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PREFACE

Integral equations occur in many fields of mechanics, mathematical physics and thermodynamics. Many physical problems which are usually solved by differential equation methods can be solved more effectively by integral equation method. Integral equation arise as representation formulas for the solutions of differential equations.

The main aim of this material is to provide:

1. *Conceptual understanding of fundamentals of integral equation.*
2. *Method of solving integral equation.*

The book has been written in very simple language with large number of worked examples and graded exercises hoping that these will be particularly useful to those studying by themselves.

The course material is based on the following books :

1. *Ram P. Kanwal, Linear integral equations, Theory and Technique, Academic press, New York (1971)*
2. *L. G. Chambers, Integral Equations : A short course, International Textbook company Ltd. (1976)*
3. *Abdul M. Wazwzr, First Course in integral equations, world scientific, Singapore (1997)*
4. *Krasnov, M. V. etal. Problems and exercises in integral equations, Mir Publishers (1971).*
5. *Abdul-Majid Wazwaz, Linear and Nonlinear Integral Equations-Method and Applications, Springer, 2011.*

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Integral Equations

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M. Sc. Mathematics

Integral Equations

CONTENTS

Each Unit begins with the section 'Objectives' -

Objectives are directive and indicative of :

1. What has been presented in the Unit and
2. What is expected from you
3. What you are expected to know pertaining to the specific Unit once you have completed working on the Unit.

The self check exercises with possible answers will help you to understand the Unit in the right perspective. Go through the possible answer only after you write your answers. These exercises are not to be submitted to us for evaluation. These are provided to you as Study Tools to help keep you in the right track as you study the Unit.

Unit – 1

INTEGRAL EQUATION

1.1 Definition :

An integral equation is an equation in which an unknown function appears under one or more integral signs.

For example, for $a \le s \le b$, $a \le t \le b$, the equations

1) $f(s) = \int_a^b k(s,t)g(t)dt$

2) $g(s) = \int_a^b k(s,t)[g(t)]^2 \, dt$

3) $g(s) = f(s) + \int_a^b k(s,t)g(t)dt$

Where the function g(s) is the unknown function while all the other functions are known, are integral equations.

1.2 Classification of Integral equation :

Integral equation are generally classified in to two main classes.

i) Linear integral equation.

ii) Non linear integral equation.

i) Linear integral equation :

An integral equation is called linear if only linear operations are performed in it upon the unknown functions. (i.e. unknown function g(s) under the integral sign occurs linearly) e.g. 1) $g(s) = 1 + \int_0^s 2t\, g(t)dt$

2) $g(s) = f(s) + \int_a^b k(s,t)\, g(t)\, dt$

ii) Non linear integral equation :

If the unknown function g(s) under the integral sign is replaced by a nonlinear function in g(s), say F{g(s)} where F is nonlinear function then the integral equation is called a non linear integral equation.

e.g. 1) $g(s) = f(s) + \lambda \int_a^s k(s,t)\left[g(t)\right]^2 dt$

2) $g(s) = f(s) + \lambda \int_0^1 k(s,t) e^{g(t)} dt$

3) $g(s) = f(s) + \lambda \int_0^1 k(s,t) \sin\left[g(t)\right] dt$

in above all integral equation [g(s)]^2, $e^{g(s)}$, sin g(s), all are non linear functions in g(s).

1.3 General form of Linear Integral equations :

The most general form of linear integral equation is

$$h(s)\, g(s) = f(s) + \lambda \int_a k(s,t) g(t)\, dt \qquad \text{-----(1)}$$

Where the upper limit may be either variable s or constant.

i) The functions, h(s), f(s), k (s, t) all are known while g(s) is unknown function to be determined.

ii) The functions h, f, k may be complex valued functions of the real variable s & t.

iii) λ is a non-zero real or complex parameter.

iv) The function k(s,t) is known as the kernel of integral equation.

1.4 Classification of Linear Integral Equations :

1.4.1 Fredholm Integral Equations :

If the upper limit in integral equation (1) is fixed say b then it reduces to

$$h(s)g(s) = f(s) + \lambda\int_a^b k(s,t)g(t)\,dt \qquad \text{-----(2)}$$

It is called Fredholm integral equation

Special types of Fredholm integral equation :

i) Put h(s) = 0 in equation (2), we get

$$0 = f(s) + \lambda\int_a^b k(s,t)\,g(t)\,dt$$

It is called Fredholm integral equation of first kind.

ii) Putting h(s) = 1 in equation (2), we get

$$g(s) = f(s) + \lambda\int_a^b k(s,t)\,g(t)\,dt \qquad \text{-----(3)}$$

It is called fredholm integral equation equation of second kind.

a) Putting f(s) = 0 in equation (3), we get

$$g(s) = \lambda\int_a^b k(s,t)\,g(t)\,dt$$

It is called homogeneous Fredholm integral equation of second kind.

b) If f(s) $\neq 0$ in equation (3), it is called nonhomogeneous Fredholm integral equation of second kind.

1.4.2 Volterra Integral equation :

If the upper limit in integral equation (1) is variable 's' then it reduces to

$$h(s)g(s) = f(s) + \lambda\int_a^s k(s,t)g(t)\,dt \qquad \text{-----(4)}$$

it is called Volterra integral equation.

Special types of Volterra integral equation :

i) Put h (s) = 0 in equation (4) we get

$$0 = f(s) + \lambda\int_a^s k(s,t)g(t)\,dt$$

it is called volterra integral equation of first kind.

ii) Putting, h (s) = 1 in equation (4), we get

$$g(s) = f(s) + \lambda\int_a^s k(s,t)g(t)\,dt \qquad \text{-----(5)}$$

it is called volterra integral equation of second kind.

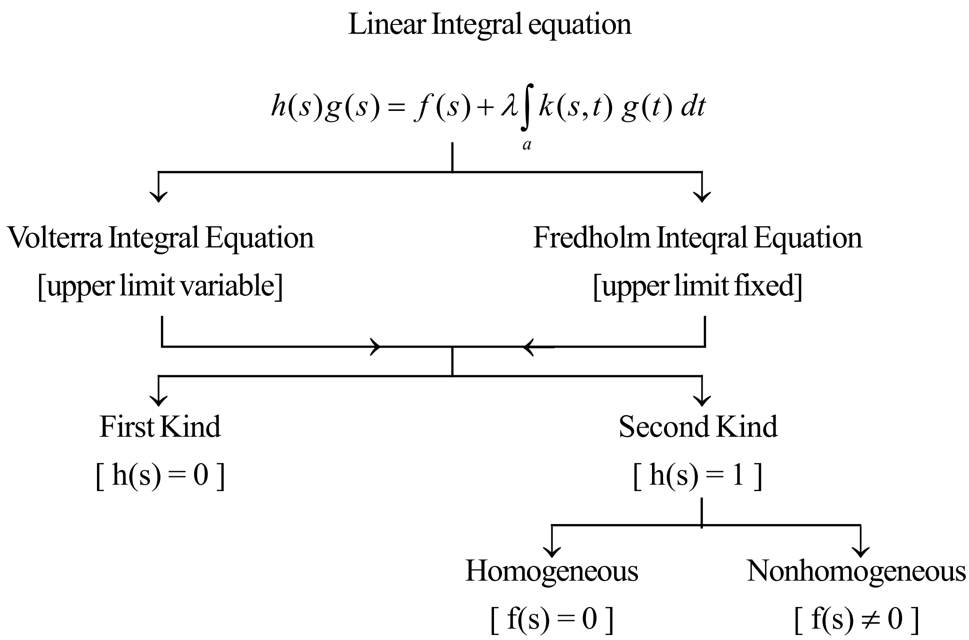
a) Putting, f(s) = 0 in equation (5), we get

$$g(s) = +\lambda\int_a^s k(s,t)g(t)\,dt$$

it is called homogeneous volterra integral equation of second kind.

b) If f(s) $\neq 0$ in equation (5) it is called nonohomogeneous volterra integral equation of second kind.

Above all discussion is given in following tree diagram.



1.4.3 Some special types of integral equations :

a) Singular integral equation :

An integral equation is called a singular integral equation if one or both the limits of integration become infinite, or if the kernel of the equation becomes infinite at one or more points in the interval of integration.

e.g. i) $g(s) = 1 + e^{-s} - \int_0^{\infty} g(t)\, dt$

ii) $g(x) = 1 + 2\sqrt{x} - \int_0^{x} \frac{1}{\sqrt{x-t}} g(t)\, dt$

Here, $k(x,t) = \frac{1}{\sqrt{x-t}}$ and k(x, t) becoming infinite as t $\rightarrow x$

iii) $g(s) = \int_0^{s} \frac{g(t)}{(s-t)\alpha} dt; 0 < \alpha < 1$

iv) $f(x)=\int_0^{\infty}\sin(x-t)g(t)dt$

b) Convolution type integral equation :

An integral equation in which the kernel k(s,t) is a function of the difference (s-t) only

i.e. k(s, t) = k(s - t)

Where k is a certain function of one variable, is called convolution type integral equation.

e.g. $g(s)=f(s)+\lambda\int_a^s k(s-t)g(t)dt$

is voltera integral equation of the convolution type.

ii) $x=\int_0^x e^{x-t}g(t)dt$

Here $k(x, t) = e^{x-t} = k(x - t)$ where $k(s) = e^s$

iii) $g(x)=1+\int_0^x \sin(x,t)\, g(t)\, dt;$ Here $k(x,t)=e^{x-t}=k(x-t)$

Where $k(s)=e^s$

c) Integro - Differential equation :

An integral equation in which various derivatives of the unknown function also present is called an integro - differential equation.

e.g. i) $g'(t)=g(t)+f(t)+\int_0^t \sin(t-x)g(x)dx$

ii) $u'(x)+\int_0^1 \exp(x-t)u(y)dy=f(x) \qquad 0\le x\le 1$

Where u(0) = 0

iii) $u''(x)=e^x-x+\int_0^1 xtu'(t)dt \qquad u(0)=1,\ u'(0)=1.$

1.5 Solution of Integral equation :

A solution of an integral equation

$$h(s)g(s)=f(s)+\lambda\int_a k(s,t)g(t)\,dt$$

on the interval of integration is the function g(s) which satisfies the given integral equation. In other words, if the given solution is substituted in the right hand side of the equation, the output of this direct substitution must yeild the left hand side.

Worked Problems :

Problems No. 1 : Show that u(x) = e^x is a solution of the volterra integral equation

$$\mathbf{u(x)=1+\int_0^x u(t)\,dt} \qquad \text{----- (1)}$$

solution : substituting u(x) = e^x in the R.H.S. of (1), we have

$$R.H.S=1+\int_0^x e^t dt=1+\left[e^t\right]_0^x$$

$$=1+e^x-1$$

$$=e^x$$

$$=u(x)$$

$$=\text{L.H.S.}$$

$\Rightarrow$ u (x) = x is solution of (1)

Problem No. 2 : Show that u(x) = x is a solution of the integral equation

$$\mathbf{u(x)=\frac{5}{6}x-\frac{1}{9}+\frac{1}{3}\int_0^1 (x+t)u(t)\,dt} \qquad \text{----- (1)}$$

Solution : Putting u(x) = x in R.H.S of (1)

$$R.H.S = \frac{5}{6}x - \frac{1}{9} + \frac{1}{3}\int_0^1 (x+t)\,t\,dt$$

$$= \frac{5}{6}x - \frac{1}{9} + \frac{1}{3}\left[\frac{xt^2}{2} + \frac{t^3}{3}\right]_0^1$$

$$= x$$

$= \text{u}(x)$

= L.H.S.

$\Rightarrow$ u(x) = x is solution of (1)

Problem 3 : Show that the function g(x) = 1 - x is a solution of the integral equation.

$$\mathbf{x = \int_0^x e^{x-t} g(t)\,dt} \qquad \text{-----(1)}$$

Solution : Put g(x) = 1 - x in R. H. S. of equation (1)

$$\therefore\ R.H.S = \int_0^x e^{x-t}(1-t)\,dt$$

$$= e^x \int_0^x e^{-t}(1-t)\,dt$$

$$= e^x \left[(1-t)\frac{e^{-t}}{(-1)} - (-1)e^{-t}\right]_0^x$$

$$= e^x\left[-(1-x)e^{-x} + 1 + e^{-x} - 1\right]$$

$$= e^x\left[-e^x + xe^{-x} + e^x\right]$$

$$= e^x\left[xe^{-x}\right]$$

$$= x$$

$$= L.H.S$$

$\Rightarrow 9(x) = 1 - x$ is the solution of equation (1)

Problem 4 : Show that $g(x) = \cos 2x$ is a solution of the integral equation

$$\mathbf{g(x) = \cos x + 3\int_0^{\pi} K(x,t)\, g(t)\, dt}$$

$$k(x,t) = \begin{cases} \sin x \cos t; 0 \le x \le t \\ \cos x \sin t; t \le x \le \pi \end{cases}$$

Solution : Put $g(x) = \cos 2x$ in RHS of equation (1)

$$\therefore \text{R.H.S.} = \cos x + 3\int_0^{\pi} k(x,t) \cos 2t\, dt$$

$$= \cos x + 3\int_0^{x} k(x,t) \cos 2t\, dt + 3\int_x^{\pi} k(x,t) \cos 2t\, dt$$

$$= \cos x + 3\int_0^{x} \cos x \sin t \cos 2t\, dt + 3\int_x^{\pi} \sin x \cos t \cos 2t\, dt$$

$$= \cos x + \frac{3}{2}\cos x \int_0^{x} (\sin 3t - \sin t)\, dt$$

$$+ \frac{3}{2}\sin x \int_x^{\pi} (\cos 3t + \cos t)\, dt$$

$$\begin{bmatrix} \therefore\ 2\sin A \cos B = \sin(A+B) + \sin(A-B) \\ 2\cos A \cos B = \cos(A+B) + \cos(A-B) \end{bmatrix}$$

$$= \cos x + \frac{3}{2}\cos x \left[-\frac{1}{3}\cos 3t + \cos t \right]_0^x$$

$$+ \frac{3}{2}\sin x \left[\frac{1}{3}\sin 3t + \sin t \right]_x^{\pi}$$

$$= \cos x + \frac{3}{2}\cos x \left[-\frac{1}{3}\cos 3x + \cos x \frac{1}{3} - 1 \right]$$

$$+\frac{3}{2}\sin x\left[0-\frac{1}{3}\sin 3x-\sin x\right]$$

$$=\cos x-\frac{1}{2}\cos x\cos 3x+\frac{3}{2}\cos^2 x-\cos x$$

$$-\frac{1}{2}\sin x\sin 3x-\frac{3}{2}\sin^2 x$$

$$=\cos x-\frac{1}{2}[\cos x\cos 3x+\sin x\sin 3x]$$

$$+\frac{3}{2}\left[\cos^2-\sin^2 x\right]-\cos x$$

$$=\cos x-\frac{1}{2}\cos(x-3x)=\frac{3}{2}\cos 2x-\cos x$$

$$=-\frac{1}{2}\cos 2x+\frac{3}{2}\cos 2x$$

$= \cos 2x$

$= g(x)$

$\Rightarrow$ $g(x)=\cos 2x$ is solution of equation (1)

Exercise :

Verify that the given function is a solution of the corresponding integral equation.

1. $u(x)=\frac{3}{2}x+\int_0^1 xtu(t)\,dt;\quad u(x)=x$

2. $u(x)=x+\frac{1}{5}x^5-\int_0^x t[u(t)]^3\,dt;\quad u(x)=x$

3. $u(x)+2\int_0^1 e^{(x-t)}u(t)\,dt;=2xe^x;\quad u(x)=\left(2x-\frac{2}{3}\right)e^x$

4. $u(x)-\int_0^{\pi}(x^2+t)\cos tu(t)\,dt=\sin x;\quad u(x)\cos x$

5. $\int_0^{x}(x-t)^2u(t)\,dt=x^3;\quad u(x)=3$

6. $u(x)=(x-1)e^{-x}+4\int_0^{\infty}e^{-(x+t)}u(t)\,dt;\quad u(x)=xe^{-x}$

7. $u(x)=\sin+2\int_0^{x}\cos(x-t)u(t)\,dt;\quad u(x)=xe^{x}$

1.7 Some problem which give rise to integral equations :

In this section we shall see some problems where integral equations can arise.

Problem 1 : Boundary Value Problem :

Consider the Laplaces equation in two limmision $\Delta u=0$ where $\Delta=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}$

This equation is of fundamental importance in many branches of physics and engineering. The equation has also played an important role in the development of pure mathematics. The method of separation of variables is of great values to the solutions of the equation. To illustrate the method, consider the Laplace's equation $\Delta_2\, u(x, y)=0$ in the half plane $y\geq o$ with the potential function $u(x, y)\to 0$ as $p=\sqrt{x^2+y^2}\to\infty$

The method of separation of variable consists of trying to find a solution of the form $u(x, y)=X(x)\,Y(y)$ Where X is a function of x alone and Y is a function of y alone.

Hence $\frac{\partial^2 u}{\partial x^2}=\frac{d^2X}{dx^2}Y$ and $\frac{\partial^2 u}{\partial y^2}=\frac{d^2Y}{dy^2}X$ and

hence $u(x, y)=X(x)\,Y(y)$ will be a solution of $\Delta_2\, u=0$ if X and Y are such that

$\frac{d^2X}{dx^2}Y+\frac{d^2Y}{dy^2}X=0$ for all (x, y) with $y\geq 0$

$$\therefore \frac{1}{X}\frac{d^2X}{dx^2} = -\frac{1}{Y}\frac{d^2Y}{dy^2} = -\zeta^2 \text{ (say) where } \zeta \text{ is constant}$$

This leads to two equation namely : $\dfrac{d^2X}{dx^2} = -\zeta^2 X$ and $\dfrac{d^2Y}{dy^2} = \zeta^2 Y$

We want a solution of Δ_2 u = 0 with u (x, y) $\to$ 0 as $\zeta = \sqrt{x^2+y^2} \to \infty$

Hence we must choose ζ to be real and a solution of the equation is

$$u(x,y) = e^{i\zeta x - |\zeta| y} \qquad (\zeta \in R)$$

Note that this is the solution of the boundary value problem;

$$\Delta_2 u(x,y) = 0, -\infty < x < \infty, y \ge 0$$

$$u(x,0) = e^{i\zeta x}, -a < x < \infty, \ (\zeta \in R)$$

$$u(x,y) \to 0 \quad \text{as} \quad \sqrt{x^2+y^2} \to \infty$$

But many problems in physics and engineering the field variable is determined not only by the partial differential equation but also by the initial and bounbary values assumed by the function. The solution of the BVP (1) can be used to find solution for the other form of u(x, y) since Δ_2 $e^{i\zeta x - |\zeta| y} = 0$ for all real ζ and Δ_2 is a linear operation, for equation choice of the funcstion F(ζ) the function.

$$u(x,0) = \int_{\infty}^{\infty} F(\zeta) e^{i\zeta x - |\zeta| y} dx$$

is also solution of Laplace equation Δ_2 u(x, y) = 0, y > 0 subject to conditions :

$$u(x,o) = \int_{-\infty}^{\infty} F(\zeta) e^{i\zeta x} dx, -\infty < x < \infty, and$$

$$u(x,y) \to 0 \ as \sqrt{x^2+y^2} \to \infty$$

The problem naturally poses a question :

Can we find a function F(ζ) such that

$$f\left(x\right)=u\left(x,o\right)=\int_{\infty}^{\infty}F\left(\zeta\right)e^{i\zeta x}d\zeta \qquad \text{-----(2)}$$

For all real values of ζ ?

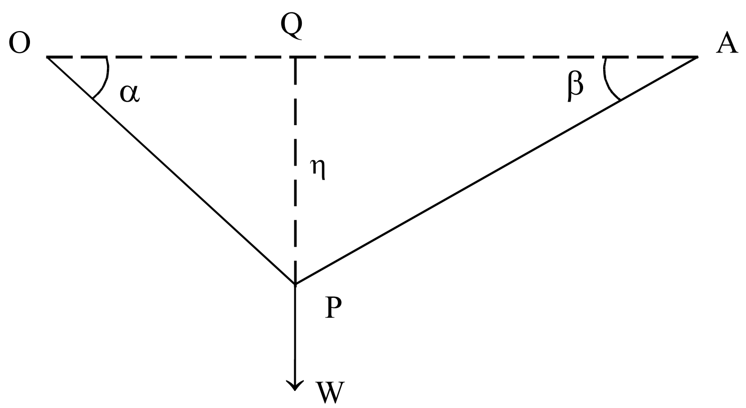
This means that f is given and F is unknown that satisfies (2). Thus (2) is a equation in which unknown function appears under the integral sign and the integral equation arises.

Once we know the solution of the integral equation (2) we get

$$u\left(x,y\right)=\frac{1}{\sqrt{2\pi}}\int_{\infty}^{\infty}F\left(\zeta\right)e^{i\zeta x-|\zeta|y}d\zeta,y>0$$

as the solution of the BVP (1)

Problem 2 : Problem of loaded elastic string. suppose a weightless elastic string is streched between two horizontal points, say O and A. Let a weight w is attached at a point Q distant ζ from O. Suppose the equillibrium occur with the depth η below OA. Suppose that the initial tension in the string T and w is small compared to T.



