SHIVAJI UNIVERSITY, KOLHAPUR

CENTRE FOR DISTANCE EDUCATION

Fuzzy Mathematics-I

(Mathematics)

For

M. Sc. Part-II : Semester-III

Paper (MT 305)

(Academic Year 2021-22 onwards)

UNIT - I

Fuzzy Sets : Basic Concepts

INTRODUCTION :

Fuzzy sets were introduced in 1965 by Lotfi Zadeh to model ambiguity and vagueness. Any realistic process is not perfect and ambiguity may arise from the interpretation of inputs or in the formulation of relationship between various attributes. Fuzzy sets is a tool which can be used to relate human reasoning capabilities to the knowledge based systems. Fuzzy logic provides mathematical base to transform certain perceptual and linguistic attributes for further computational and decision process.

1.1 Fuzzy Sets

There are three basic methods of describing a subset of X. First is listing the elements of the set. e.g. $A=\{x_1, x_2, x_3, ...\}$, $x_i \in X$ for all i = 1, 2, 3, ...

Secondly, a subset of X can be defined by the property of its members, e.g. $A=\{x \in X \mid p(x)\}$ where $p(x)$ is the property about x.

Third method is by characteristic function. If $A \subseteq X$ then it is represented by a function $c_A : X \to \{0,1\}$ such that $c_A(x)=0$ if $x \notin A$ and $c_A(x)=1$ if $x \in A$. This method of describing subsets of X can be generalied to define Fuzzy subsets of X. In characteristic function 1 represents full membership and 0 represents non-membership value of the element x in A. Characteristic function allows these two membership values to every element of X. Fuzzy sets allows the membership values between 0 and 1. We define Fuzzy subset of X formally as follows.

1. Fuzzy Set

Let, X be any set. A Fuzzy subset of X is defined by a membership function,

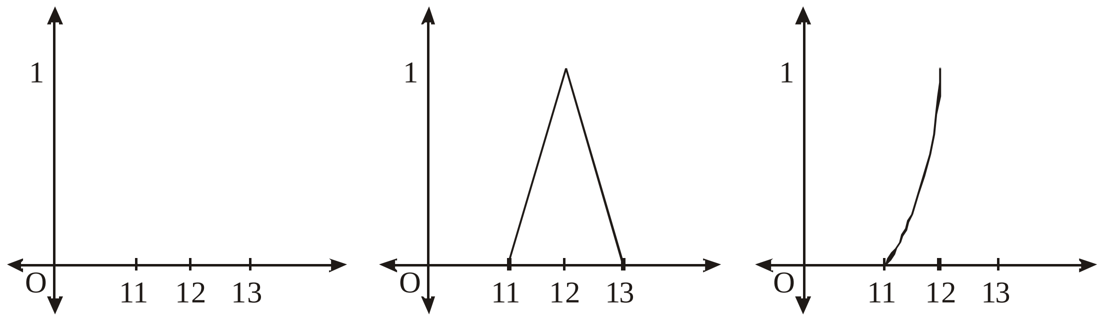
$$A: X \to [0,1]$$

The Fuzzy set is identified by its membership function.

Following are some of the examples of Fuzzy set.

2. Example

A class of real numbers close to 12 is a Fuzzy set.



$\widetilde{12} : \mathbb{R} \to I = [0,1]$ may be defined by various types of functions.

3. Example

Obtain a Fuzzy set which represents the property

(1) Highly educated, (2) Little educated

Ans. : We assume the following levels of education

0 - No education

1 - Elementary education

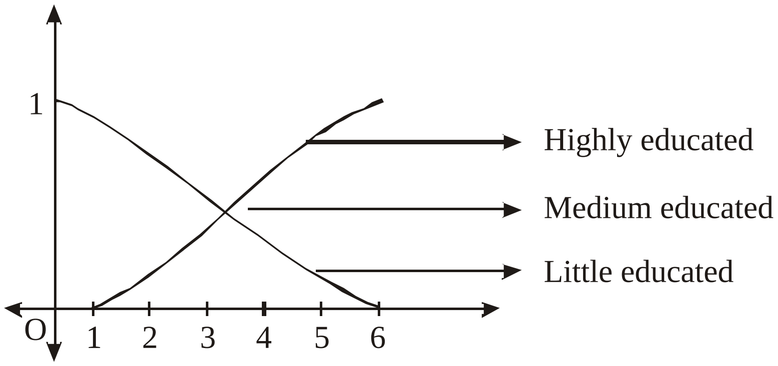
2 - Highschool education

3 - Junior college

4 - Degree college

5 - Masters degree

6 - M.Phil. / Ph.D.



4. **Example :** Represent the following linguistic concepts as fuzzy sets -

(1) Very low temperature

(2) Low temperature

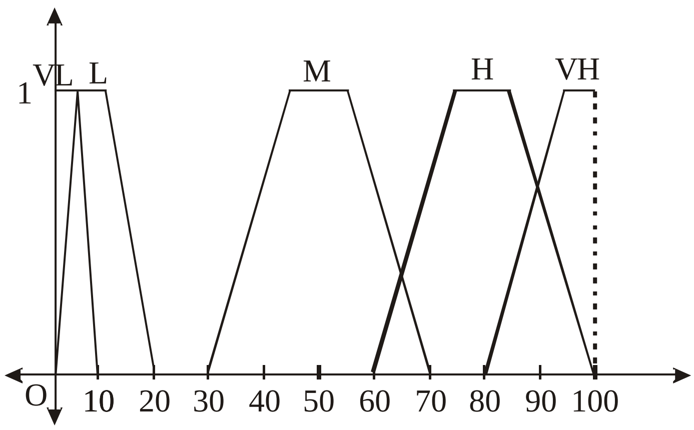
(3) Medium temperature

(4) High temperature

(5) Very high temperature

in a temperature range 0–100°C.

Ans. :



1.2 Basic Concepts

The concepts of a -cuts and strong a -cuts play an important role in the relationship between fuzzy sets and crisp sets (classical sets).

1. Definition

a -cut of a fuzzy set : Let, $A: X \to I$ be a fuzzy subset of X, then a -cut of **A** is defined as, ${}^{a}A = \{x \in X \mid A(x) \geq a\}$, $a \in [0,1]$

The strong a -cut of A is defined as, ${}^{a+}A = \{x \in X \mid A(x) > a\}$, $a \in [0,1]$

2. Note

1) ${}^{a+}A \subseteq {}^{a}A$

2) If $a = 0$, then

$${}^{0}A = \{x \in X \mid A(x) \geq 0\}$$

$$\Rightarrow {}^{0}A = X$$

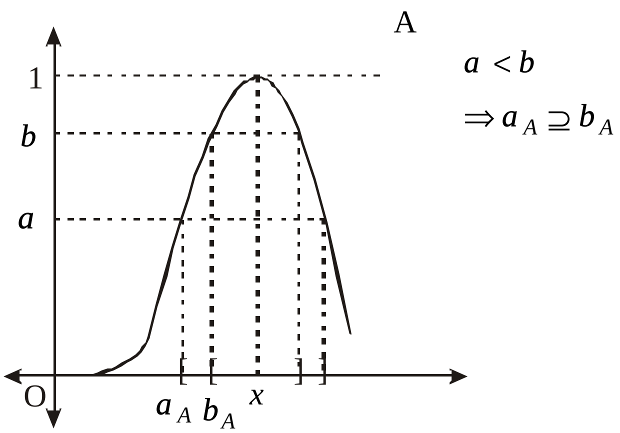
And if $a = 1$ then

$${}^{1}A = \{x \in X \mid A(x) \geq 1\}$$

$${}^{1}A = \{x \in X \mid A(x) = 1\} \qquad (\because a \in [0,1])$$

${}^{1}A$ is called the **core** of the fuzzy set A.

3. Example



4. Example

Let M be the fuzzy of middle aged persons defined by,

$$M : X \to [0,1]$$

where, X = {0, 1, 2, 3, ..., 100}

and M (x) = 0 if $x \le 20$ or $x \ge 60$

$= \frac{x-20}{15}$ if $20 \le x \le 35$

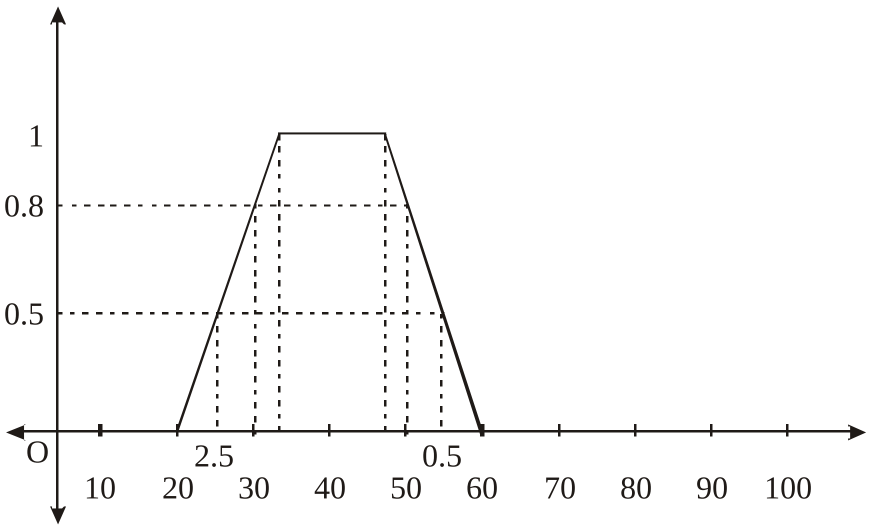
$= \frac{60-x}{15}$ if $40 \le x \le 60$

= 1 if $35 \le x \le 45$

Find, ${}^{1}M$, ${}^{0.5}M$, ${}^{0.8}M$ and ${}^{0+}M$.

Ans. : Here,

$M : X \to I$, X = {0, 1, 2, ..., 100}



For $x \in [20,35]$,

$$M(x) = \frac{x-20}{15},$$

For $x \in [45,60]$

$$M(x) = \frac{60-x}{15},$$

Now,

1) ${}^{1}M = \{x \in X \mid M(x) = 1\}$

$\Rightarrow {}^{1}M = \{35,36,37,........,45\}$

2) ${}^{0.5}M = \{x \in X \mid M(x) \geq 0.5\}$

But, $M(x) \geq 0.5$

$$\Rightarrow \frac{x-20}{15} \geq \frac{1}{2} \text{ or } \frac{60-x}{15} \geq \frac{1}{2}$$

$$\Rightarrow 2x - 40 \geq 15 \text{ or } 120 - 2x \geq 15$$

$$\Rightarrow x \geq \frac{55}{2} \text{ or } x \leq \frac{105}{2}$$

$$\Rightarrow \frac{55}{2} \leq x \leq \frac{105}{2}$$

Thus,

${}^{0.5}M = \{28,29,30,.....,51,52\}$

3) ${}^{0.8}M = \{x \in X \mid M(x) \geq 0.8\}$

But, $M(x) \geq 0.8$

$$\Rightarrow \frac{x-20}{15} \geq \frac{4}{5} \text{ or } \frac{60-x}{15} \geq \frac{4}{5}$$

$$\Rightarrow x \geq 32 \text{ or } x \leq 48$$

$$\Rightarrow 32 \leq x \leq 48$$

Thus,

${}^{0.8}M = \{32,33,34,.......,48\}$

4) $\quad {}^{0+}M = \{x \in X \mid M(x) > 0\}$

But, $M(x) > 0$

$$\Rightarrow \frac{x-20}{15} > 0 \text{ or } \frac{60-x}{15} < 0$$

$$\Rightarrow x > 20 \text{ or } x < 60$$

Thus,

$$^{0+}M = \{21, 22, 23, \ldots\ldots., 59\}$$

5. The set of all levels $x \in [0,1]$ that represents distinct a -cuts of a given fuzzy set A is called a level set of A denoted by $\wedge(A)$. i.e.

$$\wedge(A) = \{a \mid A(x) = a \text{ for some } x \in X\}$$

6. Note

For any fuzzy set A and for distinct values a_1, $a_2 \in [0,1]$, $a_1 < a_2 \Rightarrow {}^{a_1}A \supseteq {}^{a_2}A$ and ${}^{a_1+}A \supseteq {}^{a_2+}A$. Thus all a -cuts and strong a -cuts forms a families of nested crisp sets. i.e. if $a_1 < a_2 < a_3 < \ldots\ldots\ldots$ then

$$^{a_1}A \supseteq {}^{a_2}A \supseteq {}^{a_3}A \supseteq \ldots\ldots\ldots$$

7. Convex Sets

A set *A* in $\mathbb{R}^n$ is called a convex set if for any two elements $\bar{r}$, $\bar{s} \in A$, the element.

$$\bar{t} = l\,\bar{r} + (1-l)\,\bar{s} \in A \text{ for all } 0 \le l \le 1$$

i.e.for any two points $\bar{r}$ and $\bar{s}$ in *A*, the line segment joining $\bar{r}$ and $\bar{s}$ also lies in *A*.

8. Convex Fuzzy Set

Let $A: X \to I$ be a fuzzy set defined on X. The fuzzy set A is called a convex fuzzy set if every level cut of A is convex set.

9. Theorem

A fuzzy set A on $\mathbb{R}$ is convex iff

$$A\left(l\, x_1 + (1-l)\, x_2\right) \geq \min\left(A\left(x_1\right), A\left(x_2\right)\right),\ \forall x_1, x_2 \in \mathbb{R}$$

Proof :

Let $A: \mathbb{R} \to I$ be a convex fuzzy set, I = [0, 1]

Let $x_1, x_2 \in \mathbb{R}$ be arbitrary and let $A\left(x_1\right) \leq A\left(x_2\right)$

Let $A\left(x_1\right) = a$ where, $a \in [0,1]$

Then, $A\left(x_1\right) \geq a \Rightarrow x_1 \in {}^{a}A$

Also, $A\left(x_2\right) \geq A\left(x_1\right) = a$

$$\Rightarrow A\left(x_2\right) \geq a$$

$$\Rightarrow x_2 \in {}^{a}A$$

Thus, $x_1, x_2 \in {}^{a}A$ and ${}^{a}A$ is a convex set ($\because$ A is convex fuzzy set)

We get,

$$l\, x_1 + (1-l)\, x_2 \in {}^{a}A \qquad \forall\, l \in [0,1]$$

$$\Rightarrow A\left(l\, x_1 + (1-l)\, x_2\right) \geq a \qquad \forall\, l \in [0,1] \qquad \text{.... (1)}$$

Also,

$$A\left(x_2\right) \geq A\left(x_1\right) = a$$

$\Rightarrow \min\{A(x_1), A(x_2)\} = A(x_1) = a$ ..... (2)

Thus, from (1) and (2), we get,

$$A(l\,x_1 + (1-l)\,x_2) \geq \min\{A(x_1), A(x_2)\} \qquad \forall 0 \leq l \leq 1,\ \forall x_1, x_2 \in \mathbb{R}$$

Conversely,

Let, A be a fuzzy set on $\mathbb{R}$ s.t., $\forall x_1, x_2 \in \mathbb{R}$ and $\forall 0 \leq l \leq 1$

$A(l\,x_1 + (1-l)\,x_2) \geq \min\{A(x_1), A(x_2)\}$

We will prove that,

A is a convex fuzzt set.

i.e. we will show that, every a -cut of A is a convex set.

Let, a be arbitrary and $a > 0$. i.e. $0 < a \leq 1$

(For $a = 0$, ${}^0A = \mathbb{R}$ which is convex)

For, $0 < a \leq 1$ we have,

$${}^aA = \{x \in \mathbb{R} \mid A(x) \geq a\}$$

Let, $x_1, x_2 \in {}^aA$. Then,

$$A(x_1) \geq a \text{ and } A(x_2) \geq a$$

For any l, $0 \leq l \leq 1$, we have,

$A(l\,x_1 + (1-l)\,x_2) \geq \min\{A(x_1), A(x_2)\}$ (assumption)

$\geq a$

$(A(x_1) \geq a, A(x_2) \geq a \Rightarrow \min\{A(x_1), A(x_2)\} \geq a)$

Thus, $A(l\,x_1 + (1-l)\,x_2) \geq a$

$\Rightarrow l\,x_1 + (1-l)\,x_2 \in {}^aA \qquad \forall 0 \leq l \leq 1$

$\Rightarrow {}^{a}A$ is a convex set.

Thus,

$${}^{a}A \text{ is a convex set for all } a \in [0,1]$$

Hence, A is a convex fuzzy set.

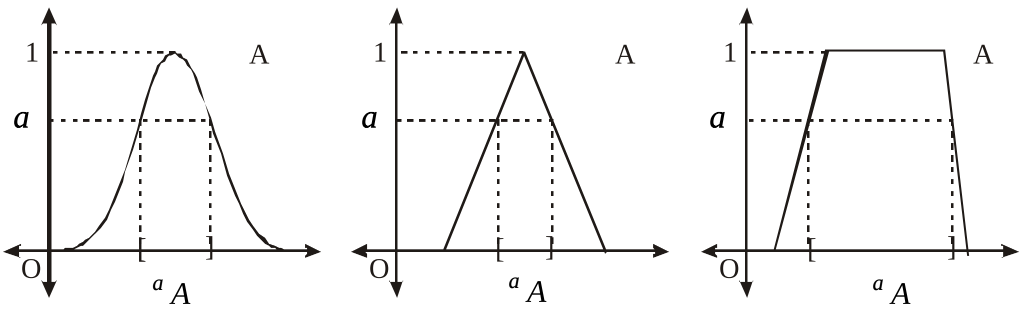
Thus,

A is a convex fuzzy set iff $\forall x_1, x_2 \in \mathbb{R}$ and $\forall l \in [0,1]$

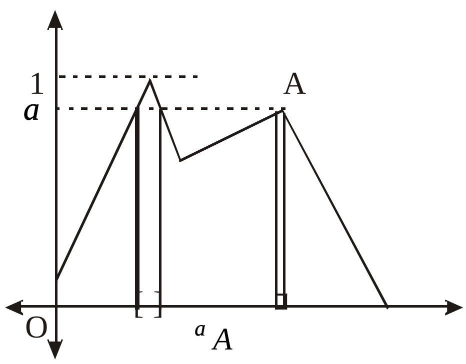
$$A(l\, x_1 + (1 - l)\, x_2) \geq \min\{A(x_1), A(x_2)\}$$

10. Examples

If A is a convex set, then we may have the following representation.



11. Example



${}^{a}A$ is not a convex set. Hence, A is not a convex fuzzy set.

12. Support of a Fuzzy Set

Let, A be a fuzzy set defined on X. The set of all elements whose "membership values are non-zero" is called the support of A.

Thus,

Support of A = Supp A

$$= \{x \in X \mid A(x) > 0\}$$

i.e. Supp A = ${}^{0+}A$

13. Core of A

The set of all elements whose "membership value is 1" is called the core of A.

Thus, core of A $= \{x \in X \mid A(x) = 1\}$

i.e. Core of A = ${}^{1}A$.

14. Height of Fuzzy Set

Let, A be a fuzzy set defined on X. The "maximal membership value" of the elements of X is called the height of A.

i.e. Height of A = h (A) $= \sup_{x \in X} A(x)$

If h(A) = 1, then A is called normal fuzzy set. Otherwise it is called subnormal fuzzy set.

15. Notation For Representation of Fuzzy Sets

Let A be a fuzzy set on X, $A : X \to I$

If $X = \{x_1, x_2, ..., x_n\}$. Then,

$$A = \frac{A(x_1)}{x_1} + \frac{A(x_2)}{x_2} ..... + \frac{A(x_n)}{x_n}$$

i.e. $A = \sum_{n} \frac{A(x_n)}{x_n}$

If $X = \{x_1, x_2, x_3, ...\}$

Then, we write

$$A = \frac{A(x_1)}{x_1} + \frac{A(x_2)}{x_2} + ...$$

$$= \sum_{n=1}^{\infty} \frac{A(x_n)}{x_n}$$

For an interval X, we denote the fuzzy set A by,

$$A = \int_{X} \frac{A(x)}{x}$$

16. **Example :**

Let, X = {0, 1, 2, ..., 10} and $A(x) = \frac{x}{x+4}$. Then,

$$A = \frac{0}{0} + \frac{1/5}{1} + \frac{2/6}{2} + \frac{3/7}{3} + ... + \frac{5/7}{10}$$

Thus,

(1) $h(A) = \frac{5}{7} = 0.71$

$\Rightarrow h(A) \neq 1$

Therefore, A is not normal fuzzy set. (subnormal)

(2) Core of $A = f$

(3) Supp (A) = {1, 2, ..., 10}

17. Scalar Cardinality of Fuzzy Set

If A is a fuzzy set defined on X, then, scalar cardinality of A is defined by

$$|A| = \sum_{x \in X} A(x)$$

18. Example

Let X = {0, 1, 2, ..., 10}.

Define $A: X \to I$ by $A(x) = \frac{x}{x+4}$. Then

$$|A| = \sum_{x \in X} A(x)$$

= A(0) + A(1) + A(2) + ... + A(10)

$$= 0 + \frac{1}{5} + \frac{2}{6} + \frac{3}{7} + ... + + \frac{10}{14}$$

= 0.2 + 0.33 + ... + 0.71 = 5.31

1.3 Operations on Fuzzy Set

1. Fuzzy Intersection

Let A and B be the two fuzzy sets defined on a set X ($A: X \to I$, $B: X \to I$).

$(A \cap B) X \to I$ defined by

$$(A \cap B)(x) = \min\{A(x), B(x)\} = A(x) \wedge B(x)$$

This is called standard Fuzzy intersection.

2. Fuzzy Union

Let A and B be the two fuzzy sets defined on a set X, then,

$(A \cup B): X \to I$ defined by

$(A \cup B)(x) = \max\{A(x), B(x)\} = A(x) \vee B(x)$

This is called standard fuzzy union.

3. Standard Fuzzy Complement

Let A be a fuzzy set on X. The standard fuzzy complement of A is defined by,

$\bar{A} : X \to I$

$\bar{A}(x) = 1 - A(x)$

4. Definition : Fuzzy Power Set

The family of all fuzzy sets defined on X is called a fuzzy power set of X and it is denoted by $\mathcal{F}$ (X).

$\mathcal{F}$ (X) is a complemented, distributive complete lattice.

5. Note

Law of contradiction and law of excluded middle are not true for the fuzzy sets.

i.e. $A \cap \bar{A} \neq f$ ..... (Law of contradiction)

and $A \cup \bar{A} \neq X$ .... (Law of excluded middle)

where,

$f : X \to I$ s.t. $f(x) = 0$ $\forall x$

and $X : X \to I$ s.t $X(x) = 1$ $\forall x$

6. Example

Prove that, for any Fuzzy sets A and B defined on X, the following properties holds,

(1) $A \cup (A \cap B) = A$

(2) $A \cap (A \cup B) = A$

} (Law of Absorption)

Proof :

(1) For any $x \in X$,

$[A \bigcup (A \bigcap B)](x) = A(x) \vee (A \bigcap B)(x)$ S.F.U. definition

$= A(x) \vee [A(x) \wedge B(x)]$ S.F.I. deinition

Now, $A(x), B(x) \in [0,1]$

Then,

either $A(x) \leq B(x)$ or $B(x) \leq A(x)$

Case (i) : If $A(x) \leq B(x)$. Then,

$A(x) \wedge B(x) = A(x)$

And $A(x) \vee [A(x) \wedge B(x)] = A(x) \vee A(x) = A(x)$

i.e. $[A \bigcup (A \bigcap B)](x) = A(x)$, $\forall x \in X$

$\Rightarrow A \bigcup (A \bigcap B) = A$

Case(ii) : Similarly,

if $B(x) \leq A(x)$. Then,

$A(x) \wedge B(x) = B(x)$

and $A(x) \vee [A(x) \wedge B(x)] = A(x) \vee B(x)$

$= A(x)$

$\Rightarrow [A \bigcup (A \bigcap B)](x) = A(x) \quad \forall x \in X$

Thus,

$A \bigcup (A \bigcap B) = A$

(2) For any $x \in X$,

$$[A \cap (A \cup B)](x) = A(x) \wedge [(A \cup B)(x)]$$

$$= A(x) \wedge [A(x) \vee B(x)]$$

Now, $A(x), B(x) \in [0,1]$. Then

either, $A(x) \leq B(x)$ or $B(x) \leq A(x)$.

Case (i) : If $A(x) \leq B(x)$. Then,

$$A(x) \vee B(x) = B(x)$$

$$\Rightarrow A(x) \wedge [A(x) \vee B(x)] = A(x) \wedge B(x)$$

$$= A(x)$$

$$\Rightarrow [A \cap (A \cup B)](x) = A(x), \quad \forall x \in X$$

Therefore, $A \cap (A \cup B) = A$

Case (ii) : If $B(x) \leq A(x)$. Then,

$$A(x) \vee B(x) = A(x)$$

$$\Rightarrow A(x) \wedge [A(x) \vee B(x)] = A(x) \wedge A(x)$$

$$= A(x)$$

$$\Rightarrow [A \cap (A \cup B)](x) = A(x) \qquad \forall x \in X$$

Therefore, $A \cap (A \cup B) = A$

7. Definition : Subset

Let, $A, B \in \mathcal{F}(X)$, we say that, $A \subseteq B$ if $A(x) \leq B(x)$, $\forall x \in X$.

8. Example

Prove that, $A \subseteq B$ iff $A \cap B = A$ and $A \cup B = B$,

where, $A, B \in \mathcal{F}(X)$.

Ans. : Let, $A \subseteq B$

i.e. $A(x) \leq B(x) \qquad \forall x \in X$

Then,

$(A \cap B)(x) = A(x) \wedge B(x)$ By definition of fuzzy intersection

i.e. $(A \cap B)(x) = A(x) \qquad \forall x \in X \ (\because A(x) \leq B(x))$

$\Rightarrow A \cap B = A$

Similarly

$(A \cup B)(x) = A(x) \vee B(x)$ By definition of fuzzy union

$\Rightarrow (A \cup B)(x) = B(x) \qquad \forall x \in X \ (\because A(x) \leq B(x))$

$\Rightarrow A \cup B = B$

Conversely,

Let, $A \cup B = B$ and $A \cap B = A$

Then, for any $x \in X$,

$(A \cup B)(x) = B(x)$ and $(A \cap B)(x) = A(x)$

$\Rightarrow A(x) \vee B(x) = B(x)$ and $A(x) \wedge B(x) = A(x)$

$\Rightarrow A(x) \leq B(x) \qquad \forall x \in X$

$\Rightarrow A \subseteq B$

9. Example

If A and B are Fuzzy sets defined on X. Then, Show that

$|A| + |B| = |A \cup B| + |A \cap B|$

Ans. : For any $x \in X$,

$$A(x), B(x) \in I$$

Since, 'I is totally ordered'

either $A(x) \leq B(x)$ or $B(x) \leq A(x)$ holds $\forall x \in X$

Thus,

Case (1) : If, $A(x) \leq B(x)$ holds, then,

$$(A \bigcup B)(x) = A(x) \vee B(x) = B(x)$$

and $$(A \bigcap B)(x) = A(x) \wedge B(x) = A(x)$$

Therefore

$$(A \bigcup B)(x) + (A \bigcap B)(x) = A(x) + B(x)$$

Case (2) : If $B(x) \leq A(x)$ holds. Then,

$$(A \bigcup B)(x) = A(x) \vee B(x) = A(x) \text{ and}$$

$$(A \bigcap B)(x) = A(x) \wedge B(x) = B(x)$$

Thus

$$(A \bigcup B)(x) + (A \bigcap B)(x) = A(x) + B(x)$$

Thus, $\forall x \in X$,

$$(A \bigcup B)(x) + (A \bigcap B)(x) = A(x) + B(x)$$

Taking summation over $x \in X$,

$$\sum_{x \in X} (A \bigcup B)(x) + \sum_{x \in X} (A \bigcap B)(x) = \sum_{x \in X} A(x) + \sum_{x \in X} B(x)$$

$$\Rightarrow |A \bigcup B| + |A \bigcap B| = |A| + |B|$$

10. Example

If A, B, C are Fuzzy sets on $\mathbb{R}^+$ defined by,

$$A(x)=\frac{1}{1+10x},\ B(x)=\left(\frac{1}{1+10x}\right)^{1/2} \text{ and } C(x)=\left(\frac{1}{1+10x}\right)^{2}$$

Order the Fuzzy sets A, B and C by inclusion.

Ans. : Since, $0\le \frac{1}{1+10x}\le 1$

$$\Rightarrow \left(\frac{1}{1+10x}\right)^{2}\le \frac{1}{1+10x}\le \left(\frac{1}{1+10x}\right)^{1/2}$$

$\Rightarrow C(x)\le A(x)\le B(x)$ for all $x\in \mathbb{R}^+$

$\Rightarrow C\subseteq A\subseteq B$

11. Definition : Degree of Subset-Hood

Let *A* and *B* be the two Fuzzy sets, the degree of subset-hood S (*A*, *B*) of *A* in *B* is defined by,

$$S(A,B)=\frac{|A\cap B|}{|A|}$$

12. Example

For a Fuzzy sets A and B defined on $X=\{x_1, x_2, x_3, x_4, x_5\}$ by,

$$A=\frac{0.1}{x_1}+\frac{0.7}{x_3}+\frac{0.9}{x_4}+\frac{1}{x_5}$$

$$B=\frac{0.3}{x_1}+\frac{0.1}{x_2}+\frac{0.6}{x_3}+\frac{1}{x_4}+\frac{0.5}{x_5}$$

Find

1) $\overline{A}$

2) $\overline{B}$

3) $A \cup B$

4) $A \cap B$

5) $\overline{A \cup B}$

6) $\overline{A \cap B}$

7) $\overline{A} \cup \overline{B}$

8) $\overline{A} \cap \overline{B}$

9) $A \cup \overline{A}$

10) $A \cap \overline{A}$

11) S (A, B)

12) S (B, A)

13) $A \Delta B \ (= (A - B) \cup (B - A) = \left(A \cap \overline{B}\right) \cup \left(B \cap \overline{A}\right))$

Ans. :

1) $$\overline{A} = \frac{0.9}{x_1} + \frac{1}{x_2} + \frac{0.3}{x_3} + \frac{0.1}{x_4} + \frac{0}{x_5}$$

2) $$\overline{B} = \frac{0.7}{x_1} + \frac{0.9}{x_2} + \frac{0.4}{x_3} + \frac{0}{x_4} + \frac{0.5}{x_5}$$

3) $$A \cup B = \frac{0.3}{x_1} + \frac{0.1}{x_2} + \frac{0.7}{x_3} + \frac{1}{x_4} + \frac{1}{x_5}$$

4) $$A \cap B = \frac{0.1}{x_1} + \frac{0}{x_2} + \frac{0.6}{x_3} + \frac{0.9}{x_4} + \frac{0.5}{x_5}$$

5) $$\overline{A \cup B} = \frac{0.7}{x_1} + \frac{0.9}{x_2} + \frac{0.3}{x_3} + \frac{0}{x_4} + \frac{0}{x_5}$$ (By 3)

6) $$\overline{A \cap B} = \frac{0.9}{x_1} + \frac{1}{x_2} + \frac{0.4}{x_3} + \frac{0.1}{x_4} + \frac{0.5}{x_5}$$ (By 4)

7) $\overline{A} \cup \overline{B} = \frac{0.9}{x_1} + \frac{1}{x_2} + \frac{0.4}{x_3} + \frac{0.1}{x_4} + \frac{0.5}{x_5}$ (By 1, 2)

8) $\overline{A} \cap \overline{B} = \frac{0.7}{x_1} + \frac{0.9}{x_2} + \frac{0.3}{x_3} + \frac{0}{x_4} + \frac{0}{x_5}$ (By 1, 2)

9) $A \cup \overline{A} = \frac{0.9}{x_1} + \frac{1}{x_2} + \frac{0.7}{x_3} + \frac{0.9}{x_4} + \frac{1}{x_5}$

10) $A \cap \overline{A} = \frac{0.1}{x_1} + \frac{0}{x_2} + \frac{0.3}{x_3} + \frac{0.1}{x_4} + \frac{0}{x_5}$

11) $S(A,B) = \frac{|A \cap B|}{|A|}$

But, |A| = 0.1 + 0.7 + 0.9 + 1 = 2.7

and, $|A \cap B|$ = 0.1 + 0 + 0.6 + 0.9 + 0.5 = 2.1

Hence, $S(A,B) = \frac{|A \cap B|}{|A|} = \frac{2.1}{2.7} = 0.77$

12) $S(B,A) = \frac{|B \cap A|}{|B|} = \frac{|A \cap B|}{|B|}$

Now, |B| = 0.3 + 0.1 + 0.6 + 1 + 0.5 = 2.5

and $|A \cap B|$ = 2.1

Thus, $S(B,A) = \frac{2.1}{2.5} = 0.84$

13) For $A \Delta B$

We know that,

$$A \Delta B = (A - B) \cup (B - A)$$

$$= (A \cap \overline{B}) \cup (B \cap \overline{A})$$

Now,

$$A \cap \overline{B} = \frac{0.1}{x_1} + \frac{0}{x_2} + \frac{0.4}{x_3} + \frac{0}{x_4} + \frac{0.5}{x_5} \qquad (\min(A, \overline{B}))$$

and

$$B \cap \overline{A} = \frac{0.3}{x_1} + \frac{0.1}{x_2} + \frac{0.3}{x_3} + \frac{0.1}{x_4} + \frac{0}{x_5} \qquad (\min(B, \overline{A}))$$

Then,

$$A \Delta B = (A \cap \overline{B}) \cup (B \cap \overline{A}) = \frac{0.3}{x_1} + \frac{0.1}{x_2} + \frac{0.4}{x_3} + \frac{0.1}{x_4} + \frac{0.5}{x_5} \qquad (\max\{(A \cap \overline{B}), (B \cap \overline{A})\})$$

13. Example

For Fuzzy sets A and B defined on X, prove that,

1) $\overline{A \cap B} = \overline{A} \cup \overline{B}$

2) $\overline{A \cup B} = \overline{A} \cap \overline{B}$

Ans. :

1) For any $x \in X$,

$$(\overline{A \cap B})(x) = 1 - (A \cap B)(x)$$

$$\Rightarrow (\overline{A \cap B})(x) = 1 - (A(x) \wedge B(x))$$

$$\Rightarrow \qquad = (1 - A(x)) \vee (1 - B(x))$$

$$\Rightarrow \qquad = \overline{A}(x) \vee \overline{B}(x)$$

$$\Rightarrow \qquad = (\overline{A} \cup \overline{B})(x)$$

Thus,

$$(\overline{A \cap B})(x) = (\overline{A} \cup \overline{B})(x) \qquad \forall x \in X$$

$$\Rightarrow \overline{A \cap B} = \overline{A} \cup \overline{B}$$

2) Now, for any $x \in X$,

$$\Rightarrow \left(\overline{A \cup B}\right)(x) = 1 - (A \cup B)(x)$$

$$\Rightarrow \qquad = 1 - \left(A(x) \vee B(x)\right)$$

$$\Rightarrow \qquad = \left(1 - A(x)\right) \wedge \left(1 - B(x)\right)$$

$$\Rightarrow \qquad = \overline{A}(x) \wedge \overline{B}(x)$$

$$\Rightarrow \qquad = \left(\overline{A} \cap \overline{B}\right)(x)$$

Thus,

$$\left(\overline{A \cup B}\right)(x) = \left(\overline{A} \cap \overline{B}\right)(x) \qquad \forall x \in X$$

$$\Rightarrow \overline{A \cup B} = \overline{A} \cap \overline{B}$$

14. Definition

If A and B are Fuzzy sets defined on X the difference A – B is defined by

$$A - B = A \cap \overline{B}$$

And the symmetric difference of A and B is define by,

$$A \Delta B = (A - B) \cup (B - A).$$

15. Example : Prove that

$$A \Delta B \Delta C = \left(\overline{A} \cap \overline{B} \cap C\right) \cup \left(\overline{A} \cap B \cap \overline{C}\right) \cup \left(A \cap \overline{B} \cap \overline{C}\right) \cup (A \cap B \cap C)$$

Proof : Consider

$$A \Delta (B \Delta C) = (B \Delta C) \Delta A$$

$$= [(B - C) \cup (C - B)] \Delta A \qquad (\because \text{ By definition})$$

$$= \left[\left(B \cap \overline{C}\right) \cup \left(C \cap \overline{B}\right)\right] \Delta A$$

$$= \left\{\left[\left(B \cap \overline{C}\right) \cup \left(C \cap \overline{B}\right)\right] - A\right\} \cup \left\{A - \left[\left(B \cap \overline{C}\right) \cup \left(C \cap \overline{B}\right)\right]\right\}$$

$$=\{[(B\cap\overline{C})\cup(C\cap\overline{B})]\cap\overline{A}\}\cup\{A\cap\overline{[(B\cap\overline{C})\cup(C\cap\overline{B})]}\}$$

$$=\{\overline{A}\cap[(B\cap\overline{C})\cup(C\cap\overline{B})]\}\cup\{A\cap[(\overline{B\cap\overline{C}})\cap(\overline{C\cap\overline{B}})]\}$$

$$=\{\overline{A}\cap[(\overline{B}\cap C)\cup(B\cap\overline{C})]\}\cup\{A\cap[(\overline{B}\cup C)\cap(\overline{C}\cup B)]\}$$

$$=\{(\overline{A}\cap(\overline{B}\cap C))(\overline{A}\cap(B\cap\overline{C}))\}\cup\{A\cap[((\overline{B}\cup C)\cap\overline{C})\cup((\overline{B}\cup C)\cap B)]\}$$

$$=\{(\overline{A}\cap\overline{B}\cap C)\cup(\overline{A}\cap B\cap\overline{C})\}\cup\{A\cap[(\overline{B}\cap\overline{C})\cup(C\cap\overline{C})\cup(\overline{B}\cap B)\cup(C\cup B)]\} \quad ..(1)$$

Case (i) $B(x)\le C(x)$, $\quad \overline{B}(x)\ge\overline{C}(x)$

$$\Rightarrow (B\cap C)(x)=B(x) \qquad (\overline{B}\cap\overline{C})(x)=\overline{C}(x)$$

$$\Rightarrow B\cap C=B, \qquad \overline{B}\cap\overline{C}=\overline{C}$$

Consider,

$$(\overline{B}\cap\overline{C})\cup(C\cap\overline{C})\cup(\overline{B}\cap B)\cup(C\cap B)$$

$$=\overline{C}\cup(C\cap\overline{C})\cup(\overline{B}\cap B)\cup B$$

$$=\overline{C}\cup B \qquad (\because A\cup(A\cap B)=A \text{ and } (C\cap\overline{C})\neq f\ .)$$

$$=(\overline{B}\cap\overline{C})\cup(B\cap C)$$

Case (ii) $B(x)\ge C(x)$, $\overline{B}(x)\le\overline{C}(x)$

$$\Rightarrow B\cap C=C,\ \overline{B}\cap\overline{C}=\overline{B}$$

$$\Rightarrow (B\cap\overline{C})\cup(C\cap\overline{C})\cup(\overline{B}\cap B)\cup(C\cap B)$$

$$=\overline{B}\cup(C\cap\overline{C})\cup(\overline{B}\cap B)\cup C$$

$$=C\cup\overline{B}$$

$$=(B\cap C)\cup(\overline{B}\cap\overline{C})$$

With this value, equation (1) becomes

$$A\Delta(B\Delta C)=\{(\overline{A}\cap\overline{B}\cap C)\cup(\overline{A}\cap B\cap\overline{C})\}\cup\{A\cap[(\overline{B}\cap\overline{C})\cup(B\cap C)]\}$$

$$=(\overline{A}\cap\overline{B}\cap C)\cup(\overline{A}\cap B\cap\overline{C})\cup(A\cap\overline{B}\cap\overline{C})\cup(A\cap B\cap C)$$

$$A\Delta(B\Delta C)=(\overline{A}\cap\overline{B}\cap C)\cup(\overline{A}\cap B\cap\overline{C})\cup(A\cap\overline{B}\cap\overline{C})\cup(A\cap B\cap C)$$

❋ ❋ ❋

Additional Properties of a -Cuts

1. Theorem

Let $A, B \in \mathcal{F}(X)$, then for any $a, b \in [0,1]$.

