ and $\| f^* \|_q = 1$.

5. Minkowski's Inequality

Let E be a measurable set and $1 < p < \infty$. If the functions f and g belongs to $L^p(E)$ then $f + g \in L^p(E)$ and $\| f + g \|_p \le \| f \|_p + \| g \|_p$

Proof : Let, $1 < p < \infty$.

Since $L^p(E)$ is a linear space, $f, g \in L^p(E)$

$$\Rightarrow f + g \in L^p(E)$$

Let $f + g \ne 0$ on E. Let $(f + g)^*$ be the conjugate function of $f + g$. Then by Holder**'s** Inequality we have,

$$\| f + g \|_p = \int\limits_E (f + g) \cdot (f + g)^*$$

And $\quad \| (f + g)^* \|_q = 1$

$$= \int\limits_E f \cdot (f + g)^* + \int\limits_E g (f + g)^*$$

$$\le \| f \|_p \cdot \| (f + g)^* \|_q + \| g \|_p \| (f + g)^* \|_q$$

$$= \left(\| f \|_p + \| g \|_p \right) \cdot \| (f + g)^* \|_q$$

$$= \| f \|_p + \| g \|_q \qquad \left(\because \| (f + g)^* \|_q = 1 \right)$$

6. Note

We have already established the Minkowski's Inequality for p = 1, p = ∞.

Hence the inequality, $\| f + g \|_p \le \| f \|_p + \| g \|_p$ holds $\forall, 1 \le p \le \infty,$.

7. Theorem : (The Cauchy -Schwarz Inequality)

Let E be a measurable set and f and g be measurable functions on E such that f^2 and g^2 are integrable over E. Then $f \cdot g$ is also integrable over E and

$$\int_E |f \cdot g| \le \left[\left(\int_E f^2\right)\left(\int_E g^2\right)\right]^{1/2}$$

Proof : Since f^2 and g^2 are integrable over E we have $f, g \in L^2(E)$. Hence by Holder inequality (p = 2, q = 2).

$$\int_E |f \cdot g| \le \|f\|_2 \cdot \|g\|_2$$

$$= \left(\int_E f^2\right)^{1/2} \cdot \left(\int_E g^2\right)^{1/2}$$

$$= \left[\left(\int_E f^2\right)\left(\int_E g^2\right)\right]^{1/2}$$

8. Corollary

Let E be a mesurable set and $1 < p < \infty$.

Let $\mathcal{F}$ be a subfamily of $L^p(E)$ such that $\|f\|_p \le M$, $\forall f \in \mathcal{F}$ and for some constant M. Then the family $\mathcal{F}$ is uniformly integrable over E.

Proof : Let $\in > 0$ and let A be any measurable subset of E of finite measure. Let p and q be the conjugates of each other consider the spaces $L^p(A)$ and $L^q(A)$. Define a function g on A by $g(x) = 1$, $\forall x \in A$. Then,

$$\int_A |g|^q = \int_A 1 = m(A) < \infty \text{. Hence, } g \in L^q(A)$$

Now $f \in \mathcal{F} \subseteq L^p(A)$. Then $\|f\|_p \le M$.

Therefore by restricting f to A and by Holder's inequality we get,

$$\int_A |f \cdot g| = \int_A |f| \le \|f\|_p \cdot \|g\|_q$$

$$\le \left(\int_A |f|^p\right)^{1/p} \cdot \left(\int_A |g|^q\right)^{1/q}$$

$$\leq \left(\int_E |f|^p \right)^{1/p} \cdot (m(A))^{1/q}$$

$$\leq M \cdot [m(A)]^{1/q}$$

Thus for all $f \in \mathcal{F}$, for given $\in > 0$, take $d = \left(\frac{\in}{M} \right)^q$ and we get,

$$m(A) < d \Rightarrow \int_A |f| \leq M\,(m(A))^q < M \cdot d^{1/q}$$

$$= M \cdot \left(\frac{\in}{M} \right)^{q \times \frac{1}{q}} = \in$$

$$\Rightarrow \int_A |f| < \in$$

Hence f is uniformly integrable over E, $\forall f \in \mathcal{F}$ i.e. The family $\mathcal{F}$ is uniformly integrable over E.