Let $\angle$AOP = α, $\angle$OAP = β. Here α and β are so small. Then the equillibrium equation is

T cos (90 - α) + T cos (90 - β) = w

i.e. T sin α + T sin β = w

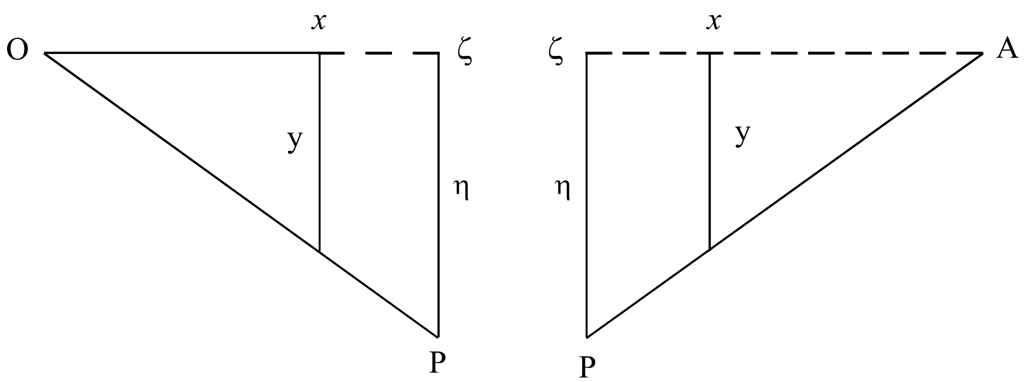
Since α and β so small, the equillibrium equation can be written as

T tan α + T tan β = w   (why)

or $T\left(\frac{\eta}{\zeta}\right)+T\left(\frac{\eta}{a-\zeta}\right)$ = w where l (OA) = a

$$\therefore \eta=\frac{w(a-\zeta)\zeta}{Ta}$$

∴ The drop y in the string at distant x from 0 is given by



$$y=\frac{x\eta}{\zeta} \qquad if\, 0\le x<\zeta$$

and $y=\dfrac{(a-x)\eta}{a-\zeta} \qquad if\, \zeta<x\le a$

Hence $y=\dfrac{wG(x,\zeta)}{T}$ where

$$G(x,\zeta)=\begin{cases}\dfrac{x(a-\zeta)}{a} & if\, 0\le x\le\zeta \\ \dfrac{\zeta(a-x)}{a} & if\, \zeta<x\le a\end{cases} \qquad \text{-----(1)}$$

Now suppose the string is loaded continuously with a weight distribution w(x) per unit length.

$\therefore$ The displacement at point x due to the weight distribution over $\zeta < x \leq \zeta + \delta\zeta$ is

$$\delta y = \frac{w(\zeta)\delta\zeta G(x,\zeta)}{T}, \begin{matrix} 0 \leq x \leq a \\ 0 \leq \zeta \leq a \end{matrix}$$

$\therefore$ The displacement at x due to the complete weight distribution is given by

$$y(x) = \frac{1}{T}\int_0^a \varsigma(x,\zeta) w(\zeta) d\zeta, 0 \leq x \leq a \qquad \text{-----(2)}$$

Thus, the displacement of the string is given in terms of the weight distribution by (2).

Now suppose we want a displacement y(x) of the string that is y(x) is given. Then what is the weight distribution which given the required displacement ? Again such weight distribution w(x) must satisfy (2). Thus (2) is a equation in which unknown function appears under the integral sign and the integral equation arise.

Problem 3 : Shop stocking problem

Suppose a shop starts selling same goods with the initial stock A of goods. That is A is the amount of goods purchased at the opening of the shop. Suppose k(t) is the proportion of goods remain unsold at time t. What is the rate at which the shop should purchase the goods so that the stock of the goods in the shop remains constant (with the assumption that all processes i.e. buying and selling to be constructed).

Suppose Q(t) be the required rate.

$\therefore$ In the time internal a $\leq$ t $\leq$ $\tau + \delta\tau$ is the amount of goods purchased.

$\therefore$ At time t, the portion of goods unsold is $k(t,\tau)\,Q(\tau)\delta\tau$

$\therefore$ The amount of goods remaining unsold upto time t is

$$Ak(t) + \int_0^t k(t-\tau) Q\tau d\tau$$

$\therefore$ The stock of goods of time t is

$$Ak(t)+\int_0^t k(t-\tau)Q\tau d\tau$$

$\therefore$ If it is to be remain constant as the initial stock A, the rate Q(t) must satisfy

$$A = Ak(t)+\int_0^t k(t-\tau)Q\tau d\tau$$

Thus the restocking rate Q (t) must satisfy the equation.

$$A(1-k(t)) = \int_0^t k(t-\tau)Q\tau d\tau \qquad \text{-----(2)}$$

Thus (2) is the equation in which unknown function appears under the integral sign and the integral equation arise.

♦♦♦

Unit – 2

CONVERSION OF ODE TO INTEGRAL EQUATIONS

In this unit we will present the technique that converts initial value problem to Volterra integeral equation, Boundary value problem to Fredholm integral equation, and integral equation to ordinary differential equation. For this we require Leibnitz rule for differentiation under integral sign (D.U.I.S.) and the formula for converting multiple integral to single ordinary integral which is discussed below.

2.1 Differentiation under integeral sign (Leibnitz Rule) :

1) Let G (x, t) and $\frac{\partial G}{\partial x}$ are continuous functions in the domain D in the xt - plane that contains the rectangular region R : $a \leq x \leq b$, $t_o \leq t \leq t_1$ and $\alpha(x)$ and $\beta(x)$ are defined functions having continuous derivatives for $a < x < b$ then,

1) $$\frac{d}{dx}\int_{\alpha(x)}^{\beta(x)} G(x,t)dt = \int_{\alpha(x)}^{\beta(x)} \frac{\partial G}{\partial x}dt + G(x,\beta(x))\frac{d\beta}{dx} - G(x,\alpha(x))\frac{d\alpha}{dx}$$

2) $$\frac{d}{dx}\int_{\alpha}^{\beta} G(x,t)dt = \int_{\alpha}^{\beta} \frac{\partial G}{\partial x}dt$$

where α, β are constants independent of x.

2.2 Identity for converting multiple integral into single ordinary integral :

Corollory : If a is constant and n $\in$ N then

$$\underbrace{\int_a^x\int_a^x\cdots\int_a^x}_{n\ times} f(t)\ dt^n = \frac{1}{(n-1)!}\int_a^x (x-t)^{n-1} f(t)\ dt$$

Proof : Let $I_n(x) = \int_a^x (x - t)^{n-1} f(t)\ dt$; $n \in N$ -----(1)

Differentiating w. r. t. x. useing Leibnitz rule

$$\therefore \frac{d}{dx} I_n(x) = \int_a^x \frac{\partial}{\partial x}\left[\left(x-t\right)^{n-1} f(t)\right]dt + \left[\left(x-t\right)^{n-1} f(t)\right]_{t=x} \frac{d}{dx}(x)$$

$$-\left[\left(x-t\right)^{n-1} f(t)\right]_{t=a} \frac{d}{dx}(a)$$

$$= \int_a^x (n-1)\ (x-t)^{n-2} f(t)dt$$

$$= (n-1)\int_a^x (x-t)^{n-2} f(t)dt$$

$$\frac{d}{dx} I_n(x) = (n-1)\ I_{n-1}(x) \qquad \text{-----(2)}$$

Differentiating w. r. t. x we get

$$\frac{d^2}{dx^n} I_n(x) = (n-1)\frac{d}{dx} I_{n-1}(x)$$

$$= (n-1)\ (n-2)\ I_{n-3}(x) \qquad (\therefore (2))$$

Similarly,

$$\frac{d^3}{dx^3} I_n(x) = (n-1)\ (n-2)\ (n-3)\ I_{n-3}(x)$$

$\therefore$ for $n > m$

$$\frac{d^m}{dx^m} I_n(x) = (n-1)\,(n-2)\,...\,(n-m) I_{n-m}(x)$$

Inparticular for m = n - 1

$$\frac{d^{n-1}}{dx^{n-1}} I_n(x) = (n-1)\,(n-2)\,....(n-(n-1) I_{n-(n-1)}(x)$$

$$= (n-1)\,(n-2)\,....1.\mathrm{I}_1(x)$$

$$= (n-1)!\ \mathrm{I}_1(x) \qquad \text{-----(3)}$$

Differentiating (3) w. r. t. x, we get

$$\frac{d^n}{dx^n} I_n(x) = (n-1)! \frac{dI_1}{dx} \qquad \text{-----(4)}$$

Now, from equation (1)

$$I_1(x) = \int_a^x f(t)\, dt$$

$$\therefore \frac{dI_1}{dx} = \int_a^x \frac{\partial}{\partial x} f(t)\, dt + \left[f(t) \right]_{t=x} \frac{d}{dx}(x) + 0$$

$$\therefore \frac{dI_1}{dx} = f(x)$$

Putting this in equation (4) we get

$$\frac{d^n}{dx_n} I_n(x) = (n-1)!\, f(x)$$

Integrating w. r. t. x from a to x, n times we get.

$$I_n(x) = (n-1)! \underbrace{\int_a^x \int_a^x ... \int_a^x}_{\text{n times}} f(t)\, dt^n \qquad \text{-----(5)}$$

Combinining equations (1) and (5) we get

$$\int_a^x \int_a^x \ldots \int_a^x f(t)\, dt^n = \frac{1}{(n-1)!} \int_a^x (x-t)^{n-1} f(t)\, dt$$

This completes the proof.

2.3 Conversion of IVP into Volterra integral Equations :

Initial value problem (Definition) :

An ordinary differential equation with the condition involving dependent variable and its derivatives at same value of the independent variable, is called initial value problem (IVP).

Problem 1 : Reduce the following IVP to volterra integral equation

y'' + y = 0,   y(0) = y' (0) = 0

Solution : Method I :

y'' + y = 0

Integrate w. r. t. *x* from o to x

$$\left[y'(x)\right]_0^x + \int_0^x y(t)\, dt = 0$$

$$y'(x) - y'(0) + \int_0^x y(t)\, dt = 0$$

$$\therefore\ y'(x) + \int_0^x y(t)\, dt = 0 \qquad\qquad (\because\ y'(0) = 0)$$

Integrating w. r. t. *x* from 0 to *x*

$$\therefore \left[y(x)\right]_0^x + \int_0^x \int_0^x y(t)\, dt^2 = 0$$

$$\therefore\ y(x) - y(0) + \frac{1}{(2-1)!} \int_0^x (x-t)\, y(t)\, dt = 0$$

$$\therefore y(x)+\int_0^x (x-t)\ y(t)\ dt=0$$

is the required integral equation.

Method II :

y'' + y = 0 -----(1)

Let y'' (x) = u (x) -----(2)

Integrating from o to *x*

$$\therefore \left[y'(x)\right]_0^x=\int_0^x u(t)\ dt$$

$$y'(x)-y'(0)=\int_0^x u(t)\ dt$$

$$\therefore y'(x)=\int_0^x u(t)\ dt \quad [\because y'(0)=0]$$

Integrating w. r. t. *x* from o to *x*

$$\therefore \left[y(x)\right]_0^x=\int_0^x\int_0^x u(t)\ dt^2$$

$$\therefore y(x)-y(0)=\int_0^x (x-t)\ u(t)\ dt$$

$$\therefore y(x)=\int_0^x (x-t)\ u(t)\ dt \qquad [\because y(0)=0] \qquad \text{-----(3)}$$

Using (2) and (3) in equation (1), We get

$$u(x)+\int_0^x (x-t)\ u(t)\ dt=0$$

is the required Integral equation.

Problem 2 : Convert IVP **y" + y = cos *x*;**

y (0) = 0, y' (0) = -1 to the volterra integeral equation

Solution : Method I :

y" + y = cos x

integrating w. r. t. x from o to x

$$\therefore y'(x) - y'(0) + \int_0^x y(t)\, dt = [\sin x]_0^x$$

$$\therefore y'(x) + 1 + \int_0^x y(t)\, dt = \sin x \qquad [\therefore y'(0) = -1]$$

Again integrating w. r. t. x from o to x

$$\therefore y(x) - y(0) + x + \int_0^x \int_0^x y(t)\, dt^2 - [\cos x]_0^x$$

$$y(x) + x + \int_0^x (x - t)\, y(t)\, dt = 1 - \cos x$$

$$\therefore \; y(x) = (1 - x - \cos x) + \int_0^x (x - t)\; y(t)\, dt$$

is the required integral equation.

Method II :

Given y" (x) + y (x) = cos x -----(1)

Let y" (x) = u (x) -----(2)

integrating w. r. t. x from o to x

$$y'(x) - y'(0) = \int_0^x u(t)\, dt$$

$$\therefore\ y'(x) + 1 = \int_0^x u(t)\, dt \qquad [\therefore\ y'(0) = -1]$$

Again, integrating w. r. t. x from o to x

$$\therefore\ y(x) - y(0) + x = \int_0^x \int_0^x u(t)\, dt^2$$

$$\therefore\ y(x) + x = \int_0^x (x-t)\, u(t)\, dt$$

$$\therefore\ y(x) = -x + \int_0^x (x-t)\, u(t)\, dt \qquad \text{-----(3)}$$

using equation (2) and (3) in equation (1)

$$\therefore u(x) - x + \int_0^x (x-t)\, u(t)\, dt = \cos \text{x}$$

$$\therefore u(x) = (x + \cos x) + \int_0^x (x-t)\, u(t)\, dt$$

is the required integral equation.

Problem 3 : Convert $y'' + \lambda s^2 y = 0$

$$y(0) = y'(0) = 0$$

to the integral equation

Solution : $y''(s) + \lambda s^2 y(s) = 0;$

integrating w. r. t. s from o t s

$$\therefore y'(s) - y'(0) + \lambda \int_0^s (s-t) t^2 y(t)\, dt = 0$$

$$\therefore y'(s) + \lambda \int_0^s (s-t) t^2 y(t)\, dt = 0$$

Integrating w. r. t. s from o to s

$$\therefore y(s) - y(0) + \lambda \int_0^s (s-t) t^2 y(t)\, dt = 0$$

$$\therefore y(s) + \lambda \int_0^x (s-t) t^2 y(t)\, dt = 0$$

is the required integral equation.

Problem 4 : Convert IVP $y'' + a_1(x) y' + a_2(x) y = f(x)$; $y(a) = y_1, y'(a) = y_2$

to the integral equation

Solution : $y'' + a_1(x) y' + a_2(x) y = f(x)$ -----(1)

Let $y''(x) = u(x)$ -----(2)

Integrating w. r. t. x from o to x

$$\therefore y'(x) - y'(a) = \int_a^x u(t)\, dt$$

$$y'(x) - y_2 = \int_a^x u(t)\, dt$$

$$y'(x) = y_2 + \int_a^x u(t)\, dt \qquad \text{------(3)}$$

Integrating again w. r. t. x from o to x

$$y(x) - y(a) - y_2 x = \int_a^x (x-t)\, u(t)\, dt$$

$$y(x) - y_1 - y_2 x = \int_a^x (x-t)\, u(t)\, dt$$

$$y(x) = y_1 + y_2 x + \int_a^x (x-t)\, u(t)\, dt \qquad \text{-----(4)}$$

using (2), (3) and (4) in equation (1)

$$u(x) + a_1(x)\left[y_2 + \int_a^x u(t)dt \right] + a_2(x)\left[y_1 + y_2 x + \int_a^x (x-t)u(t)dt \right] = f(x)$$

$$\therefore u(x) + a_1(x) y_2 + \int_a^x a_1(x)\, u(t)\, dt + a_2(x)(y_1 + y_2 x) + \int_a^x a_2(x)\, (x-t)\, u(t)\, dt = \mathrm{f}(x)$$

$$\therefore u(x) + a_1(x) y_2 + a_2(x)\, (y_1 + y_2 x) + \int_a^x [a_1(x) + a_2(x)\, (x-t)]\, u(t)\, dt = f(x)$$

i.e. $u(x) = f(x) - [a_1(x) y_2 + a_2(x)\, (y_1 + y_2 x)] - \int_a^x [a_1(x) + a_2(x)\, (x-t)]\, u(t)\, dt$

is the required integral equation.

Problem : 5 Convert I.V.P. $y'' + xy' + y = 1; y(0) = 1 \quad ; \quad y'(0) = 0$

to the integral equation

Solution : y'' (x) = u(x)

Integrating w. r. t. x from o to x

$$y'(x) = \int_0^x u(t)\, dt \qquad (y'(0) = 0)$$

Integrating again w.r.t. x from o to x

$$y(x)-1=\int_0^x (x-t)\,u(t)\,dt$$

$\therefore$ Given differential equation becomes

$$u(x)+x\left[\int_0^x u(t)dt\right]+1+\int_0^x (x-t)\,u(t)\,dt=1$$

$$u(x)+\int_0^x \left[xu(t)+(x-t)\,u(t)\right]dt=0$$

$$u(x)+\int_0^x (2x-t)\,u(t)\,dt$$

is the required integral equation.

Problem 6 : Convert the IVP $y'''+xy'+(x^2-x)y=x\,e^x+1$

with $y(0)=y''(0)=1;\ y''(0)=1$ to the volterra integral equation.

Solution : Given, $y'''+xy'+(x^2-x)y=xe^x+1$ -----(1)

Putting $y'''(x)=u(x)$ -----(2)

Integrating w.r.t. x from o to x

$$\therefore \left[y''(x)\right]_0^x=\int_0^x u(t)\,dt$$

$$y''(x)-y''(o)=\int_0^x u(t)dt$$

$$y''(x)=\int_0^x u(t)dt \qquad (\because y''(0)=0)$$

Again integrating w. r. t. x from o to x

$$\therefore y'(x)-y'(0)=\int_0^x (x-t)\,u(t)\,dt$$

$$\therefore y'(x) = 1 + \int_0^x (x-t)\, u(t)dt \qquad \text{-----(3)}$$

$$(\because y'(0) = 1)$$

Integrating again w. r. t. x from o to x

$$\therefore y(x) - y(0) = x + \frac{1}{2!}\int_0^x (x-t)^2 u(t)\, dt$$

$$\therefore y(x) = 1 + x + \frac{1}{2}\int_0^x (x-t)^2 u(t)\, dt \qquad \text{-----(4)}$$

$$(\because y(0) = 1)$$

Using equations (2), (3) and (4) in equation (1) we get

$$u(x) + x\left[\int_0^x u(t)\, dt\right] + (x^2 - x)\left[1 + x + \frac{1}{2}\int_0^x (x-t)^2 u(t)\, dt\right]$$

$$= xe^x + 1$$

$$u(x) = xe^x + 1 - x^2 + x - x^3 + x^2 \; - \int_0^x \frac{(x^2-x)(x-t)^2}{2} u(t)\, dt \; - \int_0^x x\, u(t)\, dt$$

$$\therefore u(x) = xe^x + 1 + x - x^3 \; - \int_0^x \left[\frac{(x^2-x)(x-t)^2}{2} + x\right] u(t)\, dt$$

is the required integral equation.

Problem 7 : Convert the following initial value problem $y''' - 3y''' - 6y' + 5y = 0$

Subject to the initial conditions $y(o) = y'(o) = y''(o) = 1$ to an equivalent volterra integral equation.

Solution : Given I.V.P. is

$$y''' - 3y'' - 6y' + 5y = 0 \qquad \text{-----(1)}$$

Let $y'''(x) = u(x)$ -----(2)

Integrating from o to x we get

$$\therefore [y''(x)]_0^x = \int_0^x u(t)\, dt$$

$$y''(x) - y''(0) = \int_0^x u(t)\, dt$$

$$y''(x) = 1 + \int_0^x u(t)\, dt \qquad \text{-----(3)}$$

Integrating from o to x we get

$$[y'(x)]_0^x = [x]_0^x + \int_0^x (x-t)\, u(t)\, dt$$

$$y'(x) - y'(0) = x + \int_0^x (x-t)\, u(t)\, dt$$

$$y'(x) = 1 + x + \int_0^x (x-t)\, u(t)\, dt \qquad \text{-----(4)}$$

Integrating from o to x we get

$$[y(x)]_0^x = [x]_0^x + \left[\frac{x^2}{2}\right]_0^x + \frac{1}{2!}\int_0^x (x-t)^2 u(t)\, dt$$

$$y(x) - y(0) = x + \frac{x^2}{2} + \frac{1}{2}\int_0^x (x-t)^2 u(t)\, dt$$

$$y(x) = 1 + x + \frac{x^2}{2} + \frac{1}{2}\int_0^x (x-t)^2 u(t)\, dt \qquad \text{-----(5)}$$

Using (2), (3), (4) & (5) in equation (1) we get

$$u(x)-3\left[1+\int_0^x u(t)dt\right]-6\left[1+x+\int_0^x (x-t)\,u(t)\,dt\right]+$$

$$5\left[1+x+\frac{x^2}{2}+\frac{1}{2}\int_0^x (x-t)^2 u(t)\,dt\right]=0$$

$$u(x)=3+3\int_0^x u(t)\,dt+6+6x+6\int_0^x (x-t)\,u(t)\,dt-$$

$$5-5x-\frac{5x^2}{2}-\frac{5}{2}\int_0^x (x-t)^2 u(t)\,dt$$

$$u(x)=4+x-\frac{5x^2}{2}+3\int_0^x u(t)dt+6\int_0^x (x-t)u(t)dt-\frac{5}{2}\int_0^x (x-t)^2 u(t)dt$$

$$u(x)=4+x-\frac{5x^2}{2}+\int_0^x \left[3+6(x-t)-\frac{5}{2}(x-t)^2\right]u(t)\,dt$$

which is volterra integral equation of second kind.

Problem 8 : Reduce the IVP y'' + λy = f(x); y(o) = 1, y' (o) = 0 to the volterra integral equation.

Solution : Given equation is

y'' + λy = f(x) -----(1)

Let y''(x) = u(x) -----(2)

Integrating from o to x we get

$$\left[y'(x)\right]_0^x=\int_0^x u(t)\,dt$$

$$\therefore y'(x)-y'(0)=\int_0^x u(t)\,dt$$

$$\therefore y'(x) = \int_0^x u(t)\, dt \qquad \text{-----(3)}$$

Integrating from o to x we get

$$[y(x)]_0^x = \int_0^x (x-t)\, u(t)\, dt$$

$$y(x) - y(0) = \int_0^x (x-t)\, u(t)\, dt$$

$$y(x) = 1 + \int_0^x (x-t)\, u(t)\, dt \qquad \text{-----(4)}$$

Using equation's (2), (3), (4); equation (1) becomes

$$u(x) + \lambda \left[1 + \int_0^x (x-t)\, u(t)\, dt \right] = f(x)$$

$$\therefore u(x) = f(x) - \lambda - \lambda \int_0^x (x-t)\, u(t)\, dt$$

Problem 9 : Reduce the following IVP

$y'' + y' = 0; y(1) = 0, y'(1) = 1$ to the volterra integral equation

Solution : Given equation is

$$y'' + y' = 0 \qquad \text{-----(1)}$$

Let $y''(x) = u(x)$) -----(2)

Integrating from 1 to x we get.

$$[y'(x)]_1^x = \int_1^x u(t)\, dt$$

$$\therefore y'(x) - y'(1) = \int_1^x u(t)\, dt$$

$$\therefore y'(x) = 1 + \int_1^x u(t)\, dt \qquad \text{-----(3)}$$

$$(\because y'(1) = 1)$$

Using equations (2) and (3) equation (1) becomes.

$$u(x) + 1 + \int_1^x u(t)\, dt = 0$$

$$u(x) = -1 - \int_1^x u(t)\, dt$$

is the required integral equation.