(1) ${}^{a+}A \subseteq {}^{a}A$

(2) $a \le b \Rightarrow {}^{a}A \supseteq {}^{b}A$ or ${}^{b}A \subseteq {}^{a}A$

${}^{a+}A \supseteq {}^{b+}A$ or ${}^{b+}A \subseteq {}^{a+}A$

(3) ${}^{a}(A \cap B) = ({}^{a}A) \cap ({}^{a}B)$

(4) ${}^{a}(A \cup B) = ({}^{a}A) \cup ({}^{a}B)$

(5) ${}^{a+}(A \cap B) = ({}^{a+}A) \cap ({}^{a+}B)$

(6) ${}^{a+}(A \cup B) = ({}^{a+}A) \cup ({}^{a+}B)$

(7) ${}^{a}(\overline{A}) = \overline{{}^{(1-a)+}A}$

Proof :

1) Let $a \in [0,1]$, and let $x \in X$,

Then,

$x \in {}^{a+}A$

$\Rightarrow A(x) > a$

$\Rightarrow A(x) \ge a$

$\Rightarrow x \in {}^{a+}A$

$\Rightarrow {}^{a+}A \subseteq {}^{a}A$

2) Let $a \leq b$,

Then,

$x \in {}^{b}A \Rightarrow A(x) \geq b$ (definition of b cut)

$\Rightarrow A(x) \geq b \geq a$ $(a \leq b)$

$\Rightarrow A(x) \geq a$

$\Rightarrow x \in {}^{a}A$

$\Rightarrow {}^{b}A \subseteq {}^{a}A$

Now,

$x \in {}^{b+}A \Rightarrow A(x) > b$ (definition of b + cut)

$\Rightarrow A(x) > b \geq a$ $(a \leq b)$

$\Rightarrow A(x) > a$

$\Rightarrow x \in {}^{a+}A$

$\Rightarrow {}^{b+}A \subseteq {}^{a+}A$

3) Let $x \in X$ be arbitary. Then

$x \in {}^{a}(A \cap B)$

$\Leftrightarrow (A \cap B)(x) \geq a$ $(a$ -cut)

$\Leftrightarrow A(x) \wedge B(x) \geq a$

$\Leftrightarrow A(x) \geq a$ and $B(x) \geq a$

$\Leftrightarrow x \in {}^{a}A$ and $x \in {}^{a}B$

$\Leftrightarrow x \in {}^{a}A \cap {}^{a}B$

Thus,

$x \in {}^{a}(A \cap B) \Leftrightarrow x \in {}^{a}A \cap {}^{a}B$

Hence,

${}^{a}(A \cap B) = {}^{a}A \cap {}^{a}B$

4) Let $x \in X$ be arbitrary. Then

$x \in^{a} (A \cup B)$

$\Leftrightarrow (A \cup B)(x) \geq a$ $\qquad (a-\text{cut})$

$\Leftrightarrow A(x) \vee B(x) \geq a$

$\Leftrightarrow A(x) \geq a$ or $B(x) \geq a$

$\Leftrightarrow x \in^{a} A$ or $x \in^{a} B$

$\Leftrightarrow x \in {}^{a}A \cup^{a} B$

Thus,

$x \in {}^{a}(A \cup B) \Leftrightarrow x \in {}^{a}A \cup {}^{a}B$

Hence,

${}^{a}(A \cup B) = {}^{a}A \cup {}^{a}B$

5) Let $x \in X$ be arbitrary. Then

$x \in^{a+} (A \cap B)$

$\Leftrightarrow (A \cap B)(x) > a$ $\qquad (a+\text{Cut})$

$\Leftrightarrow A(x) \wedge B(x) > a$

$\Leftrightarrow A(x) > a$ and $B(x) > a$

$\Leftrightarrow x \in^{a+} A$ and $x \in^{a+} B$

$\Leftrightarrow x \in^{a+} A \cap \in^{a+} B$

Thus

$x \in^{a+} (A \cap B) \Leftrightarrow {}^{a+}A \cap {}^{a+}B$

Hence,

${}^{a+}(A \cap B) = {}^{a+}A \cap {}^{a+}B$

6) Let $x \in X$ be arbitrary

If $x \in {}^{a+}(A \cup B)$

$\Leftrightarrow (A \cup B)(x) > a$ $\qquad\qquad$ $(a + \text{Cut})$

$\Leftrightarrow A(x) \vee B(x) > a$

$\Leftrightarrow A(x) > a$ or $B(x) > a$

$\Leftrightarrow x \in {}^{a+}A$ or $x \in {}^{a+}B$

$\Leftrightarrow x \in {}^{a+}A \cup {}^{a+}B$

Thus,

$x \in {}^{a+}(A \cup B) \Leftrightarrow x \in {}^{a+}A \cup {}^{a+}B$

Hence,

${}^{a+}(A \cup B) = {}^{a+}A \cup {}^{a+}B$

7) If $x \in {}^{a}(\overline{A})$

$\Leftrightarrow \overline{A}(x) \geq a$

$\Leftrightarrow 1 - A(x) \geq a$

$\Leftrightarrow 1 - a \geq A(x)$

$\Leftrightarrow A(x) \leq 1 - a$

$\Leftrightarrow A(x) > 1$

$\Leftrightarrow x \notin {}^{(1-a)+}A$ $\qquad\qquad$ $(x \notin A \Rightarrow x \in \overline{A}$, crisp complement)

$\Leftrightarrow x \in \overline{{}^{(1-a)+}A}$

Thus

$x \in {}^{a}(\overline{A}) \Leftrightarrow x \in \overline{{}^{(1-a)+}A}$

Hence

${}^{a}(\overline{A}) = \overline{{}^{(1-a)+}A}$

2. Theorem

Let , Then for any

(1) $A \subseteq B$ iff ${}^{a}A \subseteq {}^{a}B$

(2) $A \subseteq B$ iff ${}^{a+}A \subseteq {}^{a+}B$

(3) $A = B$ iff ${}^{a}A = {}^{a}B$

(4) $A = B$ iff ${}^{a+}A = {}^{a+}B$

$\Bigg\}\quad \forall a \in [0,1]$

Proof :

1) Let $A \subseteq B$.

Then, for any x, $A(x) \leq B(x)$

And for any $a \in [0,1]$,

$$x \in {}^{a}A \Rightarrow A(x) \geq a$$

$$\Rightarrow B(x) \geq A(x) \geq a$$

$$\Rightarrow B(x) \geq a$$

$$\Rightarrow x \in {}^{a}B$$

Hence,

$${}^{a}A \subseteq {}^{a}B \qquad \forall a \in [0,1]$$

Conversely,

Let ${}^{a}A \subseteq {}^{a}B \quad \forall a \in [0,1]$

Then, for any $x \in X$,

Let $A(x) = a_0$,

Then $A(x) = a_0$

$\Rightarrow A(x) \geq a_0$

$\Rightarrow x \in {}^{a_0}A \subseteq {}^{a_0}B$

$\Rightarrow x \in {}^{a_0}B$

$\Rightarrow B(x) \geq a_0$

$\Rightarrow B(x) \geq A(x)$

$\Rightarrow A(x) \leq B(x)$

Thus $\forall x \in X$, $A(x) \leq B(x)$

Hence $A \subseteq B$

Thus, $A \subseteq B$ iff ${}^{a}A \subseteq {}^{a}B \quad \forall a \in [0,1]$

2) Let $A \subseteq B$ holds

i.e. $A(x) \leq B(x) \quad \forall x \in X$

For any $a \in [0,1]$,

$$x \in {}^{a+}A \Rightarrow A(x) > a$$

$$\Rightarrow B(x) \geq A(x) > a$$

$$\Rightarrow B(x) > a$$

$$\Rightarrow x \in {}^{a+}B$$

Hence,

$${}^{a+}A \subseteq {}^{a+}B \qquad \forall a \in [0,1]$$

Conversely,

Let ${}^{a+}A \subseteq {}^{a+}B \qquad \forall a \in [0,1]$

Then, we show that, $A \subseteq B$ i.e. we prove that

$$A(x) \leq B(x) \qquad \forall x \in X$$

On the contrary suppose that,

$$A(x) \not\leq B(x) \text{ for some } x \in X$$

i.e. there exists $x \in X$ s.t. $A(x) > B(x)$

Thus, there exists a real number α in [0, 1] s.t.

$$A(x) > \alpha > B(x)$$

i.e. $A(x) > \alpha$ and $\alpha > B(x)$

$\Rightarrow x \in {}^{\alpha+}A$ and $B(x) \not> \alpha$

$\Rightarrow x \in {}^{\alpha+}A$ and $x \notin {}^{\alpha+}B$

$\Rightarrow {}^{\alpha+}A \not\subseteq {}^{\alpha+}B$

This is a contradiction. Since,

$${}^{\alpha+}A \subseteq {}^{\alpha+}B$$

Hence,

$$A(x) \leq B(x) \qquad \forall x \in X$$

i.e. $A \subseteq B$.

3) $A = B$ iff ${}^{\alpha}A = {}^{\alpha}B$

Here,

$A = B \Leftrightarrow A(x) = B(x) \qquad \forall x \in X$

$\Leftrightarrow A(x) \leq B(x)$ and $B(x) \leq A(x)$

$\Leftrightarrow A \subseteq B$ and $B \subseteq A \qquad \forall x \in X$

$\Leftrightarrow {}^{\alpha}A \subseteq {}^{\alpha}B$ and ${}^{\alpha}B \subseteq {}^{\alpha}A \qquad \forall \alpha \in [0,1]$

$\Leftrightarrow {}^{\alpha}A = {}^{\alpha}B \qquad \forall \alpha \in [0,1]$

i.e.

$A = B$ iff ${}^{\alpha}A = {}^{\alpha}B \qquad \forall \alpha \in [0,1]$

4) Here,

$$A = B \Leftrightarrow A(x) = B(x) \qquad \forall x \in X$$

$$\Leftrightarrow A(x) \le B(x) \text{ and } B(x) \le A(x)$$

$$\Leftrightarrow A \subseteq B \text{ and } B \subseteq A \qquad \forall x \in X$$

$$\Leftrightarrow {}^{a+}A \subseteq {}^{a+}B \text{ and } {}^{a+}B \subseteq {}^{a+}A \qquad \forall a \in [0,1]$$

$$\Leftrightarrow {}^{a+}A = {}^{a+}B$$

i.e.

$$A = B \text{ iff } {}^{a+}A = {}^{a+}B$$

3. Theorem

For any $A \in \mathcal{F}(X)$ the following holds -

1) $${}^{a}A = \bigcap_{b<a} {}^{b}A = \bigcap_{b<a} {}^{b+}A \qquad a \in [0,1]$$

2) $${}^{a+}A = \bigcup_{a<b} {}^{b}A = \bigcup_{a<b} {}^{b+}A \qquad a \in [0,1]$$

Proof :

1) For $b < a$ we have,

$${}^{b}A \supseteq {}^{a}A \text{ or } {}^{a}A \subseteq {}^{b}A \qquad \forall b < a$$

Hence,

$${}^{a}A \subseteq \bigcap_{b<a} {}^{b}A \qquad \text{... (1)}$$

On the other hand, if

$$x \in \bigcap_{b<a} {}^{b}A$$

$$\Rightarrow x \in {}^{b}A \qquad \forall b < a$$

$$\Rightarrow A(x) \ge b \qquad \forall b < a$$

Let $\in > 0$ be a small positive real number,

Then $a - \in < a$

$$\Rightarrow A(x) \geq a - \in \qquad \forall b < a$$

Since, $\in > 0$ is arbitrary (Taking $\in \to 0$)

$$\Rightarrow A(x) \geq a$$

$$\Rightarrow x \in {}^{a}A$$

Thus,

$$x \in \bigcap_{b<a} {}^{b}A \Rightarrow x \in {}^{a}A$$

i.e. $$\bigcap_{b<a} {}^{b}A \subseteq {}^{a}A \qquad \text{.... (2)}$$

Hence, we get

$${}^{a}A = \bigcap_{b<a} {}^{b}A \qquad \text{From (1) and (2)}$$

Similarly

$$x \in {}^{a}A \Rightarrow A(x) \geq a$$

$$\Rightarrow A(x) \geq a > b \qquad b < a$$

$$\Rightarrow A(x) > b \qquad \forall b < a$$

$$\Rightarrow x \in {}^{b+}A \qquad \forall b < a$$

$$\Rightarrow x \in \bigcap_{b<a} {}^{b+}A$$

Thus

$${}^{a}A \subseteq \bigcap_{b<a} {}^{b+}A \qquad \text{.... (3)}$$

Next,

$$x \in \bigcap_{b<a} {}^{b+}A$$

$$\Rightarrow x \in {}^{b+}A \qquad \forall b < a$$

$$\Rightarrow A(x) > b \qquad \forall b < a$$

$$\Rightarrow A(x) > a - \in \qquad \text{Since } a - \in < a \text{ for some } \in > 0$$

Further, $\in > 0$ is arbitrary, we get,

$$A(x) > a$$

$$\Rightarrow A(x) \geq a$$

$$\Rightarrow x \in {}^{a}A$$

Thus,

$$\bigcap_{b<a} {}^{b+}A \subseteq {}^{a}A \qquad \text{.... (4)}$$

Hence from (3) and (4) we get,

$${}^{a}A = \bigcap_{b<a} {}^{b+}A$$

2) For $a < b$ we have,

$$\Rightarrow {}^{a+}A \supseteq {}^{b+}A$$

$$\Rightarrow {}^{b+}A \subseteq {}^{a+}A \qquad \forall a < b$$

$$\Rightarrow \bigcup_{a<b} {}^{b+}A \subseteq {}^{a+}A \qquad \text{..... (1)}$$

On the other hand,

$$x \in {}^{a+}A$$

$$\Rightarrow A(x) > a \qquad\qquad a < b$$

Let, $\in > 0$ be small positive real number. Then,

$$b - \in < b$$

$$\Rightarrow A(x) > b - \in$$

Since, $\in > 0$ be arbitrary (Taking $\in \to 0$)

$$\Rightarrow A(x) > b$$

$$\Rightarrow x \in {}^{b+}A$$

$$\Rightarrow x \in \bigcup_{a<b} {}^{b+}A$$

$$\Rightarrow {}^{a+}A \subseteq \bigcup_{a<b} {}^{b+}A \qquad \text{.... (2)}$$

Then from (1) and (2)

$${}^{a+}A = \bigcup_{a<b} {}^{b+}A$$

Equivalently

Let, $x \in {}^{a+}A$

$$\Rightarrow A(x) > a$$

$$\Rightarrow A(x) \geq a$$

Let, $\in > 0$ be small positive real number such that,

$$b - \in < b$$

$$\Rightarrow A(x) \geq b - \in$$

Since, $\in > 0$ is arbitrary (Taking $\in \to 0$)

$$\Rightarrow A(x) \geq b$$

$\Rightarrow x \in {}^{b}A$

$\Rightarrow x \in \bigcup_{a<b} {}^{b}A$

$\Rightarrow {}^{a+}A = \bigcup_{a<b} {}^{b}A$ .... (3)

On the other side,

Let,

$$x \in \bigcup_{a<b} {}^{b}A$$

$\Rightarrow x \in {}^{b}A$ For some $b > a$

$\Rightarrow A(x) \geq b$ For some $b > a$

$\Rightarrow A(x) \geq b > a$ $(\because b > a)$

$\Rightarrow A(x) \geq a$

$\Rightarrow x \in {}^{a+}A$

$\Rightarrow \bigcup_{a<b} {}^{b}A \subseteq {}^{a+}A$ ..... (4)

Hence, from (3) and (4)

$\Rightarrow {}^{a+}A = \bigcup_{a<b} {}^{b}A$

Hene, the proof.

4. Theorem

If $Ai \in \mathcal{F}(X)$, $i \in I$, where I is an index sets, then following properties holds -

1) ${}^{a}\left(\bigcup_{i\in I} Ai\right) \supseteq \bigcup_{i\in I} {}^{a}Ai$

2) $\quad {}^{a}\left(\bigcap_{i\in I} Ai\right) = \bigcap_{i\in I} {}^{a}Ai$

3) $\quad {}^{a+}\left(\bigcup_{i\in I} Ai\right) = \bigcup_{i\in I} {}^{a+}Ai$

4) $\quad {}^{a+}\left(\bigcap_{i\in I} Ai\right) \supseteq \bigcap_{i\in I} {}^{a+}Ai$

Proof :

1) Let, $x \in X$ be any element.

If $\quad x \in \bigcup_{i\in I} {}^{a}Ai \Rightarrow x \in {}^{a}Ai \qquad$ for some $i \in I$

$\Rightarrow Ai(x) \geq a \qquad$ for some $i \in I$

$\Rightarrow \vee_{i\in I} Ai(x) \geq a$

$\Rightarrow \left(\bigcup_{i\in I} Ai\right) \geq a$

$\Rightarrow x \in {}^{a}\left(\bigcup_{i\in I} Ai\right)$

Hence,

$$\bigcup_{i\in I} {}^{a}Ai \subseteq {}^{a}\left(\bigcup_{i\in I} Ai\right)$$

2) Let $x \in X$ be any element.

Then,

$$x \in {}^{a}\left(\bigcap_{i\in I} Ai\right)$$

$$\Leftrightarrow \left(\bigcap_{i\in I} Ai\right)(x) \geq a$$

$$\Leftrightarrow \wedge_{i\in I} Ai(x) \geq a$$

$$\Leftrightarrow A(x) \geq a \qquad \forall i \in I$$

$$\Leftrightarrow x \in {}^{a}Ai \qquad \forall i \in I$$

$$\Leftrightarrow x \in \bigcap_{i \in I} {}^{a}Ai$$

Hence, we have

$${}^{a}\left(\bigcap_{i \in I} Ai\right) = \bigcap_{i \in I} {}^{a}Ai$$

3) Let $x \in X$ be any element

Then,

$$x \in {}^{a+}\left(\bigcup_{i \in I} Ai\right)$$

$$\Leftrightarrow \left(\bigcup_{i \in I} Ai\right)(x) > a$$

$$\Leftrightarrow \bigvee_{i \in I} Ai(x) > a$$

$$\Leftrightarrow Ai(x) > a \qquad \text{for some } i \in I$$

$$\Leftrightarrow x \in {}^{a+}Ai \qquad \text{for some } i \in I$$

$$\Leftrightarrow x \in \bigcup_{i \in I} {}^{a+}Ai$$

Hence, we have,

$${}^{a+}\left(\bigcup_{i \in I} Ai\right) = \bigcup_{i \in I} {}^{a+}Ai$$

4) Let $x \in X$ be any element. Then,

$$x \in \bigcap_{i \in I} {}^{a+}Ai$$

$$\Rightarrow x \in {}^{a+}A \qquad \text{for all } i \in I$$

$$\Rightarrow A(x) > a \qquad \text{for all } i \in I$$

$$\Rightarrow \underset{i\in I}{\wedge} Ai(x) > a$$

$$\Rightarrow \left(\bigcap_{i\in I} Ai\right)(x) > a$$

$$\Rightarrow x \in {}^{a+}\left(\bigcap_{i\in I} Ai\right)$$

Thus,

$$\bigcap_{i\in I} {}^{a+}Ai \subseteq {}^{a+}\left(\bigcap_{i\in I} Ai\right)$$

5. Note :

In the a bove theorem, the equalities in (1) and (4) need not hold.

e.g. Let, I = N = { 1, 2, ...}

Let, X = $\mathbb{R}$ and

$Ai \in \mathcal{F}(\mathbb{R})$ defined by,

$$Ai : \mathbb{R} \to I\,,\ Ai(x) = 1 - \frac{1}{i} \qquad \forall x \in \mathbb{R}$$

Then $\{Ai\}_{i\in I}$ is a family of Fuzzy sets on $\mathbb{R}$.

$$\left(\bigcup_{i\in I} Ai\right)(x) = \underset{i\in I}{\vee} Ai(x)$$

$$= \underset{i\in I}{\vee}\left(1 - \frac{1}{i}\right)$$

$$= \vee\left\{0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, ...\right\}$$

$$\Rightarrow \left(\bigcup_{i\in I} Ai\right)(x) = 1 \qquad \forall x \in \mathbb{R}$$

Hence,

$$^{1}\left(\bigcup_{i\in I} Ai\right) = \mathbb{R}$$

But,

$$^{1}(Ai) = \left\{ x \in \mathbb{R} \,\middle|\, Ai(x) \geq 1 \right\}$$

$$= \left\{ x \in \mathbb{R} \,\middle|\, 1 - \frac{1}{i} \geq 1 \right\}$$

$$\Rightarrow {}^{1}(Ai) = f \qquad\qquad \forall i$$

$$\Rightarrow \bigcup_{i\in I} \left({}^{1}Ai\right) = f$$

Hence,

$$\bigcup_{i\in I} \left({}^{1}Ai\right) \underset{\neq}{\subset} {}^{1}\left(\bigcup_{i=1} Ai\right)$$

2.2 Representation of Fuzzy Sets

1. Definition

If $_{a}A : X \to I$ is a fuzzy set defined on X.

Then, we define a special fuzzy set by

$$_{a}A(x) = \begin{cases} a & \text{if } x \in {}^{a}A \\ 0 & \text{if } x \notin {}^{a}A \end{cases}$$

i.e. $_{a}A(x) = a \cdot c_{a_A}(x)$

or $\quad _{a}A(x) = a \cdot {}^{a}A(x) \qquad\qquad \left[\because {}^{a}A = c_{a_A}\right]$

or $\quad _{a}A = a \cdot {}^{a}A$

2. Example

Let, $X = \{x_1, x_2, x_3, x_4, x_5\}$ be a finite set.

Let, $A: X \to I$ be defined by,

$$A = \frac{0.2}{x_1} + \frac{0.4}{x_2} + \frac{0.6}{x_3} + \frac{0.8}{x_4} + \frac{1}{x_5}$$

Then,

$$^{0.2}A = \{x_1, x_2, x_3, x_4, x_5\}$$

$$^{0.4}A = \{x_2, x_3, x_4, x_5\}$$

$$^{0.6}A = \{x_3, x_4, x_5\}$$

$$^{0.8}A = \{x_4, x_5\}$$

$$^{1}A = \{x_5\}$$

Then,

$_{0.2}A : X \to I$ by

$$_{0.2}A(x) = 0.2 \cdot {}^{0.2}A(x) = 0.2$$

Thus,

$$_{0.2}A = \frac{0.2}{x_1} + \frac{0.2}{x_2} + \frac{0.2}{x_3} + \frac{0.2}{x_4} + \frac{0.2}{x_5}$$

Simillarly,

$$_{0.4}A = \frac{0}{x_1} + \frac{0.4}{x_2} + \frac{0.4}{x_3} + \frac{0.4}{x_4} + \frac{0.4}{x_5}$$

$$_{0.6}A = \frac{0}{x_1} + \frac{0}{x_2} + \frac{0.6}{x_3} + \frac{0.6}{x_4} + \frac{0.6}{x_5}$$

$$_{0.8}A = \frac{0}{x_1} + \frac{0}{x_2} + \frac{0}{x_3} + \frac{0.8}{x_4} + \frac{0.8}{x_5}$$

$${}_{1}A = \frac{0}{x_1} + \frac{0}{x_2} + \frac{0}{x_3} + \frac{0}{x_4} + \frac{1}{x_5}$$

Consider, ${}_{0.2}A \cup {}_{0.4}A \cup {}_{0.6}A \cup {}_{0.8}A \cup {}_{1}A$

$$= \frac{0.2}{x_1} + \frac{0.4}{x_2} + \frac{0.6}{x_3} + \frac{0.8}{x_4} + \frac{1}{x_5}$$

3. Theorem : First Decomposition Theorem

For any $A \in \mathcal{F}(X)$,

$$A = \bigcup_{a \in [0,1]} {}_{a}A$$

where,

$${}_{a}A(x) = \begin{cases} a & \text{if } x \in {}^{a}A \\ 0 & \text{if } x \notin {}^{a}A \end{cases}$$

${}_{a}A$ is called special fuzzy set.

Proof :

Let $x \in X$ be arbitrary and let

$$A(x) = a \qquad a \in [0,1]$$

Then

$$\left(\bigcup_{a \in [0,1]} {}_{a}A \right)(x) = \bigvee_{a \in [0,1]} {}_{a}A(x), \qquad 0 \le a \le 1$$

$$= \bigvee_{a \in [0,a]} {}_{a}A(x) \vee \bigvee_{a \in (a,1]} {}_{a}A(x) \qquad \text{... (a)}$$

If $a \in [0,a]$

$$\Rightarrow a \le a$$

$$\Rightarrow a \le A(x)$$

$$\Rightarrow A(x) \geq \boldsymbol{a}$$

$$\Rightarrow x \in {}^{a}A$$

$$\Rightarrow {}_{a}A(x) = \boldsymbol{a}$$

Hence,

$$\underset{a\in[0,a]}{\vee} {}_{a}A(x) = \underset{a\in[0,a]}{\vee} \boldsymbol{a} = a$$

Next, If

$$\boldsymbol{a} \in (a,1]$$

$$\Rightarrow a < \boldsymbol{a}$$

$$\Rightarrow A(x) < \boldsymbol{a}$$

$$\Rightarrow A(x) \ngeq \boldsymbol{a}$$

$$\Rightarrow x \notin {}^{a}A$$

$$\Rightarrow {}_{a}A(x) = 0$$ (by given definition.)

Thus, $\underset{a\in[a,1]}{\vee} {}_{a}A(x) = \underset{a\in[a,1]}{\vee} 0 = 0$

Hence, (a) becomes,

$$\left(\bigcup_{a\in[0,1]} {}_{a}A\right)(x) = \underset{a\in[0,a]}{\vee} {}_{a}A(x) \vee \underset{a\in(a,1]}{\vee} {}_{a}A(x)$$

$$= a \vee 0$$ From (1) and (2)

$$= a$$

$$= A(x)$$

Thus, for all $x \in X$,

$$A(x) = \left(\bigcup_{a\in[0,1]} {}_{a}A\right)(x)$$

i.e. $A = \bigcup_{a\in[0,1]} {}_{a}A$

4. Theorem : Second Decomposition Theorem

For any $A \in \mathcal{F}(X)$

$$A = \bigcup_{a \in [0,1]} {}_{a+}A$$

where,

$$_{a+}A(x) = \begin{cases} a & \text{if } x \in {}^{a+}A \\ 0 & \text{otherwise} \end{cases}$$

Proof :

Let, $x \in X$ be arbitrary and let

$$A(x) = a\,, \quad a \in [0,1]$$

Consider,

$$\left(\bigcup_{a \in [0,1]} {}_{a+}A \right)(x) = \vee_{a \in [0,1]} {}_{a+}A(x)$$

$$\Rightarrow \left(\bigcup_{a \in [0,1]} {}_{a+}A \right)(x) = \vee_{a \in [0,a)} {}_{a+}A(x) \vee \vee_{a \in [a,1]} {}_{a+}A(x)$$

1) If $a \in [0,a)$

$$\Rightarrow a < a$$

$$\Rightarrow a > a$$

$$\Rightarrow A(x) > a$$

$$\Rightarrow x \in {}^{a+}A$$

$$\Rightarrow {}_{a+}A(x) = a$$

Thus,

$$\vee_{a \in [0,a)} {}_{a+}A(x) = \vee_{a \in [0,a)} a = a$$

Next, if $a \in [a,1]$,

$$\Rightarrow a \leq a$$

$$\Rightarrow A(x) \leq a$$

$$\Rightarrow A(x) \ngtr a$$

$$\Rightarrow x \notin {}^{a+}A$$

$$\Rightarrow {}_{a+}A(x) = 0$$

i.e. $\bigvee_{a \in [a,1]} {}_{a+}A(x) = 0$

Thus,

$$\left(\bigcup_{a \in [0,1]} {}_{a+}A\right)(x) = \bigvee_{a \in [0,a)} {}_{a+}A(x) \vee \bigvee_{a \in [a,1]} {}_{a+}A(x)$$

$= a \vee 0$ From (1) and (2)

$= a$

$= A(x)$

Hence,

$$A = \bigcup_{a \in [0,1]} {}_{a+}A$$

5. Example

Let $X = \{x_1, x_2, x_3, x_4, x_5\}$ and let $A \in \mathcal{F}(X)$ be defined by

$$A = \frac{0.2}{x_1} + \frac{0.4}{x_2} + \frac{0.6}{x_3} + \frac{0.8}{x_4} + \frac{1}{x_5}$$

Express A as a union of special Fuzzy sets and verify the decomposition theorems.

Solution : Consider the level cuts of A

$${}^{0.2}A = \{x_1, x_2, x_3, x_4, x_5\}$$

$${}^{0.4}A = \{x_2, x_3, x_4, x_5\}$$

$${}^{0.6}A = \{x_3, x_4, x_5\}$$

$^{0.8}A = \{x_4, x_5\}$

$^{1}A = \{x_5\}$

Therefore,

$${}_{0.2}A = \frac{0.2}{x_1} + \frac{0.2}{x_2} + \frac{0.2}{x_3} + \frac{0.2}{x_4} + \frac{0.2}{x_5}$$

$${}_{0.4}A = \frac{0}{x_1} + \frac{0.4}{x_2} + \frac{0.4}{x_3} + \frac{0.4}{x_4} + \frac{0.4}{x_5}$$

$${}_{0.6}A = \frac{0}{x_1} + \frac{0}{x_2} + \frac{0.6}{x_3} + \frac{0.6}{x_4} + \frac{0.6}{x_5}$$

$${}_{0.8}A = \frac{0}{x_1} + \frac{0}{x_2} + \frac{0}{x_3} + \frac{0.8}{x_4} + \frac{0.8}{x_5}$$

$${}_{1}A = \frac{0}{x_1} + \frac{0}{x_2} + \frac{0}{x_3} + \frac{0}{x_4} + \frac{1}{x_5}$$

Consider $({}_{0.2}A \cup {}_{0.4}A \cup {}_{0.6}A \cup {}_{0.8}A \cup {}_{1}A)$

$$= \frac{0.2}{x_1} + \frac{0.4}{x_2} + \frac{0.6}{x_3} + \frac{0.8}{x_4} + \frac{1}{x_5}$$

$$= A$$

Thus,

$A = {}_{0.2}A \cup {}_{0.4}A \cup {}_{0.6}A \cup {}_{0.8}A \cup {}_{1}A$

Hence, first decomposition theorem verified.

Next consider for any $a\,i \in [0,1]$ such that

$0 \le a_1 < 0.2 < a_2 < 0.4 < a_3 < 0.6 < a_4 < 0.8 < a_5 < 1 = a_6$

And,

$A(x_1) = 0.2$

$A(x_2) = 0.4$

$A(x_3) = 0.6$

$A(x_4) = 0.8$

$A(x_5) = 1$

Consider

$$\left(\bigcup_{a_i \in [0,1]} {}^{a_i+}A\right)(x) = \bigvee_{a_i \in [0,1]} a_i + A(x)$$

Therefore

$$\left(\bigcup_{a_i \in [0,1]} {}^{a_i+}A\right)(x_1) = \bigvee_{a_i \in [0,1]} a_i + A(x_1)$$

$$= \left[\bigvee_{a_1 \in [0,0.2]} a_1 + A(x_1)\right] \vee \left[\bigvee_{a_2 \in [0.2,0.4]} a_2 + A(x_1)\right]$$

$$\vee \left[\bigvee_{a_3 \in [0.4,0.6]} a_3 + A(x_1)\right] \vee \left[\bigvee_{a_4 \in [0.6,0.8]} a_4 + A(x_1)\right]$$

$$\vee \left[\bigvee_{a_5 \in [0.8,1]} a_5 + A(x_1)\right] \vee \left[1 + A(x_1)\right]$$

If $0 \le a_i < 0.2$

$\Rightarrow a_i < A(x_1)$

$\Rightarrow A(x_1) > a_i$

$\Rightarrow x_1 \in {}^{a_i+}A$, for $a_i < 0.2$.

$\Rightarrow a_i + A(x_1) = a_i$ for $a_i < 0.2$.

And if $a_i \in [0.2,1]$

$\Rightarrow 0.2 \le a_i \le 1$

$\Rightarrow A(x_1) \le a_i$

$\Rightarrow A(x_1) \not> a_i$

$\Rightarrow x_1 \notin {}^{a_i+}A$, for $a_i \in [0.2,1]$.

$\Rightarrow a_i + A(x_1) = 0$ for $a_i \in [0.2,1]$.

Thus,

$$\left(\bigcup_{a_i\in[0,1]} {}^{a_i+}A\right)(x_1) = \left[\bigvee_{a_1\in[0,0.2]} {}^{a_i+}A(x_1)\right] \vee \left[\bigvee_{a_2\in[0.2,0.4]} (0)\right]$$

$$\vee \left[\bigvee_{a_3\in[0.4,0.6]} (0)\right] \vee \left[\bigvee_{a_4\in[0.6,0.8]} (0)\right]$$

$$\vee \left[\bigvee_{a_5\in[0.8,1]} (0)\right] \vee (0)$$

$$= (0.2) \vee (0) \vee (0) \vee (0) \vee (0)$$

$$= 0.2$$

$$\Rightarrow \left(\bigcup_{a_i\in[0,1]} {}^{a_i+}A\right)(x_1) = 0.2$$

Simillarly

$$\left(\bigcup_{a_i\in[0,1]} {}^{a_i+}A\right)(x_2) = 0.4$$

$$\left(\bigcup_{a_i\in[0,1]} {}^{a_i+}A\right)(x_3) = 0.6$$

$$\left(\bigcup_{a_i\in[0,1]} {}^{a_i+}A\right)(x_4) = 0.8$$

$$\left(\bigcup_{a_i\in[0,1]} {}^{a_i+}A\right)(x_5) = 1.$$

Thus,

$$\left(\bigcup_{a_i\in[0,1]} {}^{a_i+}A\right)(x) = \frac{0.2}{x_1} + \frac{0.4}{x_2} + \frac{0.6}{x_3} + \frac{0.8}{x_4} + \frac{1}{x_5}$$

$$= A(x)$$

$$\Rightarrow A = \left(\bigcup_{a_i \in [0,1]} {}_{a_i +} A \right)$$

Hence, second decomposition theorem is varified.