9. Corollary

Let E be a measurable set of finite measure and let $1 \leq p_1 < p_2 \leq \infty$. Then $L^{p_2}(E) \subseteq L^{p_1}(E)$. And further

$$\| f \|_{p_1} \leq c \| f \|_{p_2}, \ \forall f \in L^{p_2}(E)$$

Where $C = [m(E)]^{\frac{p_2 - p_1}{p_1 - p_2}}$ if $p_2 < \infty$

and $C = [m(E)]^{\frac{1}{p_1}}$ if $p_2 = \infty$

Proof :

Let $p_2 < \infty$. Define $p = \dfrac{p_2}{p_1} > 1$ $\quad (\because p_1 < p_2)$

Let q be the conjugate of p. Let $f \in L^{p_2}(E)$.

Then $\int_E |f^{p_1}|^p = \int_E |f^{p_1}|^{\frac{p_2}{p_1}} = \int_E |f|^{p_2} < \infty$

$$\Rightarrow f^{p_1} \in L^p(E)$$

Let $g = c_E$. Then $\int_E |g|^q = \int_E 1 = m(E) < \infty$

Hence $g \in L^q(E)$. By Holder's inequality we get,

$$\int_E \left| f^{p_1} \cdot g \right| \le \left\| f^{p_1} \right\|_p \cdot \|g\|_q$$

$$\Rightarrow \int_E \left| f^{p_1} \right| \le \left(\int_E \left| f^{p_1} \right|^p \right)^{\frac{1}{p}} \cdot \left(\int_E |g|^q \right)^{\frac{1}{q}}$$

$$= \left(\int_E |f|^{p_2} \right)^{\frac{p_1}{p_2}} \cdot \left(\int_E 1 \right)^{\frac{1}{q}}$$

$$= \| f \|_{p_2}^{p_1} \cdot (m(E))^{1/q}$$

$$\Rightarrow \int_E \left| f^{p_1} \right| \le \| f \|_{p_2}^{p_1} \cdot (m(E))^{1/q}$$

Taking $\frac{1}{p_1}$ power of both sides we get

$$\left(\int_E \left| f^{p_1} \right| \right)^{1/p_1} \le \| f \|_{p_2} \cdot [m(E)]^{\frac{1}{p_1 q}}$$

$$= m(E)^{\frac{p_2 - p_1}{p_1 p_2}} \cdot \| f \|_{p_2}$$

Since $\quad \frac{1}{p_1 q} = \frac{1}{p_1}\left(1 - \frac{1}{p}\right) = \frac{1}{p_1}\left(1 - \frac{p_1}{p_2}\right) = \frac{p_2 - p_1}{p_1 p_2}$

Therefore if $C = [m(E)]^{\frac{p_2 - p_1}{p_1 p_2}}$ then

$$\| f \|_{p_1} \le C \| f \|_{p_2} < \infty$$

Hence $f \in L^{p_1}(E)$ and therefore $L^{p_2}(E) \subseteq L^{p_1}(E)$

Case II : $p_2 = \infty$ and $1 \le p_1 < p_2 = \infty$

Then $f \in L^{\infty}(E) \Rightarrow |f| \le \|f\|_{\infty}$

$$\Rightarrow |f|^{p_1} \le \|f\|_{\infty}^{p_1}$$

$$\Rightarrow \int_E |f|^{p_1} \le \int_E \|f\|_{\infty}^{p_1}$$

$$= \|f\|_{\infty}^{p_1} \cdot m(E) < \infty$$

Hence $f \in L^{p_1}(E)$. And taking $\frac{1}{p_1}$ power of both sides we get,

$$\left(\int_E |f|^{p_1}\right)^{\frac{1}{p_1}} \le \|f\|_{\infty} \cdot m(E)^{1/p_1}$$

Take $C = m(E)^{\frac{1}{p_1}}$ then

$\|f\|_{p_1} \le C\|f\|_{p_2}$, $p_2 = \infty$ and $C = m(E)^{\frac{1}{p_1}}$.