Exercise :

Derive an equivalent volterra integral equation to each of the following IVP.

1. $y'' + 5y' + 6y = 0, y(0) = 1, y'(0) = 1$
2. $y'' + y = 0, y(0) = 0, y'(0) = 1$
3. $y'' + (1 + x^2)y = \cos x; y(0) = 0, y'(0) = 2$
4. $y'' - \sin x\ y' + e^x y = x; y(0) = 1, y'(0) = -1$
5. $y''' + 4y' = x; y(0) = 0, y'(0) = 0, y''(0) = 1$
6. $y^{iv} + 2y'' + y = 3x + 4; y(0) = 0, y'(0) = 0, y''(0) = 1, y'''(0) = 1$
7. $y' + y = \sec^2 x, y(0) = 0$
8. $y''' - 2xy = 0; y(0) = \frac{1}{2}, y'(0) = 1, y''(0) = 1$
9. $y'' + 5y' + 6y = 0; y(0) = 0, y'(0) = -1$
10. $y'' - 3y'(x) + 2y(x) = 4\sin x; y(0) = 1, y'(0) = -2$

2.4 Conversion of BVP to Fredholm intergral equation.

Boundary value problem.

An ordinary differential equation with conditions, involving dependent variable and its derivatives at two different values of the independent variable is called boundary value problem (BVP)

Problem 1 : Convert $y'' + xy = 1,\ y(0) = 0, y(1) = 1$ into an integral equation.

Solution : $y'' = 1 - xy$

Integrate w. r. t. x from o to x

$$y'(x) - y'(0) = x - \int_0^x ty(t)\, dt$$

$$\therefore y'(x) = C + x - \int_0^x ty(t)\, dt$$

(Taking $y'(0) = C$)

Integrate w. r. t. x from o to x

$$y(x) - y(0) = Cx + \frac{x^2}{2} - \int_0^x (x-t)\, ty\,(t)\, dt$$

$$\therefore y(x) = Cx + \frac{x^2}{2} - \int_0^x (x-t)\, ty\,(t)\, dt \qquad \text{-----(1)}$$

$(\because y(0) = 0)$

Now from (1)

$$1 = y(1) = C + \frac{1}{2} - \int_0^1 (1-t)\, ty(t)\, dt$$

$$\therefore C = \frac{1}{2} + \int_0^1 (1-t)\, ty(t)\, dt$$

$\therefore$ Equation (1) becomes

$$y(x) = x\left[\frac{1}{2} + \int_0^1 (1-t)\, ty(t)\, dt\right] + \frac{x^2}{2} - \int_0^x (x-t)\, ty(t)\, dt$$

$$= \frac{1}{2}(x^2 + x) + \int_0^1 xt(1-t)\, y(t)\, dt - \int_0^x (x-t)\, ty(t)\, dt$$

$$= \frac{1}{2}(x^2 + x) + \int_0^1 xt(1-t)\, y(t)\, dt + \int_x^1 xt(1-t)\, ty(t)\, dt \text{ - } \int_0^x (x-t)\, ty(t)\, dt$$

$$= \frac{1}{2}x(x+1) + \int_0^x \left[x - xt - x + t\right] ty\,(t)\, dt + \int_x^1 xt(1-t)\, y(t)\, dt$$

$$= \frac{1}{2}x(x+1) + \int_0^x t^2(1-x)\, y\,(t)\, dt + \int_x^1 xt(1-t)\, y(t)\, dt$$

$$y(x) = \frac{1}{2}x(x+1) + \int_0^1 k(x,t)\, y(t)\, dt$$

where $k(x,t) = \begin{cases} t^2(1-x); 0 < t < x \\ xt(1-t); x < t < 1 \end{cases}$

is the required integral equation.

Problem 2 : Reduce the following BVP into an integral equation.

$$y''(x) + \lambda y(x) = 0; y(0) = y(l) = 0$$

Solution : $y''(x) = -\lambda y(x)$

Integrating w. r. t. x from o to x

$$y'(x) - y'(0) = -\lambda \int_0^x y(t)\, dt$$

$$y'(x) = C - \lambda \int_0^x y(t)\, dt$$

(Taking y' (o) = C)

Integrating again w. r. t. x from o to x

$$y(x) - y(0) = Cx - \lambda \int_0^x (x-t)\, y(t)\, dt$$

$$\therefore y(x) = Cx - \lambda \int_0^x (x-t)\, y(t)\, dt \qquad (\because y(0) = 0)$$

Now,

$$0 = y(l) = Cl - \lambda \int_0^l (l-t)\, y(t)\, dt$$

$$C = \frac{\lambda}{l} \int_0^l (l-t)\, y(t)\, dt$$

$\therefore$ Equation (1) becomes

$$y(x) = \frac{\lambda x}{l} \int_0^l (l-t)\, y(t)\, dt - \lambda \int_0^x (x-t)\, y(t)\, dt$$

$$= \lambda \int_0^x \frac{x(l-t)}{l} y(t)\, dt + \lambda \int_x^l \frac{x(l-t)}{l} y(t)\, dt - \lambda \int_0^x (x-t)\, y(t)\, dt$$

$$= \lambda \int_0^x \left[\frac{x(l-t)}{l} - (x-t) \right] y(t)\, dt + \lambda \int_x^l \frac{x(l-t)}{l} y(t)\, dt$$

$$= \lambda \int_0^x \frac{xl - xt - lx + lt}{l} y(t)\, dt + \lambda \int_x^l \frac{x(l-t)}{l} y(t)\, dt$$

$$= \lambda \int_0^x \frac{t(l-x)}{l} y(t)\, dt + \lambda \int_x^l \frac{x(l-t)}{l}\, y(t)\, dt$$

$$y(x) = \lambda \int_0^l k(x,t)\, y(t)\, dt$$

Where k(x, t) = $\begin{cases} \dfrac{t(l-x)}{l}; 0 < t < x \\ \dfrac{x(l-t)}{l}; x < t < l \end{cases}$

is the required fredholm integral equation

Problem 3 : Reduce the boundary value

$y'' + y = x; 0 < x < \pi$

$y(0) = 1, y(\pi) = \pi - 1$

to the fredholm integral equation.

Solution : $y'' + y = x;$

$\therefore y'' = x - y$ -----(1)

Putting $y''(x) = u(x)$ -----(2)

$\therefore$ Integrating w. r. t. x from 0 to x

$$\therefore y'(x) - y'(0) = \int_0^x u(t)\, dt$$

$$\therefore y'(x) = C + \int_0^x u(t)\, dt \qquad (\because \text{Taking } y'(0) = C)$$

Again, integrating from o to x

$$y(x) - y(0) = Cx + \int_0^x (x-t)\, u(t)\, dt$$

But y(o) = 1

$$y(x) = 1 + Cx + \int_0^x (x-t)\, u(t)\, dt \qquad \text{-----(1)}$$

Using, y (π) = π - 1 we have

$$\pi - 1 = 1 + C\pi + \int_0^\pi (x-t)\, u(t)\, dt$$

$$\therefore C = \frac{1}{\pi}\left[(\pi - 2) - \int_0^\pi (\pi - t)\, u(t)\, dt \right]$$

Putting this value of C_1 in equation (1) we get

$$y(x) = 1 + \frac{x}{\pi}\left[(\pi - 2) - \int_0^\pi (\pi - t)\, u(t)\, dt \right]$$

$$+ \int_0^x (\pi - t)\, u(t)\, dt \qquad \text{-----(3)}$$

Using (2) and (3) in eqution (1), we get

$$u(x) = x - 1 - \frac{x}{\pi}(\pi - 2) + \frac{x}{\pi}\int_0^\pi (\pi - t)\, u(t)\, dt - \int_0^x (\pi - t)\, u(t)\, dt$$

using the identity

$\int_0^\pi = \int_0^x + \int_x^\pi$ we get

$$u(x) = x - 1 - \frac{x}{\pi}(\pi - 2) + \frac{x}{\pi}\int_0^x (\pi - t)\, u(t)\, dt$$

$$+\frac{x}{\pi}\int_x^{\pi}(\pi-t)u(t)dt-\int_0^x (x-t)\,u(t)\,dt$$

$$=\frac{2x-\pi}{\pi}-\int_0^x \frac{t(x-\pi)}{\pi}\,u(t)\,dt\;-\int_x^{\pi}\frac{x(t-\pi)}{\pi}\,u(t)\,dt$$

$$u(t)=\frac{2x-\pi}{\pi}-\int_0^{\pi} k(x,t)\,u(t)\,dt$$

Where $k(x,t)=\begin{cases}\dfrac{t(x-\pi)}{\pi};\,0\le t\le x\\[2mm] \dfrac{x(t-\pi)}{\pi};\,x\le t\le \pi\end{cases}$

Problem 4 : Convert the BVP

y'' = f(x); y(0) = 0, y(1) = 0

to the integral equation

Solution : y''(x) = f(x)

Integrating from o to x

$$\therefore\, y'(x)-y'(0)=\int_0^x f(t)\,dt$$

$$\therefore\, y'(x)=C+\int_0^x f(t)\,dt \qquad \text{(Taking y' (0) = C)}$$

Again integrating w. r. t. x from o to x

$$\therefore\, y(x)-y(0)=Cx+\int_0^x (x-t)\,f(t)\,dt$$

$$\therefore\, y(x)=Cx+\int_0^x (x-t)\,f(t)\,dt \qquad \text{-----(1)}$$

[$\because$ y (0) = 0]

Using y (1) = 0

$$\therefore 0 = C + \int_0^1 (1-t)\ f(t)\ dt$$

$$\therefore C = \int_0^1 (t-1)\ f(t)\ dt$$

Putting this in equation (1) we get

$$y(x) = \int_0^1 x(t-1) f(t)\ dt + \int_0^x (x-t) f(t)\ dt$$

using the identity

$$\int_0^1 = \int_0^x + \int_x^1 \text{ we get}$$

$$y(x) = \int_0^x x(t-1) f(t)\ dt + \int_0^x (x-t) f(t)\ dt + \int_x^1 x(t-1) f(t)\ dt$$

$$= \int_0^x (xt - x + x - t) f(t)\ dt + \int_x^1 x(t-1) f(t)\ dt$$

$$= \int_0^x t(x-1) f(t)\ dt + \int_x^1 x(t-1) f(t)\ dt$$

$$\therefore y(x) = \int_0^1 k(x,t) f(t)\ dt$$

where $k(x,t) = \begin{cases} t(x-1); 0 \le t \le x \\ x(t-1); x \le t \le 1 \end{cases}$

is the required integral equation.

Exercise :

Transform the following BVP's to the integral equations :

1. $y'' + y = 0; \; y(0) = 0, \; y'(1) = 1$

2. $y'' + y = x; \; y'(0) = 0, \; y(1) = 0$

3. $y'' + y = x; \; y(0) = 1, \; y'(1) = 0$

4. $y'' + y' = 0; \; y(0) = y(1), \; y'(0) = y'(1)$

5. $y''' = 0, \; y(o) = y(1) = y''(0) = y''(1) = 0$

6. $y'' + \lambda y = ex; \; y(0) = y'(0) = 0$

7. $y'' - \lambda y = \cos x, \; y(0) = 0, y'(1) = 0$

8. $y'' = \lambda y + x^2; \; y(0) = y(\pi/2) = 0$

9. $y'' + y = x; \; y(0) = y(\pi/2) = 0.$

10. $y'' + \lambda y = x; \; y(0) = y(\pi) = 0.$

Ex. Reduce the following BVP into an integral equation $y'' + \lambda y = 0; y'(1) + V y(1) = 0$

Ans. : $y(x) = \dfrac{x}{1+V} + \lambda \int_0^{\pi} k(x,t) \, y(t) \, dt$

Where $k(x,t) = \begin{cases} \dfrac{1+V(1-x)}{1+V}; 0 < t < x \\ \dfrac{x(1+V(1-t)}{1+V}; x < t < \pi \end{cases}$

2.5 Conversion of Integral equation to ODE :

Problem 1 : If $y(x)$ is solution of the IE

$$y(x)=\lambda\int_0^1 k(x,t)\, y(t)\, dt$$

$$\text{where } k(x,t)=\begin{cases}(1-t)x; 0\le x\le t\\(1-x)t; x\le t\le 1\end{cases}$$

then show that $y(x)$ is also the solution of BVP

$$y''+\lambda y=0;\ \ y(0)=y(1)=0$$

Solution :

$$y(x)=\lambda\int_0^1 k(x,t)\,y(t)\,dt$$

$$=\lambda\int_0^1 k(x,t)\,y(t)\,dt+\lambda\int_0^1 k(x,t)\,y(t)\,dt$$

$$=\int_0^x \lambda(1-x)t\, y(t)\,dt+\int_x^1 \lambda(1-t)x\, y(t)\,dt \qquad \text{-----(1)}$$

Diff. w. r. t. x using DUIS

$$y'(x)=\int_0^x \frac{\partial}{\partial x}\left[\lambda(1-x)t\, y(t)\right]dt+\lambda(1-x)x\, y(x)\frac{d}{dx}(x)-0$$

$$+\int_x^1 \frac{\partial}{\partial x}\left[\lambda(1-t)\, x\, y(t)\right]dt\ +0-\lambda(1-x)x\, y(x)\frac{d}{dx}(x)-0$$

$$y'(x)=\int_0^x -\lambda\, t\, y(t)\,dt+\int_x^1 \lambda(1-t)\,y(t)\,dt$$

$\therefore$ Diff. w. r. t. x again

$$y''(x) = \int_0^x -\frac{\partial}{\partial x}\left[\lambda t y(t)\right] dt + \left[-\lambda x y(x)\right]\frac{d}{dx}(x) + \int_x^1 \frac{d}{dx}\left[\lambda(1-t)y(t)\right] dt$$

$$+0 - \lambda(1-x)y(x)\frac{d}{dx}(x)$$

$$\therefore y''(x) = -\lambda x y(x) - \lambda(1-x)y(x)$$

$$= -\lambda\left[x + 1 - x\right] y(x)$$

$$= -\lambda y(x)$$

$$\therefore y''(x) + \lambda y(x) = 0$$

from (1); y (0) = y(1) = 0

Problem 2 : Convert the BVP $y'' + \lambda y = x; y(0) = y'(1) = 0$

in to an fredholm integral equation. Also recover the BVP from integral equation that you obtain.

Solution : Given can be written as

$$\therefore y'' = x - \lambda y$$

Integrating w. r. t. x from o to x

$$\therefore y'(x) - y'(0) = \frac{x^2}{2} - \lambda \int_0^x y(t)\, dt$$

$$\therefore y'(x) = C + \frac{x^2}{2} - \lambda \int_0^x y(t)\, dt \qquad (\because \text{ Taking } y'(0) = C)$$

Integrate, from 0 to x

$$\therefore y(x) - y(0) = Cx + \frac{x^3}{6} - \lambda \int_0^x (x-t)\, y(t)\, dt$$

$$\therefore\ y(x) = Cx + \frac{x^3}{6} - \lambda\int_0^x (x-t)\ y(t)\ dt \qquad \text{-----(2)}$$

Now, (1) gives

$$0 = y'(1) = C + \frac{1}{2} - \lambda\int_0^1 y(t)\ dt$$

$$\therefore C = -\frac{1}{2} + \lambda\int_0^1 y(t)dt$$

$\therefore$ Equation (2) becomes

$$y(x) = -\frac{x}{2} + \lambda x\int_0^1 y(t)\ dt + \frac{x^3}{6} - \lambda\int_0^x (x-t)\ y(t)\ dt \qquad \text{-----(3)}$$

$$= \left(\frac{x^3}{6} - \frac{x}{2}\right) + \lambda\int_0^x x\ y(t)\ dt + \lambda\int_x^1 x\ y(t)\ dt - \lambda\int_0^x \left(x-t\right)\ y(t)\ dt$$

$$= \left(\frac{x^3}{6} - \frac{x}{2}\right) + \lambda\int_0^x (x-x+t)\ y(t)\ dt + \lambda\int_x^1 x\ y(t)\ dt$$

$$= \left(\frac{x^3}{6} - \frac{x}{2}\right) + \lambda\int_0^x t\ y(t)\ dt + \lambda\int_x^1 x\ y(t)\ dt$$

$$y(x) = \left(\frac{x^3}{6} - \frac{x}{2}\right) + \lambda\int_0^x k(x,t)\ y(t)\ dt \qquad \text{-----(4)}$$

where $k(x,t) = \begin{cases} t;\ 0 < t < x \\ x;\ x < t < 1 \end{cases}$

conversly, we want to convert (4) to BVP

Now equation (4) can be written as

$$y(x)=\left(\frac{x^3}{6}-\frac{x}{2}\right)+\lambda\int_0^1 x\, y(t)\, dt-\lambda\int_0^x (x-t)\, y(t)\, dt \qquad \text{-----(5)}$$

Diff. again w. r. t. x

$$y'(x)=\left(\frac{x^2}{2}-\frac{1}{2}\right)+\lambda\int_0^1 \frac{\partial}{\partial x}\left(x\, y(t)\right) dt-\lambda\int_0^1 \frac{\partial}{\partial x}\left[(x-t)\, y(t)\right] dt$$

$$+(x-x)y(x)\frac{d}{dx}(x)+(x-0)y(0)\frac{d}{dx}(0)$$

$$=\frac{x^2}{2}-\frac{1}{2}+\lambda\int_0^1 y(t)\, dt-\lambda\int_0^x y(t)\, dt \qquad \text{-----(6)}$$

Diff. again w. r. t. x

$$y''(x)=x+\lambda\int_0^1 \frac{\partial}{\partial x} y(t)\, dt-\lambda y(x)$$

$$y''(x)=x+\lambda(0)-\lambda y(x)$$

$$y''(x)+\lambda y(x)=x$$

Also, from (5) y(0) = 0

from (6) $y'(1)=-\frac{1}{2}+\frac{1}{2}=0$

Problem 3 : Convert IVP y'' - (sin x) y' + e^x y = x; y (o) = 1, y' (o) = -1 to a volterra integral equation, conversly, derive the original IVP from the integral equation obtained.

Solution : y'' - (sin$x$) y' + e^x y = x

Integrate w. r. t. x from o to x

$$\left[y'(x)\right]_0^x-\int_0^x \sin t y'(t)\, dt+\int_0^x e^t y(t)\, dt=\left[\frac{x^2}{2}\right]_0^x$$

$$y'(x) - y(0) - \left[\sin t y(t)\right]_0^x + \int_0^x \cos t y(t)\, dt + \int_0^x e^t y(t)\, dt = \frac{x^2}{2}$$

$$y'(x) + 1 - \sin x y(x) + \int_0^x \left(e^t + \cos t\right) y(t)\, dt = \frac{x^2}{2} \qquad \text{-----(3)}$$

$$(\because y'(0) = -1)$$

Again, integrating w. r. t. x from o to x

$$\left[y(x)\right]_0^x + x - \int_0^x \sin t y(t)\, dt + \int_0^x \int_0^x (e^t + \cos t)\, y(t)\, dt = \left[\frac{x^3}{6}\right]_0^x$$

$$y(x) - y(0) + x - \int_0^x \sin t\; y(t)\, dt + \int_0^x (x-t)\,(e^t + \cos t) y(t)\, dt = \frac{x^3}{6}$$

$$y(x) - 1 + x + \int_0^x \left[(x-t)(e^t + \cos t) - \sin t\right] y(t)\, dt = \frac{x^3}{6}$$

$$\therefore y(x) = \frac{x^3}{6} - x + 1 + \int_0^x \left[\sin t - (x-t)(e^t + \cos t)\right] y(t)\, dt \qquad \text{-----(2)}$$

is the required integral equation

Conversly, to obtain differential equation, Diff. (2) w. r. t. x

$$y'(x) = \frac{x^3}{2} - 1 + \int_0^x \frac{\partial}{\partial x}\left[\sin t - (x-t)\,(e^t + \cos t\right] y(t)\, dt$$

$$+ \left[\sin t - (x-t)(e^t + \cos t) y(t)\, dt\right]_{t=x} \frac{d}{dx}(x)$$

$$- \left[\sin t - (x-t)(e^t + \cos t) y(t)\, dt\right]_{t=0} \frac{d}{dx}(0)$$

$$y'(x)=\frac{x^2}{2}-1+\int_0^x\left[0-\left(e^t+\cos t\right)\right]y(t)\,dt+\sin x\;y(x)$$

$$=\frac{x^2}{2}-1-\int_0^x(e^t+\cos t)\;y(t)\,dt+\sin x\;y(x)$$

Diff. w. r. t. x again

$$y''(x)=x-0-(e^x+\cos x)y(x)+\sin x\;y'(x)+y(x)\cos x$$

$$y''(x)=x-e^x y(x)-\cos x\;y(x)+\sin x\;y'(x)+y(x)\cos x$$

$$y''(x)-(\sin x)y'(x)+e^x y(x)=0$$

From (2) y(0) = 1,

From (1) y'(0) = -1

Problem 4 : Convert IVP y'' - 3y' + 2y = 4 sin x;

y(0) = 1, y' (0) = -2 in to integral equation and derive original IVP from obtained integral equation.