2.3 Extension Principle

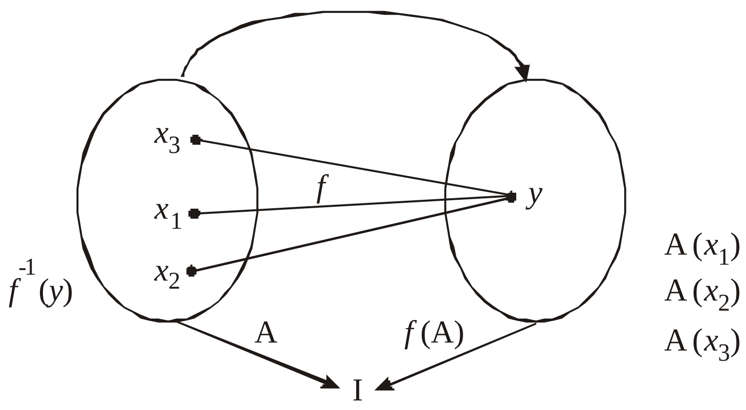
1. Definition

Let, $f : X \to Y$ be a function. Define $f : \mathcal{F}(X) \to \mathcal{F}(Y)$ by,

$$f(A)(y) = \vee_{x \in f^{-1}(y)} A(x)$$

$$\Rightarrow f(A)(y) = \vee_{y = f(x)} A(x)$$

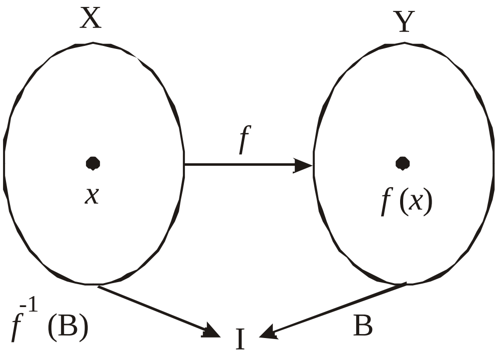
$f(A)$ is called the direct image of a fuzzy set A on X.



Definition :

For any $B \in \mathcal{F}(Y)$ we define,

$$f^{-1}(B)(x) = B(f(x))$$



$f^{-1}(B)$ is called the inverse image of the fuzzy set B defined on Y..

2. Theorem

Let, $f : X \to Y$ be a crisp function for $Ai \in \mathcal{F}(X)$ and $Bi \in \mathcal{F}(Y)$, $i \in I$, the following properties hold

1) $f(A) = f$ iff $A = f$

2) $A_1 \subseteq A_2 \Rightarrow f(A_1) \subseteq f(A_2)$

3) $f\left(\bigcup_{i \in I} Ai\right) = \bigcup_{i \in I} f(Ai)$

4) $f\left(\bigcap_{i \in I} Ai\right) \subseteq \bigcap_{i \in I} f(Ai)$

5) If $B_1 \subseteq B_2$, then $f^{-1}(B_1) \subseteq f^{-1}(B_2)$

6) $f^{-1}\left(\bigcup_{i \in I} Bi\right) = \bigcup_{i \in I} f^{-1}(Bi)$

7) $f^{-1}\left(\bigcap_{i \in I} Bi\right) = \bigcap_{i \in I} f^{-1}(Bi)$

8) $\overline{f^{-1}(B)} = f^{-1}(\overline{B})$

9) $A \subseteq f^{-1}\left(f(A)\right)$

10) $B = f\left(f^{-1}(B)\right)$

Proof :

1) Let $A = f$ i.e. $A(x) = 0 \qquad \forall x \in X$

Then, for any $y \in Y$,

$$f(A)(y) = \underset{\substack{x \in f^{-1}(y) \\ y = f(x)}}{\vee} A(x)$$

$$= 0 \qquad \text{Since, } A(x) = 0, \ \forall x .$$

Thus,

$$f(A)(y) = 0 \qquad \forall y \in Y$$

i.e. $\quad f(A) = f$

Conversely,

Let, $\quad f(A) = f$

Then, $\quad f(A)(y) = 0 \qquad \forall y \in Y$

$$\Rightarrow \underset{x \in f^{-1}(y)}{\vee} A(x) = 0 \qquad \forall y \in Y$$

$$\Rightarrow A(x) = 0 \qquad \forall x \in X$$

$$\Rightarrow A = f$$

2) Let, $\quad A_1 \subseteq A_2$

i.e. $\quad A_1(x) \leq A_2(x) \qquad \forall x \in X$

Then, for any $y \in Y$,

$$f(A_1)(y) = \underset{x \in f^{-1}(y)}{\vee} A_1(x)$$

$$\leq \bigvee_{x \in f^{-1}(y)} A_2(x) = f(A_2)(y)$$

$$\Rightarrow f(A_1)(y) \leq f(A_2)(y) \qquad \forall y \in Y$$

$$\Rightarrow f(A_1) \subseteq f(A_2)$$

3) Let $y \in Y$ be any element

Then, $$f\left(\bigcup_i Ai\right)(y) = \bigvee_{x \in f^{-1}(y)} \left(\bigcup Ai\right)(x)$$

$$= \bigvee_{x \in f^{-1}(y)} \left(\bigvee_{i \in I} A(x)\right)$$

$$= \bigvee_{i \in I} \left(\bigvee_{x \in f^{-1}(y)} A(x)\right)$$

$$= \bigvee_{i \in I} f(Ai)(y)$$

$$= \left(\bigcup_{i \in I} f(Ai)\right)(y)$$

i.e. $$f\left(\bigcup_i Ai\right) = \bigcup_i f(Ai)$$

4) For any $y \in Y$ and $i \in I$.

Consider,

$$f\left(\bigcap_i Ai\right) = \bigvee_{x \in f^{-1}(y)} \left(\bigcap_i Ai\right)(x)$$

$$= \bigvee_{x \in f^{-1}(y)} \left(\bigvee_i Ai(x)\right)$$

$$\leq \bigwedge_{i \in I} \left(\bigvee_{x \in f^{-1}(y)} Ai(x)\right)$$

$$\leq \bigwedge_{i \in I} f(Ai)(y)$$

$$=\left(\bigcap_{i\in I} f(Ai)\right)(y)$$

Hence,

$$f\left(\bigcap_{i} Ai\right)\subseteq \bigcap_{i\in I} f(Ai)$$

5) Let, $B_1 \subseteq B_2$

Then, for any $x \in X$,

$$f^{-1}(B_1)(x) = B_1(f(x)) \le B_2(f(x)) = f^{-1}(B_2)(x)$$

$$\Rightarrow f^{-1}(B_1)(x) \le f^{-1}(B_2)(x) \qquad \forall x \in X$$

$$\Rightarrow f^{-1}(B_1) \subseteq f^{-1}(B_2)$$

6) Let, $x \in X$ be any element.

Then,

$$f^{-1}\left(\bigcup_{i} Bi\right)(x) = \left(\bigcup_{i} Bi\right)(f(x))$$

$$= \vee_{i} Bi(f(x))$$

$$= \vee_{i} f^{-1}(Bi)(x)$$

$$= \left[\left(\bigcup_{i} f^{-1}(Bi)\right)\right](x)$$

Thus,

$$f^{-1}\left(\bigcup_{i} Bi\right)(x) = \left[\bigcup_{i} f^{-1}(Bi)\right](x)$$

7) Let, be any element.

Then,

$$f^{-1}\left(\bigcap_i Bi\right)(x) = \left(\bigcap_i Bi\right)\left(f(x)\right)$$

$$= \wedge_i Bi\left(f(x)\right)$$

$$= \wedge_i f^{-1}(Bi)(x)$$

$$= \left[\bigcap_i f^{-1}(Bi)\right](x)$$

Thus,

$$f^{-1}\left(\bigcap_i Bi\right) = \bigcap_i f^{-1}(Bi)$$

8) For any $x \in X$,

$$f^{-1}(B)(x) = 1 - f^{-1}(B)(x)$$

$$= 1 - B\left(f(x)\right)$$

$$= \overline{B}\left(f(x)\right)$$

$$= f^{-1}\left(\overline{B}\right)(x)$$

Hence,

$$\overline{f^{-1}(B)} = f^{-1}\left(\overline{B}\right)$$

9) For any $x \in X$,

$$f^{-1}\left(f(A)\right)(x) = f(A)\left(f(x)\right)$$

$$= f(A)(y)$$

$$= \vee_{z \in f^{-1}(y)} A(z)$$

$\geq A(x)$ $\quad (\because f(x)=y)$

i.e. $f^{-1}\left(f(A)\right)(x) \geq A(x) \Rightarrow A(x) \leq f^{-1}\left(f(A)\right)(x)$ for all $x \in X$

$\Rightarrow A \subseteq f^{-1}\left(f(A)\right)$

10) For any $y \in Y$,

$$f\left(f^{-1}(B)\right)(y) = \bigvee_{x \in f^{-1}(y)} f^{-1}(B)(x)$$

$$= \bigvee_{\substack{x \in f^{-1}(y) \\ f(x)=y}} B\left(f(x)\right)$$

$$= \bigvee_{f(x)=y} B(y) \qquad (\because f(x)=y)$$

$$= B(y)$$

Thus,

$f\left(f^{-1}(B)\right) = B$

3. Theorem

Let $f : X \rightarrow Y$ be a crisp fucntion. For any $A \in \mathcal{F}(X)$.

1) ${}^{a+}\left[f(A)\right] = f\left({}^{a+}A\right)$

2) ${}^{a}\left[f(A)\right] \supseteq f\left({}^{a}A\right)$

Proof :

1) Let $y \in Y$ be arbitrary

If $y \in {}^{a+}\left[f(A)\right]$

$\Rightarrow f(A)(y) > a$

$\Rightarrow \bigvee_{x \in f^{-1}(y)} A(x) > a$

$\Rightarrow A(x) > a$ for some $x \in X$ such that $x \in f^{-1}(y)$

$\Rightarrow x \in {}^{a+}A$ $\quad x \in f^{-1}(y)$ or $f(x) = y$

$\Rightarrow f(x) \in f({}^{a+}A)$

$\Rightarrow y \in f({}^{a+}A)$ $\quad (\because f(x) = y)$

Thus,

$${}^{a+}[f(A)] \subseteq f({}^{a+}A) \qquad \text{.... (1)}$$

Conversely, If

$$y \in f({}^{a+}A)$$

There exists some $x \in {}^{a+}A$, such that

$$y = f(x)$$

$\Rightarrow x \in {}^{a+}A$ $\quad$ where $x \in f^{-1}(y)$

$\Rightarrow A(x) > a$ $\quad$ where $x \in f^{-1}(y)$

$$\Rightarrow \bigvee_{x \in f^{-1}(y)} A(x) > a$$

$$\Rightarrow f(A)(y) > a$$

$$\Rightarrow y \in {}^{a+}[f(A)]$$

Hence,

$$f({}^{a+}A) \subseteq {}^{a+}[f(A)] \qquad \text{.... (2)}$$

Thus, we get,

$${}^{a+}[f(A)] = f({}^{a+}A) \qquad \text{From (1) and (2)}$$

2) $\quad {}^{a}[f(A)] \supseteq f({}^{a}A)$

Let $y \in Y$ be arbitrary. Then

$y \in f({}^{a}A)$

$\Rightarrow \exists$ some $x \in {}^{a}A$ s.t. $y = f(x)$

$\Rightarrow x \in {}^{a}A$ where $x \in f^{-1}(y)$

$\Rightarrow A(x) \geq a$ where $x \in f^{-1}(y)$

$$\Rightarrow \bigvee_{x \in f^{-1}(y)} A(x) \geq a$$

$\Rightarrow f(A)(y) \geq a$

$\Rightarrow y \in {}^{a}[f(A)]$

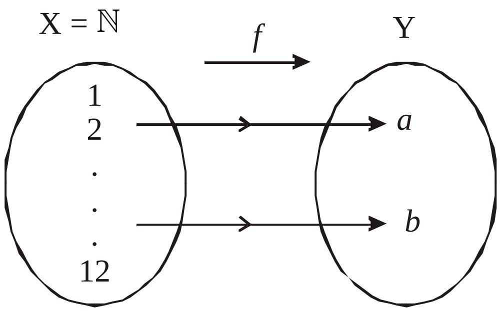
Thus,

$$f({}^{a}A) \subseteq {}^{a}[f(A)]$$

4. Note :

The equality need not hold for (2) in above theorem

Counter Example



Let, $X = \mathbb{N}$, $Y = \{a, b\}$

Define a function $f : X \to Y$ by,

$$\begin{cases} f(n) = a & \text{if } n \leq 12 \\ \quad\; = b & \text{if } n \geq 12 \end{cases}$$

Let, $A : \mathbb{N} \to I$ be a Fuzzy set defined on $\mathbb{N}$ by,

$$A(n) = 1 - \frac{1}{n} \qquad n \in \mathbb{N}, \text{ I} = [0, 1]$$

Define, $f(A) : Y \to I$,

$$f(A)(a) = \bigvee_{\substack{x \in f^{-1}(a) \\ x \in \{1, \ldots, 12\}}} A(x)$$

$$= \vee\left\{0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \ldots\ldots, \frac{11}{12}\right\}$$

$$f(A)(a) = \frac{11}{12}$$

$$f(A)(b) = \bigvee_{\substack{x \in f^{-1}(b) \\ x \in \{13,14,\ldots.\}}} A(x)$$

$$= \bigvee_{x \in \{13,14,\ldots.\}} \left(1 - \frac{1}{x}\right)$$

$$f(A)(b) = 1$$

Thus,

$f(A) : Y \to I$ is given by**,**

$f(A)(a) = \frac{11}{12}$ and

$$f(A)(b) = 1$$

Then,

$${}^{1}[f(A)] = \{y \in Y \mid f(A)(y) = 1\}$$

$$= \{b\}$$

Also,

$${}^{1}A = \{x \in X \mid A(x) \geq 1\}$$

$$= \left\{x \in X \;\middle|\; 1 - \frac{1}{x} \geq 1\right\}$$

$${}^{1}A = f$$

Then,

$$f({}^{1}A) = \{y \in Y \mid \exists x \in {}^{1}A \ni y = f(x)\}$$

$$f\left({}^{1}A\right)=f$$

Thus,

$$f\left({}^{1}A\right)\underset{\neq}{\subset}{}^{1}\left[f\left(A\right)\right]$$

5. Theorem

Let, $f:X\rightarrow Y$ be a crisp function then for any $A\in\mathcal{F}(X)$,

$$f(A)=\bigcup_{a\in[0,1]}f\left({}_{a+}A\right)$$

where, ${}_{a+}A$ is a special Fuzzy set defined by,

$${}_{a+}A(x)=a \qquad \text{if } x\in {}^{a+}A$$

$$=0 \qquad \text{if } x\notin {}^{a+}A$$

Proof : Let $y\in Y$ be arbitrary. Then,

$$\left[\bigcup_{a\in[0,1]}f\left({}_{a+}A\right)\right](y)=\bigvee_{a\in[0,1]}f\left({}_{a+}A\right)(y)$$

$$=\bigvee_{a\in[0,1]}\left\{\bigvee_{x\in f^{-1}(y)}{}_{a+}A(x)\right\}$$

$$=\bigvee_{x\in f^{-1}(y)}\bigvee_{a\in[0,1]}{}_{a+}A(x)$$

$$=\bigvee_{x\in f^{-1}(y)}\left(\bigcup_{a\in[0,1]}{}_{a+}A\right)(x)$$

$$=f\left(\bigcup_{a\in[0,1]}{}_{a+}A\right)(y)$$

But, by second decomposition theorem

$$\bigcup_{a\in[0,1]} {}_{a+}A = A$$

Hence,

$$\left(\bigcup_{a\in[0,1]} f\left({}_{a+}A\right)\right)(y) = f(A)(y)$$

$$\Rightarrow f(A) = \bigcup_{a\in[0,1]} f\left({}_{a+}A\right)$$

6. Example :

Let, X be the universal set given by,

X = {0, 1, 2, 3, ..., 10}. Let $f : X \to Y$ be a function defined by $f(x) = x^2$,

where Y = {0, 1, 4, 9, ..., 100}. Let, A, B, C be the Fuzzy sets defined on X by,

$$A(x) = \frac{x}{x+12}, \quad B(x) = \frac{1}{2^x} \text{ and } C(x) = \frac{1}{1+10(x-2)^2}$$

Find $f(A)$, $f(B)$ and $f(C)$.

Further, if B is a Fuzzy set defined on Y by,

$$D = \frac{0.5}{4} + \frac{0.6}{16} + \frac{0.7}{25} + \frac{1}{100}$$

Find $f^{-1}(D)$

Ans. : Let, $A : X \to Y$ defined by, $A(x) = \dfrac{x}{x+12}$

Define, $f(A) : Y \to I$ by**,**

$$f(A)(y) = \bigvee_{x\in f^{-1}(y)} A(x)$$

$$= \bigvee_{f(x)=y} \frac{x}{x+12}$$

But, $f(x) = x^2 \Rightarrow y = x^2$

$$\Rightarrow x = \sqrt{y}$$

Hence,

$$f(A)(y) = \bigvee_{x^2=y} \frac{x}{x+12}$$

$$= \vee \frac{\sqrt{y}}{\sqrt{y}+12}$$

$$= \frac{\sqrt{y}}{\sqrt{y}+12} \qquad y \in Y$$

Let, $B : X \to I$ defined by, $B(x) = \frac{1}{2^x}$

Define, $f(B) : Y \to I$ by,

$$f(B)(y) = \bigvee_{x \in f^{-1}(y)} B(x)$$

$$= \bigvee_{f(x)=y} \frac{1}{2^x}$$

$$= \bigvee_{x=\sqrt{y}} \frac{1}{2^x}$$

$$\Rightarrow f(B)(y) = \frac{1}{2^{\sqrt{y}}} \qquad \forall y \in Y$$

Let, $C : X \to I$ be defined by, $C(x) = \frac{1}{1+10(x-2)^2}$

Define, $f(C): Y \to I$ by,

$$\Rightarrow f(C)(y) = \underset{x \in f^{-1}(y)}{\vee} C(x)$$

$$= \underset{f(x)=y}{\vee} \frac{1}{1+10(x-2)^2}$$

$$= \underset{x=\sqrt{y}}{\vee} \frac{1}{1+10(x-2)^2}$$

$$\Rightarrow f(C)(y) = \frac{1}{1+10\left(\sqrt{y}-2\right)^2} \qquad \forall y \in Y$$

$$= \frac{1}{1+10y-40\sqrt{y}+40} \qquad \forall y \in Y$$

$$= \frac{1}{10y-40\sqrt{y}+41} \qquad \forall y \in Y$$

Next,

$$D = \frac{0.5}{4} + \frac{0.6}{16} + \frac{0.7}{25} + \frac{1}{100}$$

Define, $f^{-1}(D): X \to I$ by,

$f^{-1}(D)(x) = D\left(f(x)\right)$ where $x \in X$

Thus, $f^{-1}(D)(0) = D\left(f(0)\right)$, $f^{-1}(D)(2) = D\left(f(2)\right)$

$= D(0)$ $\qquad$ $= D(4)$

$= 0$ $\qquad$ $= 0.5$

Similarly, we can write,

$$f^{-1}(D) = \frac{0}{0} + \frac{0}{1} + \frac{0.5}{2} + \frac{0}{3} + \frac{0.6}{4} + \frac{0.7}{5} + \frac{0}{6} + \frac{0}{7} + \frac{0}{8} + \frac{0}{9} + \frac{1}{10}$$

$$\Rightarrow f^{-1}(D) = \frac{0.5}{2} + \frac{0.6}{4} + \frac{0.7}{5} + \frac{1}{10}$$

7. Example

Find $f(A)$, $f(B)$, $f(C)$ where A, B, C are as in the above example and if the function f is defined by,

$$f : X \to Y, \; f(x) = x + 7$$

where, X = {0, 1, 2, ..., 10}

Y = {7, 8, 9, ... 17}

If $D = \frac{0.5}{9} + \frac{0.6}{11} + \frac{0.7}{12} + \frac{1}{17}$ find $f^{-1}(D)$

Ans. : Let, $f(A) : Y \to I$ be defined by, $A(x) = \frac{x}{x+12}$

Define, $f(A) : Y \to I$ by,

$$f(A)(y) = \bigvee_{x \in f^{-1}(y)} A(x)$$

$$= \bigvee_{f(x)=y} \frac{x}{x+12}$$

But, $f(x) = x + 7 \Rightarrow y = x + 7$

$$\Rightarrow x = y - 7$$

Hence,

$$f(A)(y) = \bigvee_{f(x)=y} \frac{x}{x+12} = \bigvee_{x=y-7} \frac{x}{x+12}$$

$$= \vee \frac{y-7}{y-7+12}$$

$$= \vee \frac{y-7}{y+5}$$

$$= \frac{y-7}{y+5} \qquad \forall y \in Y$$

2) Let $B : X \to I$ be defined by $B(x) = \frac{1}{2^x}$

Define $f(B) : Y \to I$ by,

$$f(B)(y) = \bigvee_{x \in f^{-1}(y)} \frac{1}{2^x}$$

$$= \bigvee_{f(x)=y} \frac{1}{2^x}$$

$$= \bigvee_{x=y-7} \frac{1}{2^x}$$

$$= \vee \frac{1}{2^{y-7}}$$

$$= \frac{1}{2^{y-7}} \qquad \forall y \in Y$$

3) Let $C : X \to I$ be defined by, $C(x) = \frac{1}{1+10(x-2)^2}$

Define $f(C) : Y \to I$ by,

$$f(C)(y) = \bigvee_{x \in f^{-1}(y)} \frac{1}{1+10(x-2)^2}$$

$$= \bigvee_{f(x)=y} \frac{1}{1+10(x-2)^2}$$

$$f(C)(y) = \bigvee_{x=y-7} \frac{1}{1+10(x-2)^2}$$

$$= \vee \frac{1}{1+10(y-7-2)^2} \qquad \forall y \in Y$$

$$= \vee \frac{1}{1+10(y-9)^2} \qquad \forall y \in Y$$

$$= \frac{1}{1+10(y-9)^2} \qquad \forall y \in Y$$

4) Let $D : Y \to I$ be defined by,

$$D = \frac{0.5}{9} + \frac{0.6}{11} + \frac{0.7}{12} + \frac{1}{17}$$

Define $f^{-1}(D) : X \to I$ by,

$f^{-1}(D)(x) = D(f(x))$ where $x \in X$

$\Rightarrow f^{-1}(D)(0) = D(f(0))$ and $f^{-1}(D)(2) = D(f(2))$

$= D(0)$ $\qquad$ $= D(9)$

$= 0$ $\qquad$ $= 0.5$

Similarly for 1, 3, 6, 7, 8, 0 $\qquad$ Similarly for 4, 5 and 10

Thus, we can write

$$f^{-1}(D) = \frac{0}{0} + \frac{0}{1} + \frac{0.5}{2} + \frac{0}{3} + \frac{0.6}{4} + \frac{0.7}{5} + \frac{0}{6} + \frac{0}{7} + \frac{0}{8} + \frac{0}{9} + \frac{1}{10}$$

$$\Rightarrow f^{-1}(D) = \frac{0.5}{2} + \frac{0.6}{4} + \frac{0.7}{5} + \frac{1}{10}$$

8. Example

Let A and B be the Fuzzy sets defined on the universal set $\mathbb{Z}$, whose membership values are given by,

$$A = \frac{0.5}{-1} + \frac{1}{0} + \frac{0.5}{1} + \frac{0.3}{2}$$

$$B = \frac{0.5}{2} + \frac{1}{3} + \frac{0.5}{4} + \frac{0.3}{5}$$

Let f be a function defined by $\mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ by

(a) $f(x_1, x_2) = x_1 x_2$ and

(b) $f(x_1, x_2) = x_1 + x_2$

Find $f(A, B)$

Ans. : Here f is a function defined by,

$$f : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$$

$$(A, B) \to f(A, B) = A \cdot B$$

Thus,

$f(A, B) : \mathbb{Z} \to I$ given by**,**

$$f(A, B)(\mathbb{Z}) = \vee_{f(x,y)=z} \left[A(x) \wedge B(y) \right]$$

$$= \vee_{z=xy} \left[A(x) \wedge B(y) \right]$$

Next,

Support of A = {–1, 0, 1, 2} and Support of B = {2, 3, 4, 5}

Thus,

(Supp A). (Supp B) = { –2, –3, –4, –5, 0, 2, 3, 4, 5, 6, 8, 10}

Consider

$$f(A,B)(-2) = \underset{\substack{-2=xy\\(-1,2)}}{\vee}\left[A(x)\wedge B(y)\right]$$

$$= \underset{-2=xy}{\vee}\left(A(x)\wedge B(y)\right)$$

$$= \quad 0.5 \wedge 0.5$$

$$= 0.5$$

$$f(A,B)(-2) = 0.5$$

$$f(A,B)(4) = \underset{\substack{4=xy\\(1,4)(2,2)}}{\vee}\left(A(x)\wedge B(y)\right)$$

$$= \vee\left\{A(1)\wedge B(4), A(2)\wedge B(2)\right\}$$

$$= \vee\left\{0.5\wedge 0.5, 0.3\wedge 0.5\right\}$$

$$= \vee\left\{0.5, 0.3\right\}$$

$$= 0.5$$

$$f(A,B)(4) = 0.5$$

Next

$$f(A,B)(-3) = \underset{\substack{-3=xy\\(-1,3)}}{\vee}\left(A(-1)\wedge B(3)\right)$$

$$= \underset{-3=xy}{\vee}\left(A(-1)\wedge B(3)\right)$$

$$= 0.5 \wedge 1$$

$$= 0.5$$

$$f(A,B)(-4) = \underset{\substack{-4=xy\\(-1,4)}}{\vee}\left(A(x)\wedge B(y)\right)$$

$$= \bigvee_{-4=xy} \left(A(-1) \wedge B(4)\right)$$

$$= 0.5 \wedge 0.5$$

$$= 0.5$$

$$f(A,B)(-5) = \bigvee_{\substack{-5=xy \\ (-1,5)}} \left(A(-1) \wedge B(5)\right)$$

$$= 0.5 \wedge 0.3$$

$$= 0.3$$

$$f(A,B)(0) = \bigvee_{\substack{0=xy \\ (0,2),(0,3) \\ (0,4),(0,5)}} \left(A(x) \wedge B(y)\right)$$

$$= \vee\{A(0) \wedge B(2),\ A(0) \wedge B(3),\ A(0) \wedge B(4),\ A(0) \wedge B(5)\}$$

$$= \vee\{1 \wedge 0.5, 1 \wedge 1, 1 \wedge 0.5, 1 \wedge 0.3\}$$

$$= \vee\{0.5, 1, 0.5, 0.3\} = 1$$

$$f(A,B)(2) = \bigvee_{\substack{2=xy \\ (1,2)}} \left(A(x) \wedge B(y)\right) = A(1) \wedge B(2) = 0.5 \wedge 0.5 = 0.5$$

$$f(A,B)(3) = \bigvee_{\substack{3=xy \\ (1,3)}} \left(A(x) \wedge B(y)\right) = A(1) \wedge B(3) = 0.5 \wedge 1 = 0.5$$

$$f(A,B)(5) = \bigvee_{\substack{5=xy \\ (1,5)}} \left(A(x) \wedge B(y)\right) = A(1) \wedge B(5) = 0.5 \wedge 0.3 = 0.3$$

$$f(A,B)(6) = \bigvee_{\substack{6=xy \\ (2,3)}} \left(A(x) \wedge B(y)\right) = A(2) \wedge B(3) = 0.3 \wedge 1 = 0.3$$

$$f(A,B)(8) = \bigvee_{\substack{8=xy \\ (2,4)}} \left(A(x) \wedge B(y)\right) = A(2) \wedge B(4) = 0.3 \wedge 0.5 = 0.3$$

and

$$f\left(A,B\right)(10)=\underset{\substack{10=xy\\(2,5)}}{\vee}\left(A(x)\wedge B(y)\right)=A(2)\wedge B(5)=0.3\wedge 0.3=0.3$$

$$\Rightarrow f\left(A,B\right)=\frac{0.5}{-2}+\frac{0.5}{-3}+\frac{0.5}{-4}+\frac{0.3}{-5}+\frac{1}{0}+\frac{0.5}{2}+\frac{0.5}{3}+\frac{0.5}{4}+\frac{0.3}{5}+\frac{0.3}{6}+\frac{0.3}{8}+\frac{0.3}{10}$$

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UNIT - III

Operations of Fuzzy Sets

3.1 Fuzzy Complement

1. A function $C:[0,1]\to[0,1]$ defined by $C(x)=1-x$ is called a standard fuzzy complement :

   Thus,

   $\overline{A}(x)=C(A(x))=1-A(x)$, Clearly is a decreasing function.

   and C (0) = 1 and C (1) = 0

2. Definition

A function $C:[0,1]\to[0,1]$ which satisfies the following axioms.

C1 : C (0) = 1 and C (1) = 0

C2 : $a\le b\Rightarrow C(a)\ge C(b)$ $\forall a,b\in[0,1]$

i.e. C is a decreasing function, is called a fuzzy complement.

In addition, if C satisfies,

C3 : C is continuous

C4 : C is in volutive i.e. C (C (a)) = a $\forall a\in[0,1]$

Then, C is called Continuous involutive fuzzy complement.

3. Example

Let, $C:[0,1]\to[0,1]$ be given by, $C(x) = 1 - x$ (Standard fuzzy complement)

Then,

1) C (0) = 1 and C (1) = 0

2) $a \le b \Rightarrow -a \ge -b$

$\Rightarrow 1-a \ge 1-b$

$\Rightarrow C(a) \ge C(b)$

Thus, C is monotonic decreasing function. Hence, C is a fuzzy complement.

Further C is continuous and

$$C(C(a)) = C(1-a)$$

$$= 1-(1-a)$$

$$= a$$

Thus, standard fuzzy complement is a continuous and involutive fuzzy complement.

4. Examnple

Let $C:[0,1] \to [0,1]$ be defined by,

$$C(a) = \frac{1}{2}(1+\cos p\,a)$$

Show that, C is a continuous fuzzy complement but not involutive.

Ans. : Let,

$C:[0,1] \to [0,1]$ defined by,

$$C(a) = \frac{1}{2}(1+\cos p\,a)$$

$C1: C(0) = \frac{1}{2}(1+\cos 0) = 1$ and

$$C(1) = \frac{1}{2}(1+\cos p\,) = 0$$

$C2: a \le b$

$\Rightarrow \cos p\,a \ge \cos p\,b$

$$\Rightarrow 1+\cos p\,a \geq 1+\cos p\,b$$

$$\Rightarrow \frac{1}{2}(1+\cos p\,a) \geq \frac{1}{2}(1+\cos p\,b)$$

$$\Rightarrow C(a) \geq C(b)\cdot$$

Therefore, C is monotonic decreasing

Thus C is a fuzzy complement.

Further C is continuous and

$$C\left(C(a)\right) = C\left(\frac{1}{2}(1+\cos p\,a)\right)$$

$$= C\left(\frac{1}{2} 2\cos^2 \frac{p\,a}{2}\right)$$

$$= C\left(\cos^2 \frac{p\,a}{2}\right)$$

$$= \frac{1}{2}\left[1+\cos\left(\cos^2 \frac{p\,a}{2}\right)p\right]$$

$$\neq a$$

$\Rightarrow C$ is not involutive.

Thus, C is a continuous fuzzy complement but not involutive.

5. Sugeno's Class of Fuzzy Complements

Definition : A function $C:[0,1] \to [0,1]$ defined by**,**

$C(a) = \dfrac{1-a}{1+l\,a}$, $l \geq 0$ is called a Sugeno's class of fuzzy complements.

6. Example

Show that Sugeno's class of fuzzy complements is continous and involutive.

Ans. : Let $C:[0,1]\to[0,1]$ be defined by**,**

$$C(a)=\frac{1-a}{1+la}$$

$$C1: C(0)=\frac{1-0}{1+0}=1 \quad \text{and} \quad C(1)=\frac{1-1}{1+l}=0$$

$C2: a\le b$

$\Rightarrow -a\ge -b$

$\Rightarrow 1-a\ge 1-b$

$$\Rightarrow \frac{1-a}{1+la}\ge\frac{1-b}{1+lb}$$

$\Rightarrow C(a)\ge C(b)$

C is monotonic decreasing.

$C3:$

$$C(a)=\frac{1-a}{1+la}$$

$\Rightarrow$ C is continuous function.

$C4:$

Consider,

$$C(C(a))=C\left(\frac{1-a}{1+la}\right)$$

$$=\frac{1-\left(\frac{1-a}{1+la}\right)}{1+l\left(\frac{1-a}{1+la}\right)}$$

$$= \frac{\frac{1 + la - 1 + a}{1 + la}}{\frac{1 + la + l - la}{1 + la}}$$

$$= \frac{a(l+1)}{(l+1)}$$

$$= a$$

$$\Rightarrow C(C(a)) = a$$

Thus C is continuous and involutive fuzzy complement.

7. Example

Show that Yager's class of fuzzy complements given by,

$C:[0,1] \to [0,1]$, $C(a) = (1 - a^w)^{\frac{1}{w}}$, $w > 0$ is continuous and involutive.

Ans. : Let $C:[0,1] \to [0,1]$ be defined by,

$$C(a) = (1 - a^w)^{1/w}$$

$(C1): C(0) = (1-0)^{1/w} = 1$ and $C(1) = (1 - 1^w)^{1/w} = 0$

$(C2): a \le b$

$$\Rightarrow a^w \le b^w$$

$$\Rightarrow -a^w \ge -b^w$$

$$\Rightarrow 1 - a^w \ge 1 - b^w$$

$$\Rightarrow (1 - a^w)^{1/w} \ge (1 - b^w)^{1/w}$$

$$\Rightarrow C(a) \ge C(b)$$

Thus C is monotonic decreasing.

$(C3)$: Since

$$C(a) = (1 - a^w)^{1/w}$$

C is continuous.

$(C4)$:

Consider,

$$C(C(a)) = C\left((1 - a^w)^{1/w}\right)$$

$$= \left[1 - \left[(1 - a^w)^{1/w}\right]^w\right]^{\frac{1}{w}}$$

$$= \left[1 - (1 - a^w)\right]^{\frac{1}{w}}$$

$$= \left[a^w\right]^{\frac{1}{w}}$$

$$= a$$

$$C(C(a)) = a$$

Thus Yager's class of fuzzy complements is continuous and involutive ($w > 0$).

8. Definition

Let $C : [0,1] \rightarrow [0,1]$ be a fuzzy complement.

An element $a \in [0,1]$ is called a equilibrium point of C if $C(a) = a$.

If C is standard fuzzy complement then,

$$C\left(\frac{1}{2}\right) = 1 - \frac{1}{2} = \frac{1}{2}.$$

Hence $\frac{1}{2}$ is equilibrium point of C.

9. Example

If a fuzzy complement C is defined by,

$C(a) = \frac{1}{2}(1 + \cos p\, a)$, find the equilibriumj point.

Ans. : If $a \in [0,1]$ is the equilibrium point. Then,

$$C(a) = a$$

$$\Rightarrow \frac{1}{2}(1 + \cos p\, a) = a$$

$$\Rightarrow (1 + \cos p\, a) = 2a$$

$$\Rightarrow 2\cos^2 \frac{p\, a}{2} = 2a$$

$$\Rightarrow \cos^2 \frac{p\, a}{2} = a$$

$$\Rightarrow \cos \frac{p\, a}{2} = \sqrt{a}$$

$$\Rightarrow a = \frac{1}{2}$$

$\Rightarrow \frac{1}{2}$ is the equilibrium point of C.

We write,

$$e = \frac{1}{2}$$

10. Example

Let a fuzzy complement C is defined by,

$$C(a) = \frac{1-a}{1+l\,a}, \quad l \geq 0$$

Find the equilibrium point.

Ans. : If $e \in [0,1]$ is the equilibrium point then,

$$C(e) = e$$

$$\Rightarrow \frac{1-e}{1+l\,e} = e$$

$$\Rightarrow 1-e = e + l\,e^2$$

$$\Rightarrow l\,e^2 + 2e - 1 = 0$$

$$\Rightarrow e = \frac{-2 \pm \sqrt{4+4l}}{2l}$$

$$\Rightarrow e = \frac{-2 \pm 2\sqrt{1+l}}{2l}$$

$$\Rightarrow e = \frac{-1 \pm \sqrt{1+l}}{l}$$

But, $e \in [0,1]$

$$\Rightarrow e = \frac{-1 + \sqrt{1+l}}{l}$$

i.e. $e = \dfrac{\sqrt{1+l} - 1}{l}$

11. Example

If a fuzzy complement C is defined by

$$C(a) = (1 - a^w)^{1/w}, \quad w > 0$$

Find the equilibrium point.

Ans. : If $e \in [0,1]$ is the equilibrium point then,

$$C(e) = e$$

$$\Rightarrow (1 - e^w)^{1/w} = e$$

$$\Rightarrow 1 - e^w = e^w$$

$$\Rightarrow 1 = 2e^w$$

$$\Rightarrow \frac{1}{2} = e^w$$

$$\Rightarrow e^w = \frac{1}{2}$$

$$\Rightarrow (e^w)^{1/w} = \left(\frac{1}{2}\right)^{1/w}$$

$$\Rightarrow e = \left(\frac{1}{2}\right)^{1/w}$$

$$\Rightarrow e = \sqrt[w]{\frac{1}{2}}$$

is the equilibrium point.

12. Theorem

Every fuzzy complement has atmost one equilibrium point.

Proof : Let C be a fuzzy complement. If a is the equilibrium point of C then.

$C(a) = a$

i.e. $C(a) - a = 0$

In general, we show that, the equation, $C(a) - a = b$ has two solutions of this equation.

Then,

$C(a_1) - a_1 = b$ and $C(a_2) - a_2 = b$

$\Rightarrow C(a_1) - a_1 = C(a_2) - a_2$

Now,

$a_1, a_2 \in [0,1]$

Then, we assume that, $a_1 < a_2$

But, $a_1 < a_2 \Rightarrow C(a_1) \geq C(a_2)$

Also, $a_1 < a_2 \Rightarrow -a_1 > -a_2$

$\Rightarrow C(a_1) - a_1 > C(a_2) - a_2$

Which is a contradiction and hence, the equation, $C(a) - a = b$ cannot have two distinct solutions.

i.e. $C(a) - a = b$ has atmost one solution.

In particular, if $b = 0$, then the equation $C(a) - a = 0$ has atmost one solution.