10 Note

(1) If E is of finite mesure then $1 \le p_1 < p_2 \le \infty \Rightarrow L^{p_2}(E)$ is proper subspace of $L^{p_1}(E)$.

(2) If E is of infinite measure, then there are no inlcusion relationships among $L^p(E)$ spaces.

11. Example

Let E be a set of finite measure and let $1 \le p_1 < p_2 \le \infty$. Take $E = (0,1]$. Define a function f on E by $f(x) = x^a$ where $-\frac{1}{p_1} < a < -\frac{1}{p_2}$.

Then $f \in L^{p_1}(E)$ but $f \notin L^{p_2}(E)$. Hence $L^{p_2}(E) \underset{\neq}{\subset} L^{p_1}(E)$.

12. Example

Let $E = (1, \infty)$. Then $m(E) = \infty$. Define a function f on E by,

$$f(x) = \frac{x^{-1/2}}{1 + \ln x}, x > 1$$

Then $f \in L^2(E)$ and $f \notin L^p(E)$ for any $p \neq 2$. Hence in general there is no inclusion relationship among the $L^p(E)$ whenever E is not a set with finite measure.

8.3 L^p is Complete : The Riesz-Fischer Theorem

Since L^p spaces are normed spaces, it is possible to introduce convergence concepts in L^p, similar to the convergence in $\mathbb{R}$, which is normed by absolute value function.

1. Definition

A sequence $\{f_n\}$ in a linear space X which is normed by a norm function $\|\bullet\|$ on X is said to converge to a function f in X if

$$\lim_{n \to \infty} \| f - f_n \| = 0$$

We write $\{f_n\} \to f$ in X or $\lim_{n \to \infty} f_n = f$ in X.

2. Definition

For $1 \leq p < \infty$, $L^p(E)$ are normed linear spaces. For a sequence $\{f_n\}$ of functions in $L^p(E)$, $\{f_n\} \to f$, $f \in L^p(E)$ if $\lim_{n \to \infty} \| f_n - f \|_p = 0$ i.e.

$$\lim_{n \to \infty} \int_E | f_n - f |^p = 0$$

For $p = \infty$, the sequence $\{f_n\}$ of functions in $L^\infty(E)$ converges to a function $f \in L^\infty(E)$ if $\{f_n\} \to f$ uniformly a.e on E.

3. Definition

Let X be a normed space normed by $\|\bullet\|$. A squence $\{f_n\}$ in X is said to be Cauchy in X if for each $\in > 0$ there is a natural number N such that $\| f_n - f_m \| < \in$ for all $m, n \geq N$.

A normed linear space X is said to be **complete** if every Cauchy sequence in X converges to a function in X. A comlete normed linear space is called a **Banach Space .**

4. Proposition :

Let X be a normed linear space. Then every convergent sequence in X is Cauchy. And a Cauchy sequence in X converges it it has a convergent subsequence.

Proof : Let $\{f_n\}$ be a convergent sequence in X such that $\{f_n\} \to f$ in X. By triangle inequality for the norm on X,

$$\|f_n - f_m\| = \|f_n - f + f - f_m\|$$

$$\leq \|f_n - f\| + \|f - f_m\| \qquad \forall m, n$$

As $m \to \infty$, $f_m \to f$ and $\|f - f_m\| \to 0$. Therefore there is an integer N_1 such that

$$\|f - f_m\| < \frac{\in}{2} \qquad \forall m \geq N_1$$

Similarly there exists an integer N_2 such that

$$\|f - f_n\| < \frac{\in}{2} \qquad \forall n \geq N_2$$

Take $N = \max\{N_1, N_2\}$. Then we get

$$\|f_n - f_m\| < \frac{\in}{2} + \frac{\in}{2} = \in \qquad \forall m, n \geq N$$

Hence $\{f_n\}$ is a Cauchy sequence.