Solution : y'' - 3y' + 2y = 4sin x

Integrate w. r. t. x from o to x

$$\left[y'(x)\right]_0^x-3\left[y(x)\right]_0^x+2\int_0^x y(t)dt=-4\left[\cos x\right]_0^x$$

$$y'(x)-(-2)-3\left[y(x)-1\right]+2\int_0^x y(t)\,dt=-4\left(\cos x-1\right)$$

$$y'(x)+2-3y(x)+3+2\int_0^x y(t)\,dt=-4\left[1-\cos x\right] \qquad \text{-----(1)}$$

Integrate w. r. t. x from o to x

$$\left[y(x)\right]_0^x + 5x - 3\int_0^x y(t)\,dt + 2\int_0^x (x-t)y(t)\,dt = 4\left[x - \sin x\right]_0^x$$

$$y(x) - 1 + 5x + \int_0^x \left[2(x-t) - 3\right] y(t)\,dt = 4(x - \sin x)$$

$$y(x) = 4x - 4\sin x + 1 - 5x + \int_0^x \left[3 - 2(x-t)\right] y(t)\,dt$$

$$= 1 - x - 4\sin x + \int_0^x \left[3 - 2(x-t)\right] y(t)\,dt \qquad \text{-----(2)}$$

which is required volterra intergral equation.

Conversely to derive the original IVP Differential (2) w. r. t. x

$$y'(x) = -1 - 4\cos x + \int_0^x \frac{\partial}{\partial x}\left[3 - 2(x-t)\right] y(t)\,dt + \left[3 - 2(x-x)\right] y(x)\frac{d}{dx}(x)$$

$$+\left[3 - 2(x-0)\right] y(0)\frac{d}{dx}(0)$$

$$= -1 - 4\cos x + \int_0^x -2y(t)\,dt + 3y(x)$$

Diff. w. r. t. x again

y'' (x) = 4sin x - 2 y(x) + 3y' (x)

i.e. y'' (x) - 3y' (x) + 2y(x) = 4 sin x

Now, from (2) y(0) = 1

from (1) y'(0) + 2 - 3 + 3 = 0 $\Rightarrow$ y'(0) = -2

Thus the IVP is

y'' (x) - 3y' (x) + 2y (x) = 4sin x;

y (0) = 1, y' (0) = -2

Problem 5 : Reduce the following integral equation to an initial value problem

$$y(x) = x + \int_0^x (t-x)y(t)\, dt$$

Solution :

$$y(x) = x + \int_0^x (t-x)y(t) \qquad \text{-----(1)}$$

Differentiation w. r. t. x

$$\therefore y'(x) = 1 + \int_0^x [-y(t)]\, dt$$

$$= 1 - \int_0^x y(t)\, dt \qquad \text{-----(2)}$$

Differentiating again w. r. t. x

$\therefore$ y'' (x) = -y (x)

$\therefore$ y'' (x) + y (x) = 0

Initial conditions are obtained by putting $x = 0$ in (1) and (2)

$\therefore$ y (0) = 0

and y' (0) = 1

Required IVP is

y'' + y = 0, y(0) = 0, y' (0) = 1

Problem 6 : Find the IVP equivalent to the integral equation

$$y(x) = x^3 + \int_0^x (x-t)^2 y(t)\, dt$$

Solution : Given I.E. is

$$y(x) = x^3 + \int_0^x (x-t)^2 y(t)\, dt \qquad \text{-----(1)}$$

Differentiation (1) w. r. t. x

$$y'(x) = 3x^2 + \int_0^x 2(x-t)y(t)\, dt \qquad \text{-----(2)}$$

Diff. w. r. t. x again we get

$$y''(x) = 6x + \int_0^x 2y(t)\, dt \qquad \text{-----(3)}$$

Diff. w. r. t. x again we get

$$y'''(x) = 6 + 2\, y(x)$$

$$y'''(x) - 2y(x) - 6 = 0$$

To obtain initial condition put $x = 0$ in equations (1), (2) and (3) we get respectively

$$\therefore \; y(0) = 0$$

$$y'(0) = 0$$

$$y''(0) = 0$$

$\therefore$ The required IVP is

$$y''' - 2y - 6 = 0; \quad y(0) = y'(0) = y''(0) = 0$$

Problem 7 : Reduce the IE

$$y(x) = \frac{1}{2}x(1-x) + \int_0^1 k(x,t)\, y(t)\, dt$$

where $k(x,t) = \begin{cases} t^2(t-x); & 0 < t < x \\ xt(1-t); & x < t < 1 \end{cases}$

to the ODE

Solution : Given integral equation can be written as

$$y(x)=\frac{1}{2}x(1+x)+\int_0^x k(x,t)y(t)\,dt+\int_x^1 k(x,t)y(t)\,dt$$

$$\therefore y(x)=\frac{1}{2}x(1+x)+\int_0^x t^2(1-x)y(t)\,dt+\int_x^1 xt(1-t)y(t)\,dt \qquad \text{-----(1)}$$

Diff. w. r. t. x.

$$\therefore y'(x)=\frac{1}{2}(1+2x)+\int_0^x -t^2y\,dt+x^2(1-x)\,y(x)+\int_x^1 t(1-t)y(t)\,dt-x^2(1-x)y(x)$$

$$=\frac{1}{2}+x-\int_0^x t^2y(t)\,dt+\int_x^1 t(1-t)y(t)\,dt \qquad \text{-----(2)}$$

Differentiating w. r. t. x. again

$y''(x) = 1 - x^2\, y(x) - x\,(1 - x)\, y(x)$

$\qquad = 1 - x^2\, y(x) - x\, y(x) + x^2\, y(x)$

$\therefore\ y''(x) + x\, y(x) = 1$

from (1)

$y(0) = 0, \quad y(1) = 0$

$\therefore$ The required BVP is

$y'' + x\,y = 1; \quad y(0) = 0, \quad y(1) = 0$

Exercise :

Question1 : Reduce the Following volterra integral equation to an equivalent IVP.

i) $y(x)=e^x-\int_0^x (x-t)y(t)\,dt$

ii) $y(x)=2+3x+5x^2+\int_0^x \left[1+2(x-t)\right]y(t)\,dt$

iii) $y(x)=1+x+\frac{5}{2}x^2+\int_0^x\left[3+6(x-t)-\frac{5}{2}(x-t)^2\right]y(t)\,dt$

iv) $y(x)=x^4+x^2+2\int_0^x(x-t)^2\,y(t)\,dt$

v) $y(x)=x^2+\frac{1}{6}\int_0^x(x-t)^3\,y(t)\,dt$

Question 2 : Reduce the following fredholm intergral equation to an equivalent BVP

i) $y(x)=\int_0^1 k(x,t)y(t)\,dt$

where $k(x,t)=\begin{cases} t(x-1);\, t<x \\ x(t-1);\, t>x \end{cases}$

ii) $y(t)=\frac{2x-\pi}{\pi}-\int_0^\pi k(x,t)y(t)\,dt$

where $k(x,t)=\begin{cases} \frac{t(x-\pi)}{\pi};\, 0\le t\le x \\ \frac{x(t-\pi)}{\pi};\, x\le t\le \pi \end{cases}$

iii) $y(x)=\lambda\int_0^l k(x,t)\,y(t)\,dt$

where $k(x,t)=\begin{cases} \frac{t(l-x)}{l};\, 0<t<x \\ \frac{x(l-t)}{l};\, x<t<l \end{cases}$

♦♦♦

Unit – 3

FREDHOLM INTEGRAL EQUATIONS WITH SEPARABLE KERNELS

3.1 Solution of fredholm integral equation of the second kind with separable kernel

Consider the fredholm integral equation of the second kind.

$$g(s) = f(s) + \lambda \int k(s, t)\, g(t)\, dt \qquad \text{-----(1)}$$

As, k(s, t) is separable kernel hence written as

$$k(s,t) = \sum_{i=1}^{\eta} a_i(s) b_i(t)$$

Where the functions $a_1(s) \ldots a_n(s)$ & $b_1(t) \ldots b_n(t)$ are linearly independent functions.

$\therefore$ Then the equation (1) becomes

$$g(s) = f(s) + \lambda \int \sum_{i=1}^{\eta} a_i(s) b_i(t) g(t)\, dt$$

$$= f(s) + \lambda \sum_{i=1}^{\eta} a_i(s) \int b_i(t) g(t)\, dt \qquad \text{-----(2)}$$

Let us assume

$$\int b_i(t)\, g(t)\, dt = c_i$$

$\therefore$ Equtions (2) becomes

$$g(s) = f(s) + \lambda \sum_{i=1}^{\eta} c_i a_i(s) \qquad \text{-----(3)}$$

To find the solution of equation (1) in the form (3) we should find the value of constant C_i

Put the value of g(s) from (3) in equation (2) we get

$$f(s)+\lambda\sum_{i=1}^{\eta}c_i a_i(s)=f(s)+\lambda\sum_{i=1}^{\eta}a_i(s)\int b_i(t)\left[f(t)+\lambda\sum_{k=1}^{n}C_k a_k(t)\right]dt$$

$$\therefore\sum_{i=1}^{\eta}a_i(s)\left\{c_i-\int b_i(t)\left[f(t)+\lambda\sum_{k=1}^{n}C_k a_k(t)\right]dt\right\}=0$$

But the functions a_i (s) are linearly independent

$$\therefore c_i-\int b_i(t)\left[f(t)+\lambda\sum_{k=1}^{n}C_k a_k(t)\right]dt=0;\qquad i = 1, 2, ...,n.$$

$$c_i-\int b_i(t)f(t)dt-\lambda\sum_{k=1}^{n}C_k\int b_i(t)a_k(t)d(t)=0 \qquad \text{-----(4)}$$

Denote $\int b_i(t)f(t)dt=f_i$

$$\int b_i(t)a_k(t)dt=a_{ik}$$

equtions (4) becomes

$$\therefore c_i-\lambda\sum_{k=1}^{n}C_k a_{ik}=f_i;\quad i = 1, 2, ... n$$

$$\Rightarrow c_i-\lambda\left[C_1a_{i1}+C_2a_{i2}+.........+C_na_{in}\right]=f_i,\quad (1\le i\le n)$$

It is system of linear equations for unknown $c_1, c_2, ... c_n$ which is given by

$$\left.\begin{array}{l}c_1(1-\lambda a_{11})-c_2\lambda a_{12}-.....-c_n\lambda a_{1n}=f_1\\ -c_1\lambda a_{21}+c_2(1-\lambda a_{22})-...-c_n\lambda a_{2n}=f_2\\ \vdots\\ -c_1\lambda a_{n1}-c_2a_{22}-...+c_n(1-\lambda a_{nn})=f_n\end{array}\right\} \qquad \text{------(5)}$$

The determinant D(λ) of the system is

$$D(\lambda) = \begin{vmatrix} 1-\lambda a_{11} & -\lambda a_{12} & \dots & -\lambda a_{1n} \\ -\lambda a_{21} & 1-\lambda a_{22} & \dots & -\lambda a_{2n} \\ . & & & \\ . & & & \\ . & & & \\ -\lambda a_{n1} & -\lambda a_{22} & \dots & 1-\lambda a_{nn} \end{vmatrix}$$

Which is polynomial in λ of degree at most n. Moreover it is not identically zero since for λ = 0 it reduces to unity.

Case 1 : When atleast one right member of system (5) is non zero

i) **If D(λ) ≠ 0** for all values of λ **:** The system (5) possesses unique non zero solution, hence equation (1) have unique non-zero solution given by (3).

ii) **If D(λ) = 0** for all values of λ **:** The system (5) has no solution or they possesses infinite solution and hence (1) has no solution or infinite solutions.

Case 2 : When f (s) = 0 : The system (5) reduces to homogeneous system of equations.

i) **If D(λ) ≠ 0 :** System (5) possess trivial solution which is uniue

i.e. $c_1 = c_2 = \dots = c_n = 0$ is the only solution of (5)

∴ g(s) = 0 is the only solution for (1)

ii) **If D(λ) = 0 :** System (6) have infinite number of solution.

∴ Equation (1) has infinite solutions.

Case 3 : When f (s) ≠ 0 and f(s) is orthogonal to all $b_i(t)$,

ie. $\int_a^b f(s) b_i(t) dt = 0$, i=1, 2, ... , n

Then system (5) reduces to homogeneous system of equation.

i) **If D(λ) ≠ 0 :** System (5) possessesa unique solution $C_1 = C_2 = \dots = C_n = 0$ (trivial Sol.^{n})

∴ g(s) = f(s) is the only solution for equation (1).

ii) **If D(λ) = 0 :** System (5) possesses infinite solution.

$\therefore$ Equation (1) has infinite number of solutions.

Problem 1 : Solve the fredholm integral equation of second kind.

$$g(s) = s + \lambda \int_0^1 \left(st^2 + s^2 t\right) g(t)\, dt$$

Solution : $g(s) = s + \lambda s \int_0^1 t^2 g(t)\, dt + \lambda s^2 \int_0^1 t g(t)\, dt$

$$= s + \lambda s c_1 + \lambda s^2 c_2 \qquad \text{-----(1)}$$

where $C_1 = \int_0^1 t^2 g(t)\, dt$ -----(2)

and $C_2 = \int_0^1 t g(t)\, dt$ -----(3)

Using (1) in equation (2) we get

$$C_1 = \int_0^1 t^2 \left[t + \lambda t C_t + \lambda t^2 C_2 \right] dt$$

$$= \int_0^1 \left[t^3 + \lambda C_1 t^3 + \lambda C_2 t^4 \right] dt$$

$$= \left[\frac{t^4}{4} + \lambda C_1 \frac{t^4}{4} + \lambda C_2 \frac{t^5}{5} \right]_0^1$$

$$C_1 = \frac{1}{4} + \lambda C_1 \frac{1}{4} + \lambda C_2 \frac{1}{5}$$

$$C_1 \left(1 - \frac{\lambda}{4} \right) - \frac{\lambda}{5} C_2 = \frac{1}{4} \qquad \text{-----(4)}$$

Using (1) in equation (3) we get

$$C_2 = \int_0^1 t\left[t + \lambda t C_1 + \lambda t^2 C_2\right]dt$$

$$= \int_0^1 \left[t^2 + \lambda C_1 t^2 + \lambda C_2 t^3\right]dt$$

$$C_2 = \left[\frac{t^3}{3} + \lambda C_1 \frac{t^3}{3} + \lambda C_2 \frac{t^4}{4}\right]_0^1$$

$$= \frac{1}{3} + \lambda C_1 \frac{1}{3} + \lambda C_2 \frac{1}{4}$$

$$\therefore\ -C_1 \frac{\lambda}{3} + C_2\left(1 - \frac{\lambda}{4}\right) = \frac{1}{3} \qquad \text{-----(5)}$$

The determinant D(λ) for system (4), (5) is

$$D(\lambda) = \begin{vmatrix} 1-\frac{\lambda}{4} & -\frac{\lambda}{5} \\ -\frac{\lambda}{3} & 1-\frac{\lambda}{4} \end{vmatrix} = \left(1-\frac{\lambda}{4}\right)^2 + \frac{\lambda^2}{15} \neq 0$$

$\therefore$ system has unique solution given by

$$C_1 = \frac{60+\lambda}{240-120\lambda-\lambda^2}, C_2 = \frac{80}{240-120\lambda-\lambda^2}$$

Required solution is

$$g(s) = s + \lambda s\left[\frac{60+\lambda}{240-120\lambda-\lambda^2}\right] + \lambda s^2\left[\frac{80}{240-120\lambda-\lambda^2}\right]$$

$$= \frac{240s - 120\lambda s - \lambda^2 s + \lambda s 60 + \lambda^2 s + 80\lambda s^2}{240-120\lambda-\lambda^2}$$

$$g(s)=\frac{s(240s-60\lambda)-80\lambda s^2}{240-120\lambda-\lambda^2}$$

Problem 2 : Solve the fredholm integral equation

$$g(s)=s+\lambda\int_0^1(1+s+t)g(t)\,dt$$

Solution : $g(s)=s+\lambda\int_0^1(1+s)g(t)\,dt+\lambda\int_0^1 tg(t)\,dt$

$$=s+\lambda(1+s)\int_0^1 g(t)dt+\lambda\int_0^1 tg(t)\,dt$$

$$=s+\lambda(1+s)C_1+\lambda C_2 \qquad \text{-----(1)}$$

where $C_1=\int_0^1 g(t)\,dt$ -----(2)

$$C_2=\int_0^1 tg(t)\,dt \qquad \text{-----(3)}$$

Thus using (1) in equation (2) we get

$$C_1=\int_0^1\left[t+\lambda(1+t)C_1+\lambda C_2\right]dt$$

$$=\left[\frac{t^2}{2}+\lambda\left(1+\frac{t^2}{2}\right)C_1+\lambda C_2 t\right]_0^1$$

$$C_1=\frac{1}{2}+\lambda\left(\frac{3}{2}\right)C_1+\lambda C_2$$

$$\therefore C_1\left(1-\frac{3\lambda}{2}\right)-\lambda C_2=\frac{1}{2} \qquad \text{-----(4)}$$

Using (1) in equation (3) we get

$$C_2=\int_0^1 t\left[t+\lambda(1+t)C_1+\lambda C_2\right]dt$$

$$=\int_0^1\left[t^2+\lambda\left(t+t^2\right)C_1+\lambda C_2 t\right]dt$$

$$\left[\frac{t^3}{3}+\lambda\left(\frac{t^2}{2}+\frac{t^3}{3}\right)C_1+\lambda C_2\frac{t^2}{2}\right]_0^1$$

$$C_2=\frac{1}{3}+\frac{5\lambda}{6}C_1+C_2\frac{\lambda}{2}$$

$$-\frac{5\lambda}{6}C_1+\left(1-\frac{\lambda}{2}\right)C_2=\frac{1}{3} \qquad \text{-----(5)}$$

The determinant D(λ) for the system of equation is

$$D(\lambda)=\begin{vmatrix}\left(1-\frac{3\lambda}{2}\right) & -\lambda \\ -\frac{5\lambda}{6} & \left(1-\frac{\lambda}{2}\right)\end{vmatrix}=\left(1-\frac{3\lambda}{2}\right)\left(1-\frac{\lambda}{2}\right)+\frac{5\lambda^2}{6}\neq 0$$

$\therefore$ System has unique solution given by

$$C_1=\frac{6+\lambda}{12-24\lambda-\lambda^2},C_2=\frac{4-\lambda}{12-24\lambda-\lambda^2}$$

Putting these values of C_1 and C_2 in equation (1)

$\therefore$ We get,

$$g(s) = s + \frac{\lambda\left[10 + (6+\lambda)s\right]}{\left[12 - 24\lambda - \lambda^2\right]}$$

is the required solution

Problems 3 : Sove the integral equation

$$g(s) = f(s) + \lambda\int_0^1 (s+t)g(t)\,dt$$

Solution : $g(s) = f(s) + \lambda\int_0^1 (s+t)g(t)\,dt$

$$= f(s) + \lambda s\int_0^1 g(t)dt + \lambda\int_0^1 tg(t)dt$$

$$g(s) = f(s) + \lambda s C_1 + \lambda C_2 \qquad \text{-----(1)}$$

where $\quad C_1 = \int_0^1 g(t)\,dt \qquad$ -----(2)

and $\quad C_2 = \int_0^1 tg(t)\,dt \qquad$ -----(3)

Putting the value of g(s) in equation (2)

$$C_1 = \int_0^1 \left[f(t) + \lambda t C_1 + \lambda C_2\right]dt$$

$$= \int_0^1 f(t)\,dt + \lambda C_1\left[\frac{t^2}{2}\right]_0^1 + \lambda C_2\left[t\right]_0^1$$

$$= f_1 + \frac{\lambda C_1}{2} + \lambda C_2$$

$$\therefore \quad \left(1-\frac{\lambda}{2}\right)C_1 - \lambda C_2 = f_1 \qquad \text{-----(4)}$$

Putting the value of g(s) in equation (3)

$$\therefore C_2 = \int_0^1 \left[t\, f(t) + \lambda t^2 C_1 + \lambda C_2 t \right] dt$$

$$\int_0^1 t\, f(t)\, dt + \lambda C_1 \left[\frac{t^3}{3}\right]_0^1 + \lambda C_2 \left[\frac{t^2}{2}\right]_0^1$$

$$C_2 = f_2 + \frac{\lambda C_1}{3} + \frac{\lambda C_2}{2}$$

$$\therefore -\frac{\lambda}{3}C_1 + \left(1-\frac{\lambda}{2}\right)C_2 = f_2 \qquad \text{-----(5)}$$

$$D(\lambda) = \begin{vmatrix} 1-\frac{\lambda}{2} & -\lambda \\ -\frac{\lambda}{3} & 1-\frac{\lambda}{2} \end{vmatrix} = \frac{-\lambda^2 - 12\lambda + 12}{12}$$

If $D(\lambda) \neq 0$, then after solving (4) and (5) for C_1 and C_2 we get.

$$C_1 = \frac{-12 f_1 + 6\lambda (f_1 - 2 f_2)}{\lambda^2 + 12\lambda - 12}$$

$$= \frac{-12 f_2 + \lambda(-2 f_1 + 3 f_2)}{\lambda^2 + 12\lambda - 12}$$

and

$$C_2 = \frac{-12 f_2 - \lambda(4 f_1 - 6 f_2)}{\lambda^2 + 12\lambda - 12}$$

Putting the values of C_1 and C_2 in equation (1) we get,

$$g(s) = f(s) + \lambda s \left[\frac{-12 f_1 + 6\lambda (f_1 - 2 f_2)}{\lambda^2 + 12\lambda - 12}\right]$$

$$\left[\frac{-12f_2+6\lambda\left(f_1-2f_2\right)}{\lambda^2+12\lambda-12}\right]$$

$$g(s)=f(s)+\frac{\lambda s}{\lambda^2+12\lambda-12}\left[[6s(\lambda-2)-4\lambda\right]f_1+\left[[6(\lambda-2)-12\lambda s]f_2\right]$$

Putting the values of f_1 and f_2 we get

$$g(s)=f(s)+\lambda\int_0^1\left[\frac{6(\lambda-2)(s+t)-12\lambda st-4\lambda}{\lambda^2+12\lambda-12}\right]f(t)\,dt$$