13. Theorem

Assume that, the fuzzy complement C has an equilibrium point, e_c then,

1) $a \leq C(a)$ iff $a \leq e_c$

2) $a \geq C(a)$ iff $a \geq e_c$

Proof : If e_c is the equilibrium point of C, then

$C(e_c) = e_c$

Also, by theorem, every fuzzy complement has atmost one equilibrium point.

$\Rightarrow e_c \in [0,1]$ is unique.

Therefore, for any element $a \in [0,1]$ either,

$a < e_c$ or $a = e_c$ or $a > e_c$

1) If $a < e_c \Rightarrow C(a) \geq C(e_c)$

$\Rightarrow C(a) \geq e_c > a$ $\quad (a < e_c)$

$\Rightarrow C(a) > a$

Next, if

2) $a = e_c \Rightarrow C(a) = C(e_c) = e_c = a$

$\Rightarrow C(a) = a$

and if

3) $a > e_c \Rightarrow C(a) \leq C(e_c)$

$\Rightarrow C(a) \leq e_c < a$ $\quad (a > e_c)$

$\Rightarrow C(a) < a$

Thus, $a \leq e_c \Rightarrow C(a) \geq a$ From (1) and (2)

and $a > e_c \Rightarrow C(a) \leq a$ From (2) and (3)

Conversely, fo any $a \in [0,1]$,

$C(a) \in [0,1]$

Therefore, we have either, $a < C(a)$ or $a = C(a)$ or $a > C(a)$

1) If $a < C(a)$ or $a > C(a)$

Then clearly, $a \neq C(a)$

$\Rightarrow a \neq e_c$

Therefore, we have,

$a < e_c$ or $a > e_c$

(a) Now, if $a < C(a)$ and $a > e_c$ holds then,

$C(a) \leq C(e_c) = e_c$

$\Rightarrow C(a) \leq e_c$

$\Rightarrow a < C(a) \leq e_c$ $(\because a < C(a))$

$\Rightarrow a < e_c$

Which is a contradiction.

Thus,

$a < C(a) \Rightarrow a < e_c$

Simillarly, if

(b) $a > C(a)$ and $a < e_c$ holds

$\Rightarrow C(a) \geq C(e_c)$

$\Rightarrow C(a) \geq e_c$

$\Rightarrow a > C(a) \geq e_c$

$\Rightarrow a > e_c$

Which is a contradiction

Hence

$a > C(a) \Rightarrow a > e_c$

2) Also, if $a = C(a)$ then $a = e_C$,

Therefore,

$a \leq C(a) \Rightarrow a \leq e_c$ (by 1(a) and (2)) and $a \geq C(a) \Rightarrow a \geq e_c$ (by 1(b) and (2))

Thus, we have,

$a \leq C(a)$ iff $a \leq e_c$

$a \geq C(a)$ iff $a \geq e_c$

14. Theorem

If C is a continuous fuzzy complement, then C has unique equilibrium point.

Proof : For equilibrium point, we have,

$C(a) = a$

i.e. The equilibrium point is a solution of

$C(a) - a = 0$

We consider, a more general equation,

$C(a) - a = b$, $a \in [0,1]$

1) For $a = 0$

We get,

$C(0) - 0 = b$

$\Rightarrow 1 = b$ $(\because C(0) = 1)$

i.e. $b = 1$

2) And for $a = 1$, we get,

$C(1) - 1 = b$

$\Rightarrow 0 - 1 = b$ $(\because C(1) = 0)$

$\Rightarrow b = -1$

Thus if $a \in [0,1]$ then, $b \in [-1,1]$

Next, if C is continuous function.

then, $C(a)-a$ is also continuous.

Also, $C(a)-a$ takes the values in the interval [–1, 1].

Hence by intermediate value theorem for every $b \in [-1,1]$ there exists at least one solution $a \in [0,1]$, such that

$$C(a)-a = b$$

In particular, if $b = 0$ then, the equation $C(a)-a=0$ has at least one solution in [0, 1].

i.e. $C(a)=a$ for some $a \in [0,1]$.

If $C(a)=a$, then, a is an equilibrium point and by theorem a is unique.

Thus, if C is continuous fuzzy complement, then it has unique equilibrium point.

15. Dual Element

Let, C be a fuzzy complement. For $a \in [0,1]$, the element, ${}^{d}a \in [0,1]$ is called a dual element of a w.r.t. C. If

$$C\left({}^{d}a\right) - {}^{d}a = a - C(a)$$

1) If C is standard fuzzy complement, then, for any $a \in [0,1]$.

$$C\left({}^{d}a\right) - {}^{d}a = a - C(a)$$

$$\Rightarrow 1 - {}^{d}a - {}^{d}a = a - (1-a)$$

$$\Rightarrow 1 - 2\,{}^{d}a = 2a - 1$$

$$\Rightarrow 2\,{}^{d}a = 2 - 2a$$

$\Rightarrow 2\,{}^d a = 2(1-a)$

$\Rightarrow {}^d a = 1-a$

$\Rightarrow {}^d a = C(a)$

i.e. ${}^d a = C(a) \qquad \forall a \in [0,1]$

Hence,

$${}^d 0 = 1,\ {}^d 1 = 0,\ {}^d\left(\frac{1}{2}\right) = \frac{1}{2}.$$

16. Theorem

For any $a \in [0,1]$,

${}^d a = C(a)$ iff $C(C(a)) = a$ i.e. C is involutive.

Proof : Let,

${}^d a = C(a) \qquad \forall a \in [0,1]$

$\Rightarrow C({}^d a) = C(C(a)) \qquad \forall a \in [0,1] \qquad$ .... (1)

By definition, of dual element of a,

$C({}^d a) - {}^d a = a - C(a)$

$\Rightarrow C(C(a)) - C(a) = a - C(a) \qquad$ by (1)

$\Rightarrow C(C(a)) = a \qquad \forall a \in [0,1]$

i.e. C is involutive.

Conversely, let

$$C(C(a)) = a \qquad \forall a \in [0,1]$$

Then,

$$C(C(a)) - C(a) = a - C(a)$$

$\Rightarrow {}^{d}a = C(a)$ (definition of dual.)

Thus

${}^{d}a = C(a)$ iff $C(C(a)) = a$

17. Increasing Generator Function

A continuous function $g:[0,1] \to \mathbb{R}$ is called an increasing generator if g is strictly increasing and $g(0) = 0$.

[*g* is strictly increasing iff $a < b \Rightarrow g(a) < g(b)$]

18. Psuedo-Inverse of the Increasing Generator

Let $g:[0,1] \to \mathbb{R}$ be an increasing generator. A function $g^{(-1)}:\mathbb{R} \to [0,1]$ is called a "psuedo-inverse of *g*" if

$$g^{(-1)}(a) = \begin{cases} 0 & \text{if } a \in (-\infty, 0) \\ g^{-1}(a) & \text{if } a \in [0, g(1)] \\ 1 & \text{if } a \in (g(1), \infty) \end{cases}$$

19. Theorem

Let $C:[0,1] \to [0,1]$ be a function then C is involutive fuzzy complement iff there exists a continuous function $g:[0,1] \to \mathbb{R}$ which is a increasing generator and $C(a) = g^{(-1)}\left(g(1) - g(a)\right)$.

Proof : First, we prove the inverse implication, that means to prove C is involutive fuzzy complement.

Since *g* be a continuous function from [0, 1] to $\mathbb{R}$ such that $g(0) = 0$ and *g* is strictly increasing the pseudo inverse of *g* is defined by,

$$g^{(-1)}(a) = \begin{cases} 0 & \text{if } a \in (-\infty, 0) \\ g^{-1}(a) & \text{if } a \in [0, g(1)] \\ 1 & \text{if } a \in (g(1), \infty) \end{cases}$$

where g^{-1} is the ordinary inverse of g.

We have to prove that, C is involutive fuzzy complement, it means to show that the axioms C_1, C_2, C_3 and C_4 are satisfied.

C_1 :

$$C(0) = g^{(-1)}(g(1) - g(0)) \qquad (\because g(0) = 0)$$

$$= g^{(-1)}(g(1) - 0)$$

$$= 1$$

And

$$C(1) = g^{(-1)}(g(1) - g(1))$$

$$= g^{(-1)}(0)$$

$$= 0$$

Hence, $C(0) = 1$ and $C(1) = 0$.

C_2 : If $a < b$ $\qquad \forall a, b \in [0,1]$

Then, $g(a) < g(b)$, since g is strictly increasing,

$$\Rightarrow g(1) - g(a) > g(1) - g(b)$$

$$\Rightarrow g^{-1}[g(1) - g(a)] > g^{-1}[g(1) - g(b)]$$

$$\Rightarrow C(a) > C(b)$$

Therefore, C is strictly decreasing function.

Next,

C_3 : Since g is continuous function, C is also continuous function.

C_4 : For any $a,b \in [0,1]$

$$C(C(a)) = g^{(-1)}\left[g(1) - g(C(a))\right]$$

$$= g^{(-1)}\left[g(1) - g\left(g^{-1}(g(1) - g(a))\right)\right]$$

$$= g^{(-1)}\left[g(1) - g(1) + g(a)\right]$$

$$= g^{-1}(g(a))$$

$$= a$$

Thus, $C(C(a)) = a$

Thus, from axioms C_1, C_2, C_3 and C_4 , C is an involutive fuzzy complement.

Conversely, we have to prove that, there exists a continuous function $g:[0,1] \to \mathbb{R}$ which is increasing generator and $C(a) = g^{(-1)}\left[g(1) - g(a)\right]$.

Since C is involutive fuzzy complement. By theorem C has unique equilibrium point say e_c that is $C(e_c) = e_c$, where $e_c \in [0,1]$.

Let, $h:[0,e_c] \to [0,b]$ be any continuous, strictly increasing bijection such that $h(0) = 0$ and $h(e_c) = b$, where b any fixed (positive) real number.

Now, we define a function $g:[0,1] \to \mathbb{R}$ by

$$g(a) = \begin{cases} h(a) & a \in [o, e_c] \\ 2b - h(C(a)) & a \in [e_c, 1] \end{cases}$$

Obviously, $g(0) = h(0) = 0$ and g is continuous as well as strictly increasing, since h is continuous and strictly increasing.

Now pseudoinverse of g is given by,

$$g^{(-1)}(a)=\begin{cases}0 & \text{if } a\in(-\infty,0)\\ h^{-1}(a) & \text{if } a\in[0,b]\\ C\left(h^{-1}(2b-a)\right) & \text{if } a\in[b.2b]\\ 1 & \text{if } a\in(2b,\infty)\end{cases}$$

Next, we have to show that g satisfies

$$C(a)=g^{-1}[g(1)-g(0)]$$

Now consider for any $a\in[0,1]$, if $a\in[0,e_c]$.

$$g^{-1}[g(1)-g(a)]=g^{-1}[g(1)-h(a)]$$

$$=g^{(-1)}[2b-h(a)] \qquad (\because g(1)=2b)$$

$$=C\left(h^{-1}[2b-(2b-h(a))]\right)$$

$$\therefore g^{-1}[g(1)-g(a)]=C\left(h^{-1}[2b-2b+h(a)]\right)$$

$$=C\left(h^{-1}(h(a))\right)$$

$$=C(a)$$

Thus, $C(a)=g^{-1}[g(1)-g(a)]$

Now consider the another condition if $a\in(e_c,1]$ consider

$$g^{(-1)}[g(1)-g(a)]=g^{-1}\left[2b-\left(2(b)-h(C(a))\right)\right]$$

$$=g^{-1}\left[2b-2b+h(C(a))\right]$$

$$=g^{-1}\left[h(C(a))\right]$$

$$=h^{-1}\left[h(C(a))\right]$$

$$=C(a)$$

Thus, $C(a) = g^{-1}[g(1) - g(a)]$

Thus, in both the condition g satisfies the

$C(a) = g^{-1}[g(1) - g(a)]$

Hence, $C : [0,1] \to [0,1]$ be a function then C is involutive fuzzy complement iff there exists a continuous function $g : [0,1] \to \mathbb{R}$ which is increasing generator and

$C(a) = g^{-1}[g(1) - g(a)]$

20. Example

Let, $g : [0,1] \to \mathbb{R}$ defined by, $g(a) = a$

Then, g is continuous function such that, $g(0) = 0$

Also, $a < b \Rightarrow g(a) < g(b)$

Thus g is an increasing generator function.

Also, the psuedo-inverse of g is given by,

$g^{(-1)} : \mathbb{R} \to [0,1]$

$$g^{(-1)}(a) = \begin{cases} 0 & \text{if } a \in (-\infty, 0) \\ g^{-1}(a) & \text{if } a \in [0, g(1)] \\ 1 & \text{if } a \in (1, \infty) \end{cases} \qquad (\because g(1) = 1)$$

$$g^{(-1)}(a) = \begin{cases} 0 & \text{if } a \in (-\infty, 0) \\ a & \text{if } a \in [0,1] \\ 1 & \text{if } a \in (1, \infty) \end{cases} \qquad (\because g(a) = a)$$

Define,

$C : [0,1] \to [0,1]$ by

$$C(a) = g^{-1}[g(1) - g(a)]$$

$$= g^{-1}(1-a)$$

$$C(a) = 1 - a$$

which is a standard fuzzy complement.

21. Example

Let $g_l : [0,1] \to \mathbb{R}$ be a function defined by

$$g_l(a) = \frac{1}{l}\log(1 + l\,a), \; l > 0$$

Show that, g_l is an increasing generator and obtain the fuzzy complement C_l generated by g_l.

Ans. : Let, $g_l : [0,1] \to \mathbb{R}$ be defined by,

$$g_l(a) = \frac{1}{l}\log(1 + l\,a), \; l > 0$$

1) Then, g_l is a continuous function such that

$$g_l(0) = \frac{1}{l}\log_e(1) = 0$$

$$\Rightarrow g_l(0) = 0$$

2) If $a < b \Rightarrow l\,a < l\,b, \; \forall\, l > 0$

$$\Rightarrow 1 + l\,a < 1 + l\,b$$

$$\Rightarrow \log(1 + l\,a) < \log(1 + l\,b)$$

$$\Rightarrow \frac{1}{l}\log(1 + l\,a) < \frac{1}{l}\log(1 + l\,b)$$

$$\Rightarrow g_l(a) < g_l(b)$$

Hence, g is strictly increasing.

i.e. g_l is an increasing generator.

3) Let, $g_l^{(-1)} : \mathbb{R} \to [0,1]$ be the psuedo inverse defined by,

$$g_l^{(-1)}(a) = \begin{cases} 0 & \text{if } a \in (-\infty, g(0)) \\ g_l^{-1}(a) & \text{if } a \in [g(0), g(1)] \\ 1 & \text{if } a \in (g(1), \infty) \end{cases}$$

Here, $g_l(0) = 0$

and $g_l(1) = \frac{1}{l}\log(1+l)$

And,

Let, $g_l^{-1}(a) = b$

$\Rightarrow a = g_l(b)$

$\Rightarrow a = \frac{1}{l}\log(1+lb)$

$\Rightarrow la = \log_e(1+lb)$

$\Rightarrow e^{la} = 1+lb$

$\Rightarrow b = \frac{e^{la}-1}{l}, \; l > 0$

i.e. $g_l^{-1}(a) = \frac{e^{la}-1}{l}$

Hence, The pseudo inverse is given by,

$$g_l^{(-1)}(a) = \begin{cases} 0 & \text{if } a \in (-\infty, 0) \\ \dfrac{e^{la} - 1}{l} & \text{if } a \in [0, g(1)] \\ 1 & \text{if } a \in (g(1), \infty) \end{cases}$$

4) Now, define a function, $C : [0,1] \to [0,1]$ by,

$$C(a) = g_l^{(-1)}(g_l(1) - g_l(a))$$

$$\Rightarrow C(a) = g_l^{(-1)}\left[\frac{1}{l}\log(1+l) - \frac{1}{l}\log(1+la)\right]$$

$$\Rightarrow C(a) = g_l^{(-1)}\left[\frac{1}{l}\log\left(\frac{1+l}{1+la}\right)\right]$$

$$\Rightarrow C(a) = g_l^{(-1)}\left[\log\left(\frac{1+l}{1+la}\right)^{\frac{1}{l}}\right]$$

$$\Rightarrow C(a) = \frac{e^{l\left[\log\left(\frac{1+l}{1+la}\right)^{\frac{1}{l}}\right]} - 1}{l}$$

$$\Rightarrow C(a) = \frac{e^{\log_e\left(\frac{1+l}{1+la}\right)} - 1}{l}$$

$$\Rightarrow C(a) = \frac{\frac{1+l}{1+la} - 1}{l}$$

$$\Rightarrow C(a) = \frac{1+l-1-la}{l(1+la)}$$

$$\Rightarrow C(a) = \frac{l - l\,a}{l\,(1 + l\,a)}$$

$$\Rightarrow C(a) = \frac{l\,(1 - a)}{l\,(1 + l\,a)}$$

$$\Rightarrow C(a) = \frac{1 - a}{1 + l\,a}$$

which is a Sugeno's class of fuzzy complement.

22. Example :

Show that, a function $g_w : [0,1] \to \mathbb{R}$ defined by $g_w(a) = a^w$ ($w > 0$) is an increasing generator and obtain the fuzzy complement generated by g_w.

Ans. : We have,

$$g_w(a) = a^w \quad (w > 0)$$

1) Clearly, g_w is continuous and $g_w(0) = 0$

2) And if $a < b$

$\Rightarrow a^w < b^w$

$\Rightarrow g_w(a) < g_w(b)$

$\Rightarrow g_w$ is strictly increasing.

3) The psuedo inverse of g is defined by,

$g_w^{(-1)} : \mathbb{R} \to [0,1]$

$$g_w^{(-1)}(a) = \begin{cases} 0 & \text{if } a \in (-\infty, g(0)) \\ g_w^{-1}(a) & \text{if } a \in [g(0), g(1)] \\ 1 & \text{if } a \in (g(1), \infty) \end{cases}$$

Here, $g_w(0) = 0$, $g_w(1) = 1$.

Let,

$g_w^{-1}(a) = b$

$\Rightarrow a = g_w(b)$

$\Rightarrow a = b^w$

$\Rightarrow b = a^{\frac{1}{w}}$

$\Rightarrow g_w^{-1}(a) = a^{\frac{1}{w}}$ .... (*)

Hence, the psuedo inverse is given by,

$$g_w^{(-1)}(a) = \begin{cases} 0 & \text{if } a \in (-\infty, 0) \\ a^{\frac{1}{w}} & \text{if } a \in [0,1] \\ 1 & \text{if } a \in (1, \infty) \end{cases}$$

4) Define, $C:[0,1] \to [0,1]$ by,

$C(a) = g_w^{(-1)}\left[g_w(1) - g_w(a)\right]$

$= g_w^{(-1)}\left[1 - a^w\right]$

$\therefore C(a) = (1 - a^w)^{1/w}, \ w > 0$ by (*)

This is the Yager's class of fuzzy complements and the function, $g_w(a) = a^w, \ w > 0$ is the generator of this class.

23. Example

Show that the function $g_g(a) = \dfrac{a}{g + (1-g)a}$, $g > 0$ is an increasing generator and the class of fuzzy complements generated by g_g is

$$C_g(a) = \frac{g^2(1-a)}{a + g^2(1-a)}$$

Solution : Let, $g_g : [0,1] \to \mathbb{R}$ is defined by,

$$g_g(a) = \frac{a}{g + (1-g)a}, \; g > 0 \qquad \text{..... (1)}$$

Then, clearly g_g is continuous function such that $g_g(0) = 0$.

Now, differentiating equation (1) w.r.t. *a*, we get,

$$g_g'(a) = \frac{(g + (1-g)a)(1) - a[0 + (1-g)1]}{[g + (1-g)a]^2}$$

$$= \frac{g + (1-g)a - (1-g)a}{[g + (1-g)a]^2}$$

$$= \frac{g}{[g + (1-g)a]^2}, \; g > 0$$

$$= \frac{g}{[g + (1-g)a]^2} > 0 \qquad \text{Since } g > 0$$

Thus, $g_g'(a) > 0$

$\Rightarrow g_g$ is strictly increasing function.

OR

For any $a, b \in [0,1]$,

Case - 1 : If $a = 0$, $b \neq 0$ or b = 0 or $a \neq 0$ or a = 0, b = 0

In the above any condition,

$g_g(a) \leq g_g(b)$ if $a \leq b$.

Case - 2 : If $a, b \in [0,1]$ then,

$$a < b \Rightarrow \frac{1}{a} > \frac{1}{b}, \qquad a \neq 0 \text{ and } b \neq 0$$

$$\Rightarrow \frac{g}{a} > \frac{g}{b} \qquad g > 0$$

$$\Rightarrow \frac{g}{a} + (1-g) > \frac{g}{b} + (1-g)$$

$$\Rightarrow \frac{g + (1-g)a}{a} > \frac{g + (1-g)b}{b}$$

$$\Rightarrow \frac{a}{g + (1-g)a} < \frac{b}{g + (1-g)b}$$

$$\Rightarrow g_g(a) < g_g(b)$$

Hence, $g_g(a) < g_g(b)$ when $a < b$.

Thus, g_g is strictly increasing function.

Therefore, g_g is an increasing generator.

Now, $g_g(a) = \dfrac{a}{g + (1-g)a}$, $g > 0$

Let, $g_g(a) = b \Rightarrow a = g_g^{-1}(b)$

$$\Rightarrow \frac{a}{g + (1-g)a} = b$$

$$\Rightarrow a = gb + (1-g)ab$$

$$\Rightarrow a-(1-g)\,ab = g\,b$$

$$\Rightarrow a\left[1-(1-g)\,b\right] = g\,b$$

$$\Rightarrow a = \frac{g\cdot b}{1-(1-g)\,b},\ g>0$$

But, $a = g_g^{-1}(b)$

Therefore, $g_g^{-1}(b) = \dfrac{g\cdot b}{1-(1-g)\cdot b}, g>0$

i.e. $g_g^{-1}(a) = \dfrac{g\cdot a}{1-(1-g)\cdot a}, g>0$

Now, we define a function $C_g : [0,1]\rightarrow[0,1]$ by

$$C_g(a) = g_g^{-1}\left(g_g(1)-g_g(a)\right)$$

$$= g_g^{-1}\left[\frac{1}{g+(1-g)1}-\frac{a}{g+(1-g)\,a}\right]$$

$$= g_g^{-1}\left[\frac{1}{1}-\frac{a}{g+(1-g)\,a}\right]$$

$$= g_g^{-1}\left[\frac{g+(1-g)\,a-a}{g+(1-g)\,a}\right]$$

$$= g_g^{-1}\left[\frac{g-g\,a}{g+(1-g)\,a}\right]$$

$$= \frac{g\left[\dfrac{g-g\,a}{g+(1-g)\,a}\right]}{\left[1-(1-g)\cdot\left(\dfrac{g-g\,a}{g+(1-g)\,a}\right)\right]}$$

$$= \frac{\dfrac{g^2(1-a)}{(g+(1-g)a)}}{\dfrac{g+a-ga-(1-g)(g-ga)}{(g+(1-g)a)}}$$

$$= \frac{g^2(1-a)}{g+a-ga-g+ga+g^2-g^2a}$$

$$= \frac{g^2(1-a)}{a-g^2a+g^2}$$

$$= \frac{g^2(1-a)}{a+g^2(1-a)}, \quad g>0$$

Thus, $C_g(a) = \dfrac{g^2(1-a)}{a+g^2(1-a)}$, $g>0$

which is required class of fuzzy complement generated by the function g_g .

24. Theorem

Let $C:[0,1]\to[0,1]$ be a function, then C is a fuzzy complement iff there exist a continuous function $f:[0,1]\to\mathbb{R}$ s.t. $f(1)=0$, f is strictly decreasing and

$$C(a) = f^{(-1)}\left(f(0)-f(a)\right).$$

Proof : Let $C:[0,1]\to[0,1]$ be a fuzzy complement. Then, there exists an increasing generator g with $g(0)=0$ and $C(a)=g^{(-1)}\left(g(1)-g(a)\right)$.

Define, a function, $f:[0,1]\to\mathbb{R}$ by

$f(a)=g(1)-g(a)$ ..... (1)

Then,

1) $f(1)=g(1)-g(1)$

$\Rightarrow f(1)=0$ ..... (2)

And for,

2) $a<b$

$\Rightarrow g(a)<g(b)$

$\Rightarrow -g(a)>-g(b)$

$\Rightarrow g(1)-g(a)>g(1)-g(b)$

$\Rightarrow f(a)>f(b)$

Hence, f is strictly decreasing.

i.e. f is a decreasing generator.

3) To find psuedo inverse.

Let, $f^{-1}(a)=b$

$\Rightarrow a=f(b)$

$\Rightarrow a=g(1)-g(b)$

$\Rightarrow g(b)=g(1)-a$

$\Rightarrow b=g^{-1}\left(g(1)-a\right)$

$\Rightarrow f^{-1}(a)=g^{-1}\left(g(1)-a\right)$

But,

$f(a)=g(1)-g(a)$

$\Rightarrow f(0)=g(1)-g(0)$

$\Rightarrow f(0) = g(1)$

Hence,

$$f^{-1}(a) = g^{-1}\left(f(0) - a\right) \qquad \text{.... (3)}$$

Now, The fuzzy complement C is given by,

$$C(a) = g^{-1}\left(g(1) - g(a)\right) = g^{-1}\left(f(0) - g(a)\right)$$

$$\Rightarrow C(a) = f^{-1}\left(g(a)\right) \qquad \text{by (3)}$$

$$\Rightarrow C(a) = f^{-1}\left(g(1) - f(a)\right) \qquad \text{by (1)}$$

$$\Rightarrow C(a) = f^{-1}\left(f(0) - f(a)\right) \qquad (\because g(1) = f(0))$$

Conversely,

Suppose that there exists a stricty decreasing generator f such that

$f(1) = 0$ and $C(a) = f^{-1}\left(f(0) - f(a)\right)$

Then, we prove that the function C is a fuzzy complement which satisfies C_1, C_2, C_3 and C_4.

Now,

C (1) : Consider,

$$C(0) = f^{-1}\left(f(0) - f(0)\right)$$

$$= f^{-1}(0)$$

But, $f(1) = 0 \Rightarrow f^{-1}(0) = 1$ by (2)

$\Rightarrow C(0) = 1$

And

$$C(1) = f^{-1}\left(f(0) - f(1)\right)$$

$$= f^{-1}\left(f(0)\right) \qquad (\because f(1) = 0)$$

$C(1) = 0$

C (2) :

Let, $a_1 \le a_2$

$\Rightarrow f(a_1) \ge f(a_2)$ (∵ f is decreasing)

$\Rightarrow -f(a_1) \le -f(a_2)$

$\Rightarrow f(0) - f(a_1) \le f(0) - f(a_2)$

$\Rightarrow f^{-1}(f(0) - f(a_1)) \ge f^{-1}(f(0) - f(a_1))$ (∵ f is decreasing,

$\Rightarrow f^{-1}$ is decreasing)

$\Rightarrow C(a_1) \ge C(a_2)$

Thus, C is monotonic decreasing function.

Hence C is fuzzy complement.

C (3) :

Since, f and f^{-1} are continuous functions C is also continuous.

C (4) : Consider,

$$C(C(a)) = C\left[f^{-1}(f(0) - f(a))\right]$$

$$= f^{-1}\left[f(0) - f\left(f^{-1}(f(0) - f(a))\right)\right]$$

$$= f^{-1}[f(0) - f(0) + f(a)]$$

$$= f^{-1}[f(a)]$$

$$= a$$

$\Rightarrow C(C(a)) = a$

Hence C is continuous involutive fuzzy complement.

3.2 Fuzzy Intersection / t-norms

1. Definition

A fuzzy intersection or a t-norm *i* is a binary operation. $i:[0,1]\times[0,1]\to[0,1]$ which satisfies the following axioms.

(i1) : $i(a,1)=a$ $\quad \forall a\in[0,1]$ (Boundary condition)

(i2) : $b\le d \Rightarrow i(a,b)\le i(a,d)$ (Monotone property)

(i3) : $i(a,b)=i(b,a)$ (Commulative property)

(i4) : $i(a,i(b,d))=i(i(a,b),d)$ (Associative property)

In addition, if the following properties holds

(i5) : i is continuous

(i6) : $i(a,a)<a$ (Subidempotent)

(i7) : $a_1<a_2$, $b_1<b_2$ then, $i(a_1,b_1)<i(a_2,b_2)$ (Strict monotone property)

Then the t-norm *i* is called a strict Archimedean t-norm. If only (i5) and (i6) holds then it is called an Archimedean t-norm.

2. Theorem

The standard fuzzy intersection is the only t-norm which is idempotent.

Proof : We know that, the standard fuzzy intersection is defined by,

$$(A\cap B)(x)=\min(A(x),B(x))$$

Let, $i\equiv\min$

Then,

(i1) : $i(a,1)=\min(a,1)=a$ $\quad \forall a\in[0,1]$

(i2) : $b\le d\Rightarrow \min(a,b)\le\min(a,d)$

$$\Rightarrow i(a,b)\le i(a,d)$$

i.e. *i* is monotone.

(i3) : $i(a,b) = \min(a,b) = \min(b,a) = i(b,a)$

i.e. $i(a,b) = i(b,a) \qquad \forall a,b$

Thus, i is commutative.

(i4) : $i(a, i(b,d)) = \min\{a, \min\{b.d\}\}$

$$= a \wedge (b \wedge d)$$

$$= (a \wedge b) \wedge d$$

$$= \min\{\min\{a,b\}, d\}$$

$$= i\{i\{a,b\}, d\}$$

$$\Rightarrow i(a, i(b,d)) = i(i(a,b), d)$$

i.e. i is associative

Hence $i \equiv \min$ is a t-norm.

Also,

$$i(a,a) = \min(a,a) = a \qquad \forall a \in [0,1]$$

Hence $i \equiv \min$ is an idempotent t-norm.

Now, if i' is any other t-norm, which is idempotent

i.e. $i'(a,a) = a \qquad \forall a \in [0,1]$

Then,

1) For any $a,b \in [0,1]$ with $0 \le a \le b \le 1$.

$$a = i'(a,a) \le i'(a,b) \le i'(a,1) = a$$

$$\Rightarrow i'(a,b) = a = \min(a,b)$$

i.e. $i' = \min$

Simillarly,

2) If $0 \le b \le a \le 1$,

Then,

$$b = i'(b,b) \le i'(b,a) \le i'(b,1) = b$$

$$\Rightarrow i'(b,a) = i'(a,b) = b = \min(a,b)$$

$$\Rightarrow i' \equiv \min$$

Thus, min is the only t-norm which is idempotent.

3. Example

The following functions are examples of t-norms -

1) Standard intersection

$$i(a,b) = \min(a,b) = a \wedge b$$

2) Algebraic Product

$$i(a,b) = a \cdot b$$

3) Bounded difference

$$i(a,b) = \max\{0, a+b-1\}$$

4) Drastic Intersection

$$i(a,b) = \begin{cases} a & \text{if } b = 1 \\ b & \text{if } a = 1 \\ 0 & \text{otherwise} \end{cases}$$

This t-norm is denoted by, i_{min}.

4. Theorem

For any t-norm i, $i_{\min}(a,b) \le i(a,b) \le \min(a,b)$

Proof : Let *i* be any other t-norm, which satisfies i_1, i_2, i_3 and i_4.

Then, $\forall b \in [0,1] \Rightarrow b \le 1$

$\Rightarrow i(a,b) \leq i(a,1) = a$

$\Rightarrow i(a,b) \leq a \qquad\qquad \forall a \in [0,1]$

Similarly, $\forall a \in [0,1] \Rightarrow a \leq 1$

$\Rightarrow i(a,b) \leq i(1,b) = b$

$\Rightarrow i(a,b) \leq b \qquad\qquad \forall b \in [0,1]$

Thus, $i(a,b) \leq a$ and $i(a,b) \leq b$

$\Rightarrow i(a,b) \leq \min(a,b)$ ..... (1)

Next,

For any $a \in [0,1] \Rightarrow 0 \leq a$

$$\Rightarrow i(b,0) \leq i(b,a) = i(a,b)$$

But

$i(b,0) \leq \min(b,0) = 0 \Rightarrow i(b,0) = 0$

Thus,

$$0 \leq i(a,b) \qquad\qquad a,b \in [0,1]$$

But then,

$$\begin{aligned} i(a,b) &= i(a,1) = a && \text{if } b = 1 \\ &= i(1,b) = b && \text{if } a = 1 \\ &\geq 0 && \text{otherwise} \end{aligned}$$

But we have,

$$i_{\min}(a,b) = \begin{cases} a & \text{if } b = 1 \\ b & \text{if } a = 1 \\ 0 & \text{otherwise} \end{cases}$$ (Drastic intersection)

Hence,

$$i_{\min}(a,b) \le i(a,b) \qquad \forall a,b \in [0,1] \qquad \text{..... (2)}$$

From (1) and (2)

We get,

$$i_{\min}(a,b) \le i(a,b) \le \min(a,b)$$

5. Decreasing Generator

A decreasing generator is a continuous and strictly decreasing function $f:[0,1] \to \mathbb{R}$ s.t. $f(1)=0$ and its psuedo inverse is given by,

$$f^{(-1)}(a) = \begin{cases} 1 & \text{if } a \in (-\infty, 0) \\ f^{-1}(a) & \text{if } a \in [f(1), f(0)] \\ 0 & \text{if } a \in (f(0), \infty) \end{cases}$$

Clearly, $f^{(-1)}: \mathbb{R} \to [0,1]$

6. Example

Let, $f:[0,1] \to \mathbb{R}$ be defined by, $f(a) = 1 - a^p$, $p > 0$.

Show that, f is decreasing generator and find its psuedoinverse.

Proof : Let, $f:[0,1] \to \mathbb{R}$ be defined by,

$$f(a) = 1 - a^p, \; p > 0$$

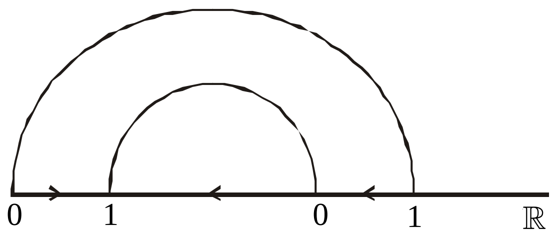
1) Clearly, f is continuous ($\because$ f is a polynomial function)

2) $f(1) = 0$

3) Further, $a < b$

$\Rightarrow a^p < b^p$

$\Rightarrow -a^p > -b^p$



$\Rightarrow 1-a^{p} > 1-b^{p}$

$\Rightarrow f(a) > f(b)$

Thus, f is strictly decreasing generator and Now,

4) The psuedoinverse of f is given by,

$f^{(-1)} : \mathbb{R} \to [0,1]$

$$f^{(-1)}(a) = \begin{cases} 1 & \text{if } a \in (-\infty, 0) \\ f^{-1}(a) & \text{if } a \in [f(1), f(0)] = [0,1] \\ 0 & \text{if } a \in (1, \infty) \end{cases}$$

Now,

$f^{-1}(a) = b$

$\Rightarrow a = f(b)$

$\Rightarrow a = 1-b^{p}$

$\Rightarrow b^{p} = 1-a$

$\Rightarrow b = (1-a)^{1/p}$

Thus,

$f^{-1}(a) = (1-a)^{1/p}$

Hence,

$$f^{-1}(a) = \begin{cases} 1 & \text{if } a \in (-\infty, 0) \\ (1-a)^{1/p} & \text{if } a \in [0,1] \\ 0 & \text{if } a \in (1, \infty) \end{cases}$$

7. Example

Show that, the function $f:[0,1]\to\mathbb{R}$ is given by $f(a)=-ln\ a$, $a\in[0,1]$ and $f(0)=\infty$ is a decreasing generator and also find its psuedoinverse.

Proof : Clearly,

1) f is continuous function and

2) $f(1)=-ln(1)=0$

Further,

3) $a<b$

$\Rightarrow ln\ a<ln\ b$

$\Rightarrow -ln\, a>-ln\ b$

$\Rightarrow f(a)>f(b)$

Thus f is strictly decreasing.

Hence f is decreasing generator.

4) Next,

The psuedoinverse of f is given by,

$$f^{(-1)}(a)=\begin{cases}1 & \text{if } a\in(-\infty,0)\\ f^{-1}(a) & \text{if } a\in\left[f(1),f(0)\right]\\ 0 & \text{if } a\in\left(f(0),\infty\right)\end{cases}$$

Now,

$f^{-1}(a)=b$

$\Rightarrow a=f(b)=-\ln_e(b)$

$\Rightarrow \ln_e(b)=-a$

$\Rightarrow b=e^{-a}$

Thus,

$f^{-1}(a) = e^{-a}$

We have, $f(0) = \infty$

Then,

The psuedo inverse is given by,

$$f^{-1}(a) = \begin{cases} 1 & \text{if } a \in (-\infty, 0) \\ e^{-a} & \text{if } a \in [0, \infty) \end{cases}$$

8. Note

If $f : [0,1] \to \mathbb{R}$ is a decreasing generator with its psuedoinverse $f^{(-1)}$, then,

$$f\left(f^{(-1)}(a)\right) = f\left(e^{-a}\right), \qquad a \in [a, \infty]$$

$$= -ln\left(e^{-a}\right)$$

$$= a\ ln_e e \qquad \because e^{-a} \in [0,1]$$

$$= a$$

Thus, $f\left(f^{(-1)}(a)\right) = a$

Also, $f^{(-1)}\left(f(a)\right) = f^{(-1)}\left(-ln_e a\right)$

$$= e^{-(-ln_e a)}$$

$$= e^{ln_e a}$$

$\Rightarrow f^{(-1)}\left(f(a)\right) = a \qquad \Rightarrow f\left(f^{-1}(a)\right) = f^{-1}\left(f(a)\right)$

9. Increasing Generator

A continuous strictly increasing function,

$g : [0,1] \to \mathbb{R}$ s.t. $g(0) = 0$ is called an increasing generator and it's psuedo inverse is given by

$$g^{(-1)}(a)=\begin{cases}0 & \text{if } a\in(-\infty,g(0))\\ g^{-1}(a) & \text{if } a\in[g(0),g(1)]\\ 1 & \text{if } a\in(g(1),\infty)\end{cases}$$

where g (0) = 0

10. Example

Show that, a function, $g:[0,1]\to\mathbb{R}$ defined by $g(a)=-ln(1-a)$ and $g(1)=\infty$ is an increasing generator and find its psuedoinverse.