Now let $\{f_n\}$ be a Cuchy sequence in X that has a subsequence $\{f_{n_k}\}$ which converges to f in X. Let $\in > 0$ be given. Since $\{f_n\}$ is Cauchy we can choose an integer N such that

$$\|f_n - f_m\| < \frac{\in}{2} \qquad \forall m, n \geq N$$

Since $\{f_{n_k}\}$ converges to f we can choose k such that $n_k > N$ and $\|f_{n_k} - f\| < \frac{\in}{2}$.

Using triangle inequality for the norm we get

$$\|f_n - f\| = \|f_n - f_{n_k} + f_{n_k} - f\|$$

$$\leq \|f_n - f_{n_k}\| + \|f_{n_k} - f\| < \frac{\in}{2} + \frac{\in}{2}$$

$$\Rightarrow \|f_n - f\| < \in \text{ for all } n \geq N.$$

Therefore the sequence converges to f in X.

5. Definition : Rapidly Cauchy Sequence

Let X be a linear space normed by $\|\bullet\|$. A sequence $\{f_n\}$ in X is said to be rapidly Cauchy if there is a convergent series of positive numbers $\sum_{k=1}^{\infty} \in_k$ for which

$$\|f_{k+1} - f_k\| \leq \in_k^2 \text{ for all k.}$$

6. Note

If $\{f_n\}$ is a sequence in normed linear space and if there is a sequence of non-negative numbers $\{a_k\}$ such that

$$\|f_{k+1} - f_k\| \leq a_k \text{ for all k.}$$

Then $\quad f_{n+k} - f_n = [f_{n+k} - f_{n+k-1}] + [f_{n+k-1} - f_{n+k-2}] + ... + [f_{n+1} - f_n]$

$$= \sum_{j=n}^{n+k-1} [f_{j+1} - f_j] \text{ for all n, k.}$$

Therefore, $\|f_{n+k} - f_n\| = \left\| \sum_{j=n}^{n+k-1} [f_{j+1} - f_j] \right\|$

$$\leq \sum_{j=n}^{n+k-1} \|f_{j+1} - f_j\|$$

$$\leq \sum_{j=n}^{n+k-1} a_j \leq \sum_{j=n}^{\infty} a_j, \quad \forall n, k$$

Thus $\quad \|f_{n_k} - f_n\| = \sum_{j=n}^{\infty} a_j, \ \forall n, k$

7. Proposition

Let X be a normed linear space. Then every rapidly Cauchy sequence in X is Cauchy. Further every Cauchy sequence has rapidly Cauchy subsequence.

Proof : Let $\{f_n\}$ be a rapidly Cauchy sequence in X and let $\sum_{k=1}^{\infty} \in_k$ be a convergent series of non-negative numbers for which

$$\|f_{k+1} - f_k\| < \in_k^2 \text{ for all k.}$$

$$\Rightarrow \|f_{n+k} - f_n\| \leq \sum_{j=n}^{\infty} \in_j^2$$

Since the series $\sum_{k=1}^{\infty} \in_k$ converges, the series $\sum_{k=1}^{\infty} \in_k^2$ also converges. Hence the sequence $\{f_n\}$ is a Cauchy sequence.

Conversely assume that $\{f_n\}$ is a Cauchy sequence in X. We can choose strictly increasing sequence of natural numbers $\{n_k\}$ such that

$$\|f_{n_{k+1}} - f_{n_k}\| \leq \left(\frac{1}{2}\right)^k, \ \forall k$$

Take $a_k = \left(\frac{1}{2}\right)^{\frac{k}{2}} = \left(\frac{1}{\sqrt{2}}\right)^k$. Then $\|f_{n_{k+1}} - f_{n_k}\| \leq \left(\frac{1}{2}\right)^k = a_k^2, \forall k$

and $\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty}\left(\frac{1}{\sqrt{2}}\right)^k$ converges

Hence $\{f_{n_{k+1}}\}$ is a rapidly Cauchy subsequence.