Problem 4 : Solve $g(s)=\cos s+\lambda\int_0^{\pi}\sin(s-t)\,g(t)\,dt$

Solution : $g(s)=\cos s+\lambda\int_0^{\pi}\sin(s-t)\,g(t)\,dt$

$$=\cos s+\lambda\sin s\int_0^{\pi}\cos t g(t)\,dt-\lambda\cos s\int_0^{\pi}\sin t g(t)\,dt$$

$$g(s)=\cos s+C_1\lambda\sin s-C_2\lambda\cos s \qquad \text{-----(1)}$$

Where,

$$C_1=\int_0^{\pi}\cos t g(t)\,dt \qquad \text{-----(2)}$$

$$C_2=\int_0^{\pi}\sin t g(t)\,dt \qquad \text{-----(3)}$$

Putting (1) in (2) we get

$$C_1=\int_0^{\pi}[\cos^2 t+C_1\lambda\sin t\cos t-C_2\lambda\cos^2 t]\,dt$$

$$=\int_0^\pi \left[\left(\frac{1+\cos 2t}{2}\right)+\frac{\lambda C_1}{2}\sin 2t-\frac{\lambda C_2}{2}\left(\frac{1+\cos 2t}{2}\right)\right]dt$$

$$=\frac{1}{2}\left[t+\frac{\sin 2t}{2}\right]_0^\pi+\frac{\lambda C_1}{2}\left[\frac{-\cos 2t}{2}\right]_0^\pi-\frac{\lambda C_2}{2}\left[t+\frac{\sin 2t}{2}\right]_0^\pi$$

$$=\frac{\pi}{2}-\frac{\lambda C_1}{2}[1-1]-\frac{\lambda C_2}{2}(\pi)$$

$$=\frac{\pi}{2}-\frac{\lambda\pi}{2}C_2$$

$$\therefore C_1+\frac{\lambda\pi}{2}C_2=\frac{\pi}{2}$$

$$\therefore 2C_1+\lambda\pi C_2=\pi \qquad \text{-----(4)}$$

Uusing (1) in (2) we get

$$C_2=\int_0^\pi\left[\sin t\cos t+C_1\lambda\sin^2 t-C_2\lambda\sin t\cos t\right]dt$$

$$=\int_0^\pi\left[\frac{\sin 2t}{2}+\frac{C_1\lambda}{2}(1-\cos 2t)-\frac{\lambda C_1}{2}\sin 2t\right]dt$$

$$=\left[\frac{-\cos 2t}{4}\right]_0^\pi+\frac{\lambda C_1}{2}\left[t-\frac{\sin 2t}{2}\right]_0^\pi+\frac{\lambda C_2}{2}\left[\frac{\cos 2t}{2}\right]_0^\pi$$

$$=\frac{\lambda C_1}{2}(\pi)$$

$$\therefore -\frac{\lambda C_1}{2}(\pi)+C_2=0$$

$$-\lambda\pi C_1+2C_2=0 \qquad \text{-----(5)}$$

$$\therefore D(\lambda) = \begin{vmatrix} 2 & \lambda\pi \\ -\lambda\pi & 2 \end{vmatrix} = 4 + \lambda^2\pi^2$$

$$D_1 = \begin{vmatrix} \pi & \lambda\pi \\ 0 & 2 \end{vmatrix} = 2\pi$$

$$D2 = \begin{vmatrix} 2 & \pi \\ -\lambda\pi & 0 \end{vmatrix} = \lambda\pi^2$$

$$C_1 = \frac{D_1}{D} = \frac{2\pi}{4 + \lambda^2\pi^2}$$

$$C_2 = \frac{D_2}{D} = \frac{4\lambda^2}{4 + \lambda^2\pi^2}$$

$$\therefore \text{If } \; D(\lambda) \neq 0 \;\; i.e. \quad \lambda \neq \frac{-2i}{\pi}$$

then

$$C_1 = \frac{D_1}{D} = \frac{2\pi}{4 + \lambda^2\pi^2}$$

$$C_2 = \frac{D_2}{D} = \frac{\lambda\pi^2}{4 + \lambda^2\pi^2}$$

Putting these values of C_1 and C_2 in equation (1) we get

$$g(s) = \cos s + \lambda \sin s\left[\frac{2\pi}{4 + \lambda^2\pi^2}\right] - \lambda \cos s\left[\frac{\lambda\pi^2}{4 + \lambda^2\pi^2}\right]$$

$$= \frac{4\cos s + 2\pi\lambda \sin s}{4 + \lambda^2\pi^2}; \lambda \neq \pm\frac{2i}{\pi}$$

is the required solution.

Problem 5 : Solve $\phi(x) = e^x + \lambda\int_0^1 2e^{x+t}\phi(t)\,dt$

Solution : $\phi(x) = e^x + \lambda\int_0^1 2e^{x+t}\phi(t)\,dt$

$$= e^x + 2\lambda e^x \int_0^1 e^t\phi(t)\,dt$$

$$= e^x + 2\lambda e^x (c) \qquad \text{-----(1)}$$

where $C = \int_0^1 e^t\phi(t)\,dt$ -----(2)

Putting (1) in (2) we get

$$C = \int_0^1 e^t\left[e^t + 2\lambda e^t C\right]dt$$

$$= \int_0^1 e^{2t}\left[1 + 2\lambda C\right]dt$$

$$= \left(1 + 2\lambda C\right)\left[\frac{e^{2t}}{2}\right]_0^1$$

$$= \left(1 + 2\lambda c\right)\left(\frac{e^2 - 1}{2}\right)$$

$$\therefore C = \frac{e^2 - 1}{2} + C\left[e^2 - 1\right]\lambda$$

$$\Rightarrow C = \frac{e^2 - 1}{2\left[1 - \lambda\left(e^2 - 1\right)\right]}; \lambda \neq \frac{1}{e^2 - 1}$$

Putting the value of c in (1) we get

$$\phi(x) = e^x + \frac{2\lambda e^x (e^2 - 1)}{2\left[1 - \lambda(e^2 - 1)\right]}; \lambda \neq \frac{1}{e^2 - 1}$$

Problem 6 : Consider the equation

$$g(s) = f(s) + \lambda \int_0^1 k(s,t) g(t) \, dt$$

and show that, for the kernel given in Table the function D(λ) has the given expression.

Table :

Case	Kernel	D(λ)
1	± 1	$1 \mp \lambda$
2	st	$1 - \frac{\lambda}{3}$
3	$s^2 + t^2$	$1 - \frac{2\lambda}{3} - \frac{4\lambda^2}{45}$

Solution : $g(s) = f(s) + \lambda \int_0^1 k(S,t) g(t) \, dt$ -----(1)

Case 1 : a) k(s, t) = 1

∴ Equation (1) becomes

$$g(s) = f(s) + \lambda \int_0^1 g(t) \, dt$$

$$\therefore g(s) = f(s) + \lambda C_1 \qquad \text{-----(2)}$$

where $C_1 = \int_0^1 g(t) \, dt$ -----(3)

Using (2) in (3) we get

$$C_1 = \int_0^1 f(t)\,dt + \lambda C_1 \int_0^1 dt$$

$$= \int_0^1 f(t)\,dt + \lambda C_1$$

$$\therefore C_1(1-\lambda) = \int_0^1 f(t)\,dt$$

$$\Rightarrow D(\lambda) = 1-\lambda$$

(b) Now, if k(s, t) = -1

Proceed as above we get

$$D(\lambda) = 1+\lambda$$

Case 2 : k(s, t) = st

$\therefore$ Equation (1) becomes

$$g(s) = f(s) + \lambda \int_0^1 st g(t)\,dt$$

$$= f(s) + \lambda s \int_0^1 t g(t)\,dt$$

$$= f(s) + \lambda s C_2 \qquad \text{-----(4)}$$

where $C_2 = \int_0^1 t g(t)\,dt$ -----(5)

Using (4) in (5) we get

$$C_2 = \int_0^1 t[f(t) + \lambda t C_2]\,dt$$

$$\therefore C_2 = \int_0^1 tf(t)\,dt + \int_0^1 \lambda C t^2 dt$$

$$= \int_0^1 tf(t)\,dt + \frac{\lambda C}{3}$$

$$\therefore \left(1 - \frac{\lambda}{3}\right) C_2 = \int_0^1 tf(t)\,dt$$

$$\Rightarrow D(\lambda) = 1 - \frac{\lambda}{3}$$

Case 3 : $k(s,t) = s^2 + t^2$

$\therefore$ Equation (1) becomes

$$g(s) = f(s) + \lambda \int_0^1 (s^2 + t^2)\, g(t)\,dt$$

$$= f(s) + \lambda s^2 \int_0^1 g(t)\,dt + \lambda \int_0^1 t^2 g(t)\,dt$$

$$= f(s) + \lambda s^2 C_1 + \lambda C_2 \qquad \text{-----(6)}$$

where $C_1 = \int_0^1 g(t)\,dt$ -----(7)

$$C_2 = \int_0^1 t^2 g(t)\,dt \qquad \text{-----(8)}$$

Using (6) in (7) we get

$$C_1 = \int_0^1 [f(t) + \lambda C_1 t^2 + \lambda C_2]\,dt$$

$$= \int_0^1 f(t)\,dt + \frac{\lambda C_1}{3} + \lambda C_2$$

$$C_1\left(1-\frac{\lambda}{3}\right) - \lambda C_2 = \int_0^1 f(t)\,dt \qquad \text{-----(9)}$$

Using (8) in (6) we get

$$C_2 = \int_0^1 t^2[f(t) + \lambda C_1 t^2 + \lambda C_2]dt$$

$$= \int_0^1 t^2 f(t)\,dt + \lambda C_1 \int_0^1 t^4 dt + \lambda C_2 \int_0^1 t^2 dt$$

$$= \int_0^1 t^2 f(t)\,dt + \frac{\lambda C_1}{5} + \frac{\lambda C_2}{3}$$

$$C_2\left(1-\frac{\lambda}{3}\right) - \frac{\lambda}{3} C_1 = \int_0^1 t^2 f(t)\,dt \qquad \text{-----(10)}$$

from (9) and (10)

$$D(\lambda) = \begin{vmatrix} \left(1-\frac{\lambda}{3}\right) & -\lambda \\ \frac{-\lambda}{5} & 1-\frac{\lambda}{3} \end{vmatrix}$$

$$= \left(1-\frac{\lambda}{3}\right)^2 - \frac{\lambda^2}{5}$$

$$= 1 - \frac{2\lambda}{3} - \frac{4\lambda^2}{45}$$

Exercise :

Solve the following integral equations using method of separable kernels.

1. $g(s) = s + \lambda \int_0^1 (1 + s + t) g(t)\, dt$

2. $g(s) = f(s) + \lambda \int_{-1}^{1} (st + s^2 t^2) g(t)\, dt$

3. $g(s) = s + \lambda \int_0^{2\pi} |\pi - t| \sin s g(t)\, dt$

4. $g(s) = \sin s + \lambda \int_0^{\frac{\pi}{2}} \sin s \cos t g(t)\, dt$

5. $g(s) = f(s) + \lambda \int_0^1 st\, g(t)\, dt$

6. $g(s) = \tan^{-1} s + \int_{-1}^{1} e^{\sin^{-1} s} g(t)\, dt$

7. $\phi(x) = \cos x + \lambda \int_0^{\pi} \sin(x - t)\, \phi(t)\, dt$

8. $\phi(x) = s + \lambda \int_0^1 (st^2 + s^2 t)\, \phi(t)\, dt$

9. $\phi(x) = f(s) + \lambda \int_0^1 (s^2 + t^2)\, g(t)\, dt$

10. In the integral equation

$$g(s) = s^2 + \int_0^1 \sin st \; g(t)\; dt$$

Replace sinst by the first two terms of its power series development.

$$\sin st = st - \frac{(st)^3}{3!} + ...$$

and obtain an approximate solution.

3.2 Fredholm Theorem

Theorem (Fredholm Theorem) : The inhomogeneous fredholm integral equation

$$g(s) = f(s) + \lambda \int k(s,t) g(t) dt$$

with the separable kernel has one and only one solution given by

$$g(s) = f(s) + \lambda \int \Gamma(s,t;\lambda) f(t) dt$$

where $\Gamma(s,t;\lambda) = \dfrac{D(s,t;\lambda)}{D(\lambda)}$. for $D(\lambda) \mp 0$ is termed as resolvent (or reciprocal) kernel and it is ratio of two polynomials in λ .

Proof : Consider inhomogeneous fredholm integral equation

$$g(s) = f(s) + \lambda \int k(s,t) g(t) dt \qquad \text{-----(1)}$$

with separable kernel

$$k(s,t) = \sum_{i=1}^{n} a_i(s) b_i(t)$$

Eqution (1) becomes

$$g(s) = f(s) + \lambda \int \left[\sum_{i=1}^{n} a_i(s) b_i(t) \right] g(t) dt$$

$$= f(s) + \lambda \sum_{i=1}^{n} a_i(s) \int b_i(t) g(t) dt$$

$$g(s) = f(s) + \lambda \sum_{i=1}^{n} C_i a_i(s)$$

where $C_i = \int b_i(t) g(t) dt$ -----(3)

i = 1, 2, 3, ... n

Using (2) in (3) we get (for fixed i)

$$C_i = \int b_i(t)[f(t) + \lambda \sum_{k=1}^{n} c_k a_k(t)]dt$$

$$= \int b_i(t) f(t) dt + \lambda \sum_{\kappa=1}^{n} c_k \int b_i(t) a_k(t) dt$$

Denote $f_i = \int b_i(t) f(t) dt$ -----(4)

$$a_{ik} = \int b_i(t) a_k(t) dt$$

$$\therefore C_i - \lambda \sum_{k=1}^{n} c_k a_{ik} = f_i \quad ; i = 1, 2, \ldots n$$

Which is system of linear equations with n knowns C_i; i = 1, 2, ... n

The Determinour $D(\lambda)$ of the system is

$$D(\lambda) = \begin{vmatrix} 1-\lambda a_{11} & -\lambda a_{12} \ldots & -\lambda a_{1n} \\ -\lambda a_{21} & 1-\lambda a_{22} & -\lambda a_{2n} \\ . & & \\ . & & \\ -\lambda a_{n1} & -\lambda a_{n2} & 1-\lambda a_{nn} \end{vmatrix}$$

Which is polynomial of degree at most n.

For D(λ) $\neq$ 0, the system has only one solution (unique solution), given by cramers rule.

$$C_i = \frac{1}{D(\lambda)}\left[D_{1i}\ f_1 + D_{2i}\ f_2 + \ldots + D_{ni}\ f_n\right]$$

i = 1, 2, ... n

Where D_{hi} demotes the cofactor of the (h, i) th element of the determinant D(λ).

The integral equation (1) has unique solution given by using (2) as

$$g(s) = f(s) + \lambda \sum_{i=1}^{n} \frac{D_{1i} f_1 + D_{2i} f_2 + \ldots + D_{ni} f_n}{D(\lambda)} a_i(s)$$

Substituting f_i from equation (4) we get

$$g(s) = f(s) + \frac{\lambda}{D(\lambda)} \int \left\{ \sum_{i=1}^{n} \left[D_{1i} b_1(t) + D_{2i} b_2(t) + \ldots\ldots + D_{ni} b_n(t)\right] a_i(s) \right\} f(t)\,dt$$

Consider the determinat of $(n+1)^{th}$ order

$$D(s,t;\lambda) = -\begin{vmatrix} 0 & a_1(s) & a_2(s) & \ldots & a_n(s) \\ b_1(t) & 1-\lambda a_{11} & -\lambda a_{12} & \ldots & -\lambda a_{1n} \\ b_2(t) & -\lambda a_{21} & 1-\lambda a_{22} & \ldots & -\lambda a_{2n} \\ \vdots & & & & \\ b_n(t) & -\lambda a_{n1} & -\lambda a_{n2} & \ldots & 1-\lambda a_{nn} \end{vmatrix}$$

By developing it by the elements of the first row and corresponding minor by the elemtns of first column, we find that $D(s,t;\lambda) = \sum_{i=1}^{n} \left[D_{1_i}\ b_1(t) + \ldots\ldots\ldots + D_{n_i} b_n(t)\right]$

Therefore,

$$g(s) = f(s) + \frac{\lambda}{D(\lambda)} \int D(s,t;\lambda) f(t)\,dt$$

$$= f(s) + \lambda \int \frac{D(s,t;\lambda)}{D(\lambda)} f(t)\,dt$$

$$= f(s) + \lambda \int \Gamma(s,t;\lambda) f(t)\, dt$$

$$where \quad \Gamma(s,t;\lambda) = \frac{D(s,t;\lambda)}{D(\lambda)}$$

Which is ratio of two polynomial in λ is called resolvent (or reciprocal) kernel

3.3 Transpose or adjoint of integral equation :

The integral equation

$$\psi(s) = f(s) + \lambda \int k(t,s) \psi(t)\, dt$$

is called the transpose or adjoint of the integral equation.

$g(s) = f(s) + \lambda \int k(s,t) g(t)\, dt$, and vice versa

Theorem : Then transposed equation

$\psi(s) = f(s) + \lambda \int k(t,s) \psi(t)\, dt$ does -----(1)

is also possessess a unique solution

whenever,

$g(s) = f(s)t + \lambda \int k(s,t) g(t) dt$ -----(2)

Proof : Let $k(s,t) = \sum_{i=1}^{n} a_i(s) b_i(t)$

where the functions $a_1(s) \ldots a_n(s)$ and $b_1(t) \ldots b_n(t)$ are linearly independent, then

$$k(t,s) = \sum_{i=1}^{n} a_i(t) b_i(s)$$

equation (1) becomes

$$\psi(s) = f(s) + \lambda \int \left[\sum_{i=1}^{n} a_i(t) b_i(s) \right] \Psi(t)\, dt$$

$$= f(s) + \lambda \sum_{i=1}^{n} b_i(s) \int a_i(t) \psi(t) dt$$

Let $C_i = \int a_i(t) \psi(t) dt, \quad i = 1, 2, \ldots, n$ -----(3)

$$\psi(s) = f(s) + \lambda \sum_{i=1}^{n} c_i b_i(s)$$ -----(4)

and the problem reduces to finding the unknown C_i

Putting (4) in (3) we get

$$C_i = \int a_i(t)[f(t) + \lambda \sum_{k=1}^{n} c_k b_k(t)]dt; \ i = 1, 2, 3, \ldots, n$$

$$= \int a_i(t) f(t) dt + \lambda \sum_{k=1}^{n} c_k \int a_i(t) b_k(t) dt; \qquad i = 1, 2, \ldots, n$$

Put $\int a_i(t) f(t) dt = f_i$

$\int a_i(t) b_k(t) dt = a_{ki}$

$$\therefore c_i = f_i + \lambda \sum_{k=1}^{n} c_k a_{ki} \quad ; i = 1, 2, \ldots. n$$

i.e. $c_i - \lambda \sum_{k=1}^{n} c_k a_{ki} = f_i \quad ; i = 1, 2, \ldots. n$

it is the system of equation given by

$$(1 - \lambda a_{11}) c_1 - \lambda a_{21} c_2 - \lambda a_{31} c_3 - \ldots - \lambda a_{n1} c_n = f_1$$

$$-\lambda a_{12} c_1 + (1 - \lambda a_{22}) c_2 - \lambda a_{32} c_3 - \ldots - \lambda a_{n2} c_n = f_2$$

....

$$-\lambda a_{1n} c_1 - \lambda a_{2n} c_2 - \lambda a_{3n} c_3 - \ldots - (1 - \lambda a_{nn}) c_n = f_n$$

whose determinant is

$$D(\lambda) = -\begin{vmatrix} 1-\lambda a_{11} & -\lambda a_{21} & \ldots -\lambda a_{n1} \\ -\lambda a_{12} & 1-\lambda a_{22} & \ldots -\lambda a_{n2} \\ \vdots & & \\ -\lambda a_{n1} & -\lambda a_{n2} & \ldots 1-\lambda a_{nn} \end{vmatrix} = |I-\lambda A^{T}| = D^{T}(\lambda)$$

Which is just a transpose of fredholm determinant $D\ (\lambda) = |I-\lambda A|$ of equation (2). This means that the transposed IE possesses unique solution whenever the original IE.

Thus eigen values of transposed equation are same as those of the original equation.

A necessary and sufficient condition for the in homogeneous integral equation $g(\mathrm{s}) = f(\mathrm{s}) + \lambda\int_a^b k(s,t)g(t)\ dt$ to have solution for $\lambda = \lambda_o$ a root of $\mathrm{D}(\lambda) = \mathrm{O}$ is

that f (s) is orthogonal to the eigen functions corresponds to $\lambda = \lambda_o$ of the transposed

equation $\psi(\mathrm{S}) = \lambda\int_a^b k(t,s)\ \psi(t)\ dt$

Proof : Let g(s) be the solution of (1) corresponding to λ where $D(\lambda) = 0$

Let $\psi_i(s); (i = 1, 2, \ldots r)$ be all the eigen functions of transported equation (2) then,

$$\int_a^b f(s)\psi_i(s)ds$$

$$= \int_a^b \psi_i(s)\left[g(s) - \lambda\int_a^b K(s,t)g(t)dt\right]ds \qquad (\because (1))$$

$$= \int_a^b g(s)\psi_i(s)ds - \lambda\int_a^b \psi_i(s)\left[\int_a^b k(s,t)g(t)dt\right]ds$$

$$= \int_a^b g(s)\psi_i(s)\,ds - \lambda\int_a^b g(t)\left[\int_a^b \psi_i(s)k(s,t)\,ds\right]dt$$

$$= \int_a^b g(s)\psi_i(s)\,ds - \lambda\int_a^b g(s)\left[\int_a^b \psi_i(t)k(t,s)\,dt\right]ds$$

$$= \int_a^b g(s)\psi_i(s)\,ds - \int_a^b g(s)\left[\lambda\int_a^b \psi_i(t)k(t,s)\,dt\right]ds$$

$$= \int_a^b g(s)\psi_i(s)\,ds - \int_a^b g(s)\psi_i(s)\,ds \quad [\because (2)]$$

$$= 0$$

$$\therefore \int_a^b f(s)\psi_i(s)\,ds = 0 \qquad (i = 1,2,\ldots r)$$

This shows that, if g(s) is the solution of (1) then f(s) is orthogonal to all eigen functions of transposed equation (2)

Conversely, let f (s) be orthogonal to all eigen functions of (2) then the algebraic system

$$C_i - \lambda\sum_{i=1}^{n} a_{ik}c_k = f_i \quad ;1 \le i \le n \qquad \text{-----(3)}$$

$i.e.[I-\lambda A]c = f$, where $A = [aij]$, $C = [C_1, \ldots\ldots\ldots, C_n]^T$, $f = [f_1, f_2, \ldots\ldots., f_n]^T$

I is unit matrix of order n obtained from equation (1), reduces to only n-r independent equations. This means that the rank of matric $[I-\lambda A]$ is exactly P = n-r and therefore the system (3) is soluble, hence the integral equation (1) is soluble.