Proof : Here,

$$g(a)=-\ell n(1-a)$$

Clearly, g is continuous and $g(0)=0$ and

If $a < b$

$\Rightarrow -a > -b$

$\Rightarrow 1-a > 1-b$

$\Rightarrow \ell n(1-a) > \ell n(1-b)$

$\Rightarrow -\ell n(1-a) < -\ell n(1-b)$

$\Rightarrow g(a) < g(b)$

Thus, g is strictly increasing.

Hence, g is an increasing generator.

Next, its psuedo inverse is given by,

$$g^{(-1)}:\mathbb{R}\to[0,1]$$

and

$$g^{(-1)}(a)=\begin{cases}0 & \text{if } a\in(-\infty,g(0))\\ g^{-1}(a) & \text{if } a\in[g(0),g(1)]\\ 1 & \text{if } a\in(g(1),\infty)\end{cases}$$

Here,

$g(0)=0$ and $g(1)=\infty$ – given

and if $g^{(-1)}(a)=b$

$\Rightarrow a=g(b)$

$\Rightarrow a=-ln_e(1-b)$

$\Rightarrow -a=ln_e(1-b)$

$\Rightarrow e^{-a}=(1-b)$

$\Rightarrow b= 1-e^{-a}$

$\Rightarrow g^{(-1)}(a)=1-e^{-a}$

Hence,

$$g^{(-1)}(a)=\begin{cases}0 & \text{if } a\in(-\infty,0)\\ 1-e^{-a} & \text{if } a\in[0,\infty)\end{cases}$$

11. Example

Show that, the function $g:[0,1]\rightarrow\mathbb{R}$ defined by $g(a)=a^p, p>0$ is an increasing generator and find its psuedo inverse.

Proof : Let a function $g:[0,1]\rightarrow\mathbb{R}$ be defined by,

$g(a)=a^p, p>0$

1) Clearly, g is continuous and

2) $g(0)=0$

3) If $a<b$

$\Rightarrow a^p<b^p$

$\Rightarrow g(a)<g(b)$

$\Rightarrow$ 'g' is strictly increasing

$\Rightarrow$ 'g' is increasing generator.

4) Now, its psuedo-inverse is given by,

$g^{(-1)}:\mathbb{R}\rightarrow[0,1]$ and

$$g^{(-1)}(a)=\begin{cases}0 & \text{if } a\in(-\infty,g(0)) \\ g^{-1}(a) & \text{if } a\in[g(0),g(1)] \\ 1 & \text{if } a\in(g(1),\infty)\end{cases}$$

Here,

$g(0)=0$ and $g(1)=1$

and

if $g^{(-1)}(a)=b$

$\Rightarrow a=g(b)$

$\Rightarrow a=b^{p}$

$\Rightarrow b=a^{1/p}$

$\Rightarrow g^{(-1)}(a)=a^{1/p}$

Hence,

$$g^{(-1)}(a)=\begin{cases}0 & \text{if } a\in(-\infty,0) \\ a^{1/p} & \text{if } a\in[0,1] \\ 1 & \text{if } a\in(1,\infty)\end{cases}$$

12. Theorem

Let, f be a decreasing generator. A function g defined by $g(a)=f(0)-f(a)$, $\forall a\in[0,1]$ is an increasing generator with and $g(1)=f(0)$ it's psuedo inverse is given by,

$$g^{(-1)}(a) = f^{(-1)}\left(f(0) - a\right)$$

Proof : Given that, f is a decreasing generator. Therefore, f is strictly decreasing continuous function s.t. $f(1) = 0$.

Now, $g : [0,1] \to \mathbb{R}$ is given by,

$$g(a) = f(0) - f(a)$$

$$\Rightarrow g(0) = f(0) - f(0) = 0$$

Also, g is continuous ($\because$ f is continuous)

And for $a, b \in [0,1]$,

$$a < b$$

$$\Rightarrow f(a) > f(b) \qquad (\because f \text{ is strictly decreasing})$$

$$\Rightarrow -f(a) < -f(b)$$

$$\Rightarrow f(0) - f(a) < f(0) - f(b)$$

$$\Rightarrow g(a) < g(b)$$

i.e. g is strictly increasing.

Thus, g is strictly increasing, continuous function with $g(0) = 0$.

Hence, g is an increasing generator.

Further, the psuedo inverse of g is given by,

$g^{(-1)} : \mathbb{R} \to [0,1]$ and

$$g^{(-1)}(a) = \begin{cases} 0 & \text{if } a \in (-\infty, 0) \\ g^{-1}(a) & \text{if } a \in [g(0), g(1)] \\ 1 & \text{if } a \in (1, \infty) \end{cases}$$

Now, if

$$g^{-1}(a) = b$$

$\Rightarrow a = g(b)$

$\Rightarrow a = f(0) - f(b)$

$\Rightarrow f(b) = f(0) - a$ .... (*)

$\Rightarrow b = f^{-1}\left(f(0) - a\right)$

Therefore,

$g^{-1}(a) = f^{-1}\left(f(0) - a\right)$

Thus,

$$g^{(-1)}(a) = \begin{cases} 0 & \text{if } a \in (-\infty, 0) \\ f^{-1}\left(f(0) - a\right) & \text{if } a \in [0, g(1)] \\ 1 & \text{if } a \in \left(g(1), \infty\right) \end{cases}$$

Also,

$g(a) = f(0) - f(a)$

$\Rightarrow g(1) = f(0) - f(1)$

$\Rightarrow g(1) = f(0)$ $(\because f(1) = 0)$

Then,

$$g^{(-1)}(a) = \begin{cases} 0 & \text{if } a \in (-\infty, 0) \\ f^{-1}\left(f(0) - a\right) & \text{if } a \in [0, f(0)] \\ 1 & \text{if } a \in \left(f(0), \infty\right) \end{cases}$$

We have, given that *f* is a decreasing generator. Then, psuedo inverse of *f* is given by,

$$f^{(-1)}(a) = \begin{cases} 1 & \text{if } a \in (-\infty, 0) \\ f^{-1}(a) & \text{if } a \in [f(1), f(0)] \\ 0 & \text{if } a \in \left(f(0), \infty\right) \end{cases}$$

Now replace, a by $f(0) - a$

$$f^{(-1)}\left(f(0)-a\right)=\begin{cases}1 & f(0)-a\in(-\infty,0)\\ f^{-1}\left(f(0)-a\right) & f(0)-a\in\left[f(1),f(0)\right]\\ 0 & f(0)-a\in\left(f(0),\infty\right)\end{cases}$$

Now,

1) $f(0)-a\in(-\infty,0)$

$$\Rightarrow -\infty< f(0)-a<0$$

$$\Rightarrow -\infty<-a<-f(0)-f(0)$$

$$\Rightarrow -\infty< a< f(0)$$

$$\Rightarrow f(0)<a<\infty$$

$$\Rightarrow a\in\left(f(0),\infty\right)$$

Next,

2) $f(0)-a\in\left[f(1),f(0)\right]$

$$\Rightarrow f(0)-a\in\left[0,f(0)\right]$$

$$\Rightarrow 0\le f(0)-a\le f(0)$$

$$\Rightarrow -f(0)\le -a\le 0$$

$$\Rightarrow f(0)\ge a\ge 0$$

$$\Rightarrow 0\le a\le f(0)$$

$$\Rightarrow a\in\left[0,f(0)\right]$$

Also,

3) $f(0)-a\in\left(f(0),\infty\right)$

$$\Rightarrow f(0)<f(0)-a<\infty$$

$\Rightarrow 0 < -a < \infty$ $\qquad\qquad -f(0)$

$\Rightarrow 0 > a > -\infty$

$\Rightarrow -\infty < a < 0$

$\Rightarrow a \in (-\infty, 0)$

Hence,

$$f^{-1}\left(f(0)-a\right) = \begin{cases} 0 & \text{if } a \in \left(f(0), \infty\right) \\ f^{-1}\left(f(0)-a\right) & \text{if } a \in \left[f(1), f(0)\right] \\ 1 & \text{if } a \in (-\infty, 0) \end{cases}$$

where, $f(1) = 0$

Therefore, we must have,

$$g^{(-1)}(a) = f^{(-1)}\left(f(0)-a\right)$$

13. Theorem

Let, g be increasing generator. A function, $f:[0,1] \to \mathbb{R}$ defined by, $f(a) = g(1) - g(a)$ is a decreasing generator and its psuedo inverse is given by

$$f^{(-1)}(a) = f^{(-1)}\left(g(1)-a\right)$$

Proof : Given that, g is an increasing generator. Thus, g strictly increasing continuous function and $g(0) = 0$.

Now, $f:[0,1] \to \mathbb{R}$ is

$$f(a) = g(1) - g(a) \qquad\qquad \forall a \in [0,1]$$

Then,

$$f(0) = g(1) - g(0) = g(1)$$

$$\Rightarrow f(0) = g(1)$$

and $f(1)=0$

Also, *f* is continuous $(\because\ g$ is continuous)

And for any $a,b\in[0,1]$,

$a<b$

$$\Rightarrow g(a)<g(b)$$

$$\Rightarrow -g(a)>-g(b)$$

$$\Rightarrow g(1)-g(a)>g(1)-g(b)$$

$$\Rightarrow f(a)>f(b)$$

$\therefore f$ is strictly decreasing.

Hence, f is decreasing generator and the psuedo inverse of *f* is given by.

$$f^{(-1)}(a)=\begin{cases}0 & \text{if } a\in(-\infty, f(1)) \\ f^{-1}(a) & \text{if } a\in[f(1), f(0)] \\ 1 & \text{if } a\in(f(0),\infty)\end{cases}$$

Now, if $f^{-1}(a)=b$

$$\Rightarrow a=f(b)$$

$$\Rightarrow a=g(1)-g(b) \qquad \text{– given}$$

$$\Rightarrow g(b)=g(1)-a$$

$$\Rightarrow b=g^{-1}(g(1)-a)$$

$$\Rightarrow f^{-1}(a)=g^{-1}(g(1)-a)$$

Hence,

$$f^{(-1)}(a)=\begin{cases}1 & \text{if } a\in(-\infty,0) \\ g^{-1}(g(1)-a) & \text{if } a\in[0, f(0)] \\ 0 & \text{if } a\in(f(0),\infty)\end{cases}$$

Also, psuedo inverse of g is given by,

$$g^{(-1)}(a)=\begin{cases}0 & \text{if } a\in(-\infty,0)\\ g^{-1}(a) & \text{if } a\in[0,g(1)]\\ 1 & \text{if } a\in(g(1),\infty)\end{cases}$$

Hence, replacing a by $g(1)-a$ we get,

$$g^{(-1)}(g(1)-a)=\begin{cases}0 & \text{if } g(1)-a\in(-\infty,0)\\ g^{-1}(g(1)-a) & \text{if } g(1)-a\in[0,g(1)]\\ 1 & \text{if } g(1)-a\in(g(1),\infty)\end{cases}$$

and $f(1)=0$, $f(0)=g(1)$

Now,

1) If $g(1)-a\in(-\infty,0)$

$$\Leftrightarrow -\infty < g(1)-a<0$$

$$\Leftrightarrow -\infty < -a < -g(1)$$

$$\Leftrightarrow \infty > a > g(1)$$

$$\Leftrightarrow g(1) < a < \infty$$

$$\Leftrightarrow a\in(g(1),\infty)$$

2) Next, if $g(1)-a\in[0,g(1)]$

$$\Leftrightarrow 0\le g(1)-a\le g(1)$$

$$\Leftrightarrow -g(1)\le -a\le 0$$

$$\Leftrightarrow g(1)\ge a\ge 0$$

$$\Leftrightarrow 0\le a\le g(1)$$

$$\Leftrightarrow a\in(0,g(1))$$

and also,

3) If $g(1)-a\in(g(1),\infty)$

$$\Leftrightarrow g(1)<g(1)-a<\infty$$

$$\Leftrightarrow 0<-a<\infty$$

$$\Leftrightarrow 0>a>-\infty$$

$$\Leftrightarrow -\infty<a<0$$

$$\Leftrightarrow a\in(-\infty,0)$$

Thus,

$$g^{(-1)}(g(1)-a)=\begin{cases}0 & \text{if } a\in(g(1),\infty)\\ g^{-1}(g(1)-a) & \text{if } a\in[0,g(1)]\\ 1 & \text{if } a\in(-\infty,0)\end{cases}$$

But, $g(1)=f(0)$

$$\Rightarrow g^{(-1)}(g(1)-a)=\begin{cases}0 & \text{if } a\in(-\infty,0)\\ g^{-1}(g(1)-a) & \text{if } a\in[0,f(0)]\\ 1 & \text{if } a\in(f(0),\infty)\end{cases}$$

Thus, we must have,

$$f^{(-1)}(a)=g^{(-1)}(g(1)-a)$$

Hence, the proof.

14. Construction of t-norms

Let $i:[0,1]\times[0,1]\to[0,1]$ be a function. Then *i* is an Archimedean t-norms iff there exists a decreasing generator *f* such that

$$i(a,b)=f^{(-1)}(f(a)+f(b)) \qquad a,b\in[0,1]$$

15. Example

Let $f_p : [0,1] \rightarrow \mathbb{R}$ by $f_p(a) = 1 - a^p$, $a \in [0,1]$ and $p > 0$.

Show that, f_p is a decreasing generator and obtain a t-norm generated by f_p.

Proof : $f_p(a) = 1 - a^p$, $\quad p > 0$

Clearly, f_p is a continuous function and $f_p(1) = 0$ and for $a < b$.

$$\Rightarrow a^p < b^p$$

$$\Rightarrow -a^p < -b^p$$

$$\Rightarrow 1 - a^p > 1 - b^p$$

$$\Rightarrow f_p(a) > f_p(b)$$

Thus, f_p is a strictly decreasing continuous function s.t. $f_p(1) = 0$.

Thus, f_p is a decreasing generator.

Next, psuedo inverse of f_p is given by,

$$f_p^{(-1)}(a) = \begin{cases} 0 & \text{if } a \in (-\infty, 0) \\ f_p^{-1}(a) & \text{if } a \in [0, f_p(0)] \\ 1 & \text{if } a \in (f_p(0), \infty) \end{cases}$$

Now, $f_p(0) = 1$ and $f_p(1) = 0$.

and $f_p^{-1}(a) = b$

$$\Rightarrow a = f_p(b)$$

$$\Rightarrow a = 1 - b^p$$

$$\Rightarrow b^p = 1 - a$$

$$\Rightarrow b = (1-a)^{1/p}$$

$$\Rightarrow f_p^{-1}(a) = (1-a)^{1/p}$$

Hence, the psuedo inverse is given by,

$$f_p^{(-1)}(a) = \begin{cases} 1 & \text{if } a \in (-\infty, 0) \\ (1-a)^{1/p} & \text{if } a \in [0,1] \\ 0 & \text{if } a \in (1, \infty) \end{cases} \quad \text{.... (1)}$$

Define,

$i : [0,1] \times [0,1] \to [0,1]$ by,

$$i(a,b) = f_p^{(-1)}\left(f_p(a) + f_p(b) \right) \qquad a,b \in [0,1]$$

$$= f_p^{(-1)}\left[1 - a^p + 1 - b^p\right]$$

$$= f_p^{(-1)}\left[2 - a^p - b^p\right]$$

$$= \begin{cases} 1 & \text{if } 2 - a^p - b^p < 0 \\ \left[1 - \left(2 - a^p - b^p\right)\right]^{\frac{1}{p}} & \text{if } 2 - a^p - b^p \in [0,1] \\ 0 & \text{if } 2 - a^p - b^p > 1 \end{cases} \quad \text{by (1)}$$

Since,

$$2 - a^p - b^p \geq 0 \qquad \forall a, b \in [0,1]$$

We get,

$$i(a,b) = \left[a^p + b^p - 1\right] \qquad 2 - a^p - b^p \in [0,1]$$

$$= 0 \qquad 2 - a^p - b^p > 1$$

Now, if

$$2 - a^p - b^p \in [0,1]$$

$$\Rightarrow 0 \le 2 - a^p - b^p \le 1 \Rightarrow 1 \le a^p + b^p \Rightarrow 0 \le a^p + b^p - 1$$

$$\Rightarrow 0 \le a^p + b^p - 1$$

$$\Rightarrow 0 \le \left(a^p + b^p - 1\right)^{1/p}$$

$$\Rightarrow 0 \le i(a,b)$$

And, if

$$2 - a^p - b^p > 1$$

$$\Rightarrow 0 > a^p + b^p - 1$$

$$\Rightarrow a^p + b^p - 1 < 0$$

$$\Rightarrow i(a,b) = 0$$

Thus,

$$i(a,b) = \begin{cases} \left(a^p + b^p - 1\right)^{1/p} & a^p + b^p - 1 \ge 0 \\ 0 & a^p + b^p - 1 < 0 \end{cases}$$

$$\Rightarrow i(a,b) = \max\left\{0, \left(a^p + b^p - 1\right)^{1/p}\right\}, \quad a,b \in [0,1]$$

If $p = 1$,

We get,

$$i(a,b) = \max\{0, (a + b - 1)\}$$

This is a bounded difference t-norm.

16. Example

Let, $f_w(a) = (1-a)^w$, $w > 0$. Show that, f_w is a decreasing generator. Find its psuedo inverse and obtain a t-norm generated by f_w.

Proof : Let, $f_w : [0,1] \to \mathbb{R}$ defined by

$$f_w(a) = (1-a)^w,\ w > 0$$

Clearly, f_w is continuous and $f_w(1) = 0$, and for $a < b$,

$\Rightarrow -a > -b$

$\Rightarrow 1-a > 1-b$

$\Rightarrow (1-a)^w > (1-b)^w$

$\Rightarrow f_w(a) > f_w(b)$

Thus, f_w is strictly decreasing. Therefore, f_w is a decreasing generator.

Further, the psuedo inverse is given by,

$$f_w^{(-1)}(a) = \begin{cases} 1 & \text{if } a \in (-\infty, 0) \\ f_w^{-1}(a) & \text{if } a \in \left[0, f_w(0)\right] \\ 0 & \text{if } a \in \left(f_w(0), \infty\right) \end{cases}$$

Now, let,

$$b = f_w^{-1}(a) \Rightarrow a = f_w(b) = (1-b)^w$$

$$\Rightarrow a^{1/w} = 1-b$$

$$\Rightarrow b = 1 - a^{1/w}$$

Thus,

$f_w^{-1}(a) = 1 - a^{1/w}$. Also $f_w(0) = 1$

Therefore,

$$f_w^{(-1)}(a) = \begin{cases} 1 & \text{if } a \in (-\infty, 0) \\ 1 - a^{1/w} & \text{if } a \in [0,1] \\ 0 & \text{if } a \in (1, \infty) \end{cases}$$

Next, define a t-norm i by

$i:[0,1]\times[0,1]\to[0,1]$

$$i(a,b)=f_w^{(-1)}\left(f_w(a)+f_w(b)\right)$$

$$= f_w^{(-1)}\left((1-a)^w+(1-b)^w\right)$$

$$=\begin{cases} 1 & \text{if } (1-a)^w+(1-b)^w\in(-\infty,0) \\ 1-\left[(1-a)^w+(1-b)^w\right]^{1/w} & \text{if } (1-a)^w+(1-b)^w\in[0,1] \\ 0 & \text{if } (1-a)^w+(1-b)^w\in(1,\infty) \end{cases}$$

Since,

$$(1-a)^w+(1-b)^w\geq 0 \qquad \forall a,b\in[0,1]$$

We get,

$$i_w[a,b]=1-\left[(1-a)^w+(1-b)^w\right]^{1/w},\quad (1-a)^w+(1-b)^w\leq 1$$

$$=0 \qquad (1-a)^w+(1-b)^w\geq 1$$

$$\Rightarrow i(a,b)=1-\min\left\{1,\left[(1-a)^w+(1-b)^w\right]^{1/w}\right\}$$

This t-norm is called a Yager's class of t-norms denoted by i_w.

i.e.

$$i_w(a,b)=1-\min\left\{1,\left[(1-a)^w+(1-b)^w\right]^{1/w}\right\},\ \forall a,b\in[0,1]$$

17. Note

1) For $w\to 0$, $i_w\to$ bounded difference t-norm.

2) For $w=1$, $i_w\to i_{\min}$

3) For $w\to\infty$, $i_w\to$ min

18. Theorem

Let, i_w be the Yager's class of t-norms defined by

$$i_w(a,b) = 1 - \min\left\{1, \left[(1-a)^w + (1-b)^w\right]^{1/w}\right\}$$

Then,

$$i_{\min}(a,b) \le i_w(a,b) \le \min(a,b)$$

Proof :

1) If $b = 1$, then

$$i_w(a,b) = 1 - \min\left\{1, \left[(1-a)^w + (1-b)^w\right]^{1/w}\right\}$$

$$= 1 - \min\{1, (1-a)\}$$

$$= 1 - (1-a)$$

$$= a$$

2) Simillarly, if $a = 1$ then,

$$i_w(a,b) = b$$

3) And, if $a \ne 1$ and $b \ne 1$,

Then, consider,

$$\lim_{w\to\infty} i_w(a,b) = \lim_{w\to\infty}\left\{1 - \min\left[1, \left[(1-a)^w + (1-b)^w\right]^{1/w}\right]\right\}$$

$$= 1 - \min\left[1, \lim_{w\to\infty}\left[(1-a)^w + (1-b)^w\right]^{1/w}\right]$$

$$= 1 - \min\{1, (1+1)^{\infty}\}$$

$$= 1 - 1$$

$$= 0$$

Thus,

$i_w(a,b) = a$, if $b = 1$

$= b$, if $a = 1$ ..... (A)

≥ 0 , if $a \neq 1,\ b \neq 1$ as $w \to 0$.

But,

$i_{\min}(a,b) = a$ if $b = 1$ ...... (B)

$= b$ if $a = 1$

$= 0$ otherwise

Hence, from (A) and (B)

$i_{\min}(a,b) \leq i_w(a,b)$ $\forall w > 0$ .... (1)

Next,

We will prove that,

$i_w(a,b) \leq \min(a,b)$

First, we will show that,

$$\lim_{w\to\infty} \min\left\{1, \left[(1-a)^w + (1-b)^w\right]^{1/w}\right\}$$

$= \max\{1-a, 1-b\}$

Case (1) :

$a \neq b \neq 1$

Let, $Q = \left[(1-a)^w + (1-b)^w\right]^{1/w}$

Since, $a \neq b$, we assume that, $a < b$

Consider,

$$\lim_{w\to\infty} \log Q = \lim_{w\to\infty} \log\left[(1-a)^w + (1-b)^w\right]^{1/w}$$

$$= \lim_{w\to\infty} \frac{1}{w}\log\left[(1-a)^w + (1-b)^w\right]$$

$$= \lim_{w\to\infty} \frac{\log\left[(1-a)^w + (1-b)^w\right]}{w}$$

$$= \lim_{w\to\infty} \frac{(1-a)^w \log(1-a) + (1-b)^w \log(1-b)}{1\cdot\left[(1-a)^w + (1-b)^w\right]}$$

$$= \lim_{w\to\infty} \frac{(1-a)^w \log(1-a) + (1-b)^w \log(1-b)}{\left[(1-a)^w + (1-b)^w\right]}$$

Now, $a < b \Rightarrow 1-a > 1-b$

Divide N & D by, $(1-a)^w$

$$= \lim_{w\to\infty} \frac{\log(1-a) + \left[\frac{(1-b)}{(1-a)}\right]^w \log(1-b)}{1+\left(\frac{1-b}{1-a}\right)^w}$$

$$= \frac{\log(1-a)+0}{1+0}$$

$$= \log(1-a)$$

Thus,

$$\lim_{w\to\infty} \log Q = \log(1-a)$$

$\Rightarrow \lim_{w\to\infty} Q = 1-a$ where $a < b$

Also, $a < b$,

$\Rightarrow 1-a > 1-b$

$\Rightarrow \max\{1-a, 1-b\} = 1-a$

$$\Rightarrow \lim_{w\to\infty} Q = \max\{1-a, 1-b\}$$

Therefore,

$$\lim_{w\to\infty} \min\left\{1, \left[(1-a)^w + (1-b)^w\right]^{1/w}\right\}$$

$$= \min\{1, \max\{1-a, 1-b\}\}$$

$$= \max\{1-a, 1-b\} \qquad (\because a \neq b \neq 1)$$

Case - (2)

$a = b\ (\neq 1)$

Consider,

$$\lim_{w\to\infty} Q = \lim_{w\to\infty} \left[(1-a)^w + (1-b)^w\right]^{1/w}$$

$$= \lim_{w\to\infty} \left[2(1-a)^w\right]^{1/w} \qquad (\because a = b)$$

$$= \lim_{w\to\infty} 2^{1/w}(1-a)$$

$$= 2^0 (1-a)$$

$$= (1-a) = (1-b)$$

Thus,

$$\lim_{w\to\infty} Q = 1-a = 1-b$$

$$= \max\{1-a, 1-b\}$$

Hence,

$$\lim_{w\to\infty} \min\left\{1, \left[(1-a)^w + (1-b)^w\right]^{1/w}\right\}$$

$$= \min\{1, \max\{1-a, 1-b\}\}$$

$$= \max\{1-a, 1-b\} \qquad a = b \neq 1$$

Case (3) :

Either a = 1 or b = 1

Consider,

$$\lim_{w\to\infty} \min\left\{1, \left[(1-a)^w + (1-b)^w\right]^{1/w}\right\}$$

$$= \begin{cases} 1-b & \text{if } a=1 \\ 1-a & \text{if } b=1 \end{cases}$$

$= \max\{1-a, 1-b\}$ (either a = 1 or b = 1)

Case (4) :

$a = b = 1$

Then, $1 - a = 0 = 1 - b$

$\therefore \max\{1-a, 1-b\} = 0$

And,

$$\lim_{w\to\infty} \min\left\{1, \left[(1-a)^w + (1-b)^w\right]^{1/w}\right\}$$

= min (1, 0) ($\because$ $a = b = 1$)

= 0

Hence,

$$\lim_{w\to\infty} \min\left\{1, \left[(1-a)^w + (1-b)^w\right]^{1/w}\right\}$$

$= \max\{1-a, 1-b\}$

Thus, for $\forall a, b \in [0,1]$,

$$\lim_{w\to\infty} \min\left\{1, \left[(1-a)^w + (1-b)^w\right]^{1/w}\right\}$$

$= \max\{1-a, 1-b\}$ .... (*)

Therefore,

$$\lim_{w\to\infty} i_w(a,b) = \lim_{w\to\infty}\left[1-\min\left\{1,\left[(1-a)^w+(1-b)^w\right]^{1/w}\right\}\right]$$

$$= 1-\max\{1-a, 1-b\} \qquad \text{by } (*)$$

$$= \min\{a,b\}$$

i.e.

$$\lim_{w\to\infty} i_w(a,b) = \min(a,b)$$

Hence,

$$i_w(a,b) \le \min(a,b) \qquad \text{.... (2)}$$

From (1) and (2) we get,

$$i_{\min}(a,b) \le i_w(a,b) \le \min(a,b)$$

19. Note

There are several methods of defining t-norms. The following is one of the methods of constructing t-norms.

Let i be a given t-norm and let $g:[0,1]\to[0,1]$ be a strictly increasing continuous function with $g(0)=0$, $g(1)=1$.

Then, a function,

$$i^g[a,b] = g^{(-1)}\left[i\left(g(a),g(b)\right)\right] \qquad \forall a,b\in[0,1]$$

i^g is also called a t-norm generated by g.

21. Example

Obtain a t-norm generated by a function

$g:[0,1]\to[0,1]$ by,

$$g(a) = \frac{a+1}{2} \qquad , \ a \neq 0$$

$$= 0 \qquad , \ a = 0$$

And $i(a,b) = a \cdot b$,

Proof : $g : [0,1] \to [0,1]$ is defined by,

$$g(a) = \frac{a+1}{2} \qquad , \ a \neq 0$$

$$= 0 \qquad , \ a = 0$$

1) Then, $g(0) = 0$, $g(1) = 1$

2) Also for $a, b \in [0,1]$

$a < b$

$$\Rightarrow a + 1 < b + 1$$

$$\Rightarrow \frac{a+1}{2} < \frac{b+1}{2}$$

$$\Rightarrow g(a) < g(b)$$

Thus, g is strictly increasing and clearly it is continuous and g is an increasing generator.

3) The psuedo inverse of g is given by,

$$g^{(-1)}(a) = \begin{cases} 0 & \text{if } a = 0 \\ g^{-1}(a) & \text{if } a \in (0,1] \end{cases}$$

Let,

$g^{-1}(a) = b$

$$\Rightarrow a = g(b)$$

$$\Rightarrow a = \frac{b+1}{2} \Rightarrow b = 2a - 1$$

$$\Rightarrow g^{-1}(a) = 2a - 1$$

Thus,

$$g^{(-1)}(a)=\begin{cases}0 & \text{if } a=0\\ 2a-1 & \text{if } a\in(0,1]\end{cases}$$

But, $g^{(-1)}(a)\geq 0$. Hence,

$$g^{(-1)}(a)=\begin{cases}0 & 0\leq a\leq \frac{1}{2}\\ 2a-1 & a\in\left[\frac{1}{2},1\right]\end{cases}$$

4) Define, a function i^g by,

$$i^g(a,b)=g^{(-1)}\left[i\left(g(a),g(b)\right)\right]$$

$$=g^{(-1)}\left(i\left(\frac{a+1}{2},\frac{b+1}{2}\right)\right)$$

$$=g^{(-1)}\left(\frac{a+1}{2}\cdot\frac{b+1}{2}\right) \qquad (\because i(a,b)=ab)$$

$$=g^{(-1)}\left(\frac{a+b+ab+1}{4}\right)$$

$$=\begin{cases}0 & \text{if } \frac{a+b+ab+1}{4}\in\left[0,\frac{1}{2}\right]\\ 2\left(\frac{a+b+ab+1}{4}\right)-1 & \text{if } \frac{a+b+ab+1}{4}\in\left[\frac{1}{2},1\right]\end{cases}$$

Now,

$$\frac{a+b+ab+1}{4}\in\left(\frac{1}{2},1\right]$$

$$\Rightarrow \frac{1}{2}<\frac{a+b+ab+1}{4}\leq 1$$

$$\Rightarrow 1 < \frac{a+b+ab+1}{2} \leq 2$$

$$\Rightarrow 0 < \frac{a+b+ab+1}{2} - 1 \leq 1$$

$$\Rightarrow 0 < \frac{a+b+ab-1}{2} \leq 1$$

Thus,

$$i^g(a,b) = \frac{a+b+ab-1}{2} \qquad \text{if } \frac{a+b+ab-1}{2} \in (0,1]$$

$$= 0 \qquad \text{otherwise}$$

Thus,

$$i^g(a,b) = \max\left\{0, \frac{a+b+ab-1}{2}\right\}$$

21. Example

Obtain a t-norm generated by,

$$g(a) = \frac{a}{2} \qquad a \neq 1$$

$$= 1 \qquad a = 1$$

and $i(a,b) = a \cdot b$

Proof : Let, $g : [0,1] \rightarrow [0,1]$ be defined by,

$$g(a) = \begin{cases} \frac{a}{2} & \text{if } a \neq 1 \\ 1 & \text{if } a = 1 \end{cases}$$

Then,

1) $g(0) = 0$ and $g(1) = 1$

2) For $a, b \in [0,1]$

if $a < b$

$$\Rightarrow \frac{a}{2} < \frac{b}{2}$$

$$\Rightarrow g(a) < g(b)$$

$\Rightarrow$ g is strictly increasing and clearly it is continuous.

$\Rightarrow$ g is an increasing generator.

3) The psuedo inverse of g is given by,

$$g^{(-1)}(a) = \begin{cases} g^{-1}(a) & \text{if } a \in [0,1] \\ 1 & \text{if } a = 1 \end{cases}$$

Let, $g^{-1}(a) = b$

$$\Rightarrow a = g(b)$$

$$\Rightarrow g(b) = a$$

$$\Rightarrow \frac{b}{2} = a$$

$$\Rightarrow b = 2a$$

$$\Rightarrow g^{-1}(a) = 2a$$

Thus

$$g^{(-1)}(a) = \begin{cases} 2a & \text{if } a \in [0,1] \\ 1 & \text{if } a = 1 \end{cases}$$

Hence,

$$g^{(-1)}(a) = \begin{cases} 2a & \text{if } a \in \left[0, \frac{1}{2}\right] \\ 1 & \text{if } a \in \left[\frac{1}{2}, 1\right] \end{cases}$$

Next,

4) Define a function i^g by,

$$i^g(a,b) = g^{(-1)}\left(i\left(g(a),g(b)\right)\right)$$

$$= g^{(-1)}\left(i\left(\frac{a}{2},\frac{b}{2}\right)\right)$$

$$= g^{(-1)}\left(\frac{ab}{4}\right) \qquad (\because i(a,b)=ab)$$

$$= \begin{cases} 2\left(\frac{ab}{4}\right) & \frac{ab}{4}\in\left[0,\frac{1}{2}\right) \\ 1 & \frac{ab}{4}\in\left[\frac{1}{2},1\right] \end{cases}$$

Thus,

$$i^g(a,b) = \begin{cases} \frac{ab}{2} & \text{if } \frac{ab}{4}\in\left[0,\frac{1}{2}\right) \\ 1 & \text{if } \frac{ab}{4}\in\left[\frac{1}{2},1\right] \end{cases}$$

Now, $\frac{ab}{4}\in\left[0,\frac{1}{2}\right)$

$$\Rightarrow 0 \le \frac{ab}{4} < \frac{1}{2}$$

$$\Rightarrow 0 \le \frac{ab}{2} < 1$$

Then,

$$i^g(a,b) = \begin{cases} \frac{ab}{2} & \text{if } \frac{ab}{2}\in[0,1) \\ 1 & \text{otherwise} \end{cases}$$

Thus,

$$i^{g}(a,b)=\min\left\{\frac{ab}{2},1\right\}$$

23. Example

$f_s(a)=-ln\dfrac{s^a-1}{s-1}$ $(s>0,\ s\neq 1)$ is a decreasing generator and find the class of t-norms.

Proof : Let $f_s(a)=-ln\dfrac{s^a-1}{s-1}$

Clearly, f_s is continuous.

Now, for $a,b\in[0,1]$, let $a<b$

$\Rightarrow s^a<s^b$ $\quad (s>0$ and $s\neq 1)$

$\Rightarrow s^a-1<s^b-1$

$$\Rightarrow \frac{s^a-1}{s-1}<\frac{s^b-1}{s-1}$$

$$\Rightarrow \ell n\left(\frac{s^a-1}{s-1}\right)<\ell n\left(\frac{s^b-1}{s-1}\right)$$

$$\Rightarrow -\ell n\left(\frac{s^a-1}{s-1}\right)>-\ell n\left(\frac{s^b-1}{s-1}\right)$$

$f_s(a)>f_s(b)$

Hence, f_s is strictly decreasing function.

and $f_s(1)=-\ell n\dfrac{s-1}{s-1}=-\ell n(1)=0$

Thus f_s is continuous, strictly decreasing function with $f_s(1)=0$.

Hence, f_s is decreasing generator.

Now to find the class of t-norms we have to find the pseudo inverse of f_s.

Let,

$$f_s^{(-1)}(a)=\begin{cases}1 & \text{if } a\in(-\infty,0)\\ f_s^{(-1)}(a) & \text{if } a\in\left[0, f_s(1)\right]\\ 0 & \text{if } a\in\left(f_s(1),\infty\right)\end{cases}$$

Now consider, $f_s^{(-1)}(a)=b$

$\Rightarrow a= f_s(b)$

$$\Rightarrow a=-\ell n\frac{s^b-1}{s-1}$$

$$\Rightarrow -a=\ell n\frac{s^b-1}{s-1}$$

$$\Rightarrow e^{-a}=\frac{s^b-1}{s-1}$$

$\Rightarrow s^b=1+(s-1)e^{-a}$

$\Rightarrow \log_s s^b=\log_s\left[1+(s-1)e^{-a}\right]$

$\Rightarrow b=\log_s\left[1+(s-1)e^{-a}\right]$

i.e. $f_s^{(-1)}(a)=\log_s\left[1+(s-1)e^{-a}\right]$

Then pseudo inverse of f_s is

$$f_s^{(-1)}(a)=\begin{cases}1 & \text{if } a\in(-\infty,0)\\ \log_s\left[1+(s-1)e^{-a}\right] & \text{if } a\in\left[0, f_s(1)\right]\\ 0 & \text{if } a\in\left(f_s(1),\infty\right)\end{cases}$$

Thus,

$$f_s^{(-1)}(a) = \log_s\left[1+(s-1)e^{-a}\right]$$

Now we define $i_s : [0,1]\times[0,1] \to [0,1]$ by

$$i_s(a,b) = f_s^{(-1)}\left(f_s(a) + f_s(b)\right)$$

$$= f_s^{(-1)}\left[-\ell n\frac{s^a-1}{s-1} - \ell n\frac{s^b-1}{s-1}\right]$$

$$= f_s^{(-1)}\left[-\ell n\left(\frac{(s^a-1)\cdot(s^b-1)}{(s-1)^2}\right)\right]$$

$$= \log_s\left\{1+(s-1)e^{-\left[-\ell n\left(\frac{(s^a-1)\cdot(s^b-1)}{(s-1)^2}\right)\right]}\right\}$$

$$= \log_s\left\{1+\cancel{(s-1)}\cdot\frac{(s^a-1)\cdot(s^b-1)}{(s-1)^{\cancel{2}}}\right\}$$

$$i_s(a,b) = \log_s\left\{1+\frac{(s^a-1)\cdot(s^b-1)}{(s-1)}\right\}$$

24. Show that, $g_s(a) = -\ell n\dfrac{s^{1-a}-1}{s-1}$ $(s>0,\ s\neq 1)$ is an increasing generator. Find the t-conorms generated by g_s.