8. Theorem

Let E be a measurable set and $1 \leq p \leq \infty$. Then every rapidly Cauchy sequence in $L^p(E)$ converges, with respect to $L^p(E)$ norm, pointwise a.e on E to a function in $L^p(E)$.

Proof : We assume that $1 \leq p < \infty$. Let $\{f_n\}$ be a rapidly convergent subsequence in $L^p(E)$. Then for a convergent series $\sum_{k=1}^{\infty} \in_k$ of positive numbers we have,

$$\| f_{k+1} - f_k \|_p \le \in_k^{\,2}, \forall k$$

$$\Rightarrow \left(\int_E | f_{k+1} - f_k |^p \right)^{1/p} \le \in_k^{\,2} \text{ for all k.}$$

$$\Rightarrow \int_E | f_{k+1} - f_k |^p \le \in_k^{\,2p} \text{ for all k.}$$

For a fixed natural number k we have,

$$| f_{k+1}(x) - f_k(x) | \ge \in_k \text{ if and only if } | f_{k+1}(x) - f_k(x) |^p \ge \in_k^{\,p}$$

Therefore by Chebychev's inequality we get,

$$m\left\{ x \in E \,\middle|\, | f_{k+1}(x) - f_k(x) | \ge \in_k \right\} = m\left\{ x \in E \,|\, | f_{k+1}(x) - f_k(x) |^p \ge \in_k^{\,p} \right\}$$

$$\le \frac{1}{\in_k^{\,p}} \int_E | f_{k+1} - f_k |^p \le \frac{1}{\in_k^{\,p}} \cdot \in_k^{\,2p}$$

$$\le \in_k^{\,p}$$

Let $\quad E_k = \left\{ x \in E \,|\, | f_{k+1}(x) - f_k(x) | \ge \in_k \right\}$

Then $m(E_k) \le \in_k^{\,p}$. And

$$m\left(\bigcup_{k=1}^{\infty} E_k \right) \le \sum_{k=1}^{\infty} m(E_k) \le \sum_{k=1}^{\infty} \in_k^{\,p}$$

But $p \ge 1$, hence the series $\sum_{k=1}^{\infty} \in_k^{\,p}$ converges. Therefore

$$\sum_{k=1}^{\infty} m(E_k) < \infty$$

$$\Rightarrow \lim_{n \to \infty} \sum_{k=n}^{\infty} m(E_k) = 0$$

Hence by Borel-Cantelli lemma almost all $x \in E$ belongs to atmost finitely many E_k's. i.e. $\exists$ a set $E_0 \subseteq E$ such that $m(E_0) = 0$ and for all $x \in E - E_0$, there exists an integer $K(x)$ such that

$$| f_{k+1}(x) - f_k(x) | < \in_k, \ \forall k \ge K(x)$$

(since x belongs to atmost finitely many E_k's)

Thus if $x \in E - E_0$ then

$$\left| f_{k+1}(x) - f_k(x) \right| < \in_k, \ \forall k \geq K(x)$$

$$\Rightarrow \left| f_{n+k}(x) - f_n(x) \right| \leq \sum_{j=n}^{n+k-1} \left| f_{j+1}(x) - f_j(x) \right|$$

$$\leq \sum_{j=n}^{n+k-1} \in_j \qquad \text{for all } n \geq K(x) \text{ and for all k.}$$

$$\leq \sum_{j=n}^{\infty} \in_j \qquad \text{for all } n \geq K(x) \text{ and } \forall k.$$

Since the series $\sum_{j=n}^{\infty} \in_j$ converges, the sequence $\{ f_k(x) \}$ is Cauchy. Since the set $\mathbb{R}$ of real numbers is complete, the sequence $\{ f_k(x) \}$ converges in $\mathbb{R}$. Let $f_k(x) \to f(x)$.