Problem 1 : Solve the integral equation

$$g(s) = f(s) + \lambda\int_0^1 (1-3st)\,g(t)\,dt$$

by discussing all the possible cases.

Solution : $g(s) = f(s) + \lambda \int_0^1 (1-3st)\, g(t)\, dt$ -----(1)

$$= f(s) + \lambda \int_0^1 g(t)\, dt - \lambda 3s \int_0^1 t g(t)\, dt$$

$\therefore g(s) = f(s) + \lambda c_1 - 3s\lambda\, c_2$ -----(2)

where $c_1 = \int_0^1 g(t)\, dt$ -----(3)

and $c_2 = \int_0^1 t g(t)\, dt$ -----(4)

Using equation (2) in (3) we get

$$c_1 = \int_0^1 \left[f(t) + \lambda c_1 - 3t\lambda c_2 \right] dt$$

$$= \int_0^1 f(t)\, dt + \lambda c_1 - \frac{3}{2}\lambda c_2$$

$c_1(1-\lambda) + \frac{3}{2}\lambda c_2 = \int_0^1 f(t)\, dt$ -----(5)

Using equation (2) in (4) we get

$$c_2 = \int_0^1 \left[t\, f(t) + t\lambda c_1 - 3\lambda c_2 t^2 \right] dt$$

$$= \int_0^1 t\, f(t) + \frac{\lambda}{2} c_1 - c_2 \lambda$$

$$\therefore -\frac{\lambda}{2}c_1 + (1+\lambda)c_2 = \int_0^1 t\, f(t)\,dt \qquad \text{-----(6)}$$

Thus

$$\left.\begin{aligned} (1-\lambda)c_1 + \frac{3\lambda}{2}c_2 &= \int_0^1 f(t)\,dt \\ -\frac{\lambda}{2}c_1 + (1+\lambda)c_2 &= \int_0^1 t\, f(t)\,dt \end{aligned}\right\} \qquad \text{-----(7)}$$

$$D(\lambda) = \begin{vmatrix} (1-\lambda) & \dfrac{3\lambda}{2} \\ \dfrac{-\lambda}{2} & (1+\lambda) \end{vmatrix} = (1-\lambda^2) + \frac{3}{4}\lambda^2$$

$$D(\lambda) = 1 - \frac{\lambda^2}{4}$$

Case I : If f(s) = 0

Then system (7) becomes homogeneous system of linear equatins.

i) If $D(\lambda) \neq 0 \quad \Rightarrow 1 - \frac{\lambda^2}{4} \neq 0 \;\Rightarrow \lambda \neq \pm 2$

system (7) possesses unique solution given by $C_1 = C_2 = 0$

Integral equation (1) has solution g(s) = 0

ii) If $D(\lambda) = 0 \quad \Rightarrow \lambda = \pm 2$

a) For $\lambda = 2$ system (7) takes the term

$$\left.\begin{aligned} -c_1 + 3c_2 &= 0 \\ -c_1 + 3c_2 &= 0 \end{aligned}\right\} \Rightarrow c_1 = 3c_2$$

Equation (2) reduces to

$g(s) = 6C_2(1-s)$

$\Rightarrow g(s) = A(1-s)$

Where A is arbitrary constant is the solution of (1)

b) For $\lambda = -2$ system (7) reduces to

$$\left.\begin{aligned} 3c_1 - 3c_2 &= 0 \\ \frac{3}{2}c_1 - c_2 &= 0 \end{aligned}\right\} \Rightarrow c_1 = c_2$$

Equation (2) becomes

$g(s) = -2c_1(1+3s)$

$= B(1+3s)$

Where B is arbitrary constants is the solution.

Case II : When $f(s) \neq 0$

i) If $D(\lambda) \neq 0$, the system (7) has unique solution in which C_1 and C_2 can be determined by solving (2) simultaneously.

In this case (1) has unique solution.

ii) If $D(\lambda) = 0 \quad \Rightarrow \lambda = \pm 2$

i.e. $\lambda = 2, -2$ are the roots of $D(\lambda) = 0$

Examine the eigen functions of the transposed homogeneous equation

$$\psi(s) = \lambda \int_0^1 (1-3st)\psi(t)\,dt$$

By case I (ii) the eigen functions corrosponding to $\lambda = 2$ and $\lambda = -2$ are $\psi_1(s) = (1-s)$ *and* $\psi_2(s) = 1-3s$ respectively.

It follows that from the theorem that the integral equation

$$g(s) = f(s) + 2\int_0^1 (1-3st)\, g(t)\, dt$$

will have a solution if f(s) is orthogonal to 1 - s i.e. if f(s) satisfy the condition.

$$\int_0^1 (1-3s) f(s)\, ds = 0$$

While the integral equation.

$$g(s) = f(s) - 2\int_0^1 (1-3st)\, g(t)\, dt$$

Will have a solution if f(s) is orthogonal to (1-3s)

i.e. if $\int_0^1 (1-3s) f(s)\, ds = 0$

Problem : 2 : Show that the integral equation

$$g(s) = f(s) + \frac{1}{\pi}\int_0^{2\pi} [\sin(s+t)]\, g(t)\, dt$$

Possesses no solution for f(s) = s, but that possesses infinitely many solutions when f(s) = 1.

Solution : Consider $g(s) = f(s) + \lambda\int_0^{2\pi} [\sin(s+t)]\, g(t)\, dt$ -----(1)

$$= f(s) + \lambda \sin s \int_0^{2\pi} [\cos t]\, g(t)\, dt + \lambda \cos s \int_0^{2\pi} [\sin t]\, g(t)\, dt$$

$$g(s) = f(s) + \sin s \lambda c_1 + \cos \lambda c_2 \qquad \text{-----(2)}$$

where

$$c_1 = \int_0^{2\pi} g(t) \cos t\, dt \qquad \text{-----(3)}$$

$$c_2 = \int_0^{2\pi} g(t)\sin t dt \qquad \text{-----(4)}$$

Using (2) in (3) we get

$$c_1 = \int_0^{2\pi} f(t)\cos t\, dt + \lambda c_1 \int_0^{2\pi} \sin t \cos t\, dt + \lambda c_2 \int_0^{2\pi} \cos^2 t\, dt$$

$$= \int_0^{2\pi} f(t)\cos t dt + \frac{\lambda c_1}{2}\int_0^{2\pi} \sin 2t dt + \frac{\lambda c_2}{2}\int_0^{2\pi} (1+\cos 2t)\, dt$$

$$= \int_0^{2\pi} f(t)\cos t dt + \frac{\lambda c_1}{2}\left[\frac{-\cos 2t}{2}\right]_0^{2\pi} + \frac{\lambda c_2}{2}\left[t + \frac{\sin 2t}{2}\right]_0^{2\pi}$$

$$= \int_0^{2\pi} f(t)\cos t dt + \pi\lambda c_2$$

$$\therefore c_1 - \pi\lambda c_2 = \int_0^{2\pi} f(t)\cos t dt \qquad \text{-----(5)}$$

Using (2) in (4) we get

$$c_2 = \int_0^{2\pi} f(t)\sin t dt + \lambda c_1 \int_0^{2\pi} \sin^2 t dt + \lambda c_2 \int_0^{2\pi} \sin(t)\cos(t)\, dt$$

$$= \int_0^{2\pi} f(t)\sin t dt + \frac{\lambda c_1}{2}\int_0^{2\pi} (1-\cos 2t)\, dt + \frac{\lambda c_2}{2}\int_0^{2\pi} \sin 2\, t$$

$$= \int_0^{2\pi} f(t)\sin t dt + \frac{\lambda c_1}{2}\left[t - \sin 2t\right]_0^{2\pi} + \frac{\lambda c_2}{2}\left[\frac{-\cos 2t}{t}\right]_0^{2\pi}$$

$$= \int_0^{2\pi} f(t)\sin t dt + \pi\lambda c_1$$

$$\therefore -\pi\lambda c_1 + c_2 = \int_0^{2\pi} f(t)\sin t dt \qquad \text{-----(6)}$$

$\therefore$ we have system of equation

$$\left.\begin{aligned} c_1 - \pi\lambda c_2 &= \int_0^{2\pi} f(t)\cos t dt \\ -\pi\lambda c_1 + c_2 &= \int_0^{2\pi} f(t)\sin t dt \end{aligned}\right\} \qquad \text{-----(7)}$$

$$\therefore D(\lambda) = \begin{vmatrix} 1 & -\pi\lambda \\ -\pi\lambda & 1 \end{vmatrix} = 1 - \pi^2\lambda^2$$

Let, $D(\lambda) = 0 \Rightarrow 1 - \pi^2\lambda^2 = 0 \Rightarrow \lambda = \pm\dfrac{1}{\pi}$

Now we determine the eigen functions of transposed homogeneous equation (note that kernel is symetric)

$$\psi(s) = \lambda\int_0^{2\pi} \sin(s+t)\psi(t)dt \qquad \text{-----(8)}$$

The algebraic system corrosponding to (8)

$$\left.\begin{aligned} c_1 - \pi\lambda c_2 &= 0 \\ -\pi\lambda c_1 + c_2 &= 0 \end{aligned}\right\} \qquad \text{-----(9)}$$

i) For $\lambda_1 = \dfrac{1}{\pi}$ system (9) becomes

$$\left.\begin{aligned} c_1 - c_2 &= 0 \\ -c_1 + c_2 &= 0 \end{aligned}\right\} \Rightarrow c_1 = c_2$$

$\therefore$ from equation (2)

$$\psi(s) = \frac{c_1}{\pi}\sin s + \frac{c_1}{\pi}\cos s$$

$$= \frac{c_1}{\pi}(\sin s + \cos s)$$

$\therefore$ The eigen function corrosponding to

$$\lambda_1 = \frac{1}{\pi} \quad is \quad \psi_1(s) = \sin s + \cos s$$

ii) For $\lambda_2 = -\frac{1}{\pi}$ system (9) becomes

$$\left.\begin{aligned} c_1 + c_2 &= 0 \\ c_1 + c_2 &= 0 \end{aligned}\right\} \Rightarrow c_1 = -c_2$$

Using in equation (2) we get

$$\psi(s) = \frac{-c_1}{\pi}\sin s + \frac{c_1}{\pi}\cos s$$

$$= \frac{-c_1}{\pi}(\sin s - \cos s)$$

$\therefore$ The eigen function corrosponding to

$$\lambda_2 = -\frac{1}{\pi} \quad is \quad \psi_2(s) = \sin s - \cos s$$

Case 1 : f(s) = s

$$\int_0^{2\pi} f(s)\psi_1(s)\,ds = \int_0^{2\pi} s(\sin s + \cos s)\,ds$$

$$= \left[s(-\cos s) - (-\sin s)\right]_0^{2\pi} + \left[s(\sin s) - (-\cos s)\right]_0^{2\pi}$$

$$= -2\pi$$

$$\therefore \int_0^{2\pi} f(s)\psi_1(s)\,ds = -2\pi \neq 0$$

$\Rightarrow$ All eigen functions of transposed equations are not orthogonal to f(s) = s

$\Rightarrow$ The integral equation (1) has no solution.

Case 2 : f(s) = 1

$$\int_0^{2\pi} f(s)g_1(s)\,ds = \int_0^{2\pi} (\sin s + \cos s)\,ds$$

$$= [-\cos s + \sin s]_0^{2\pi}$$

$$= 0$$

and

$$\int_0^{2\pi} f(s)\psi_2(s)\,ds = \int_0^{2\pi} (\sin s - \cos s)\,ds$$

$$= [-\cos s - \sin s]_0^{2\pi}$$

$$= 0$$

$\Rightarrow$ f(s) = 1 is orthogonal to all the eigen functions of transposed equation.

$\Rightarrow$ The integral equation (1) is possesses solution.

Problem 3 : Solve the integral equation

$$g(s) = 1 + \lambda \int_{-\pi}^{\pi} e^{iw(s-t)} g(t)\,dt$$

considering separately all the exceptional cases.

Solution : $g(s) = 1 + \lambda \int_{-\pi}^{\pi} e^{iw(s-t)} g(t)\,dt$ -----(1)

$$= 1 + \lambda e^{iws} \int_{-\pi}^{\pi} \overline{e}^{iwt} g(t)\,dt$$

$$g(s) = 1 + \lambda c e^{iws}$$

where,

$$c = \int_{-\pi}^{\pi} e^{-iwt} g(t)\, dt$$

$$\therefore c = \int_{-\pi}^{\pi} e^{-iwt} \left[1 + \lambda c e^{iwt}\right] dt$$

$$= \int_{-\pi}^{\pi} \left[e^{-iwt} + \lambda c\right] dt$$

$$= \left[\frac{e^{-iw\pi}}{-iw}\right]_{-\pi}^{\pi} + \lambda c [t]_{-\pi}^{\pi}$$

$$\frac{e^{-iw\pi} - e^{iw\pi}}{-iw} + 2\lambda\pi c$$

$$\frac{2}{\omega}\left[\frac{e^{iw\pi} - e^{-iw\pi}}{2i}\right] + 2\lambda\pi c$$

$$c = \frac{2}{w}\sin \pi w + 2\lambda\pi c$$

$$c(1 - 2\lambda\pi) = \frac{2}{w}\sin \pi w$$

$$\therefore c = \frac{2}{w(1-2\lambda\pi)}\sin \pi w \;\; ; \; if\, \lambda \neq \frac{1}{2\pi} \,\&\, w \neq 0$$

Thus $g(s) = 1 + \frac{\lambda \sin \pi w}{w(1-2\lambda\pi)} e^{iws} \;\; ; \; \lambda \neq \frac{1}{2\pi}, w \neq 0$ is the solution

If $\lambda = \frac{1}{2\pi}$

Then we want to determine eigen function of the transposed equation.

$$\psi(s) = \lambda \int_{-\pi}^{\pi} e^{iw(t-s)} \psi(t)\, dt$$

this equation can be written as

$$\psi(s) = \lambda e^{-iws} c$$

where

$$c = \int_{\pi}^{\pi} e^{iwt} \psi(t)\, dt$$

Putting $\psi(s)$ in C we get

$$c = \int_{\Pi}^{\pi} \lambda c e^{iwt} e^{-iwt} dt$$

$$= \lambda c [t]_{-\pi}^{\pi}$$

$$= 2\pi\lambda c$$

$$\therefore c(1 - 2\pi\lambda) = 0$$

If c = o then $\psi(s) = 0$ which is not eigen function.

Let $c \neq 0 \quad \Rightarrow 1 - 2\pi\lambda = 0 \quad \Rightarrow \lambda = \dfrac{1}{2\pi}$

Which is an eigen values and the corrosponding eigen function is

$$\psi(s) = e^{-iws} \qquad \left[\because \text{Taking } \frac{c}{2\pi} = 1\right]$$

Now the equation (1) has solution for $\lambda = \dfrac{1}{2\pi}$

$\therefore \psi(s)$ is orthogonal to f(s) = 1 on $[-\pi, \pi]$

Now $\int_{-\pi}^{\pi} \psi(s) f(s) ds = \int_{-\pi}^{\pi} e^{-iws} ds$

$$= \left[\frac{e^{-iws}}{-iw} \right]_{-\pi}^{\pi}$$

$$= \frac{2}{w} \sin \pi w \quad ; w \neq 0$$

Equation (1) has solution if

$$\frac{2}{w} \sin \pi w = 0$$

$$\Rightarrow \sin \pi w = 0 \quad \Rightarrow w = n$$

Thus for $\lambda = \frac{1}{2\pi}$ equation (1) has solution if w = n otherwise not.

Exercise :

1) Solve the integral equation :

$$g(s) = f(s) + \lambda \int_{0}^{2\pi} \cos(s+t) g(t) dt$$

and find the condition that f(s) must satisfy in order that this equation has a solution when λ is an eigen value.

3.4 Eigen values and Eigen functions :

The values of λ for which the homogeneous fredholm integral equation.

$$g(s) = \lambda \int_{a}^{b} k(s,t) g(t) dt \qquad \text{-----(1)}$$

has a non zero solution (i.e. g(s) # 0) are called eigen value of (1) or of the kernal k(s, t) and every non zero solution of (1) corrosponding to the eigen value λ is called eigen function.

1) The number $\lambda = 0$ is not eigen value. Since for $\lambda = 0$ if follows from (1) that g(s)=0

2) If g(s) is an eigen function of (1) then g(s) where c is an arbitrary constant different from zero is also on eigen function of (1) which corrospond to same eigen value of λ

Problem 1 : Describe the procedure of finding the eigen values and eigen function for the homogeneous Fredholm integral equation $g(s) = \lambda \int k(s,t)\, g(t)\, dt$ of the second kind with separable kernel.

Solution : Consider a homogeneous fredholm integral equation of second kind.

$$g(s) = \lambda \int k(s,t) g(t) dt \qquad \text{-----(1)}$$

With separable kernel

$$\therefore k(x,t) = \sum_{i=1}^{n} a_i(s) b_i(t)$$

Where $a_1(s), a_2(s) \ldots a_n(s)$ and $b_1(t), \ldots b_n(t)$ are linearly independent function.

Equation (1) becomes

$$g(s) = \lambda \int \sum_{i=1}^{n} a_i(s) b_i(t) g(t) dt$$

$$= \lambda \sum_{i=1}^{n} a_i(s) \int b_i(t) g(t) dt \qquad \text{-----(2)}$$

For each *i* (*i* =1, 2, ..., n)

let $\int b_i(t) g(t) dt = c_i$

Then

$$g(s) = \lambda \sum_{i=1}^{n} c_i a_i(s) \qquad \text{-----(3)}$$

Is the solution of equation (1) in which we have to determine the constants C_i ; i = 1, ..., n

Put the value of g (s) from (3) in (2)

we obtain,

$$\lambda\sum_{k=1}^{n} C_i, a_i(s) = \lambda\sum_{i=1}^{n} a_i(s) \int_a^b b_i(t)\ \text{g}(t)\ dt$$

$$\sum_{k=1}^{n} C_i, a_i(s) = \sum_{i=1}^{n} a_i(s) \int_a^b b_i(t)\left(\lambda\sum_{i=1}^{n} C_k a_k(t)\right)dt$$

$$\sum_{k=1}^{n} a_i(s)\{C_i - \int_a^b b_i(t)\left(\lambda\sum_{i=1}^{n} C_k a_k(t)\right)dt = 0$$

Since, $a_1(s), a_2(s), ...a_n(s)$ are linearly independant, we must have

$$C_i - \int_a^b b_i(t)\left(\lambda\sum_{i=1}^{n} C_k a_k(t)\right)dt = 0, \quad \text{i} = 1, 2, ..., \text{n}$$

$$C_i - \lambda\sum_{i=1}^{n} C_k \int_a^b b_i(t)\ a_k(t)\ dt = 0, \quad \text{i} = 1, 2, ..., \text{n}$$

Denote $\int b_i(t) a_k(t) dt = a_{ik}$

we get,

$$C_i - \lambda\sum_{i=1}^{n} C_k a_{ik} = 0, \quad (\text{i} = 1, 2, ..., \text{n})$$

$$C_i - \lambda\left(C_i a_{i1} + C_2 a_{i2} + ... + C_n a_{in}\right) = 0 \quad (\text{i} = 1, 2, ..., \text{n})$$

It is homogeneous system of linear equations for 'n' unknown which is given by

$$\left.\begin{aligned} c_1(1-\lambda a_{11}) - c_2\lambda a_{12} ... - c_n\lambda a_{1n} &= 0 \\ -c_1\lambda a_{21} + c_2(1-\lambda a_{22}) ... - c_n\lambda a_{2n} &= 0 \\ -c_1\lambda a_{n1} - c_2\lambda a_{22} ... - c_n\lambda a_{nn} &= 0 \end{aligned}\right\} \quad \text{-----(4)}$$

The determinant D(λ) of the system is

$$D(\lambda)=\begin{vmatrix}(1-\lambda a_{11}) & -\lambda a_{12} & -\lambda a_{13}\ldots-\lambda a_{1n}\\ -\lambda a_{21} & (1-\lambda a_{22}) & -\lambda a_{23}\ldots-\lambda a_{2n}\\ \vdots & & \\ -\lambda a_{n1} & \lambda a_{n2} & -\lambda a_{n3}\ldots(1-\lambda a_{nn})\end{vmatrix}$$

Case 1 : If $D(\lambda)\neq 0$: The system (4) has trivial solution given by $C_1=C_2=\ldots=C_n=0$ which is unique.

From (3) g(s) = o

For any λ (1) has only zero solution i.e. g(s) = 0

In this case, equation (1) does not possess any eigen value and eigen function.

Case 2 : If D(λ) = 0 : The system (5) has infinite solution, hence integral equation has infinite solution.

[$\because$ In this case atleast one of the Ci's can be assigned arbitrarly and the remaining C_i'S can be determined accordingly.]