Proof : Clearly g_s is continuous.

Now, for $a,b\in[0,1]$ consider,

$a<b$

$\Rightarrow -a > -b$

$\Rightarrow 1-a > 1-b$

$$\Rightarrow s^{1-a} > s^{1-b}$$

$$\Rightarrow s^{1-a} - 1 > s^{1-b} - 1$$

$$\Rightarrow \frac{s^{1-a}-1}{s-1} > \frac{s^{1-b}-1}{s-1}$$

$$\Rightarrow \ell n \frac{s^{1-a}-1}{s-1} > \ell n \frac{s^{1-b}-1}{s-1}$$

$$\Rightarrow -\ell n \frac{s^{1-a}-1}{s-1} < -\ell n \frac{s^{1-b}-1}{s-1}$$

$$\Rightarrow g_s(a) < g_s(b)$$

$\Rightarrow g_s$ is strictly increasing function.

Thus, g_s is continuous, strictly increasing function with $g_s(0) = 0$.

Hence g_s is an increasing generator.

We find the pseudo inverse of g_s.

Let,

$$g_s^{(-1)}(a) = \begin{cases} 1 & \text{if } a \in (-\infty, 0) \\ g_s^{-1}(a) & \text{if } a \in [0, g_s(1)] \\ 0 & \text{if } a \in (g_s(1), \infty) \end{cases}$$

Now, let

$$g_s^{-1}(a) = b$$

$$\Rightarrow a = g_s(b)$$

$$\Rightarrow a = -ln \frac{s^{1-b}-1}{s-1}$$

$$\Rightarrow -a = ln \frac{s^{1-b}-1}{s-1}$$

$$\Rightarrow e^{-a} = \frac{s^{1-b}-1}{s-1}$$

$$\Rightarrow s^{1-b} - 1 = (s-1)e^{-a}$$

$$\Rightarrow s^{1-b} = 1 + (s-1)e^{-a}$$

$$\Rightarrow 1 - b = \log_s\left[1 + (s-1)e^{-a}\right]$$

$$\Rightarrow b = 1 - \log_s\left\{1 + (s-1)e^{-a}\right\}$$

i.e. $g_s^{-1}(a) = 1 - \log_s\left\{1 + (s-1)e^{-a}\right\}$

Then, pseudo inverse of g_s is

$$g_s^{(-1)}(a) = 1 - \log_s\left\{1 + (s-1)e^{-a}\right\}$$

We define a t-conorm, $u_s : [0,1]\times[0,1] \to [0,1]$ by

$$u_s(a,b) = g_s^{(-1)}\left(g(a) + g(b)\right)$$

$$= g_s^{(-1)}\left[-\ell n \frac{\left(s^{1-a}-1\right)\left(s^{1-b}-1\right)}{(s-1)^2}\right]$$

$$= 1 - \log_s\left\{1 + \cancel{(s-1)} \cdot \frac{\left(s^{1-a}-1\right)\left(s^{1-b}-1\right)}{(s-1)^{\cancel{2}}}\right\}$$

$$u_s(a,b) = 1 - \log_s\left\{1 + \frac{\left(s^{1-a}-1\right)\left(s^{1-b}-1\right)}{(s-1)}\right\}$$

25. $g_{l,w}(a) = \frac{1}{l} ln\left(1 + l\,a^w\right)$, show that $g_{l,w}$ is increasing generator and obtain Fuzzy complement generated by $g_{l,w}$.

Proof : $g_{l,w}(a) = \frac{1}{l} ln\left(1 + l\,a^w\right)$

Clearly $g_{l,w}$ is continuous and $g_{l,w}(0)=0$.

Now for $a,b\in[0,1]$ consider,

$a<b$

$a^w<b^w$

$\Rightarrow l\,a^w< l\,b^w$

$\Rightarrow ln(1+l\,a^w)<ln(1+l\,b^w)$

$$\Rightarrow \frac{1}{l}ln(1+l\,a^w)<\frac{1}{l}ln(1+l\,b^w)$$

$\Rightarrow g_{l,w}(a)<g_{l,w}(b)$

$\Rightarrow g_{l,w}$ is an increasing function also continuous with $g_{l,w}(0)=0$.

Hence, $g_{l,w}$ is an increasing generator.

Now we find pseudo inverse of $g_{l,w}$.

$$g_{l,w}^{(-1)}(a)=\begin{cases}1 & \text{if } a\in(-\infty,0)\\ g_{l,w}^{-1}(a) & \text{if } a\in\left[0,g_{l,w}(1)\right]\\ 0 & \text{if } a\in\left(g_{l,w}(1),\infty\right)\end{cases}$$

Let, $g_{l,w}^{(-1)}(a)=b$

$\Rightarrow a=g_{l,w}(b)$

$$\Rightarrow a=\frac{1}{l}ln(1+l\,b^w)$$

$\Rightarrow l\,a=ln(1+l\,b^w)$

$\Rightarrow e^{l\,a}=1+l\,b^w$

$\Rightarrow l\,b^w=e^{l\,a}-1$

$$\Rightarrow b^{w} = \frac{1}{l}\left(e^{la} - 1\right)$$

$$\Rightarrow b = \left[\frac{1}{l}\left(e^{la} - 1\right)\right]^{\frac{1}{w}}$$

$$\Rightarrow g_{l,w}{}^{(-1)}(a) = \left\{\frac{1}{l}\left(e^{la} - 1\right)\right\}^{\frac{1}{w}}$$

Hence, pseudo inverse is given by,

$$g_{l,w}{}^{(-1)}(a) = \begin{cases} 0 & \text{if } a \in (-\infty, 0) \\ \left[\frac{1}{l}\left(e^{la} - 1\right)\right]^{\frac{1}{w}} & \text{if } a \in \left[0, \frac{1}{l}\ell n(1+l)\right] \\ 1 & \text{if } a \in \left(\frac{1}{l}(\ell n(1+l)), \infty\right) \end{cases}$$

Now if C is fuzzy complement generated by g then,

$$C_{l,w}(a) = g^{(-1)}\left(g(1) - g(a)\right)$$

$$= g^{(-1)}\left\{\frac{1}{l}ln(1+l) - \frac{1}{l}ln\left(1 + l a^{w}\right)\right\}$$

$$= g^{(-1)}\left\{\frac{1}{l}ln\left(\frac{1+l}{1+l a^{w}}\right)\right\}$$

$$= \left\{\frac{1}{l}\cdot\left(e^{l\cdot\left[\frac{1}{l}\ell n\frac{1+l}{1+l a^{w}}\right]} - 1\right)\right\}^{\frac{1}{w}}$$

$$= \left\{\frac{1}{l}\left[\frac{1+l}{1+l a^{w}} - 1\right]\right\}^{\frac{1}{w}}$$

$$= \left\{ \frac{1}{l} \left[\frac{1+l-1-l a^w}{1+l a^w} \right] \right\}^{\frac{1}{w}}$$

$$= \left(\frac{1}{\not{l}} \cdot \frac{\not{l}(1-a^w)}{1+l a^w} \right)^{\frac{1}{w}}$$

$$C_{l,w}(a) = \left(\frac{1-a^w}{1+l a^w} \right)^{\frac{1}{w}}$$

Then the Fuzzy complement $C_{l,w}$ generated by $g_{l,w}$ is

$$C_{l,w}(a) = \left(\frac{1-a^w}{1+l a^w} \right)^{\frac{1}{w}}$$

which is the Sugen's class of Fuzzy complement for $w = 1$ and Yagar's class of fuzzy complement for $l = 0$ as special sub-classes.

3.3 Fuzzy Union / t-Conorm

1. Definition

A fuzzy t-conorm u is a binary operation on [0, 1], which satisfies the following axioms and $u:[0,1]\times[0,1]\rightarrow[0,1]$

(u_1) : (Boundary condition)

$u(a, 0) = a \qquad \forall a \in [0,1]$

(u_2) : (Monotone property)

$b \le d$

$\Rightarrow u(a,b) \le u(a,d) \qquad \forall a$

(u_3) : (Commutative property)

$u(a,b) = u(b,a)$

(u_4) : (Associative property)

$$u(a, u(b,d)) = u(u(a,b), d)$$

The function u is called a **t-conorm** or **fuzzy union**.

In addition, the function u may satisfy the following

(u_5) : u is continuous.

(u_6) : $u(a,a) > a$ [Superidempotent]

(u_7) : $a_1 < a_2$, $b_1 < b_2$

$\Rightarrow u(a_1, b_1) < u(a_2, b_2)$ [Strict monotone]

A t-conorm which is continuous and superidempotent is called "Archimedian t-conorm". Further, if it is strictly monotone, then, it is called a "strict Archimedian t-conorm".

2. Theorem

The standard fuzzy union is the only idempotent t-conorm.

Proof : The standard fuzzy union is given by, max operator.

$$(A \cup B)(x) = \max(A(x), B(x))$$

i.e. $u \equiv \max$

Then,

(u_1) : $\max(a,0) = a \quad \forall a \in [0,1]$

(u_2) : $b \le d \Rightarrow \max(a,b) \le \max(a,d)$

(u_3) : $\max(a,b) = \max(b,a)$

(u_4) : $\max(a, \max(b,d))$

$$= a \vee (b \vee d)$$

$$= (a \vee b) \vee d$$

$$= \max\{\max(a,b), d\}$$

Thus, 'max' satisfies u_1, u_2, u_3, u_4.

and hence, it is a t-conorm.

Also, $\max(a,a) = a \qquad \forall a \in [0,1]$

Thus, max is an idempotent t-conorm.

Next if u is any other t-conorm, which is idempotent.

i.e. $u'(a,a) = a$

Then, for $a,b \in [0,1]$

either $a \le b$ or $b \le a$

1) If $0 \le a \le b \le 1$ then,

$b = u'(0,b) \le u'(a,b) \le u'(b,b) = b$

$\Rightarrow u'(a,b) = b = \max(a,b) \qquad (\because a \le b)$

Similarly,

2) If $0 \le b \le a \le 1$,

Then,

$a = u'(a,0) \le u'(a,b) \le u'(a,a) = a$

$\Rightarrow u'(a,b) = a = \max\{a,b\}$

Thus,

$\forall a,b \in [0,1]$

$u'(a,b) = \max(a,b)$

i.e. $u' \equiv \max \equiv u \Rightarrow u \equiv u'$

Thus, "$\max \equiv u$" which is the standard fuzzy union is the only t-conorm, which is idempotent.

3. Examples :

Following are some examples of t-conorms.

1) Standard fuzzy union

$$u(a,b) = \max(a,b) = a \vee b$$

2) Algebraic sum

$$u(a,b) = a + b - ab$$

3) Bounded sum

$$u(a,b) = \min(1, a+b)$$

4) Drastic union

$$u(a,b) = \begin{cases} a & \text{if } b = 0 \\ b & \text{if } a = 0 \\ 1 & \text{otherwise} \end{cases}$$

Drastic union is denoted by u_{max}.

4) Theorem

For all $a, b \in [0,1]$.

$$\max(a,b) \le u(a,b) \le u_{\max}(a,b)$$

Proof :

1) We have, u, which is given t-conorm.

Then, $\forall a, b \in [0,1]$.

$\Rightarrow 0 \le a$, $0 \le b$

$\Rightarrow a = u(a,0) \le u(a,b)$

and $b = u(b,0) = u(0,b) \le u(a,b)$

Thus

$a \le u(a,b)$ and $b \le u(a,b)$

$\Rightarrow \max(a,b) \le u(a,b) \qquad \forall a,b \in [0,1] \qquad$ .... (1)

2) Next,

When $b = 0$,

$u(a,b) = u(a,0) = a$

and when $a = 0$.

$u(a,b) = u(0,b) = b$

Also

$u(a,b) \le 1 \qquad \forall a,b$

Thus

$$u(a,b) = \begin{cases} a & \text{if } b = 0 \\ b & \text{if } a = 0 \\ \le 1 & \text{Otherwise} \end{cases}$$

But

$$u_{\max}(a,b) = \begin{cases} a & \text{if } b = 0 \\ b & \text{if } a = 0 \\ 1 & \text{Otherwise} \end{cases}$$

Hence,

$u(a,b) \le u_{\max}(a,b)$ ..... (2)

Thus, from (1) and (2),

$\max(a,b) \le u(a,b) \le u_{\max}(a,b) \qquad \forall a,b \in [0,1]$

5. Note

Let, u be a binary operation on the unit interval [0, 1]. Then u is an Archimedian t-conorm iff there exists an increasing generator g such that,

$$u(a,b) = g^{(-1)}\left(g(a)+g(b)\right)$$

6. Example

Show that, a function $g_p : [0,1] \to \mathbb{R}$ defined by $g_p(a) = 1-(1-a)^p$, $p > 0$ is an increasing generator. Find its psuedo inverse and obtain a class of t-conorms generated by g_p.

Proof : We have,

$$g_p(a) = 1-(1-a)^p$$

$$g_p(0) = 0$$

Also, g is continuous function

and, $a < b$

$$\Rightarrow -a > -b$$

$$\Rightarrow 1-a > 1-b$$

$$\Rightarrow (1-a)^p > (1-b)^p$$

$$\Rightarrow -(1-a)^p < -(1-b)^p$$

$$\Rightarrow 1-(1-a)^p < 1-(1-b)^p$$

$$\Rightarrow g_p(a) < g_p(b)$$

i.e. g_p is an strictly increasing function and g_p is an increasing generator.

Thus, g_p is an increasing generator.

The pseudo inverse of g_p is given by**,**

$$g_p^{(-1)} : \mathbb{R} \to [0,1]$$

$$g_p^{(-1)}(a) = \begin{cases} 0 & a \in (-\infty, g(0)) \\ g_p^{-1}(a) & a \in [g(0), g(1)] \\ 1 & a \in (g_p(1), \infty) \end{cases}$$

Here, $g_p(0) = 0$, $g_p(1) = 1$

Let,

$$g_p^{-1}(a) = b \Rightarrow a = g_p(b)$$

$$\Rightarrow a = 1 - (1-b)^p$$

$$\Rightarrow a - 1 = -(1-b)^p$$

$$\Rightarrow (1-a) = (1-b)^p$$

$$\Rightarrow (1-a)^{1/p} = (1-b)$$

$$\Rightarrow b = 1 - (1-a)^{1/p}$$

$$\Rightarrow g_p^{-1}(a) = 1 - (1-a)^{1/p}$$

Thus

$$g_p^{(-1)}(a) = \begin{cases} 0 & a \in (-\infty, 0) \\ 1 - (1-a)^{1/p} & a \in [0,1] \\ 1 & a \in (1, \infty) \end{cases} \qquad \text{.... (*)}$$

Define, a t-conorm 'u' by,

$$u(a,b) = g_p^{(-1)}\left(g_p(a) + g_p(b)\right)$$

$$= g_p^{(-1)}\left[1-(1-a)^p + 1-(1-b)^p\right]$$

$$= g_p^{(-1)}\left[2-(1-a)^p-(1-b)^p\right]$$

Thus,

$$u(a,b)=1-\left[1-2+(1-a)^p+(1-b)^p\right]^{1/p} \quad \text{if } 2-(1-a)^p-(1-b)^p \in [0,1]$$

$$= 1 \quad \text{otherwise}$$

$$\Rightarrow u(a,b)=1-\left[(1-a)^p+(1-b)^p-1\right]^{1/p} \quad \text{if } 2-(1-a)^p-(1-b)^p$$

$$= 1 \quad \text{otherwise}$$

Then,

$$u(a,b)=1-\max\left\{0,\left[(1-a)^p+(1-b)^p-1\right]^{1/p}\right\}$$

This t-conorm is called Schweizer and Sklar t-conorm for p = 1, we get,

$$u(a,b)=1-\max\{0,(1-a-b)\}$$

$$= \min(1,1)+\min\{0,a+b-1\}$$

$$= \min\{1,a+b\}$$

which is bounded sum t-conorm.

7. Example (Yager's class of t-conorms)

If $g_w(a)=a^w$, w > 0, show that g_w is an increasing generator and t-conorm generated by g_w is given by**,**

$$u_w(a,b)=\min\left(1,\left(a^w+b^w\right)^{1/w}\right),\ w>0$$

Proof : $g_w(a)=a^w$, $w > 0$

1) Clearly, g_w is continuous and $g_w(0)=0$.

2) Next, for $a,b\in[0,1]$.

and $a < b$

$\Rightarrow a^w < b^w$

$\Rightarrow g_w(a) < g_w(b)$

$\Rightarrow g_w$ is strictly increasing.

$\Rightarrow g_w$ is an increasing generator function.

3) Then, the pseudo inverse of g_w is given by,

$$g_w^{(-1)}(a) = \begin{cases} 0 & a \in (-\infty, 0) \\ g_w^{-1}(a) & a \in [0, g_w(1)] \\ 1 & a \in (g_w(1), \infty) \end{cases}$$

Here, $g_w(1) = 1$.

Next, let $g_w^{(-1)}(a) = b$

$\Rightarrow a = g_w(b)$

$\Rightarrow a = b^w$

$\Rightarrow b = a^{1/w}$

$\Rightarrow g_w^{(-1)}(a) = a^{1/w}$

Hence,

$$g_w^{(-1)}(a) = \begin{cases} 0 & a \in (-\infty, 0) \\ a^{1/w} & a \in [0,1] \\ 1 & a \in (1, \infty) \end{cases}$$

4) Define, t-conorm u by,

$$u(a,b) = g_w^{(-1)}\left(g_w(a) + g_w(b)\right)$$

$$= g_w^{(-1)}\left(a^w + b^w\right)$$

$$= \left(a^w + b^w\right)^{1/w}$$

Hence,

$$u(a,b) = \begin{cases} (a^w + b^w)^{1/w} & \text{if } a^w + b^w \in [0,1] \\ 1 & \text{otherwise} \end{cases}$$

Then,

$$u(a,b) = \min\left\{1, (a^w + b^w)^{1/w}\right\},\ w > 0$$

Thus, the t-conorm

$$u(a,b) = \min\left(1, (a^w + b^w)^{1/w}\right),\ w > 0$$

is a Yager's class of t-conorms.

Note : If $a \le b$ then $u_a\ (a,b) \ge u_b\ (a,b)$ i.e. $\{u_w\}$ is decreasing as $w \to \infty$.

8. Theorem

Let, u_w denotes the Yager's class of t-conorms defined by,

$$u_w\ (a,b) = \min\left(1, (a^w + b^w)^{1/w}\right),\ w > 0$$

Then,

$$\max(a,b) \le u_w\ (a,b) \le u_{\max}\ (a,b)$$

Where, max is the standard fuzzy union and $u_{\max}$ is the drastic union t-conorm.

Proof :

1) First we prove that,

$$\lim_{w \to 0} u_w\ (a,b) = u_{\max}\ (a,b)$$

$a = 0$ or $b = 0$

Then,

$$u_{\max}\ (a,b) = \begin{cases} b & \text{if } a = 0 \\ a & \text{if } b = 0 \end{cases}$$

Now for $a = 0$

$$\lim_{w\to 0} u_w(a,b) = \lim_{w\to\infty} \min\left(1, \left(a^w + b^w\right)^{1/w}\right)$$

$$= \lim_{w\to\infty} \min(1, b)$$

$$= b$$

Simillarly for $b = 0$

$$\lim_{w\to 0} u_w(a,b) = a$$

Next,

$$\lim_{w\to 0} u_w(a,b) \le 1$$

and $u_{\max}(a,b) = 1$ if $a \ne 0$ and $b \ne 0$

Thus,

$$\lim_{w\to 0} u_w(a,b) = \begin{cases} b & \text{if } a = 0 \\ a & \text{if } b = 0 \\ \le 1 & \text{otherwise} \end{cases}$$

And

$$u_{\max}(a,b) = \begin{cases} b & \text{if } a = 0 \\ a & \text{if } b = 0 \\ 1 & \text{otherwise} \end{cases}$$

Hence, we get,

$$\lim_{w\to 0} u_w(a,b) \le u_{\max}(a,b) \qquad \forall a,b \in [0,1]$$

i.e. $u_w(a,b) \le u_{\max}(a,b)$ $\forall\, w > 0$ ...... (1)

To prove that, $\max(a,b) \le u_w(a,b)$

We will show that,

$$\lim_{w\to\infty} u_w(a,b) = \max(a,b)$$

1) $a = 0$ or $b = 0$

If $a = 0$

Then, $\max(a,b) = b$

And $\lim_{w\to\infty} u_w(a,b) = \lim_{w\to\infty} \min\left\{1,\left(a^w + b^w\right)^{1/w}\right\}$

$= b$

Simillarly, if $b = 0$ then, we get,

$$\max(a,b) = a = \lim_{w\to\infty} u_w(a,b)$$

2) $a \neq 0$ and $b \neq 0$ and $a = b$

Then,

$\max(a,b) = a$ (or b)

And $\lim_{w\to\infty} u_w(a,b) = \lim_{w\to\infty} \min\left\{1,\left(a^w + b^w\right)^{1/w}\right\}$

$= \lim_{w\to\infty} \min\left\{1,\left(2a^w\right)^{1/w}\right\}$

$= \lim_{w\to\infty} \min\left\{1, 2^{1/w} a\right\}$

$= \min\{1, a\}$

$= a$

i.e. $\max(a,b) = \lim_{w\to\infty} u_w(a,b)$

3) $a \neq 0$, $b \neq 0$ and $a \neq b$.

We assume that, $a < b$

Then, $\max(a,b) = b$ .... (*)

And $\lim_{w\to\infty} u_w(a,b) = \lim_{w\to\infty} \min\left\{1, \left(a^w + b^w\right)^{1/w}\right\}$

$$= \min\left\{1, \lim_{w\to\infty} \left(a^w + b^w\right)^{1/w}\right\}$$

Let, $Q = \left(a^w + b^w\right)^{1/w}$

$\Rightarrow \log Q = \log\left(a^w + b^w\right)^{1/w}$

$\Rightarrow \log Q = \frac{1}{w}\log\left(a^w + b^w\right)$

$\Rightarrow \lim_{w\to\infty} \log Q = \lim_{w\to\infty} \frac{\log\left(a^w + b^w\right)}{w}$ .... $\left[\frac{\infty}{\infty}\right]$ .... $(a \neq 1,\ b \neq 1)$

$$= \lim_{w\to\infty} \frac{\frac{1}{a^w + b^w}\left[a^w \log a + b^w \log b\right]}{1}$$

$$= \lim_{w\to\infty} \frac{a^w \log a + b^w \log b}{a^w + b^w}$$

$$= \lim_{w\to\infty} \frac{\frac{a^w}{b^w}\log a + \log b}{\frac{a^w}{b^w} + 1}$$

$$= \lim_{w\to\infty} \frac{\left(\frac{a}{b}\right)^w \log a + \log b}{\left(\frac{a}{b}\right)^w + 1}$$

$$= \frac{\log b}{1}$$

$\Rightarrow \lim_{w\to\infty} \log Q = \log b$ $\left(\because \log\left(\lim_{w\to\infty} Q\right) = \log b\right)$

$\Rightarrow \lim_{w\to\infty} Q = b$

i.e. $\lim_{w\to\infty} \left(a^w + b^w\right)^{1/w} = b$

$\Rightarrow \lim_{w\to\infty} u_w(a,b) = \lim_{w\to\infty} \min\left\{1, \left(a^w + b^w\right)^{1/w}\right\}$

$= \min\{1, b\}$

$= b$

Thus if, $a < b$,

Then, $\lim_{w\to\infty} u_w(a,b) = b = \max\{a,b\}$ - by (*)

4) If $a = 1$ or $b = 1$ and $a \neq b$.

Then,

$a^w + b^w = 1 + b^w$ or $a^w + 1$

≥ 1

$\Rightarrow \left(a^w + b^w\right)^{1/w} \geq 1$

And, hence,

$\min\left\{1, \left(a^w + b^w\right)^{1/w}\right\} = 1$

i.e. $u_w(a,b) = 1$

And also,

$\max(a,b) = 1$

i.e. $u_w(a,b) = \max(a,b)$ $\forall w > 0$

Hence we get

$\lim_{w\to\infty} u_w(a,b) = \max(a,b)$

Further, $u_w(a,b)$ is a decreasing function and hence,

$$u_w(a,b) \geq \max(a,b)$$

i.e. $\max(a,b) \leq u_w(a,b) \qquad \forall w > 0 \qquad \text{..... (2)}$

Hence, we have from (1) and (2)

$$\max(a,b) \leq u_w(a,b) \leq u_{\max}(a,b) \qquad \forall w > 0$$

3.4 Dual Triple

1. Definition

A t-norm i and t-conorm u are said to be dual w.r.t. fuzzy complement c if

$$c(i(a,b)) = u(c(a), c(b))$$

$$c(u(a,b)) = i(c(a), c(b))$$

The triplet < *i, u, c* > is called a 'dual triple'.

2. Note

The triplet < *i, u, c* > is called a dual triple if they satisfies DeMorgan's laws.

3. Example

Show that < *ab, a + b – ab, c* > is a dual triple where c is a standard fuzzy complement.

Proof : Let,

$$i(a,b) = ab$$

$$u(a,b) = a + b - ab \qquad \text{.... (1)}$$

$$c(a) = 1 - a$$

be the three operators.

Consider,

$$c(u(a,b)) = c(a + b - ab)$$

$$= 1 - a - b + ab$$

$$=1(1-a)-b(1-a)$$

$$=(1-a)\cdot(1-b)$$

$$=c(a)\cdot c(b)$$

$$=i(c(a),c(b)) \qquad \text{.... (2)}$$

Similarly,

$$c(i(a,b))=c(a\cdot b)$$

$$=1-ab \qquad \text{..... (3)}$$

And

$$u(c(a),c(b))=u(1-a,1-b)$$

$$=(1-a)+(1-b)-(1-a)(1-b)$$

$$=2-a-b-(1-b-a+ab)$$

$$=2-a-b-1+b+a-ab$$

$$=1-ab \qquad \text{... (4)}$$

Thus by (3) and (4)

$$c(i(a,b))=u(c(a),c(b)) \qquad \text{.... (5)}$$

Thus, by (2) and (5)

$< i, u, c >$ is a dual triple.

4. Example

Show that $< \max(0, a+b-1), \min(1, a+b), c >$ is a dual triple, where c is standard fuzzy complement.

Proof : Let,

$$i(a,b)=\max(0,a+b-1)$$

$$u(a,b)=\min(1,a+b) \qquad \text{.... (1)}$$

$c(a) = 1 - a$

Let, $a, b \in [0,1]$ be arbitrary.

Case (1) : $a + b > 1$,

Then,

$c(i(a,b)) = c(\max(0, a+b-1))$

$= c(a+b-1)$

$= 1 - a - b + 1$

$\Rightarrow c(i(a,b)) = 2 - a - b$ .... (2)

And

$u(c(a), c(b)) = u(1-a, 1-b)$

$= \min(1, 1-a+1-b)$

$= \min(1, 2-(a+b))$

$= 2 - (a+b)$

$= 2 - a - b$ ..... (3)

Hence, from (2) and (3)

$c(i(a,b)) = u(c(a), c(b))$

Case (2) : $a + b \leq 1$

Then,

$c(i(a,b)) = c(\max(0, a+b-1))$

$= c(0)$

$= 1 - 0$

$\Rightarrow c(i(a,b)) = 1$ .... (4)

And

$u(c(a), c(b)) = u(1-a, 1-b)$

$$= \min(1, 1 - a + 1 - b)$$

$$= \min(1, 2 - a - b)$$

$$= 1 \qquad (\because 2 - a - b \geq 1) \qquad \text{.... (5)}$$

Hence, from (4) and (5)

$$c(i(a,b)) = u(c(a), c(b)) \qquad \text{.... (6)}$$

Next, we prove that

$$c(u(a,b)) = i(c(a), c(b))$$

Case (1) :

$$a + b > 1$$

$$c(u(a,b)) = c(\min(1, a + b))$$

$$= c(1)$$

$$= 1 - 1 = 0 \qquad \text{... (7)}$$

And

$$i(c(a), c(b)) = i(1 - a, 1 - b)$$

$$= \max(0, 1 - a + 1 - b - 1)$$

$$= \max(0, 1 - (a + b))$$

$$= 0 \qquad \text{... (8)}$$

Hence, from (7) and (8)

$$c(u(a,b)) = i(c(a), c(b))$$

Case (2) : $a + b \leq 1$

$$c(u(a,b)) = c(\min(1, a + b))$$

$$= c(a + b)$$

$$= 1 - a - b \qquad \text{.... (9)}$$

And

$$i(c(a), c(b)) = i(1-a, 1-b)$$

$$= \max\{0, 1-a+1-b-1\}$$

$$= \max\{0, 1-(a+b)\}$$

$$= 1-(a+b)$$

$$= 1-a-b \qquad \text{.... (10)}$$

Thus, from (9) and (10)

$$c(u(a,b)) = i(c(a), c(b)) \qquad \text{...... (11)}$$

Hence, from (6) and (11)

We can say that,

$\langle \max(0, a+b-1), \min(1, a+b), c \rangle$ is a dual triple.

5. Theorem

The following triples are dual w.r.t. standard fuzzy complement 'c'.

(1) $\langle \min, \max, c \rangle$

(2) $\langle i_{\min}, u_{\max}, c \rangle$

Proof :

(1) Let, $a, b \in [0,1]$

Case (1) : $a \geq b$

Let, $i(a,b) = \min(a,b)$

$$u(a,b) = \max(a,b) \qquad \text{.... (1)}$$

And $c(a) = 1-a$

Consider,

$$c(i(a,b)) = c(\min(a,b))$$

$$= c(b)$$

$=1-b$

And, $u\left(c(a),c(b)\right)=u(1-a,1-b)$

$= \max(1-a,1-b)$

$=1-b$

$\Rightarrow c\left(i(a,b)\right)=u\left(c(a),c(b)\right)$

And $c\left(u(a,b)\right)=c\left(\max(a,b)\right)$

$=c(a)$

$=1-a$

And $i\left(c(a),c(b)\right)=i(1-a,1-b)$

$= \min(1-a,1-b)$

$=1-a$

$\Rightarrow c\left(u(a,b)\right)=i\left(c(a),c(b)\right)$

Case (2) : $a<b$

Then,

$c\left(i(a,b)\right)=c\left(\min(a,b)\right)$

$=c(a)$

$=1-a$

And

$u\left(c(a),c(b)\right)=u(1-a,1-b)$

$= \max(1-a,1-b)$

$=1-a$

$\Rightarrow c\left(i(a,b)\right)=u\left(c(a),c(b)\right)$

Simillalry,

$$c\left(u\left(a,b\right)\right)=c\left(\max\left(a,b\right)\right)$$

$$=c\left(b\right)$$

$$=1-b$$

And,

$$i\left(c\left(a\right),c\left(b\right)\right)=i\left(1-a,1-b\right)$$

$$=\min\left(1-a,1-b\right)$$

$$=1-b$$

$$c\left(u\left(a,b\right)\right)=i\left(c\left(a\right),c\left(b\right)\right)$$

2. Let, $a,b\in[0,1]$ be arbitrary, we have

$$u_{\max}\left(a,b\right)=a \quad \text{if } b = 0$$

$$= b \quad \text{if } a = 0 \qquad \text{.... (2)}$$

$$= 1 \quad \text{otherwise}$$

And

$$i_{\min}\left(a,b\right)=a \quad \text{if } b = 1$$

$$= b \quad \text{if } a = 1 \qquad \text{.... (3)}$$

$$= 0 \quad \text{otherwise}$$

Case (1) :

We are considering $a = 1$

Thus, $c\left(u_{\max}\left(a,b\right)\right)=c(1)=0$

And $i_{\min}\left(c\left(a\right),c\left(b\right)\right)=i_{\min}\left(0,c\left(b\right)\right)=0$

Therefore,

$$c\left(u_{\max}\left(a,b\right)\right)=i_{\min}\left(c\left(a\right),c\left(b\right)\right)$$

Simillarly

$$c\left(i_{\min}\left(a,b\right)\right)=c\left(b\right)=1-b$$

And

$$u_{\max}\left(c(a),c(b)\right)=u_{\max}\left(0,c(b)\right)=c(b)=1-b$$

$$\therefore c\left(i_{\min}(a,b)\right)=u_{\max}\left(c(a),c(b)\right)$$

Case (2) :

we are considering $a = 0$

Then,

$$c\left(u_{\max}(a,b)\right)=c(b)$$

$$=1-b$$

And

$$i_{\min}\left(c(a),c(b)\right)=i_{\min}\left(1,c(b)\right)$$

$$=c(b)$$

$$=1-b$$

Thus

$$c\left(u_{\max}(a,b)\right)=i_{\min}\left(c(a),c(b)\right)$$

Simillarly

$$c\left(i_{\min}(a,b)\right)=c(0)=1$$

And,

$$u_{\max}\left(c(a),c(b)\right)=u_{\max}\left(1,c(b)\right)=1$$

Thus,

$$c\left(i_{\min}(a,b)\right)=u_{\max}\left(c(a),c(b)\right)$$

Case (3) :

$a \neq 0$, 1 and $b \neq 0$,1

Then,

$c\left(u_{\max}(a,b)\right)=c(1)=0$

And

$i_{\min}\left(c(a),c(b)\right)=0$

i.e.

$c\left(u_{\max}(a,b)\right)=i_{\min}\left(c(a),c(b)\right)$

Simillarly

$c\left(i_{\min}(a,b)\right)=c(0)=1$

And

$u_{\max}\left(c(a),c(b)\right)=1$

Thus,

$c\left(i_{\min}(a,b)\right)=u_{\max}\left(c(a),c(b)\right)$

Thus, for all $a,b\in[0,1]$, we have,

6. Example

Show that $\left\langle ab,\frac{a+b}{ab+1},\frac{1-a}{1+a}\right\rangle$ is a dual triple

Proof : Let, $i(a,b)=ab$

$$u(a,b)=\frac{a+b}{ab+1} \quad \text{.... (1)}$$

And $c(a)=\frac{1-a}{1+a}$

Consider,

$c\left(i(a,b)\right)=c(a\cdot b)$

$$= \frac{1-ab}{1+ab} \qquad \text{.... (2)}$$

$$u\left(c(a), c(b)\right) = u\left(\frac{1-a}{1+a} \cdot \frac{1-b}{1+b}\right)$$

$$= \frac{\frac{1-a}{1+a} + \frac{1-b}{1+b}}{1 + \left(\frac{1-a}{1+a}\right)\left(\frac{1-b}{1+b}\right)}$$

$$= \frac{\frac{(1-a)(1+b) + (1+a)(1-b)}{(1+a)(1+b)}}{\frac{(1+a)(1+b) + (1-a)(1-b)}{(1+a)(1+b)}}$$

$$= \frac{2-2ab}{2+2ab}$$

$$= \frac{1-ab}{1+ab} \qquad \text{.... (3)}$$

Thus, from (2) and (3)

$$c\left(i(a,b)\right) = u\left(c(a), c(b)\right) \qquad \text{.... (4)}$$

Simillarly

$$c\left(u(a,b)\right) = c\left(\frac{a+b}{ab+1}\right)$$

$$= \left(1 - \frac{a+b}{ab+1}\right) \Big/ \left(1 + \frac{a+b}{ab+1}\right)$$

$$= \frac{ab+1-a-b}{ab+1+a+b} = \frac{(1-a) - b(1-a)}{(1+a) + b(1+a)}$$

$$= \frac{(1-a)(1-b)}{(1+a)(1+b)} \qquad \text{... (5)}$$

Now,

$$i(c(a), c(b)) = i\left(\frac{1-a}{1+a}, \frac{1-b}{1+b}\right)$$

$$= \frac{(1-a)}{(1+a)} \cdot \frac{(1-b)}{(1+b)}$$

$$= \frac{(1-a)(1-b)}{(1+a)(1+b)} \qquad \text{... (6)}$$

Thus, from (5) and (6)

$$c(u(a,b)) = i(c(a), c(b)) \qquad \text{.... (7)}$$

Thus, from (4) and (7)

$\left\langle ab, \dfrac{a+b}{ab+1}, \dfrac{1-a}{1+a} \right\rangle$ is a dual triple.

7. Theorem

Let i be a t-norm and c be an involutive fuzzy complement. The binary operation u defined on [0, 1] by,

$u(a,b) = c\langle i(c(a), c(b))\rangle$ is a t-conorm such that

$\langle i, u, c\rangle$ is a dual triple.

Proof : Given that

$$u : [0,1] \times [0,1] \rightarrow [0,1]$$

$$u(a,b) = c\langle i(c(a), c(b))\rangle$$

We'll show that, u satisfies the axioms u_1, u_2, u_3 and u_4.