Then $f_k \to f$ a.e on E. (since $x \in E - E_0$ and $m(E_0) = 0$)

Now we have,

$$\| f_{n+k} - f_n \|_p = \left\| \sum_{j=n}^{n+k-1} \left| f_{j+1}(x) - f_j(x) \right| \right\|$$

$$\leq \sum_{j=n}^{n+k-1} \left\| f_{j+1}(x) - f_j(x) \right\|_p$$

$$\leq \sum_{j=n}^{n+k-1} \in_j^2 \leq \sum_{j=n}^{\infty} \in_j^2$$

$$\Rightarrow \left(\int_E \left| f_{n+k} - f_n \right|^p \right)^{\frac{1}{p}} \leq \sum_{j=n}^{\infty} \in_j^2$$

$$\Rightarrow \int_E \left| f_{n+k} - f_n \right|^p \leq \left(\sum_{j=n}^{\infty} \in_j^2 \right)^p, \ \forall n, k$$

Since $f_n \to f$ pointwise a.e on E, taking limit as $k \to \infty$, we get (By Fatous Lemma)

$$\int_E \left| f - f_n \right|^p \leq \underline{\lim} \int_E \left| f_{n+k} - f_n \right|^p \leq \left(\sum_{j=n}^{\infty} \in_j^2 \right)^p$$

$$\Rightarrow \int_E |f - f_n|^p \le \left(\sum_{j=n}^{\infty} \in_j{}^2\right)^p \text{, for all n.}$$

Now $\sum_{j=1}^{\infty} \in_j{}^2$ converges. Hence $\left(\sum_{j=1}^{\infty} \in_j{}^2\right)^p < \infty$.

$$\Rightarrow \int_E |f - f_n|^p < \infty$$

$\Rightarrow |f - f_n|^p$ is integrable over E.

$$\Rightarrow f - f_n \in L^p(E)$$

But $f_n \in L^p(E)$ for all n, and $L^p(E)$ is a linear space. Hence $f_n + (f - f_n) \in L^p(E)$, $\forall n$.

$$\Rightarrow f \in L^p(E)$$

where *f* is a pointwise limit of $\{f_n\}$ a.e on E. Thus $\{f_n\} \in L^p(E)$ and $f_n \to f$ a.e on E, pointwise then $f \in L^p(E)$. i.e. $L^p(E)$ is complete.

9. Riesz-Fischer Theorem

Let E be a measurable set and $1 \le p \le \infty$. Then $L^p(E)$ is a Banach space. Moreover if $\{f_n\} \to f$ in $L^p(E)$, then a subsequence of $\{f_n\}$ converges pointwise a.e on E to *f*.

Proof : We know that $L^p(E)$, $1 \le p \le \infty$ is a normed linear space. We prove that $L^p(E)$ is complete.

Consider a Cauchy sequence $\{f_n\}$ in $L^p(E)$. Then there exists a subsequence $\{f_{n_k}\}$ of $\{f_n\}$ which is rapidly Cauchy. By previous theorem every rapidly Cauchy sequence converges pointwise to a function in $L^p(E)$. Let $\{f_{n_k}\} \to f$ pointwise a.e on E where $f \in L^p(E)$. And by proposition, if a subsequence of a Cauchy sequence converges then the Cauchy sequence converges in the normed linear space.

Hence the given Cauchy sequence $\{f_n\}$ converges to the function *f* in $L^p(E)$.

10. Note

If a sequence $\{f_n\}$ in $L^p(E)$ converges pointwise a.e. on E to a function f in $L^p(E)$ then $\{f_n\}$ may not converge ingeneral in $L^p(E)$.

For example : E = [0, 1], $1 \le p < \infty$. For each natural number n define $f_n = n^{1/p} \cdot c_{\left(0, 1/n\right]}$. Then the sequence $\{f_n\}$ converges pointwise a.e on E to a function f = 0. But the sequence $\{f_n\}$ does not converge to f = 0 w.r.t. $L^p[0,1]$ norm.