In fact, the equation D(λ)=0 is the polynomial in λ having degree $m\leq n$

$\therefore$ Integral equation (1) has m eigen values say $\lambda_1, \lambda_2, \ldots \lambda_m$

There corresponds following non zero solution for each λ given as below.

$$\begin{array}{llll} c_1^{(1)}c_2^{(1)}\ldots c_n^{(1)} & for\ \lambda_1 \\ c_1^{(1)}c_2^{(2)}\ldots c_n^{(2)} & for\ \lambda_2 \\ \ldots\ \ldots & \ldots\ \ldots \\ c_1^{(m)}c_2^{(m)}\ldots c_n^{(m)} & for\ \lambda_m \end{array}$$

$\therefore$ Non zero solution corresponding to each λ_k; k =1, 2, 3 ... m is given by

$$g_k(s)=\lambda_k\sum_{i=1}^{n}c_i^{(k)}a_i(s);\quad k=1,2\ldots m$$

Which are the required eigen functions corresponding to λ_k

Problem 1 : Find the eigen values and eigen function of the homogenous integral equation.

$$g(s) = \lambda \int_0^{2\pi} \sin(s+t) g(t) dt$$

Solution :

$$g(s) = \lambda \int_0^{2\pi} (\sin s \cos t + \cos s \sin t) g(t) dt$$

$$= \lambda \sin s \int_0^{2\pi} \cos(t) g(t) dt + \lambda \cos s \int_0^{2\pi} \sin t g(t) dt$$

$$g(s) = \lambda \sin s c_1 + \lambda \cos c_2 \qquad \text{-----(1)}$$

where,

$$c_1 = \int_0^{2\pi} \cos t g(t) dt \qquad \text{-----(2)}$$

$$c_2 = \int_0^{2\pi} \sin t g(t) dt \qquad \text{-----(3)}$$

Putting (1) in equation (2)

$$c_1 = \int_0^{2\pi} \cos t \left[\lambda \sin t c_1 + \lambda \cos t c_2\right] dt$$

$$= \lambda c_1 \int_0^{2\pi} \sin t \cos t \, dt + \lambda c_2 \int_0^{2\pi} \cos^2 t \, dt$$

$$= \frac{\lambda c_1}{2} \int_0^{2\pi} \sin 2t dt + \frac{\lambda c_2}{2} \int_0^{2\pi} (1 + \cos 2t) dt$$

$$= \frac{-\lambda c_1}{2}\left[\frac{\cos 2t}{2}\right]_0^{2\pi} + \frac{\lambda c_2}{2}\left[t + \frac{\sin 2t}{2}\right]_0^{2\pi}$$

$$= \frac{-\lambda c_1}{4}[1-1] + \frac{\lambda c_2}{2}\left[(2\pi - 0) + \frac{0-0}{2}\right]$$

$$\therefore c_1 - \lambda \pi c_2 = 0 \qquad \text{-----(4)}$$

Putting (1) in eqution (3) we get

$$c_2 = \int_0^{2\pi} \sin t\left[\lambda \sin t c_1 + \lambda \cos t c_2\right] dt$$

$$= \frac{\lambda c_1}{2}\int_0^{2\pi} (1 - \cos 2t)\, dt + \frac{\lambda c_2}{2}\int_0^{2\pi} \sin 2t dt$$

$$= \frac{\lambda c_1}{2}\left[t - \frac{\sin 2t}{2}\right]_0^{2\pi} - \frac{\lambda c_2}{2}\left[\frac{\cos 2t}{2}\right]_0^{2\pi}$$

$$= \frac{\lambda c_1}{2}\left[(2\pi - 0)\right] - \frac{\lambda c_2}{2}\left[\frac{1-1}{2}\right]$$

$$c_2 = \lambda c_1 \pi$$

$$\lambda c_1 \pi - c_2 = 0 \qquad \text{-----(5)}$$

for non zero solution, we must have

$$D(\lambda) = 0$$

$$\begin{vmatrix} 1 & -\lambda\pi \\ \lambda\pi & -1 \end{vmatrix} = 0 \;\Rightarrow -1 + \lambda^2\pi^2 = 0$$

$$\Rightarrow \lambda^2 = \frac{1}{\pi^2}$$

$$\Rightarrow \lambda = \pm\frac{1}{\pi}$$

Thus $\lambda = \frac{1}{\pi}, -\frac{1}{\pi}$ are the required eigen values.

To determine eigen function :

1) for $\lambda = \frac{1}{\pi}$ equation (4) and (5) becomes

$$\left.\begin{aligned} c_1 - c_2 &= 0 \\ c_1 - c_2 &= 0 \end{aligned}\right\} \Rightarrow c_1 = c_2$$

Thus equation (1) becomes

$$\therefore g(s) = \frac{c_1}{\pi}\sin s + \frac{c_1}{\pi}\cos s$$

$$= \frac{c_1}{\pi}(\sin s - \cos s)$$

Required eigen function is $g(s) = \sin s + \cos s$ corresponding to $\lambda = -\frac{1}{\pi}$

for $\lambda = -\frac{1}{\pi}$ equation (4) and (5) becomes

$$\left.\begin{aligned} c_1 + c_2 &= 0 \\ c_1 + c_2 &= 0 \end{aligned}\right\} \Rightarrow -c_1 = c_2$$

Thus equation (1) becomes

$$\therefore g(s) = \frac{-c_1}{\pi}\sin s + \frac{c_2}{\pi}\cos s$$

$$= \frac{-c_1}{\pi}(\sin s + \cos s)$$

$\therefore$ Required eigen function corresponding to $\lambda = -\frac{1}{\pi}$ is $g(s) = \sin s - \cos s$

Problem : 2 Find the eigen values and eigen functions for the homogeneous integral equation. (or solve homogeneous integral equation)

$$\phi(x)=\lambda\int_{-1}^{1}\left(5xt^3+4x^2t+3tx\right)\phi(t)\,dt$$

$$\phi(x)=\lambda\int_{-1}^{1}\left[5xt^3+\left(4x^2+3x\right)t\right]\phi(t)\,dt$$

$$=\lambda 5x\int_{-1}^{1}t^3\phi(t)\,dt+\left(4x^2+3x\right)\lambda\int_{-1}^{1}t\phi(t)\,dt$$

$$\phi(x)=5x\lambda c_1+\left(4x^2+3x\right)\lambda c_2 \qquad \text{-----(1)}$$

where $c_1=\int_{-1}^{1}t^3\phi(t)\,dt$ -----(2)

and $c_2=\int_{-1}^{1}t\phi(t)\,dt$ -----(3)

Putting (1) in equation (2) we get

$$c_1=\int_{-1}^{1}t^3\left[5t\lambda c_1+\left(4t^2+3t\right)\lambda c_2\right]dt$$

$$=5\lambda c_1\int_{-1}^{1}t^4\,dt+\lambda c_2\int_{-1}^{1}\left(4t^5+3t^4\right)dt$$

$$=10\lambda c_1\int_{0}^{1}t^4\,dt+\lambda c_2\int_{-1}^{1}\left(4t^5+3t^4\right)dt$$

$$=10\lambda c_1\left[\frac{t^s}{5}\right]_0^1+\lambda c_2 6\left[\frac{t^s}{5}\right]_0^1$$

$$c_1=2\lambda c_1+\frac{6}{5}\lambda c_2$$

$$\left(1-2\lambda\right)c_1+\frac{5}{6}\lambda c_2=0 \qquad \text{-----(4)}$$

Putting (1) in equation (2) we get

$$c_2=\int_{-1}^{1}t\left[5t\lambda c_1+\left(4t^2+3t\right)\lambda c_2\right]dt$$

$$=\int_{-1}^{1}\left[5\lambda c_1t^2+\lambda c_2\left(4t^3+3t^2\right)\right]dt$$

$$=10\lambda c_1\int_{0}^{1}t^2dt+6\lambda c_2\int_{0}^{1}t^2dt$$

$$=10\lambda c_1\left[\frac{t^3}{3}\right]_0^1+6\lambda c_2\left[\frac{t^3}{3}\right]_0^1$$

$$=\frac{10\lambda c_1}{3}+\frac{6\lambda c_2}{3}=\frac{10\lambda c_1}{3}+2\lambda c_2$$

$$i.e\left(1-\frac{10\lambda}{3}\right)c_1+2\lambda c_2=0$$

$$\frac{-10}{3}\lambda c_1+\left(1-2\lambda\right)c_2=0 \qquad \text{-----(5)}$$

For non zero soluation, we have

$$D(\lambda)=0$$

$$\begin{vmatrix}1-2\lambda & -\frac{6}{5}\lambda \\ -\frac{10\lambda}{3} & 1-2\lambda\end{vmatrix}=0$$

$$\therefore\left(1-2\lambda\right)^2-4\lambda^2=0\Rightarrow 1-4\lambda+4\lambda^2-4\lambda^2=0$$

$\therefore \lambda = \frac{1}{4}$

Putting $\lambda = \frac{1}{4}$ in (4) and (5), we get

$$\frac{1}{2}c_1 - \frac{3}{10}c_2 = 0$$

$$-\frac{5}{6}c_1 - \frac{1}{2}c_2 = 0$$

Bothe equation reduces to

$$c_1 = \frac{3}{5}c_2$$

From (1), Required eigen function is

$$\phi(x) = \frac{5x}{4}\left(\frac{3}{5}\right)c_2 + \left(4x^2 + 3x\right)\frac{1}{4}c_2$$

$$= \frac{c_2}{4}\left[3x + 4x^2 + 3x\right]$$

$$= \frac{c_2}{4}\left[4x^2 + 6x\right]$$

$\therefore$ Required function corresponding to $\lambda = \frac{1}{4}$

$$\phi(x) = 4x^2 + 6x = 2\left(x^2 + 2x\right)$$

Problem : 3 Find the eigen values and eigen functions of the homogeneous integral equation.

$$g(s) = \lambda \int_1^2 \left(st + \frac{1}{st}\right) g(t)\, dt$$

Solution :

$$g(s) = \lambda s\int_1^2 tg(t) + \frac{\lambda}{s}\int_1^2 \frac{1}{t} g(t)\,dt$$

$$g(s) = \lambda s c_1 + \frac{\lambda}{s} c_2 \qquad \text{-----(1)}$$

where,

$$c_1 = \int_1^2 tg(t)\,dt \qquad \text{-----(2)}$$

$$c_2 = \int_1^2 \frac{1}{t} g(t)\,dt \qquad \text{-----(3)}$$

Now, Putting (2) in equation (3) we get

$$c_1 = \int_1^2 t\left[\lambda t c_1 + \frac{\lambda}{t} c_2\right] dt$$

$$= \lambda c_1 \int_1^2 t^2 dt + \lambda c_2 \int_1^2 dt$$

$$= \lambda c_1 \left[\frac{t^3}{3}\right]_1^2 + \lambda c_2 \left[t\right]_1^2$$

$$= \lambda c_1 \left[\frac{8-1}{3}\right] + \lambda c_2 \left[2-1\right]$$

$$0 = c_1\left(\frac{7\lambda}{3} - 1\right) + \lambda c_2 \qquad \text{-----(4)}$$

Now, Putting (3) in equation (1) we get

$$c_2 = \int_1^2 \frac{1}{t}\left[\lambda t c_1 + \frac{\lambda}{t} c_2\right] dt$$

$$= \int_1^2 \left[\lambda c_1 + \frac{\lambda c_2}{t^2} \right] dt$$

$$= \lambda c_1 \left(t \right)_1^2 + \lambda c_2 \left[-\frac{1}{t} \right]_1^2$$

$$= \lambda c_1 \left(2-1 \right) + \lambda c_2 \left[-\frac{1}{2} + 1 \right]$$

$$\lambda c_1 + c_2 \left(\frac{\lambda}{2} - 1 \right) = 0 \qquad \text{-----(5)}$$

For non zero solution

$$D(\lambda) = 0$$

$$\therefore \begin{vmatrix} \left(\frac{7\lambda}{3} - 1 \right) & \lambda \\ \lambda & \left(\frac{\lambda}{2} - 1 \right) \end{vmatrix} = 0$$

$$\left(\frac{7\lambda}{3} - 1 \right)\left(\frac{\lambda}{2} - 1 \right) - \lambda^2 = 0 \Rightarrow \frac{1}{6}\lambda^2 - \frac{17}{6}\lambda + 1 = 0$$

$$\therefore \lambda^2 - 17\lambda + 6 = 0$$

$$\therefore \lambda = \frac{17 \pm \sqrt{289 - 24}}{2} = \frac{1}{2}\left(17 \pm \sqrt{265} \right)$$

$$\therefore \lambda = \frac{17}{2} + \frac{\sqrt{265}}{2}, \frac{17}{2} - \frac{\sqrt{265}}{2}$$

$$= 16.6394,\ 0.3606$$

To find eigen function :

(i) For $\lambda = 16.6394$

(4), (5) beomes

$$\left[1-\frac{7}{3}(16.6394)\right]c_1 - 16.6394c_2 = 0$$

$$-16.6394c_1 + \left[1-\frac{1}{2}(16.6394)\right]c_2 = 0$$

Solving, both equation reduces to

$c_2 = -2.2732C_1$

Thus equation (1) becomes

$$g(s) = 16.6394sc_1 + \frac{16.6394}{s}(-2.2732)c_1$$

$$= 16.6394c_1\left[1-\frac{2.2732}{s}\right]$$

The required eigen function corresponding to $\lambda = 16.6394$

$$g(s) = 1-\frac{2.2732}{s} \qquad \text{(Taking 16.63944=1)}$$

(ii) For $\lambda = 0.3606$: Equation (4) and (5) becomes

$$\left[1-\frac{7}{3}(0.3609)\right]c_1 - 0.3606c_2 = 0$$

$$-0.3606c_1 + \left[1-\frac{1}{2}(0.3636)\right]c_2 = 0$$

Solving both equation reduces to

$C_2 \cong 0.4399\, C_1$

Thus equation (1) becomes

$$g(s) = 0.3609Sc_1 + \frac{0.3609}{s}(0.4399C_1)$$

$$= 0.3609c_1\left[1+\frac{0.4399}{S}\right]$$

Taking $0.36094 = 1$

Required eigen function corresponding to $\lambda = 0.3609$ is

$$g(s) = 1+\frac{0.4399}{5}$$

Problem 4 : Solve the homogeneous fredholm integral equation

$$g(s) = \lambda\int_0^1 e^{s+t} g(t)\,dt$$

Solution : $g(s) = \lambda\int_0^1 e^s e^t g(t)\,dt = \lambda e^s \int_0^1 e^t g(t)\,dt$

$$g(s) = \lambda e^s c \qquad \text{-----(1)}$$

where $c = \int_0^1 e^t g(t)\,dt$ -----(2)

using (1) in (2) we get

$$c = \int_0^1 e^t \lambda e^t c\,dt$$

$$= \lambda c\int_0^1 e^{2t} dt = \lambda c\left[\frac{e^{2t}}{2}\right]_0^1$$

$= \lambda c\left[\frac{e^2-1}{2}\right]$, To find eigen function we must have $c \neq o$

$$\Rightarrow \lambda\left(\frac{e^2-1}{2}\right)=1 \quad \Rightarrow \lambda = \frac{2}{e^2-1}$$

From equation (1) we get

$$g(s) = \frac{2c}{e^2 - 1} e^s$$

Required eigen function is corresponding to $\lambda = \dfrac{2}{e^2 - 1}$ is

$$g(s) = e^s \qquad \left(\because taking\ \frac{2c}{e^2 - 1} = 1 \right)$$

Problem : 5 Show the integral equation $g(s) = \lambda \int_0^\pi (\sin s \sin 2t)\, g(t)\, dt$

has no eigen values

Solution : $g(s) = \lambda \sin s \int_0^\pi \sin 2t g(t)\, dt$

$$= \lambda \sin sc \qquad \text{-----(1)}$$

where $c = \int_0^\pi \sin 2t g(t)\, dt$ -----(2)

using (1) in (2) we get

$$c = \int_0^\pi \sin 2t [\lambda \sin tc]\, dt$$

$$c = c\lambda \int_0^\pi \sin 2t \sin t dt$$

$$\lambda = \frac{1}{\int_0^\pi \sin^2 t \cos t dt}$$

put sint = $x \Rightarrow$ cos t dt = dx

Limit

t	0	π
x	0	0

Therefore, $\int_o^{\pi} \sin^2 t \text{ cost dt} = 0$. Hence, λ does not ext.

$\therefore$ The integral equation has no eigen values.

Problem 6 : Find the eigen values and eigen function of the integral equation.

$$g(s) = \lambda \int_0^1 (\sin \pi s \cos \pi t)\, g(t)\, dt$$

Solution : $g(s) = \lambda \int_0^1 (\sin \pi s \cos \pi t)\, g(t)\, dt$ -----(1)

$$\Rightarrow g(s) = \lambda \sin \pi s \int_0^1 \cos \pi t g(t)\, dt$$

$$= \lambda c \sin \pi s \qquad \text{-----(2)}$$

$$c = \int_0^1 \cos \pi t g(t)\, dt \qquad \text{-----(3)}$$

using (2) in (3) we get

$$c = \int_0^1 \cos \pi t \lambda c (\sin \pi t)\, dt$$

$$= \lambda c \int_0^1 \sin \pi t \cos \pi t dt$$

$$= \frac{\lambda c}{2} \int_0^1 \sin 2\pi t dt$$

$$= \frac{\lambda c}{2}\left[\frac{-\cos 2\pi t}{2\pi}\right]_0^1$$

$$= -\frac{\lambda c}{2}\left[\frac{\cos 2\pi - \cos(0)}{2\pi}\right]$$

$$= -\frac{\lambda c}{2}\left[\frac{1-1}{2\pi}\right] = 0$$

c = 0 ; Thus g(s) = 0

If $C \neq 0$, then $\lambda = \dfrac{1}{\dfrac{-1}{2}\int_0^1 \sin 2\pi t dt}$

As $\int_0^1 \sin 2\pi t \; dt = 0$

∴ λ does not exist.

Thus for any λ (1) has only zero solution g(s) = 0

Equation (1) does not posses any eigen values and eigen function.

Problem : 7 Solve the homogeneous Fredholm integral equation

$$g(s) = \frac{2}{\pi}\lambda\int_0^\pi \cos(s+t) g(t)\, dt$$

Solution : $g(s) = \frac{2}{\pi}\lambda\int_0^\pi \cos(s+t) g(t)\, dt$

$$= \frac{2}{\pi}\lambda \cos s\int_0^\pi \cos t g(t)\, dt - \frac{2}{\pi}\lambda \sin s\int_0^\pi \sin t g(t)\, dt$$

$$g(s) = \frac{2}{\pi}\lambda\left[c_1 \cos s - c_2 \sin s\right] \qquad \text{-----(1)}$$

where

$$c_1 = \int_0^{\pi} \cos t g(t)\, dt \qquad \text{-----(2)}$$

$$c_2 = \int_0^{\pi} \sin t g(t)\, dt \qquad \text{-----(3)}$$

substiuting equation (1) in to eqution (2) and (3) and integrating we get

$$c_1 - \lambda c_1 = 0 \Rightarrow c_1(1-\lambda) = 0 \qquad \text{-----(4)}$$

$$c_2 + \lambda c_2 = 0 \Rightarrow c_2(1+\lambda) = 0 \qquad \text{-----(5)}$$

For λ =1

C_1 is arbitrary and $C_2 = 0$

therefore $g(s) = \frac{2}{\pi} C_1 \cos s$

For λ =-1

C_2 is arbitrary and $C_1 = 0$

$$g(s) = -\frac{2}{\pi} c_2 \sin s$$

The eigen funtion corresponding to λ =1 is $g(s) = \cos s$

The eigen funtion corresponding to λ =-1 is $g(s) = \sin s$

Problem : 8 Find the eigen values and eigen function of the homogeneous integral equation.

$$g(s) = \lambda \int_0^1 (6s - 2t)\, g(t)\, dt$$

Solution : $g(s)=\lambda\int_0^1(6s-2t)g(t)\,dt$

$$=6\lambda s\int_0^1 g(t)\,dt-2\lambda\int_0^1 tg(t)\,dt$$

$$g(s)=6\lambda sc_1-2\lambda c_2 \quad \text{-----(1)}$$

$$c_1=\int_0^1 g(t)\,dt \quad \text{-----(2)}$$

$$c_2=\int_0^1 tg(t)\,dt \quad \text{-----(3)}$$

Putting equation (1) into eqution (2) and (3) and integration we get.

$$(1-3\lambda)c_1+\lambda c_2=0 \quad \text{-----(4)}$$

$$-4\lambda c_2+(1+\lambda)c_2=0 \quad \text{-----(5)}$$

For $\lambda=1$ eqution (4) and (5) becomes

$$\left.\begin{matrix}-2c_1+c_2=0\\-4c_1+2c_2=0\end{matrix}\right\}\Rightarrow c_2=2c_1$$

Thus equation (1) becomes

$$g(s)=6sc_1-4c_1=2c_1(3s-2)$$

Thus the eigen function corresponding $\lambda=1$ is g(s) = 3s - 2

Exercise

Q: 1 Find the eigen values and eigen function of the following homogeneous fredholm integral equation.

1. $g(s)=\lambda\int_0^\pi\left[\cos^2 s\cos 2t+\cos 3s\cos^3 t\right]g(t)\,dt$

2. $\phi(x)=\lambda\int_0^1\left(4x^2\log t-gt^2\log x\right)\phi(t)\,dt$

3. $\phi(x)=\lambda\int_0^{\pi/4}\sin^2 x\phi(t)\,dt$

4. $\phi(x)=\lambda\int_{-1}^1\left(x\cosh t-t\sinh x\right)\phi(t)\,dt$

5. $g(s)=\lambda\int_0^1 2tg(t)\,dt$

6. $g(s)=\lambda\int_0^1 se^t g(t)\,dt$

7. $g(s)=\lambda\int_0^1 stg(t)\,dt$

8. $g(s)=\lambda\int_1^2\left(st+\frac{1}{st}\right)g(t)\,dt$

9. $g(s)=\lambda\int_0^{\pi/2}\cos s\sin tg(t)\,dt$

10. $g(s)=\frac{2}{\pi}\lambda\int_0^{\pi}\sin(s+t)g(t)\,dt$

11. $g(s)=\lambda\int_0^{\pi/3}\sec x\tan tg(t)\,dt$

12. $g(s)=\lambda\int_0^1\left(s^2+\frac{1}{2}s^3t^2\right)g(t)\,dt$

13. $g(s)=\lambda\int_0^1 \sin^{-1} s g(t) d$

14. $g(s)=\lambda\int_{-1}^1 \left(s+e^{s^2t^2}\right) g(t)\,dt$

Q : 2 Show that following homogeneous integral equation has no eigen function.