(u_1) : Consider,

$$u(0,a) = c\langle i(c(0), c(a))\rangle$$

$$= c\langle i(1, c(a))\rangle$$

$$= c\langle c(a)\rangle$$

$$= a$$

i.e. $u(0,a) = a$ $\forall a \in [0,1]$

(u_2) : Let, $a,b,d,e[0,1]$ s.t. $b \le d$

Then, $b \le d$

$\Rightarrow c(b) \ge c(d)$

$\Rightarrow i(c(a),c(b)) \ge i(c(a),c(b))$

$\Rightarrow c(i(c(a),c(b))) \le c(i(c(a),c(b)))$

$\Rightarrow u(a,b) \le u(a,d)$

i.e. u is monotone.

(u_3) : We have,

$$u(a,b) = c\langle i(c(a),c(b))\rangle$$

$$= c\langle i(c(b),c(a))\rangle$$

$$= u(b,a)$$

$\Rightarrow$ u is commutative.

(u_4) : Let, $a,b,d,e[0,1]$

Consider

$$u(a,u(b,d)) = c\left(i\langle c(a),c(u(b,d))\rangle\right)$$

$$= c\left(i\left(c(a),c\left(c\left(i(c(b),c(d))\right)\right)\right)\right)$$

$$= c\left(i\left(c(a),i(c(b),c(d))\right)\right)$$

$$= c\left(i\left(i(c(a),c(b),c(d))\right)\right)$$

$$= c\left(i\left\langle c\left(c\left(i(c(a),c(b))\right)\right),c(d)\right\rangle\right)$$

$$= c\left(i\left\langle c\left(u\left(a,b\right)\right), c\left(d\right)\right\rangle\right)$$

$$= u\left(u\left(a,b\right), d\right)$$

Thus, u is associative.

Thus, u satisfies u_1, u_2, u_3 and u_4.

Therefore, u is a t-conorm.

Next, we will show that, $\langle i, u, c\rangle$ is a dual triple.

$$c\left(u\left(a,b\right)\right) = c\left\{c\left\langle i\left(c\left(a\right), c\left(b\right)\right)\right\rangle\right\}$$

$$\Rightarrow c\left(u\left(a,b\right)\right) = i\left\langle\left(c\left(a\right), c\left(b\right)\right)\right\rangle \qquad \text{... (1)}$$

And, $u\left(c\left(a\right), c\left(b\right)\right)$

$$= c\left\{i\left\langle c\left(c\left(a\right)\right), c\left(c\left(b\right)\right)\right\rangle\right\}$$

Therefore,

$$u\left(c\left(a\right), c\left(b\right)\right) = c\left(i\left(a,b\right)\right) \qquad \text{.... (2)}$$

Hence from (1) and (2)

$\langle i, u, c\rangle$ is a dual triple.

8. Example

Let, $i\left(a,b\right) = ab$ and let, $c_l\left(a\right) = \dfrac{1-a}{1+l\,a}$, $l \geq 0$ be the Sugeno's class of fuzzy complements, obtain the t-conorm u such that $\langle i, u, c\rangle$ is a dual triple.

Proof : Define a t-conorm u by,

$$u\left(a,b\right) = c\left\{i\left(c\left(a\right), c\left(b\right)\right)\right\}$$

$$= c_l\left\{i\left(\frac{1-a}{1+l\,a}, \frac{1-b}{1+l\,b}\right)\right\} \qquad l \geq 0$$

$$= c_l \left\{ \frac{(1-a)(1-b)}{(1+la)(1+lb)} \right\}$$

$$= \frac{1 - \left\{ \frac{(1-a)(1-b)}{(1+la)(1+lb)} \right\}}{1 + l \left\{ \frac{(1-a)(1-b)}{(1+la)(1+lb)} \right\}}$$

$$= \frac{(1+la)(1+lb) - (1-a)(1-b)}{(1+la)(1+lb) + (1-a)(1-b)}$$

$$= \frac{a(l+1) + b(l+1) + ab\left(l^2 - 1\right)}{(1+l) + lab(1+l)}$$

$$= \frac{a + b + (l-1)ab}{1 + lab} \qquad \text{dividing by } l+1.$$

Thus, $\left\langle a \cdot b, \frac{a+b+(l-1)ab}{1+lab}, \frac{1-a}{1+la} \right\rangle$, $l \geq 0$ is a dual triple.

1) For, $l = 0$, we get

$$\langle a \cdot b, a + b - ab, 1 - a \rangle$$

2) For, $l = 1$, we get

$$\left\langle a \cdot b, \frac{a+b}{1+ab}, \frac{1-a}{1+a} \right\rangle$$

9. Theorem

Given a t-conorm u and an involutive fuzzy complement c. Then, the binary operation i on [0, 1] defined by,

$$i(a,b) = c\left(u\left(c(a), c(b)\right)\right), \ \forall a, b \in [0,1]$$

is a t-norm such that $\langle i, u, c \rangle$ is a dual triple.

Proof : Given that,

$i : [0,1] \times [0,1] \to [0,1]$ by

$$i(a,b) = c\left(u\left\langle c(a), c(b)\right\rangle\right)$$

(i_1) $\quad i(a,1) = c\left(u\left\langle c(a), c(1)\right\rangle\right)$

$$= c\left(u\left\langle c(a), 0\right\rangle\right)$$

$$= c(c(a))$$

$$= a \qquad \forall a \in [0,1]$$

(i_2) : Let, $a, b, d \in [0,1]$ such that, $b \le d$

Then, we have,

$$i(a,b) = c\left\langle u\left(c(a), c(b)\right)\right\rangle$$

Now,

$b \le d$

$$\Rightarrow c(b) \ge c(d)$$

$$\Rightarrow u\left(c(a), c(b)\right) \ge u\left(c(a), c(d)\right)$$

$$\Rightarrow c\left\langle u\left(c(a), c(b)\right)\right\rangle \le c\left\langle u\left(c(a), c(d)\right)\right\rangle$$

$$\Rightarrow i(a,b) \le i(a,d)$$

$\Rightarrow$ i is monotonic.

(i_3) : $\quad i(a,b) = c\left\langle u\left(c(a), c(b)\right)\right\rangle$

$$= c\left\langle u\left(c(b), c(a)\right)\right\rangle$$

$$= i(b,a)$$

$\Rightarrow$ i is commutative.

(i_4) : For $a,b,d \in [0,1]$,

Consider

$$i(a,i(b,d))$$

$$= c\left\langle u\left[c(a), c\left\langle i(b,d)\right\rangle\right]\right\rangle$$

$$= c\left\langle u\left[c(a), c\left\langle c\left(u\left(c(b),c(d)\right)\right)\right\rangle\right]\right\rangle$$

$$= c\left\langle u\left[c(a), u\{c(b),c(d)\}\right]\right\rangle$$

$$= c\left\langle u\left[u\left\langle c(a),c(b)\right\rangle, c(d)\right]\right\rangle$$

$$= c\left\langle u\left\langle c\left\{c\left[u\left\langle c(a),c(b)\right\rangle\right]\right\}, c(d)\right\rangle\right\rangle$$

$$= c\left\langle u\left\{c\left\langle i(a,b)\right\rangle\right\}, c(d)\right\rangle$$

$$= i\left(i(a,b),d\right)$$

Thus,

$$i(a,i(b,d)) = i\left(i(a,b),d\right)$$

$\Rightarrow$ i is associative.

Thus i satisfies i_1, i_2, i_3 and i_4.

Hence i is a t-norm.

Next, we will prove that $\langle i,u,c\rangle$ is a dual triple.

Consider,

$$c\left(i(a,b)\right) = c\left\langle c\left(u\left(c(a),c(b)\right)\right)\right\rangle$$

$$= u\left\langle c(a),c(b)\right\rangle \quad \text{.... (1)}$$

and,

$$i(c(a), c(b)) = c\langle u(c(c(a)), c(c(b)))\rangle$$

$$= c\langle u(a, b)\rangle \qquad .... (2)$$

Hence, from (1) and (2)

$\langle i, u, c\rangle$ is a dual triple.

10. Example :

Show that, 1) $\langle \min, \max, c_l\rangle$

2) $\langle \min, \max, c_w\rangle$

are dual triples, where c_l is Sugeno's class of fuzzy complements and c_w is a Yager**'s** class of fuzzy complements.

Proof :

1) We know that, Sugeno's class of fuzzy complements is given by,

$$c_l(a) = \frac{1-a}{1+la}, \quad l > 0$$

Case (1) : $a \le b$

$$c_l(i(a, b)) = c_l(\min(a, b))$$

$$= c_l(a)$$

$$= \frac{1-a}{1+la}$$

And,

$$u(c_l(a), c_l(b)) = \max(c_l(a), c_l(b))$$

$$= \max\left\{\frac{1-a}{1+la}, \frac{1-b}{1+lb}\right\}$$

$$= \frac{1-a}{1+la}$$

$$\Rightarrow c_l\left(i(a,b)\right)=u\left(c_l(a),c_l(b)\right)$$

And,

$$c_l\left(u(a,b)\right)=c_l\left(\max(a,b)\right)$$

$$=c_l(b)$$

$$=\frac{1-b}{1+lb}$$

And

$$i\left(c_l(a),c_l(b)\right)$$

$$=\min\left(c_l(a),c_l(b)\right)$$

$$=\min\left\{\frac{1-a}{1+la},\frac{1-b}{1+lb}\right\}$$

$$=\frac{1-b}{1+lb}$$

Thus,

$$c_l\left(u(a,b)\right)=i\left(c_l(a),c_l(b)\right) \qquad \text{.... (1)}$$

Case (2) : $a \geq b$

$$c_l\left(i(a,b)\right)=c_l\left(\min(a,b)\right)$$

$$=c_l(b)$$

$$=\frac{1-b}{1+lb}$$

And,

$$u\left(c_l(a),c_l(b)\right)=\max\left(c_l(a),c_l(b)\right)$$

$$=\max\left\{\frac{1-a}{1+la},\frac{1-b}{1+lb}\right\}$$

$$= \frac{1-b}{1+lb}$$

$$\Rightarrow c_l\left(i(a,b)\right) = u\left(c_l(a), c_l(b)\right)$$

And,

$$c_l\left(u(a,b)\right) = c_l\left(\max(a,b)\right)$$

$$= c_l(a)$$

$$= \frac{1-a}{1+la}$$

And,

$$i\left(c_l(a), c_l(b)\right)$$

$$= \min\left(c_l(a), c_l(b)\right)$$

$$= \min\left\{\frac{1-a}{1+la}, \frac{1-b}{1+lb}\right\}$$

$$= \frac{1-a}{1+la}$$

Thus,

$$c_l\left(u(a,b)\right) = i\left(c_l(a), c_l(b)\right) \quad \text{...... (2)}$$

Therefore from (1) and (2) (min, max, C_l) is dual triple.

2) We know that,

Yager's class of fuzzy complements is given by,

$$c_w(a) = \left(1-a^w\right)^{1/w},\ w > 0$$

Case (1) : $a \le b$

$$c_w\left(i(a,b)\right) = c_w\left(\min(a,b)\right)$$

$$= c_w(a)$$

$$= (1-a^w)^{1/w}, \ w > 0$$

And,

$$u(c_w(a), c_w(b)) = \max(c_w(a), c_w(b))$$

$$= \max\{(1-a^w)^{1/w}, (1-b^w)^{1/w}\}$$

$$= (1-a^w)^{1/w}$$

Then,

$$c_w(i(a,b)) = u(c_w(a), c_w(b))$$

and,

$$c_w(u(a,b)) = c_w(\max(a,b))$$

$$= c_w(b)$$

$$= (1-b^w)^{1/w}$$

and,

$$i(c_w(a), c_w(b)) = \min\{c_w(a), c_w(b)\}$$

$$= \min\{(1-a^w)^{1/w}, (1-b^w)^{1/w}\}$$

$$= (1-b^w)^{1/w}$$

$$\Rightarrow c_w(u(a,b)) = i(c_w(a), c_w(b)) \qquad \text{.... (3)}$$

Case (2) : $a \geq b$

Consider,

$$c_w(u(a,b)) = c_w(\max(a,b))$$

$$= c_w(a)$$

$$= (1-a^w)^{1/w}$$

And,

$$i\left(c_w(a), c_w(b)\right) = \min\{c_w(a), c_w(b)\}$$

$$= \min\left\{(1-a^w)^{1/w}, (1-b^w)^{1/w}\right\}$$

$$= (1-a^w)^{1/w}$$

$$\Rightarrow c_w\left(u(a,b)\right) = i\left(c_w(a), c_w(b)\right)$$

And,

$$c_w\left(i(a,b)\right) = c_w\left(\min(a,b)\right)$$

$$= c_w(b)$$

$$= (1-b^w)^{1/w}$$

And,

$$u\left(c_w(a), c_w(b)\right) = \max\{c_w(a), c_w(b)\}$$

$$= \max\left\{(1-a^w)^{1/w}, (1-b^w)^{1/w}\right\}$$

$$= (1-b^w)^{1/w}$$

$$\Rightarrow c_w\left(i(a,b)\right) = u\left(c_w(a), c_w(b)\right) \qquad \text{.... (4)}$$

Thus from (3) and (4) $\langle \min, \max, c_w \rangle$ is a dual triple.

11. Definition :

For crisp sets,

$A \cup \bar{A} = X$ - law of excluded middle

$A \cap \bar{A} = f$ - law of contradiction.

12. Theorem :

Let $\langle i, u, c \rangle$ be a dual triple where c is an involutive fuzzy complement and the t-norm i and the t-conorm u are generated by an increasing generator g. Then the fuzzy operations i, u, c satisfies the law of excluded middle and the law of contradiction.

Proof : c is an involutive fuzzy complement.

Hecne there exist an increasing generator g such that,

$$c(a) = g^{(-1)}\left(g(1) - g(a)\right)$$

Also i and u are generated by g.

$$i(a,b) = g^{(-1)}\left(g(a) + g(b) - g(1)\right)$$

$$u(a,b) = g^{(-1)}\left(g(a) + g(b)\right)$$

Consider, for any $a \in [0,1]$,

$$u\left(a, c(a)\right) = u\left(a, g^{(-1)}\left(g(1) + g(a)\right)\right)$$

$$= g^{(-1)}\left[g(a) + g\left(g^{(-1)}\left(g(1) - g(a)\right)\right)\right]$$

$$= g^{(-1)}\left[g(a) + g(1) - g(a)\right]$$

$$= g^{(-1)}\left(g(1)\right)$$

$$u\left(a, c(a)\right) = 1$$

This shows that law of excluded middle is satisfied,

Next consider for $a \in [0,1]$,

$$i\left(a, c(a)\right) = i\left(a, g^{(-1)}\left(g(1) - g(a)\right)\right)$$

$$= g^{(-1)}\left[g(a) + g\left(g^{(-1)}\left(g(1) - g(a)\right)\right) - g(1)\right]$$

$$= g^{(-1)}\left[g(a) + g(1) - g(a) - g(1)\right]$$

$= g^{(-1)}(0)$ $\quad (\because g(0) = 0)$

$= 0$

13. Theorem :

Let $\langle i, u, c \rangle$ be a dual triple that satisfies the law of excluded middles and the law of contradiction. Then $\langle i, u, c \rangle$ does not satisfy distributive law.

Proof : $\langle i, u, c \rangle$ is a dual triple and if *i* and *u* satisfies distributive law then we have,

$$i(a, u(b, d)) = u(i(a, b), i(a, d)) \qquad \forall a, b, d \in [0,1]$$

Let *e* be an equilibrium of *c*.

i.e. $c(e) = e$

Since $c(0) = 1$ and $c(1) = 0$.

$\Rightarrow e \neq 0$ and $e \neq 1$

Let $a = b = d = e$

Then $u(e, e) = u(e, c(e))$ (By law of excluded middle)

$= 1$

And $i(e, e) = i(e, c(e))$ (By law of contradiction)

$= 0$

Then, $i(e, u(e, e)) = i(e, 1) = e$

$u(i(e, e), i(e, e)) = u(0, 0) = 0$

Hence $i(e, u(e, e)) \neq u(i(e, e), i(e, e))$.

This shows that, *i* and *u* does not satisfy distributive law.

3.5 Aggregation Operation

Aggregation operation on n fuzzy sets ($n > 2$) is defined by $h:[0,1]^n \to [0,1]$ which satisfies following axioms.

1) $h(0,0,0....0)=0$
$h(1,1,1....1)=1$ (Boundary condition for h)

2) For $(a_1, a_2, ...a_n)$ and $(b_1, b_2, ...b_n)$ such that $a_i, b_i \in [0,1]$ such that $a_i \le b_i \quad \forall i$

$\Rightarrow h(a_1, a_2, ...a_n) \le h(b_1, b_2, ...b_n)$

i.e. h is monotonic increasing in all it's arguments.

3) h is continuous.

If h satisfies the above three axioms then it is called an aggregation operator.

In addition the aggregation operator h may satisfy.

4) h is symmetric in all it's arguments

i.e. $h(a_1, a_2, ...a_n) = h(a_{a1}, a_{a2}, ...a_{an})$

5) h is idempotent.

i.e. $h(a, a, a....a) = a$ for all $a \in [0,1]$

If the aggregation operator h satisfies idempotent property then we get,

$$\min(a_1, a_2, ...a_n) \le h(a_1, a_2, ...a_n) \le \max(a_1, a_2, ...a_n)$$

2. Example :

Define a function $h_a : [0,1]^n \to [0,1]$ by

$$h_a(a_1, a_2, ...a_n) = \left(\frac{a_1^a + a_2^a + ... + a_n^a}{n} \right)^{1/a}$$

where a_1's are not all zero and $a \in \mathbb{R}$ and $a \neq 0$.

Show that,

$$h_a = \begin{cases} \text{Arithmatic mean if } a = 0 \\ \text{Geometric mean if } a \to 0 \\ \text{Harmonic mean if } a = -1 \\ \max(a_1, a_2, ...a_n) \text{ if } a \to \infty \\ \min(a_1, a_2, ...a_n) \text{ if } a \to -\infty \end{cases}$$

Proof : Given

$$h_a(a_1, a_2, ...a_n) = \left(\frac{a_1^a + ... + a_n^a}{n}\right)^{1/a} \qquad .... (1)$$

1) If $a = 1$ then,

$$h_a(a_1, ...a_n) = \frac{a_1 + a_2 + ... + a_n}{n}$$

which is the Arithmetic mean.

2) To determine Geometric Mean.

Taking log to the base *e*, on both sides of equation (1) we get,

$$ln\ h_a = \frac{1}{a} ln\left(\frac{a_1^a + ... + a_n^a}{n}\right)$$

$$\Rightarrow ln\ h_a = \frac{ln\left(a_1^a + ... + a_n^a\right) - ln\ n}{a} \qquad ... (2)$$

Taking lim as $a \to 0$ we have,

$$\lim_{a \to 0} ln\ h_a = \lim_{a \to 0} \frac{ln\left(a_1^a + ... + a_n^a\right) - l\,n\,n}{a} \qquad \left[\frac{0}{0}\right]$$

Using L. Hospital rule, we have,

$$\Rightarrow \lim_{a\to 0} ln\ h_a = \lim_{a\to 0} \frac{a_1^a\, ln\ a_1 + ... + a_n^a\, ln\ a_n}{a_1^a + ... + a_n^a}$$

$$= \frac{ln\ a_1 + ... + l\,n\, a_n}{1+1+.....+1}$$

$$= \frac{1}{n} ln\left(a_1, a_2, ...a_n\right)$$

$$= ln\left(a_1, a_2, ...a_n\right)^{1/n}$$

$$\Rightarrow \lim_{a\to 0} h_a = \left(a_1, a_2, ...a_n\right)^{1/n}$$

which is geometric mean.

(iii) Put $a = -1$ in equation (1) we have,

$$h_{-1}\left(a_1, ..., a_n\right) = \left(\frac{a_1^{-1} + ... + a_n^{-1}}{n}\right)^{-1}$$

$$= \frac{n}{a_1^{-1} + a_2^{-1} + ... + a_n^{-1}}$$

$$= \frac{n}{\frac{1}{a_1} + \frac{1}{a_2} + ... + \frac{1}{a_n}}$$

which is Harmonic mean.

(iv) Taking lim as $a \to \infty$ on both side of equation (2) we have,

$$\lim_{a\to\infty} l\,n\, h_a = \lim_{a\to\infty} \frac{\ell n\ \left(a_1^a + a_2^a + ... + a_n^a\right) - \ell n\ n}{a}$$

By L Hospital rule we have,

$$= \lim_{a\to\infty} \frac{a_1^a\, l\,n\, a_1 + ... + a_n^a\, ln\ a_n}{a_1^a + a_2^a + ... + a_n^a}$$

let $a_i = \max(a_1, a_2, ...a_n)$

Divide Numerator and Denominator by a_i^{a} , we have,

$$\lim_{a\to\infty} ln\, h_a = \lim_{a\to\infty} \frac{\left(\frac{a_1}{a_i}\right)^a ln\ a_1 + ... + 1\cdot ln\ a_i + ... + \left(\frac{a_n}{a_i}\right)^a ln\, a_n}{\left(\frac{a_1}{a_i}\right)^a + ... + 1 + ... + \left(\frac{a_n}{a_i}\right)^a}$$

$$= \frac{0 + 0 + ... + ln\ a_i + 0 + ... + 0}{0 + 0 + ... + 0 + 1 + 0 + ... + 0}$$

$$= ln\, a_i$$

$$\Rightarrow \lim_{a\to\infty} h_a = a_i = \max(a_1, a_2, ...a_n)$$

(v) Taking lim as $a \to -\infty$ on both sides of equation (2).

We get

$$\lim_{a\to-\infty} ln\ h_a = \lim_{a\to-\infty} \frac{ln\ \left(a_1^a + a_2^a + ... + a_n^a\right) - ln\ n}{a}$$

by L Hospital rule,

$$= \lim_{a\to-\infty} \frac{a_1^a ln\ a_1 + ... + a_n^a ln\, a_n}{a_1^a + a_2^a + ... + a_n^a}$$

let $a = -t$ then $t \to \infty$.

$$\lim_{t\to\infty} ln\, h_{-t} = \lim_{t\to\infty} \frac{(a_1)^{-t} ln\ a_1 + (a_2)^{-t} ln\, a_2 + ... + (a_n)^{-t} ln\ a_n}{(a_1)^{-t} + ... + (a_n)^{-t}}$$

$$= \lim_{t\to\infty} \frac{\left(\frac{1}{a_1}\right)^t ln\ a_1 + \left(\frac{1}{a_2}\right)^t ln\ a_2 + ...+ \left(\frac{1}{a_n}\right)^t ln\ a_n}{\left(\frac{1}{a_1}\right)^t + ...+ \left(\frac{1}{a_n}\right)^t}$$

let $a_i = \min(a_1, a_2, ...a_n)$

Divide N and D by $\left(\frac{1}{a_i}\right)^t$, we have,

$$\lim_{t\to\infty} l\,n\,h_{-t} = \lim_{t\to\infty} \frac{\left(\frac{a_i}{a_1}\right)^t ln\ a_1 + ...+ 1\cdot ln\ a_i + ...+ \left(\frac{a_i}{a_n}\right)^t l\,n\,a_n}{\left(\frac{a_i}{a_1}\right)^t + ...+ 1 + ...+ \left(\frac{a_i}{a_n}\right)^t}$$

$$= \frac{0+0+....0+\ell n a_i + 0 + ...+0}{0+0+....+0+1+0+....+0}$$

$$\lim_{t\to\infty} ln\ h_{-t} = ln\ a_i$$

Then $\lim_{t\to\infty} h_{-t} = a_i$

$$\Rightarrow \lim_{a\to-\infty} h_a = a_i = \min(a_1, a_2, ...a_n)$$

Then from (i), (ii), (iii), (iv) and (v) we get required result.

3. Definition :

Ordered Weighted Averaging Operation (OWA operations)

If $\overline{w} = (w_1, w_2, ...w_n)$ be a weighting vector such that $w_i \in [0,1]$ such that

$$\sum_{i=1}^{n} w_i = 1$$

Then OWA operator associated with $\overline{w}$ is defined by,

$$h_w(a_1, a_2, \ldots. a_n) = w_1 b_1 + w_2 b_2 + \ldots. + w_n b_n$$

where, $b_1, b_2, \ldots. b_n$ is a permutation of $(a_1, a_2, \ldots. a_n)$ in which the elements are arranged in decreasing order .

4. Example :

For any 4-vector (0.7, 0.9, 0.5, 0.1) and $\overline{w}$ = (0.3, 0.1, 0.2, 0.4).

Then,

h_w = (0.7, 0.9, 0.5, 0.1) = 0.9 × (0.3) + 0.7 (0.1) + 0.5 (0.2) + 0.1 (0.4)

= 0.27 + 0.07 + 0.10 + 0.04

= 0.48

5. Note :

If h_w is ordered weighted averaging operator then for w_* = (0, 0, 0, ... 0, 1) we have,

$$hw_*(a_1, a_2, \ldots. a_n) = b_n = \min(a_1, a_2, \ldots. a_n)$$

And for $w^* = (1,0,0,\ldots 0)$

$$h_w{}^* = (a_1, a_2, \ldots. a_n) = b_1 = \max(a_1, a_2, \ldots. a_n)$$

and for any other weight vector

$$\min(a_1, a_2, \ldots. a_n) \le h_w(a_1, a_2, \ldots. a_n) \le \max(a_1, a_2, \ldots. a_n)$$

6. Theorem

Let $h : [0,1]^n \to \mathbb{R}^{+1}$ be a function that satisfies boundary conditions and which is monotonic increasing.

Let h satisfies the property given by

$$h(a_1 + b_1, a_2 + b_2, \ldots. a_n + b_n) = h(a_1, a_2, \ldots. a_n) + h(b_1, b_2, \ldots. b_n)$$

where $a_i, b_i, a_i + b_i \in [0,1] \quad \forall i$

Then there exist $w_i > 0$, $\forall i$ such that

$$h(a_1, a_2, \ldots a_n) = \sum_{i=1}^{n} w_i a_i$$

Proof : Let $h_i(a_i) = h(0,0,\ldots a_i, 0,\ldots 0)$

Then $\forall a, b$ with $a + b \in [0,1]$

$$h_i(a+b) = h_i(0,0,\ldots a+b, 0,\ldots 0)$$

$$= h_i(0,0,\ldots a, 0,\ldots 0) + h_i(0,0,\ldots b, 0\ldots 0)$$

$$h_i(a+b) = hi(a) + hi(b)$$

Thus h_i are functions which satisfies compositional property.

$$h_i(a+b) = h_i(a) + h_i(b)$$

where a, b, $a + b \in [0,1]$

Therefore by theorem, there exist a real number w_i for each $a_i \in [0,1]$ such that

$$h_i(a) = w_i \; a$$

Hence,

$$h(a_1, a_2, \ldots a_n) = h(a, 0, 0, \ldots 0) + h(0, a_2, 0 \ldots 0) + \ldots + h(0, 0, \ldots a_n)$$

$$= h_1(a_1) + h_2(a_2) + \ldots + h_n(a_n)$$

$$= w_1 a_1 + w_2 a_2 + \ldots + w_n a_n$$

$$h(a_1, a_2, \ldots a_n) = \sum_{i=1}^{n} w_i a_i$$

i.e. $h(a_1, a_2, a_3 \ldots a_n) = \sum_{i=1}^{n} w_i a_i$

7. **Note :** If the function *h* in the above theorem also satisfies idempotent property then *h*, become a weighted average

For $a = a_1 = a_2 = ... = a_n$

Then, $h(a_1, a_2, ...a_n) = h(a, a, a...a) = a$

But $h(a_1, a_2, ...a_n) = \sum_{i=1}^{n} w_i a_i$

$$= \sum_{i=1}^{n} w_i a$$

$$\Rightarrow a = a \sum_{i=1}^{n} w_i$$

$$\Rightarrow \sum_{i=1}^{n} w_i = 1$$

i.e. *h* is ordered weighted operation.

❄ ❄ ❄

UNIT - IV

Fuzzy Arithmetic

4.1 Fuzzy Numbers

1. Definition

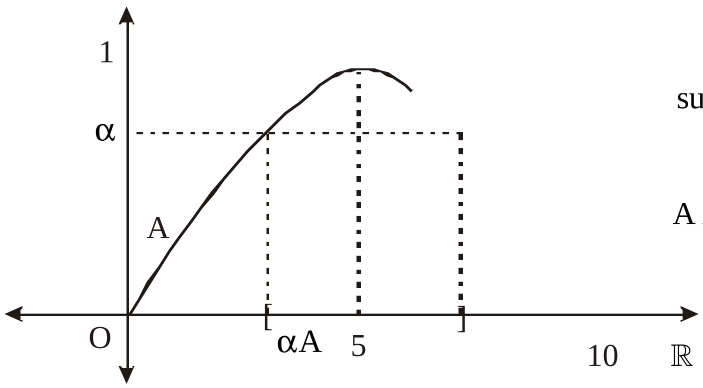
A fuzzt set A and $\mathbb{R}$ is called a fuzzy number if

1) A is a normal fuzzy set.

2) ${}^{a}A$ is a closed interval for $a \in [0,1]$.

3) Support of A i.e. 0 + A is a bounded set.

2. Example

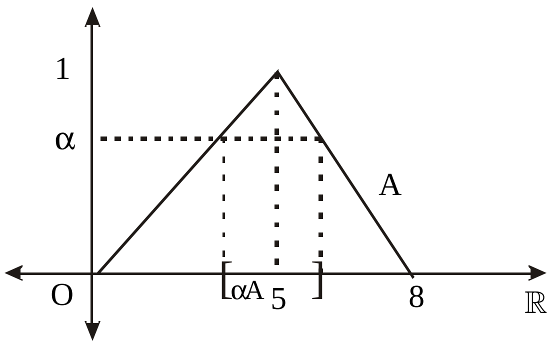


$\text{supp } A = \{x \in \mathbb{R} \mid A(x) > 0\}$

$= (0, 10)$

A is a fuzzy number 5.

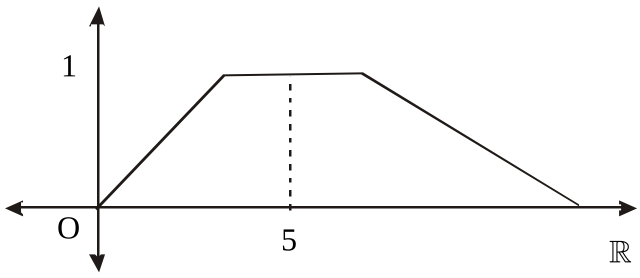
3. Example

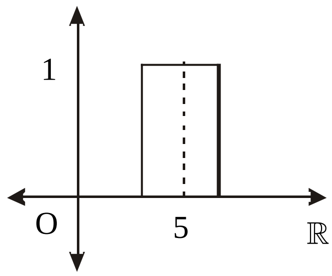


supp A = (0, 8)

A is a triangular fuzzy number.

4. Example





5. Chractarization of fuzzy numbers

Theorem : Let $A \in \mathcal{F}(\mathbb{R})$. Then A is a fuzzy number iff there exists a closed interval $[a,b] \neq f$ such that

$$A(x) = 1 \quad \text{if } x \in [a,b]$$
$$= \ell(x) \quad \text{if } x \in (-\infty, a)$$
$$= r(z) \quad \text{if } x \in (b, \infty)$$

where $\ell : (-\infty, a) \to [0,1]$ which is monotonic, increasing, continuous from the right and such that $\ell(x) = 0$ for $x \in (-\infty, w_1)$. And r is a function $r : (b, \infty) \to [0,1]$ that is monotonic, decreasing continuous from the left such that $r(x) = 0$ for $x \in (w_2, \infty)$.

Proof : Let A be a fuzzy number. Then by definition a_{A} is a closed interval for each $a \in [0,1]$.

Thus for $a = 1$, ^{1}A is a closed interval.

Let ^{1}A = $[a, b]$

i.e. $A(x) = 1$ for all $x \in [a,b]$.

and $A(x) < 1$ if $x \notin [a,b]$.

Define a function $\ell : (-\infty, a) \to [0,1]$ by

$\ell(x) = A(x)$

Then $0 \leq \ell(x) < 1$. Since $0 \leq A(x) < 1 \quad \forall x \in (-\infty, a)$

If $x \le y < a$ then, $y = l\,(x) + (1 - l\,)a$, $0 \le l < 1$.

$\therefore A(y) = A(l\,x + (1 - l\,)a)$

$\ge \min(A(x), A(a))$

$\ge \min(A(x), 1) = A(x)$

$A(y) \ge A(x)$ or $A(x) \le A(y)$

Thus, $x \le y \Rightarrow A(x) \le A(y)$

$\Rightarrow \ell(x) \le \ell(y)$

$\therefore \ell$ is monotonic increasing function. Next on the contrary suppose that ℓ is not continuous from the right. Therefore for some $x_0 \in (-\infty, a)$ there exist a seq^n $\{x_n\}$ such that

$x_0 \le x_n \;\; \forall n$ and $x_n \to x_0$ as $n \to \infty$

i.e. $\lim\limits_{x \to \infty} x_n = x_0$

But $\lim\limits_{n \to \infty} \ell(x_n) = a > \ell(x_0)$. For some a .

$\Rightarrow \lim\limits_{n \to \infty} A(x_n) = a > A(x_0)$ $\qquad (\because \ell(n) = A(x))$

$\Rightarrow A(x_n) \ge a$, $\forall n$

$\Rightarrow x_n \in a_A$, $\forall n$

Since a_A is closed interval and $\{x_n\} \subseteq a_A$ and $x_n \to x_0$

Hence $x_0 \in a_A$ $\qquad [\because a_A$ is closed]

$\Rightarrow A(x_0) \ge a$

Which is a contradiction because

$$A(x_0) < a$$

Hence, ℓ is continuous from right.

Similarly, we can show that r is monotonic decreasing from the left.

We have to prove that *r* is monotonic decreasing.

Define a function $r:(b,\infty)\to[0,1]$ by $r(x)=A(x)$

Then $0\le r(x)<1 \qquad \forall x\in(b,\infty)$

If $b<x\le y\le w_2$ for some $w_2\in\mathbb{R}$ then *x* can be written as,

$x=l\,y+(1-l)b$ for some $l\in[0,1]$

$$A(x)=A(l\,y+(1-l)b)$$

$$\ge \min(A(y),A(b))$$

$$=\min(A(y),1)$$

$$=A(y)$$

i.e. $A(x)\ge A(y)$

i.e. $r(x)\ge r(y)$

Thus $\qquad x\le y\Rightarrow r(x)\ge r(y)$.

Hence *r* is monotonic decreasing.

Next on the contrary we assume that *r* is not continuous from left.

Therefore for some $x_0\in(b,\infty)$ there exist a seq.[n] $\{x_n\}$ such that

$x_0<x_n$ and $x_n\to x_0$ as $n\to\infty$

but $\lim_{n\to\infty} r(x_n)=a>r(x_0)$ for some a.

But $r(x)=A(x)$

$$\lim_{n\to\infty} A(x_n)=a>A(x_0)$$

$$\Rightarrow A(x_n)\ge a>A(x_0),\quad \forall n$$

$$\Rightarrow x_n\in a_A,\quad \forall n$$

But α_A is a closed interval and $\{x_n\} \in \alpha_A$.

Also $x_n \to x_0$

Hence, $x_0 \in \alpha_A$

$\Rightarrow A(x_0) \geq \alpha$

But $A(x_0) < \alpha$. Hence we get a contradiction.

Therefore r is continuous from the left.

Thus r is monotonic decreasing and continious from the left.

Further A is a fuzzy number. Hence

${}^{0+}$A is a bounded set. Hence $\exists w_1, w_2 \in \mathbb{R}$. such that

$$^{0+}A(x_0) = (w_1, w_2)$$

$\Rightarrow \; A(x) = 0$ if $x \leq w_1$, or $x \geq w_2$

$\Rightarrow \; \ell(x) = 0$ if $x \leq w_1$ and $r(x) = 0$ if $x \geq w_2$

Thus if A is a fuzzy number then we have $\ell : (-\infty, a) \to [0,1]$ such that

$$\begin{aligned} \ell(x) &= A(x), & x &\in (\infty, a) \\ &= 0 & x &\in (-\infty, w_1) \end{aligned}$$

ℓ is increasing and continuous from right.

And $r : (b, \infty) \to [0,1]$

$$\begin{aligned} r(x) &= A(x) & x &\in (b, w_2) \\ &= 0 & x &\in (w_2, \infty) \end{aligned}$$

where r is decreasing and continuous from left.

Conversely, suppose that, A is a fuzzy set defined by

$$A(x) = 1, \quad x \in [a,b]$$

$$= \ell(x), \quad x \in (-\infty, a)$$

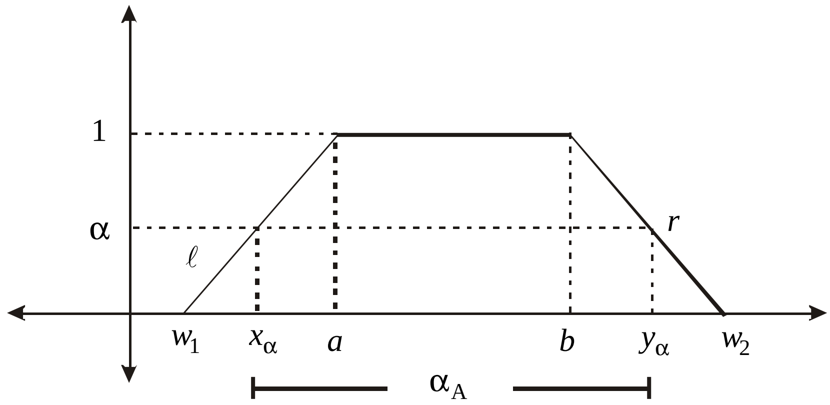
$$= r(x), \quad x \in (b, \infty)$$

where ℓ is monotonic increasing and continuous from right and r is monotonic decreasing continuous from left and ℓ and r are non-zero over a finite interval.

Since $A(x) = 1$ for $x \in [a,b]$ A is a normal fuzzy set.

Also ${}^{0+}A = (w_1, w_2)$ which is a bounded set.

Next we prove that ${}^{\alpha}A$ is a closed interval $\forall \alpha \in [0,1]$



Let $x_\alpha = \inf \{x | \ell(x) \geq \alpha, x < a\}$

and $y_\alpha = \sup\{x | r(x) \geq \alpha, x > b\}$

Let, $x_0 \in {}^{\alpha}A$ then if $x_0 < a$

$$\Rightarrow A(x_0) \geq \alpha$$

$$\Rightarrow \ell(x_0) \geq \alpha, \; x_0 < a$$

$$\Rightarrow x_0 \in \{x | \ell(x) \geq \alpha, x < a\}$$

$\Rightarrow x_0 \geq \inf\{x|\ell(x) \geq \alpha, x < a\}$

$\Rightarrow x_0 \geq x_\alpha$

Similarly, if $x_0 \in {}^\alpha A$ and $x_0 > b$ then,

$$x_0 \leq y_\alpha$$

Also, $x_0 \in {}^\alpha A$ and $a \leq x_0 \leq b$ then

$$x_\alpha \leq x_0 \leq y_\alpha$$

Thus, $x_0 \in {}^\alpha A \Rightarrow x_0 \in [x_\alpha, y_\alpha]$

$$\Rightarrow {}^\alpha A \subseteq [x_\alpha, y_\alpha] \qquad \text{.... (1)}$$

On the other hand,

$$x_\alpha = \inf\{x|\ell(x) \geq \alpha, x < a\}$$

Then there exist a sequence $\{x_n\}$ in $\{x|\ell(x) \geq \alpha, x < a\}$ such that

$$x_n \to x_\alpha \text{ and } x_n \geq x_\alpha, \ \forall n$$

Since ℓ is continuous from right, we have,

$$\ell(x_n) \to \ell(x_\alpha)$$

i.e. $\lim_{n\to\infty} \ell(x_n) = \ell(x_\alpha)$

But $\ell(x_n) \geq \alpha$

$$\Rightarrow \lim_{n\to\infty} \ell(x_n) \geq \alpha$$

$$\Rightarrow \ell(x_\alpha) \geq \alpha$$

$$\Rightarrow A(x_\alpha) \geq \alpha$$

$$\Rightarrow x_\alpha \in {}^\alpha A$$

On the other hand

$$y_a = \sup\{x \mid r(x) \geq a , x > b\}$$

then there exist a seq.[n] $\{y_n\}$ in $\{x | r(x) \geq a , x > b\}$ such that

$$y_n \to y_a \text{ and } y_n \geq y_a , \ \forall n$$

Since *r* is continuous from the left.

We have, $\quad r(y_n) \to r(y_a)$

$$\Rightarrow \lim_{n\to\infty} r(y_n) = r(y_a)$$

But $\quad y_n \in \{x | r(x) \geq a , x > b\}$

Hence $\quad r(y_n) \geq a , \ \forall n$

$$\Rightarrow \lim_{n\to\infty} r(y_n) \geq a$$

$$\Rightarrow r(y_a) = a$$

$$\Rightarrow A(y_a) = a$$

$$\Rightarrow y_a \in {}^a A$$

Thus $\quad x_a \in {}^a A$ and $y_a \in {}^a A$

Hence, $\quad [x_a , y_a] \subseteq {}^a A \qquad$ .... (2)

From (1) and (2) we get

$${}^a A = [x_a , y_a] \qquad \forall a \in [0,1]$$

Thus A is a normal fuzzy set with ${}^{0+}A$ is bounded and ${}^a A$ is a closed interval $\forall a \in [0,1]$.

Therefore, A is a fuzzy number.

6. Arithmetic Operations on Intervals

If * denoted any of the four operations on closed intervals [+, –, •, /] then

$$[a,b]*[d,e]=\{f*g \,/\, a\le f\le b, d\le g\le e\}$$

Hence,

$$[a,b]+[d,e]=\{f+g \mid a\le f\le b, d\le g\le e\}$$

$$=[a+d, b+e]$$

$$[a,b]-[d,e]=[a-e, b-d]$$

$$[a,b]\cdot[d,e]=\left[\max\{ad,ae,bd,be\},\max\{ad,ae,bd,be\}\right]$$

$$\frac{[a,b]}{[d,e]}=\left[\max\left\{\frac{a}{d}\cdot\frac{a}{e}\cdot\frac{b}{d}\cdot\frac{b}{e}\right\},\max\left\{\frac{a}{d}\cdot\frac{a}{e}\cdot\frac{b}{d}\cdot\frac{b}{e}\right\}\right]$$

provided $0\notin[d,e]$.

7. Examle :

1) $[2,5]+[-4,9]=[-2,14]$

2) $[2,5]-[-4,9]=[-7,9]$

3) $[-3,-1]-[6,11]=[-14,-7]$

4) $[2,5]\cdot[-4,9]=\left[\min\{-8,-20,18,45\},\max\{-8,-20,18,45\}\right]$

$$=[-20,45]$$

5) $$\frac{[-3,-1]}{[5,8]}=\left[\min\left\{\frac{-3}{5},\frac{-3}{8},\frac{-1}{5},\frac{-1}{8}\right\},\max\left\{\frac{-3}{5},\frac{-3}{8},\frac{-1}{5},\frac{-1}{8}\right\}\right]$$

$$=\left[\frac{-3}{5},\frac{-1}{8}\right]$$

6) $[-1,1]\cdot\left[-2,-\frac{1}{2}\right]=[-2,2]$

7) $\dfrac{[4,10]}{[1,2]} = [2,10]$

8) $[3,4]\cdot[2,2] = [6,8]$

9) $\dfrac{[-1,1]}{\left[-2,-\frac{1}{2}\right]} = \left[\min\left\{\frac{-1}{2},2,\frac{-1}{2},-2\right\},\max\left\{\frac{-1}{2},2,\frac{-1}{2},-2\right\}\right] = [-2,2]$

8. Arithmetic Operations on Fuzzy Numbers

Let A and B be the continuous fuzzy numbers and let * be a binary operation on

$\mathbb{R}\ [* = +, -, \cdot, /]$

Then we define $A*B : \mathbb{R} \to I$ by

$$(A*B)(z) = \sup_{z=x*y}\left[A(x) \cap B(y)\right]$$

If A * B is a fuzzy number then for each $a \in [0,1]$ we have,

$${}^{a}(A*B) = {}^{a}A * {}^{a}B$$

9. Method of Evaluating A * B

If A and B are triangular fuzzy numbers then following procedure may be helpful in evaluating the fuzzy set A * B.

1) Find ${}^{a}A = [a, a_2]$, ${}^{a}B = [b, b_2]$ for $a \in [0,1]$

2) Consider, $S = \{a_1 * b_1, a_1 * b_2, a_2 * b_1, a_2 * b_2\}$

Then ${}^{a}A * {}^{a}B = [\min S, \max S] = [S_1, S_2]$

$\Rightarrow {}^{a}A * B = [S_1, S_2], \quad \forall a \in [0,1]$

3) Find and if

Then A * B is zero outside [c, d]

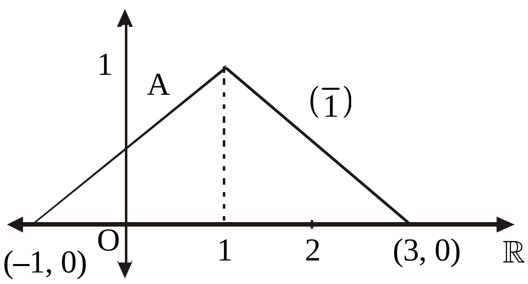
4) Find (A * B) by using $A * B = \bigcup_{a} a\,(A * B)$.

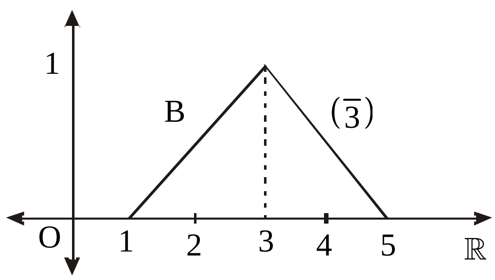
10. Example

For the following fuzzy numbers A and B find A + B and A – B where,

$$A(x) = \begin{cases} 0 & x \le -1, x \ge 3 \\ \dfrac{(x+1)}{2} & -1 \le x \le 1 \\ \dfrac{3-x}{2} & +1 \le x \le 3 \end{cases}$$

$$B(x) = \begin{cases} 0 & x \le -1, x \ge 5 \\ \dfrac{(x-1)}{2} & 1 \le x \le 3 \\ \dfrac{5-x}{2} & 3 \le x \le 5 \end{cases}$$





Now if $x \in {}^{a}A \Rightarrow A(x) \ge a$

$$\Rightarrow \frac{x+1}{2} \ge a \quad \text{or} \quad \frac{3-x}{2} \ge a$$

$$\Rightarrow x \ge 2a - 1 \quad \text{or} \quad x \le 3 - 2a$$

$$\Rightarrow x \in [2a - 1, 3 - 2a]$$

Hence ${}^{\alpha}A = [2\alpha - 1, 3 - 2\alpha], \quad \forall \alpha \in [0,1]$

Next if $x \in {}^{\alpha}B \Rightarrow B(x) \geq \alpha$

$$\Rightarrow \frac{x-1}{2} \geq \alpha \quad \text{or} \quad \frac{5-x}{2} \geq \alpha$$

$$\Rightarrow x \geq 2\alpha - 1 \quad \text{or} \quad x \leq 5 - 2\alpha$$

$$\Rightarrow x \in [2\alpha + 1, 5 - 2\alpha]$$

Hence ${}^{\alpha}B = [2\alpha + 1, 5 - 2\alpha]$

$\Rightarrow {}^{\alpha}A + {}^{\alpha}B = [2\alpha - 1, 3 - 2\alpha] + [2\alpha + 1, 5 - 2\alpha]$

${}^{\alpha}A + {}^{\alpha}B = [4\alpha, 8 - 4\alpha]$

${}^{\alpha}(A + B) = [4\alpha, 8 - 4\alpha], \quad \forall \alpha \in [0,1]$

${}^{0+}(A + B) = (0, 8)$ i.e.

Support of $(A + B) = (0, 8)$

Next if $x \in [4\alpha, 8 - 4\alpha]$

$\Rightarrow 4\alpha \leq x \leq 8 - 4\alpha$

$\Rightarrow \alpha \leq \frac{x}{4}$ and $\alpha \leq \frac{8-x}{4}, \quad \alpha \in [0,1]$

For, $\alpha = 0$, $0 \leq x \leq 8$

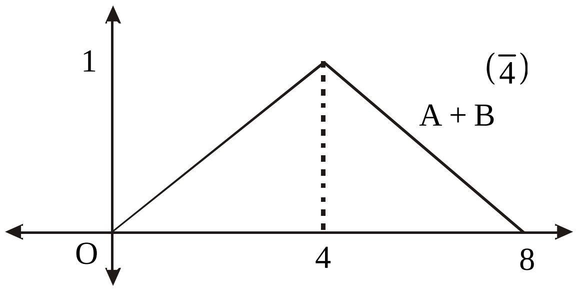
$\alpha = 1$, $4 \leq x \leq 4 \Rightarrow x = 4$

For $0 \leq x \leq 4$, where $(A + B)(x) = \frac{x}{4}$

and for $4 \leq x \leq 8$, where $(A + B)(x) = \frac{8-x}{4}$

Hence we have,

$$(A+B)(x)=\begin{cases}0 & 0\le x, x\ge 8\\ \dfrac{x}{4} & 0\le x\le 4\\ \dfrac{8-x}{4} & 4\le x\le 8\end{cases}$$



$[(\bar{1})+(\bar{3})=(\bar{4})]$

Now consider,

$${}^{a}A-{}^{a}B=[2a-1,3-2a]-[2a+1,5-2a]$$

$$\Rightarrow {}^{a}(A-B)=[4a-6,2-4a]$$

$$\Rightarrow {}^{0+}(A-B)=(-6,2)$$

Hence, support $\sup(A-B)=(-6,2)$

Next if $x\in[4a-6,2-4a]$

$4a-6\le x$ or $z\le 2-4a$

$a\le\dfrac{x+6}{4}$ or $a\le\dfrac{2-x}{4}$ $\quad a\in[0,1]$

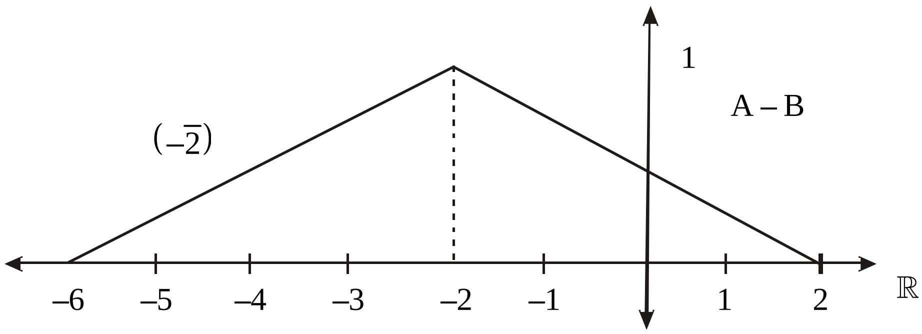
Also for a = 1, x = – 2. Hence,

For $-6\le x\le -2$, $\quad (A-B)(x)=\dfrac{x+6}{4}$

And for $-2\le x\le 2$, $\quad (A-B)(x)=\dfrac{2-x}{4}$

Thus we have,

$$(A-B)(x)=\begin{cases}0 & x\le -6, x\ge 2\\ \dfrac{x+6}{4} & -6\le x\le -2\\ \dfrac{2-x}{4} & -2\le x\le 2\end{cases}$$



$\overline{1}-\overline{3}=-\overline{2}$

11. Example

Which of the following is a fuzzy number

1) $A(x)=\sin x \quad 0\le x\le p$

$= 0$ otherwise

2) $B(x)=x \quad 0\le x\le 1$

$= 0$ otherwise

3) $C(x)=1 \quad 0\le x\le 10$

$= 0$ otherwise

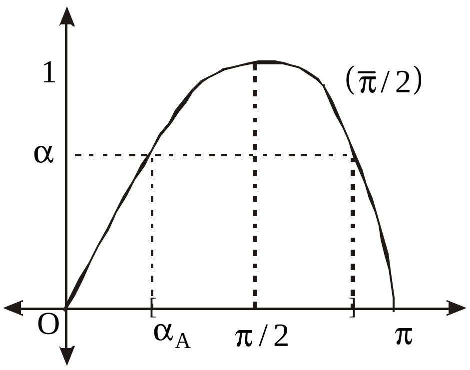
4) $D(x)=\min(1,x) \quad x\ge 0$

$= 0 \quad x<0$

5) $E(x)=1 \quad x=5$

$= 0$ otherwise

1) $A(x) = \sin x$



$A\left(\frac{p}{2}\right) = 1$

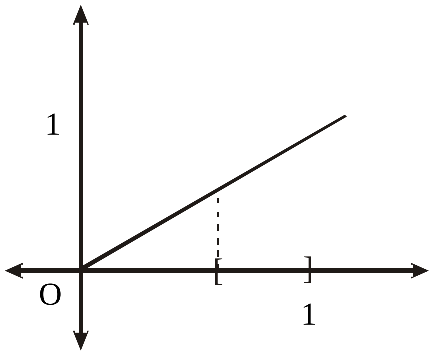
$\therefore$ A is normal

$\sup A = (0, p)$

${}^{a}A$ is closed interval in $(0, p)$

$\therefore$ A is a fuzzy number $\left(\frac{\bar{p}}{2}\right)$.

2) $B(x) = x$ $\quad 0 \le x \le 1$



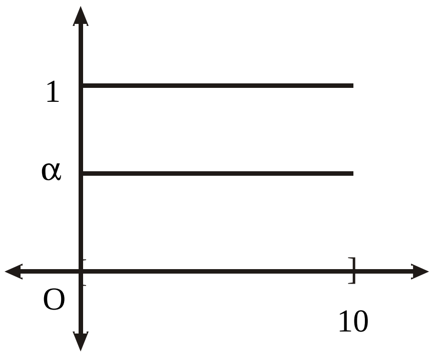
1) $B(1) = 1$ $\quad \therefore$ B is normal

2) $\sup B = [0,1]$

3) ${}^{a}B$ is closed for all .

$\therefore$ B is a fuzzy number.

3) $C(x) = 1$ $\quad 0 \le x \le 10$

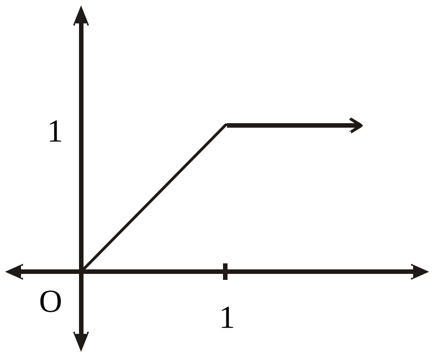


1) $C(x)$ is normal

2) $\sup C = [0,10]$

3) ${}^{a}C = [0,10]$, $\forall a$

4) $D(x) = \min(1, x)$



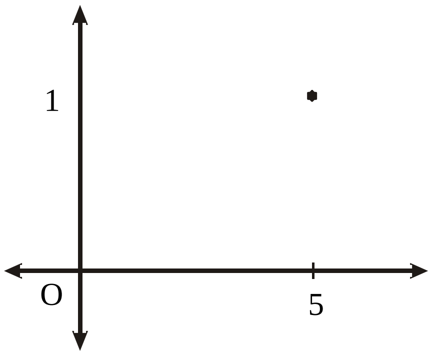
1) D is normal

2) $\sup D = (0, \infty)$

which is not bounded.

$\therefore$ D is not a fuzzy number.

5) $E(x) = 1$ $\quad x = 5$



1) E (5) = 1 $\therefore$ E is normal

2) sup E = {5}

3) ${}^{a}E = \{5\}$

Which is not closed interval.

$\therefore$ E is not a fuzzy number.

12. Example

If $A(x) = \dfrac{x+2}{2}$ $\quad -2 < x \leq 0$

$= \dfrac{2-x}{2}$ $\quad 0 < x < 2$

$= 0$ otherwise

And $B(x) = \dfrac{x-2}{2}$ $\quad 2 \leq x \leq 4$

$= \dfrac{6-x}{2}$ $\quad 4 < x \leq 6$

$= 0$ otherwise

Calculate A + B and A – B.

Solution : $A(x) \geq \alpha$

$\Rightarrow \frac{x+2}{2} \geq \alpha$ or $\frac{2-x}{2} \geq \alpha$

$x+2 \geq 2\alpha$ or $2-x \geq 2\alpha$

$x \geq 2\alpha - 2$ or $x \leq -2\alpha + 2$

$x \in {}^{\alpha}A \Rightarrow -2+2\alpha \leq x \leq -2\alpha + 2$

i.e. ${}^{\alpha}A = [2\alpha - 2, 2 - 2\alpha], \ \forall \alpha \in [0,1]$

for $\alpha = 0$

${}^{0+}A = (2,2)$

sup A = (–2, 2) which is bounded.

Similarly, if $B(x) \geq \alpha$

$\Rightarrow \frac{x-2}{2} \geq \alpha$ $\frac{6-x}{2} \geq \alpha$

$\Rightarrow x-2 \geq 2\alpha$ $6-x \geq 2\alpha$

$\Rightarrow x \geq 2\alpha + 2$ $-x \geq 2\alpha - 6$ or $x \leq -2\alpha + 6$

Hence, ${}^{\alpha}B = [2\alpha + 2, 6 - 2\alpha]$

Next, ${}^{\alpha}A - {}^{\alpha}B = [2\alpha - 2, 2 - 2\alpha] - [2\alpha + 2, 6 - 2\alpha]$

$= [2\alpha - 2 - (6 - 2\alpha), 2 - 2\alpha - (2\alpha - 2)]$

${}^{\alpha}(A-B) = [4\alpha - 8, -4\alpha]$

${}^{0+}(A-B) = (-8, 0)$

Now, $x \in {}^{\alpha}(A-B)$

$\Rightarrow x \in [4\alpha - 8, -4\alpha]$

$\Rightarrow x \geq 4\alpha - 8$ and $x \leq -4\alpha$

$$\Rightarrow a \le \frac{x+8}{4} \text{ and } -\frac{x}{4} \ge a$$

$$\Rightarrow (A-B)(x) = \frac{x+8}{4} \text{ and } (A-B)(x) = -\frac{x}{4}$$

Thus, $(A-B)(x) = 0$ if $x \in [-8,0]$

$= \frac{x+8}{4}$ if $-8 \le x \le -4$

$= -\frac{x}{4}$ if $-4 \le x \le 0$

13. Definition

Let * denote any of the four basic arithmetic operations and let A and B denote fuzzy numbers. Then a fuzzy number $A * B$ on $\mathbb{R}$ is given by

$$(A*B)(z) = \sup_{z=x*y} \min\big(A(x), B(y)\big)$$

$$= \bigvee_{z=x*y} \big(A(x) \wedge B(y)\big)$$

14. Theorem

Let $* \in \{+,-,\cdot,/\}$ and let A, B denote continuous fuzzy numbers. Then $A * B$ is also continuous.

Prrof : We know that A * B is a fuzzy number. We show that A * B is continuous. On the contrary assume that A * B is not continuous at some point say $z_0 \in \mathbb{R}$. Then

$$\lim_{z \to z_0^-} (A*B)(z) < (A*B)(z_0)$$

$$= \sup_{z_0 = x*y} \big(A(x) \wedge B(y)\big)$$

Therefore there exists x_0 and y_0 such that

$z_0 = x_0 * y_0$ and

$$\lim_{z \to z_0^-} (A * B)(z) < \left(A(x_0) \wedge B(y_0)\right)$$

Since the binary operations +, –, •, / are monotonic w.r.t. first and second argument, we can find two sequences $\{x_n\}$ and $\{y_n\}$.

Such that $x_n \to x_0$, $y_n \to y_0$ and $x_n * y_n < x_0 * y_0 = z_0$ for all n.

Let $z_n = x_n * y_n$. Then

$$z_n = z_0^- \text{ as } n \to \infty \ (\because z_n < z_0)$$

Thus

$$\lim_{z \to z_0^-} (A * B)(z) = \lim_{n \to \infty} (A * B)(z_n)$$

$$= \lim_{n \to \infty} \sup_{z_n = x_n * y_n} \left(A(x_n) \wedge B(y_n)\right)$$

$$\geq \lim_{n \to \infty} \left[A(x_n) \wedge B(y_n)\right]$$

$$= \left[A\left(\lim_{n \to \infty} x_n\right) \wedge B\left(\lim_{n \to \infty} y_n\right)\right]$$

$$= \left[A(x_0) \wedge B(y_0)\right]$$

i.e. $\lim_{z \to z_0^-} (A * B)(z) \geq \left[A(x_0) \wedge B(y_0)\right]$

which is a contradiction. Hence A * B is a continuous.

4.2 Lattice of Fuzzy Numbers

We extend the lattice operations min and max on set of real numbers to the corresponding operations on Fuzzy numbers. Only difference between the set of real numbers $\mathbb{R}$ and the set of fuzzy real numbers $\mathcal{R}$, is that the set $\mathbb{R}$ is totally ordered but the set $\mathcal{R}$ is not totally ordered.

1. Definition :

For any two fuzzy numbers A and B we define MIN and MAX by,

$$\text{MIN}(A,B)(z) = \sup_{z=\min(x,y)} \min\left(A(x), B(y)\right)$$

$$= \bigvee_{z=\min(x,y)} \left[A(x) \wedge B(y)\right]$$

And,

$$\text{MAX}(A,B)(z) = \sup_{z=\max(x,y)} \min\left(A(x), B(y)\right)$$

$$= \bigvee_{z=\max(x,y)} \left[A(x) \wedge B(y)\right]$$

where $z \in \mathbb{R}$. MIN (A, B) and MAX (A, B) are fuzzy numbers on $\mathbb{R}$.

2. Note

1. The expressions for MIN and MAX are obtained by using extension principle.
2. The operations MIN and MAX are totally different from the standard fuzzy intersection and union.

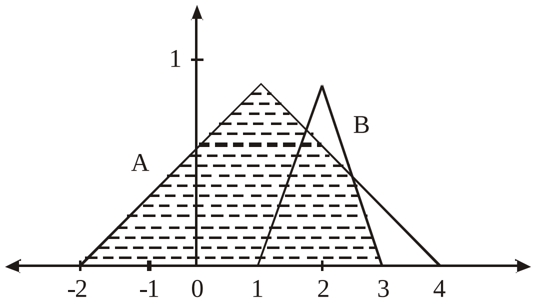
3. Example

For fuzzy numbers A and B defined by

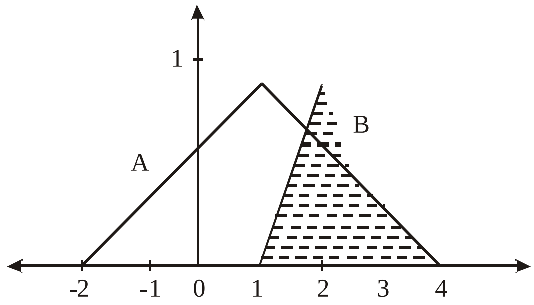
$$A(x) = \begin{cases} 0 & x \le -2 \text{ and } x \ge 4 \\ \dfrac{x+2}{3} & -2 < x \le 1 \\ \dfrac{4-x}{3} & 1 \le x < 4 \end{cases}$$

$$B(x) = \begin{cases} 0 & x \le 1 \text{ and } x \ge 3 \\ x-1 & 1 < x \le 2 \\ 3-x & 2 \le x < 3 \end{cases}$$

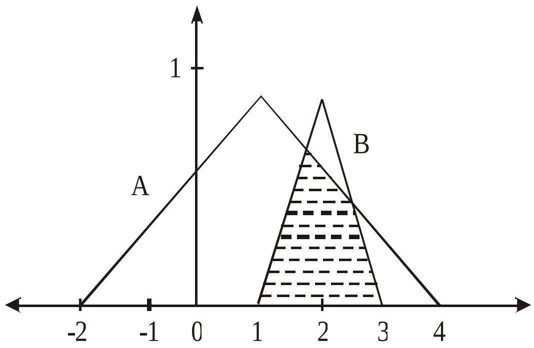
Then MIN (A, B), MAX (A, B), $A \cap B$ and $A \cup B$ are given by,



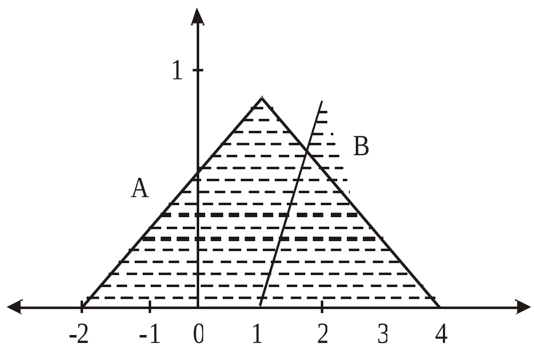
MIN (A, B)



MAX (A, B)



$A \cap B$



$A \cup B$

4. Theorem

Let $\mathcal{R}$ denote the set of all fuzzy numbers. Then for any A, B, C $\in \mathcal{R}$ the following properties hold.

(a) Commutativity

MIN (A, B) = MIN (B, A)

MAX (A, B) = MAX (B, A)

(b) Associativity

MIN (MIN (A, B), C) = MIN (A, MIN (B, C))

MAX (MAX (A, B), C) = MAX (A, MAX (B, C))

(c) Idempotency

MIN (A, A) = A, MAX (A, A) = A

(d) Absorption Law

MIN (A, MAX (A, B)) = A

MAX (A, MIN (A, B)) = A

(e) Distributivity

MIN (A, MAX (B, C)) = MAX (MIN (A, B), MIN (A, C))

MAX (A, MIN (B, C)) = MIN (MAX (A, B), MAX (A, C))

Proof :

(a) For $z \in \mathbb{R}$,

$$\text{MIN}(A,B)(z) = \sup_{z=\min(x,y)} \min\left(A(x), B(y)\right)$$

$$= \vee_{z=\min(x,y)} \left[A(x) \wedge B(y)\right]$$

$$= \vee_{z=\min(x,y)} \left[B(y) \wedge A(x)\right]$$

= MIn (B, A) (z)

Similarly,

$$\text{MAX}(A,B)(z) = \sup_{z=\max(x,y)} \max\left(A(x), B(y)\right)$$

$$= \vee_{z=\max(x,y)} \left[A(x) \wedge B(y)\right]$$

$$= \vee_{z=\max(x,y)} \left[B(y) \wedge A(x)\right]$$

= MAX (B, A) (z)

(b) For any $z \in \mathbb{R}$.

$$\text{MIN}\left(A, \text{MIN}(B,C)\right) = \vee_{z=x\wedge y} \left[A(x) \wedge \text{MIN}(B,C)(y)\right]$$

$$= \vee_{z=x\wedge y} \left[A(x) \wedge \vee_{y=u\wedge v} B(u) \wedge C(v)\right]$$

$$= \vee_{z=x\wedge u\wedge v} A(x) \wedge B(u) \wedge C(v) \qquad \text{.... (i)}$$

Since $\wedge$ and $\vee$ are distributive in $\mathbb{R}$. Also,

$$\text{MIN}\big(\text{MIN}(A,B),C\big)(z) = \underset{z=x\wedge y}{\vee}\big[\text{MIN}(A,B)(x)\wedge C(y)\big]$$

$$= \underset{z=x\wedge y}{\vee}\Big[\Big(\underset{x=u\wedge v}{\vee} A(u)\wedge B(v)\Big)\wedge C(y)\Big]$$

$$= \underset{z=u\wedge v\wedge y}{\vee} A(u)\wedge B(v)\wedge C(y)$$

$$= \underset{z=x\wedge u\wedge v}{\vee} A(x)\wedge B(u)\wedge C(v) \qquad \text{.... (ii)}$$

From (i) and (ii) we get,

MIN (A, MIN (B, C) = MIN (MIN (A, B), C)

Similarly MAX is associative.

(c) For $z\in\mathbb{R}$.

$$\text{MIN}(A,A)(z) = \underset{z=x\wedge y}{\vee}\big[A(x)\wedge A(y)\big]$$

$$= \underset{z=z\wedge z}{\vee}\big[A(z)\wedge A(z)\big]$$

$$= \underset{z=z\wedge z}{\vee} A(z)$$

$$= A(z)$$

Similarly MAX (A, A) = A holds.

(d) For any $z\in\mathbb{R}$.

$$\text{MIN}\big(A,\text{MAX}(A,B)\big)(z) = \underset{z=x\wedge y}{\vee}\big[A(x)\wedge \text{MAX}(A,B)(y)\big]$$

$$= \underset{z=x\wedge y}{\vee}\Big[A(x)\wedge \underset{z=u\vee v}{\vee}\big(A(u)\wedge B(v)\big)\Big]$$

$$= \underset{z=x\wedge(u\vee v)}{\vee}\big[A(x)\wedge\big(A(u)\wedge B(v)\big)\big]$$

$$= \text{M say}$$

Since B is a fuzzy number $B(v_0)=1$ for some v_0.

Therefore for $z = z \wedge (z \vee v_0)$ we have,

$$M \geq A(z) \wedge A(z) \wedge B(v_0) \geq A(z) \wedge A(z) = A(z)$$

Also, $z = x \wedge (u \vee v) = (x \wedge u) \vee (x \wedge v)$

$\Rightarrow z \leq x \leq x \vee v$ and $z \geq x \wedge u$

$\Rightarrow x \wedge u \leq z \leq x \vee u$

$\Rightarrow z = l\,(x \wedge u) + (1 - l)(x \vee u)$ for some l , $l \in [0,1]$

$\Rightarrow A(z) = A[\,l\,(x \wedge u) + (1 - l)(x \vee u)]$ $\quad$ ($\because$ A is convex)

$\geq A(x \wedge u) \wedge A(x \vee u)$

If $x \leq u$ then $x \wedge u = x$ and $x \vee u = u$,And if

$u \leq x$ then $x \wedge u = u$, $x \vee u = x$

Therefore we have,

$$A(x \wedge u) \wedge A(x \vee u) = A(x) \wedge A(u)$$

Therefore,

$$A(z) \geq A(x) \wedge A(u) \geq A(x) \wedge A(u) \wedge B(v)$$

$$\Rightarrow A(z) \geq \bigvee_{z = x \wedge (u \vee v)} A(x) \wedge A(u) \wedge B(v) = M$$

Thus $\quad A$ (z) = M = MIN (A, MAX (A, B)) (z)

i.e. $\quad$ MIN (A, MAX (A, B) = A

Similarly we can show that

MAX (A, MIN (A, B)) = A

(e) $\quad$ For any $z \in \mathbb{R}$ consider,

$$\text{MIN}(A, \text{MAX}(B, C))(z) = \bigvee_{z = x \wedge y} [A(x) \wedge \text{MAX}(B, C)(y)]$$

$$= \bigvee_{z = x \wedge y} \left[A(x) \wedge \bigvee_{y = u \vee v} (B(u) \wedge C(v))\right]$$

$$= \bigvee_{z = x \wedge (u \vee v)} [A(x) \wedge B(u) \wedge C(v)]$$

And

$$\mathrm{MAX}\left(\mathrm{MIN}(A,B),\mathrm{MIN}(A,C)\right)(z) = \underset{z=x\vee y}{\vee}\left[\mathrm{MIN}(A,B)(x)\wedge \mathrm{MIN}(A,C)(y)\right]$$

$$= \underset{z=x\vee y}{\vee}\left[\underset{x=m\wedge n}{\vee}\left(A(m)\wedge B(n)\right)\wedge \underset{y=s\wedge t}{\vee}\left(A(s)\wedge C(t)\right)\right]$$

$$= \underset{z=(m\wedge n)\vee(s\wedge t)}{\vee}\left[A(m)\wedge B(n)\wedge A(s)\wedge C(t)\right]$$

Let $E=\left\{A(x)\wedge B(u)\wedge C(v) \mid z=x\wedge(u\vee v)\right\}$

$F=\left\{A(m)\wedge B(n)\wedge A(s)\wedge C(t) \mid z=(m\wedge n)\vee(s\wedge t)\right\}$

Let $a\in E \Rightarrow a=A(x)\wedge B(u)\wedge C(v)$ for some *x*, *v*, *w* such that

$z=x\wedge(u\vee v)$

Take $m=s=x$, $n=u$, $t=v$. Then,

$$(m\wedge n)\vee(s\wedge t) = (x\wedge u)\vee(x\wedge v)$$

$$= x\wedge(u\vee v)$$

$$= z$$

And $A(m)\wedge B(n)\wedge A(s)\wedge C(t) = A(x)\wedge B(u)\wedge A(x)\wedge C(u)$

$$= A(x)\wedge B(u)\wedge C(v)$$

$$= a$$

Hence, $a\in F$. Thus $E\subseteq F \Rightarrow \sup E \le \sup F$

Next $b\in F \Rightarrow b=A(m)\wedge B(n)\wedge A(s)\wedge C(t)$

where $(m\wedge n)\vee(s\wedge t)=z$

Now $z=(m\wedge n)\vee(s\wedge t)$

$$=[(m\wedge n)\vee s]\wedge[(m\wedge n)\vee t]$$

$$=[(m\vee s)\wedge(n\vee s)]\wedge[(m\vee t)\wedge(n\vee t)]$$

$$=(m\vee s)\wedge(n\vee s)\wedge(m\vee t)\wedge(n\vee t)$$

Let $x = (m \vee s) \wedge (n \vee s) \wedge (m \vee t)$, $u = n$ and $v = t$. Then

$$z = x \wedge (u \vee v) \text{ and } x \leq (m \vee s).$$

If $z = x$ then we can prove that $m \wedge s \leq x$. And if

$$z = u \vee v \text{ we can prove that } m \wedge s = x.$$

Thus $m \wedge s \leq x \leq m \vee s$. By convex combination we have

$$x = l\,(m \vee s) + (1 - l)(m \wedge s), \text{ for some } l \in [0,1].$$

But A is convex fuzzy set. Hence we have,

$$A(x) = A(l\,(m \vee s) + (1 - l)(m \wedge s) \geq A(m \vee s) \wedge A(m \wedge s))$$

$$\Rightarrow A(x) \geq A(m) \wedge A(s)$$

Let $\quad a = A(x) \wedge B(u) \wedge C(v).$

Then we have $z = x \wedge (u \vee v)$ and

$$a = A(x) \wedge B(u) \wedge C(v) \geq A(m) \wedge A(s) \wedge B(u) \wedge C(v)$$

$$= A(m) \wedge B(n) \wedge A(s) \wedge C(t)$$

$$= b$$

i.e. $a \geq b$

Thus for every $b \in F$ there exists $a \in E$ such that $a \geq b$. i.e. $\sup E \geq \sup F$

Hence, $\sup E = \sup F$ and this implies

MIN (*A*, MAX (*B*, *C*)) (*z*) = MAX (MIN (*A*, *B*), MIN (*A*, *C*)) (*z*)

Since $z \in \mathbb{R}$ is arbitrary we have,

MIN (*A*, MAX (*B*, *C*)) = MAX (MIN (*A*, *B*), MIN (*A*, *C*))

Similarly we can prove that MAX distributes over MIN i.e.

MAX (*A*, MIN (*B*, *C*)) = MIN (MAX (*A*, *B*), MAX (*A*, *C*))

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