1. $\phi(x)=-\int_0^1 \phi(t)\,dt$

2. $\phi(x)=\frac{1}{2}\int_0^{\pi} \sin x\phi(t)\,dt$

3. $\phi(x)=\frac{1}{50}\int_0^{10} tg(t)\,dt$

3.5 Eigenvalues and Eigenfunction of the homogeneous fredholm integral equation by reducing it to strum liouville problems.

Whenever homogeneous fredho/m integral equation is given in the form.

$$g(s)=\lambda\int_a^b k(s,t)g(t)\,dt$$

where

$$k(s,t)=\begin{cases} a(s)b(t) & ; \ a\le s\le t \\ a(t)b(s) & : \ t\le s\le b \end{cases}$$

Then to find eigen values and eigen function reduce it to BVP with two boundary conditions.

Then solve the resulting BVP to determine eigen values and eigen function.

Infact above Integral equation reduces to Sturm Liouville problems.

Strum Liouville Problems (Definition) :

A second order differential equation of the form.

$$\left[p(x)y'\right]' + \left[q(x) + \lambda r(x)\right]y = 0; x_1 \le x \le x_2$$

$$a_1 y(x_1) + b_1 y'(x_1) = 0$$

$$a_2 y(x_2) + b_2 y'(x_2) = 0$$

Where p(x), q(x) and r(x) are continues functions in the interval $x_1 \le x \le x_2$ in addition p(x) has continuous derivative and not vanish λ is real parameter, a_1, b_1, a_2, b_2 are real constants such that atleast one in each condition (1) is non zero.

Problem 1 : Determine eigen values and cigen function for the homogeneous Integral equation $g(x) = \lambda \int_0^1 k(x,t)\, g(t)\, dt$

where

$$k(x,t) = \begin{cases} x(t-1); & 0 \le x \le t \\ t(x-1); & t \le x \le 1 \end{cases}$$

Solution : Given $g(x) = \lambda \int_0^1 k(x,t)\, g(t)\, dt$

where $k(x,t) = \begin{cases} x(t-1); & 0 \le x \le t \\ t(x-1); & t \le x \le 1 \end{cases}$

$$\therefore g(x) = \lambda \int_0^x k(x,t)\, g(t)\, dt + \lambda \int_x^1 k(x,t)\, g(t)\, dt$$

$$= \int_0^x \lambda t(x-1)\, g(t)\, dt + \int_x^1 \lambda x(t-1)\, g(t)\, dt \qquad \text{-----(1)}$$

Differentiating w. r. t. x. we get

$$g'(x) = \int_0^x \lambda t g(t)\,dt + \lambda x(x-1)g(x) - 0 + \int_x^1 \lambda(t-1)g(t)\,dt + 0 - \lambda x(x-1)g(x)$$

$$g'(x) = \int_0^x \lambda t g(t)\,dt + \int_x^1 \lambda(t-1)g(t)\,dt \qquad \text{-----(2)}$$

$$g''(x) = \left[0 + \lambda x g(x) - 0\right] + \left[0 + 0 - \lambda(x-1)g(x)\right]$$

$$g''(x) = \lambda x g(x) - \lambda x g(x) + \lambda g(x)$$

$$g''(x) - \lambda g(x) = 0 \qquad \text{-----(3)}$$

$$\left.\begin{aligned} g(0) &= 0 \\ g(1) &= 0 \end{aligned}\right\} \qquad \text{-----(4)}$$

Case 1 : $\lambda = 0$

$\therefore g''(x) = 0$

$\therefore g(x) = Ax + B$

$\therefore 0 = g(0) = B$

$0 = g(1) = A + B = B(\because A = 0$

$A = B = 0$

g(s) = 0 which is not eigen function so $\lambda = 0$ is not eigen values.

Case 2 : $\lambda > 0$

$\lambda = \mu^2; \mu \neq 0$

then $g'' - \mu^2 g = 0$

$0 = D^2 - \mu^2 = (r-\mu)(r+\mu)$

$\Rightarrow D = \mu, -\mu$

$\therefore g(x) = Ae^{\mu x} + Be^{-\mu x}$

$0 = g(0) = A + B \Rightarrow A = -B$

$0 = g(1) = Ae^{\mu} + Be^{-\mu}$

$= Ae^{\mu} - Ae^{-\mu}$

$= A\left(e^{\mu} - e^{-\mu}\right)$

$\Rightarrow A = 0 \quad \left(\because e^{\mu} - e^{-\mu} \neq 0, \text{ since } \mu \neq 0\right)$

$\therefore A = B = 0$

g(s) = 0 which is not eigen function therefore $\lambda = \mu^2$ is not eigen values.

Case 3 : $\lambda < 0$

let $\lambda = -\mu^2 \;\; ; \mu \neq 0$

Then $g'' + \mu^2 g = 0$

Auxillary equation $D^2 + \mu^2 = 0 \Rightarrow D = \pm i\mu$

Solution is

$g(x) = A\cos\mu x + B\sin\mu x$

$\therefore 0 = g(1) = A\cos\mu + B\sin\mu$

$0 = B\sin\mu$

$\therefore B \neq 0 \quad and\ \sin\mu = 0 \Rightarrow \mu = n\pi$

$n = \pm 1, 2, 3......$

$\therefore \lambda = -\mu^2 = -n^2\pi^2 \quad ; n = 1, 2, 3, .\left[-ve\ not\ \cos ider\ \right]$

$$g(x) = B\sin\mu x$$

$$= B\sin n\pi x$$

Required eigen functions are

$$g(x) = \sin n\pi x$$

Corrosponding to eigen values $\lambda = -n^2\pi^2, n = 1,2,3...$

Problem 2 : Determine the eigen values and eigen functions of the homogeneous equation.

$$\phi(x) = \lambda\int_0^\pi k(x,t)\phi(t)\,dt$$

where,

$$k(x,t) = \begin{cases} \cos x\sin t & ;0 \le x \le t \\ \cos t\sin x; & t \le x \le \pi \end{cases}$$

Solution : $\phi(x) = \int_0^x \lambda k(x,t)\phi(t)dt + \int_x^\pi \lambda k(x,t)\phi(t)\,dt$

$$= \int_0^x \lambda\cos t\sin x\phi(t)\,dt + \int_x^\pi \lambda\cos x\sin t\phi(t)\,dt \quad \text{-----(1)}$$

Differentiating w. r. t. x.

$$\phi'(x) = \int_0^x \lambda\cos t\cos x\phi(t)\,dt + \lambda\cos x\sin x\phi(t)\,dt$$

$$+\int_0^x \lambda(-\sin x)\sin t\phi(t)\,dt + 0 - \lambda\cos x\sin x\phi(x)$$

$$= \int_0^x \lambda\cos t\cos x\phi(t)\,dt - \int_x^\pi \lambda\sin x\sin t\phi(t)\,dt \quad \text{-----(2)}$$

Differentiating w. r. t. x. again

$$\phi^{'}(x) = -\int_0^x \lambda \cos t \sin x \phi(t)\, dt + \lambda \cosh^2 x \phi(x) - 0$$

$$-\int_0^x \lambda \cos x \sin t \phi(t)\, dt + 0 + \lambda \sin^2 x \phi(x)$$

$$= -\phi(x) + \lambda\phi(x)$$

$$\phi(x) - (\lambda - 1)\phi(x) = 0$$

Case 1 : If $(\lambda - 1) = 0 \Rightarrow \lambda = 1$

then,

$$\phi''(x) = 0 \Rightarrow \phi(x) = Ax + B$$
$$\therefore \phi'(x) = B$$
Now
$$0 = \phi'(0) \Rightarrow B = 0$$
$$0 = \phi(\pi) = A\pi + B \Rightarrow A = 0$$

A = B = 0 given $\phi(x) = 0$ which is not eigen function

so $\lambda = 1$ is not eigen value.

Case 2 : $\lambda - 1 > 0$

Let $\lambda - 1 = \mu^2; \mu \neq 0$

Then

$$\phi'' - \mu^2\phi = 0$$

A. E. is

$$D^2 - \mu^2 = 0 \Rightarrow D = \pm\mu$$

$$\therefore \phi(x) = Ae^{\mu x} + Be^{-\mu x}$$

and $\phi'(x) = \mu A e^{\mu x} - \mu B e^{-\mu x}$

$0 = \phi(\pi) = A + B \quad \Rightarrow A = -B$

$0 = \phi'(0) = A\mu - B\mu$

$A\mu + A\mu = 0 \Rightarrow 2A\mu = 0$

$\Rightarrow A = 0 \ \because \mu \neq 0$

$\therefore A = B = 0$

$\therefore \phi(x) = 0$ which is not eigen function

$\therefore \lambda = \mu^2 + 1$ is not eigen value.

Case 3 :

Let $\lambda - 1 = -\mu^2 \ ; \mu \neq 0$

then $\phi'' + \mu^2 \phi = 0$

$\therefore A.E$ is $D^2 + \mu^2 = 0$

$D = \pm i\mu$

$\therefore \phi(x) = A\cos \mu x + B\sin \mu x$

and $\phi'(x) = -A\mu \sin \mu x + B\mu \cos \mu x$

$0 = \phi(\pi) = A\cos \mu\pi + B\sin \mu\pi$

$0 = \phi'(0) = A\mu(0) + B\mu(1)$

$\therefore B\mu = 0 \ \Rightarrow B = 0(\because \mu \neq 0)$

$(\because n = 0 \ \Rightarrow d(x) = B\sin x = 0)$

$\therefore \mu \cos \mu\pi = 0$

$if\ A=0 \quad \phi(x)=0$

which is not eigen function.

for eigen function, A $\neq$ 0 is must

$$\therefore \cos\mu\pi = 0 \Rightarrow \mu\pi = (2n+1)\pi/2;\ \ n=\pm1,\pm2,...$$

$$\therefore \mu = \left(\frac{2n+1}{2}\right)$$

Thus,

$$\phi(x) = A\cos\left(\frac{2n+1}{2}\right)x$$

$$\lambda - 1 = -\mu^2 = -\frac{(2n+1)^2}{4}$$

Required eigen functions are

$$\phi(x) = \cos\left(n+\frac{1}{2}\right)x$$

Corresponding to,

$$\lambda_n = 1-\left(n+\frac{1}{2}\right)^2 \quad ;n\pm1,\pm2....$$

Problem 3 : Determine the eigen values and eigen functions of the homogeneous integral equation.

$$g(x) = \lambda\int_0^1 k(x,t)g(t)\,dt$$

Where, $k(x,t) = \begin{cases} t(x+1); 0\le x\le t \\ x(t+1); t\le x\le 1 \end{cases}$

Solution : Given $g(x)=\lambda\int_0^1 k(x,t)\,g(t)\,dt$ -----(1)

Where, $k(x,t)=\begin{cases} t(x+1);\, 0\le x\le t \\ x(t+1);\, t\le x\le 1 \end{cases}$

Equation (1) can be written as

$$g(x)=\lambda\int_0^x k(x,t)\,g(t)\,dt+\lambda\int_x^1 k(x,t)\,g(t)\,dt$$

$$\therefore g(x)=\int_0^x \lambda x(t+1)\,g(t)\,dt+\int_x^1 \lambda t(x+1)\,g(t)\,dt \qquad \text{-----(2)}$$

Differentiating w.r.t.x. we get

$$g'(x)=\int_0^x \lambda(t+1)\,g(t)\,dt+\lambda x(x+1)\,g(x)$$

$$+\int_x^1 \lambda t g(t)\,dt-\lambda x(x+1)\,g(x)$$

$$g'(x)=\int_0^x \lambda(t+1)\,g(t)\,dt+\int_x^1 \lambda t g(t)\,dt \qquad \text{-----(3)}$$

Differentiating w.r.t.x again we get

$$g''(x)=\lambda(x+1)\,g(x)-\lambda x g(x)$$

$$=\lambda x g(x)+\lambda g(x)-\lambda x g(x)$$

$$=\lambda g(x)$$

$$g''(x)-\lambda g(x)=0 \qquad \text{-----(4)}$$

Putting $x=0$ in (2) and (3) we get

$$g(0)=\int_0^1 \lambda t g(t)\,dt$$

$$g'(0)=\int_0^1 \lambda t g(t)\,dt$$

$$\Rightarrow g(0)=g'(0) \qquad \text{-----(5)}$$

putting x = 1 in (2) and (3) we get

$$g(1)=\int_0^1 \lambda(t+1)g(t)\,dt$$

$$g'(1)=\int_0^1 \lambda(t+1)g(t)\,dt$$

$$\Rightarrow g(1)=g'(1) \qquad \text{-----(6)}$$

Equation (1) is equivalent to BVP

$$g''(x)-\lambda g(x)=0$$

$$g(0)=g'(0)\ ;g(1)=g'(1)$$

which is sturm Liourille problem.

case 1 : *if* $\lambda=0$ then

$$g(x)=Ax+B \Rightarrow g'(x)=A$$

Now,

$$g(0)=B \quad ,g'(0)=A$$

$$\therefore g(0)=g'(0)\Rightarrow A=B$$

and $g(1)=g'(1)\Rightarrow A+B=A\Rightarrow B=0$

$$A=B=0$$

$\therefore g(x)=0$ which is not an eigen function.

Case 2 : *if* $\lambda > 0$

Let $\lambda = \mu^2$, where $\mu \neq 0$

then

$$g(x) = Ae^{\mu x} + Be^{-\mu x}$$

$$g'(x) = Aue^{\mu x} - Bue^{-\mu x}$$

$$g(0) = g'(0) \Rightarrow A + B = A\mu - B\mu$$

$$\Rightarrow A(1-\mu) + B(1+\mu) = 0 \qquad \text{-----(7)}$$

$$g(1) = g'(1) \Rightarrow Ae^{\mu} + Be^{-\mu} = Aue^{\mu} - Bue^{-\mu}$$

$$\Rightarrow Ae^{\mu}(1-\mu) + Be^{-\mu}(1+\mu) = 0 \qquad \text{-----(8)}$$

for non trivial solution of system of equation (7) and (8) we must have

$$\begin{vmatrix} 1-\mu & 1+\mu \\ e^{\mu}(1-\mu) & e^{-\mu}(1+\mu) \end{vmatrix} = 0$$

$$\Rightarrow (1-\mu^2)e^{-\mu} - (1-\mu^2)e^{\mu} = 0$$

$$\Rightarrow 2(1-\mu^2)\sinh\mu = 0$$

Since, $\mu \neq 0 \Rightarrow \sinh \mu \neq 0$

Thus, $1-\mu^2 = 0 \ \Rightarrow \mu = \pm 1$

For $\mu = 1$ equation (7) and (8) becomes

$$A(0) + 2B = 0 \ \Rightarrow B = 0$$

$$A(0) + 2Be^{-1} = 0 \ \Rightarrow B = 0$$

Thus B = 0 and A is arbitrary.

$g(x) = Ae^x$

Similarly when μ = -1we get A = 0 and B is arbitrary

$\therefore g(x) = Be^x$

Thus g(x) = e^x is eigen function corrosponding to $\lambda = \mu^2 = (\pm 1)^2 = 1$

Case 3 : $\lambda < 0$

Let $\lambda = -\mu^2; \mu \neq 0$

$\therefore g(x) = A\cos\mu x + B\sin\mu x$

$\Rightarrow g'(x) = -A\mu\sin\mu x + B\mu\cos\mu x$

Now,

$$g(0) = g'(0) \Rightarrow A\cos\mu + B\sin\mu = -A\mu\sin\mu + B\mu\cos\mu \qquad \text{-----(10)}$$

Using (9) in (10) we get

$B\mu\cos\mu + B\sin\mu = -B\mu^2\sin\mu + B\mu\cos\mu$

$\therefore B(1+\mu^2)\sin\mu = 0$

Since,

$B \neq 0. \quad 1+\mu^2 \neq 0$

$\Rightarrow \sin\mu = 0 \Rightarrow \mu = n\pi \quad ; n \in N$

$\lambda = -\mu^2 = -n^2\pi^2 \quad ; n \in N$

$\therefore g(x) = B[\mu\cos\mu x + \sin\mu x]$

$\qquad = B[n\pi\cos n\pi x + \sin n\pi x]$

Then eigen function corrosponding to $\lambda = -n^2\pi^2$ are $g(x) = n\pi\cos n\pi x + \sin n\pi x$

Exercise :

Find the eigen values and eigen functions of the following homogeneous equation.

$$g(x) = \int k(x,t)\, g(t)\, dt$$

where,

1. $k(x,t) = \begin{cases} (x+1)\ (t-2) & ;0 \le x \le t \\ (t+1)\ (x-2) & ;t \le x \le 1 \end{cases}$

2. $k(x,t) = \begin{cases} \sin x \cos t & ;0 \le x \le t \\ \cos x \sin t & ;t \le x \le \pi/2 \end{cases}$

3. $k(x,t) = \begin{cases} \sin x \sin(t-1) & ;-\pi \le x \le t \\ \sin t \sin(x-1) & ;t \le x \le \pi \end{cases}$

4. $k(x,t) = \begin{cases} -e^{-t} \sinh x & ;0 \le x \le t \\ -e^{-x} \sinh t & ;t \le x \le 1 \end{cases}$

5. $k(x,t) = \begin{cases} e^{x-t} \sinh x & ;0 \le x \le t \\ e^{t-x} \sinh t & ;t \le x \le 1 \end{cases}$

6. $k(x,t) = e^{-|1x-t|}\ ;0 \le x \le 1\ ,0 \le t \le 1$

Definition : Two function f(x) and g(x) are said to be orthogonal on [a, b] if

$$\int_a^b f(x)\, g(x)\, dx = 0$$

Theorem : Eigen functions g(s) and $\psi(s)$ Corrosponding to distinct eigen values λ_1 and λ_2 respectively, of the homogeneous integral equation and its transpose are orthogonal.

Proof : Consider the homogeneous integral equation.

$$g(s) = \lambda_1 \int_a^b k(s,t)\, g(t)\, dt \qquad \text{-----(1)}$$

and its transpose

$$\psi(s) = \lambda_2 \int_a^b k(t,s)\, \psi(t)\, dt \qquad \text{-----(2)}$$

Multiply equation (2) by $\lambda_1 g(s)$ and integrate from a to b

$$\int_a^b \lambda_1 \psi(s)\, g(s)\, ds = \lambda_1 \lambda_2 \int_a^b g(s) \left[\int_a^b k(t,s)\, \psi(t)\, dt \right] ds$$

Changing the order of integration on R.H.S.

$$\lambda_1 \int_a^b \psi(s)\, g(s)\, ds = \lambda_1 \lambda_2 \int_a^b \psi(t) \left[\int_a^b k(t,s)\, g(s)\, ds \right] dt \qquad \text{-----(3)}$$

Now from (1)

$$g(x) = \lambda_1 \int_a^b k(x,t)\, g(t)\, dt$$

$$= \lambda_1 \int_a^b k(x,s)\, g(s)\, ds$$

($\therefore$ property of definite integration)

$$g(t) = \lambda_1 \int_a^b k(t,s)\, g(s)\, ds$$

Equation (3) becomes

$$\lambda_1 \int_a^b \psi(t)\, g(s)\, ds = \lambda_2 \int_a^b \psi(t)\, g(t)\, dt$$

$$= \lambda_2 \int_a^b \psi(s) g(s) ds$$

$$(\lambda_1 - \lambda_2) \int_a^b \psi(s) g(s) ds = 0$$

Since,

$\lambda_1 \neq \lambda_2 \quad \Rightarrow (\lambda_1 - \lambda_2) \neq 0$

$$\therefore \int_a^b \psi(s) g(s) ds = 0$$

$\psi(s)$ and g(s) are orthogenal.

Theoram : If $\lambda = \lambda_0$ is root of multiplicity m>1 of the equation $D(\lambda) = 0$ then the homogeneous integral equation $g(s) = \lambda \int_a^b k(s,t) g(t) dt$

has r linearly independent solutions : r is the index of the eigen value such that $l \leq r \leq m$

Proof : Consider the homogeneous integral equation

$$g(s) = \lambda \int_a^b k(s,t) g(t) dt \qquad \text{-----(1)}$$

with separable kernel $k(s,t)$

i.e. $k(s,t) = \sum_{i=1}^{n} a_i(s) b_i(t)$

where $a_1(s), \ldots, a_n(s), b_1(t), \ldots, b_n(t)$ are linearly independent functions.

Equation (1) becomes

$$g(s) = \lambda \int_a^b \left[\sum_{i=1}^{n} a_i(s) b_i(t) \right] g(t) dt$$

$$= \lambda \sum_{i=1}^{n} a_i(s) \int_a^b b_i(t) g(t) dt \qquad \text{-----(2)}$$

$$\text{Let } \int_a^b b_i(t) g(t) dt = C_i ; (i = 1, \ldots n) \qquad \text{-----(3)}$$

$$\therefore\ g(s) = \lambda \sum_{i=1}^{n} C_i a_i(s) \qquad \text{-----(4)}$$

To determine $C_i;(i=1,...n)$ putting the value of g(s) given by (4) in equation (2) we get.

$$\lambda \sum_{i=1}^{n} C_i a_i(s) = \lambda \sum_{i=1}^{n} a_i(s) \int_a^b b_i(t) \left[\lambda \sum_{k=1}^{n} C_k a_k(t) \right] dt$$

$$\therefore \sum_{i=1}^{n} a_i(s) \left\{ C_i - \lambda \int_a^b b_i(t) \left[\sum_{k=1}^{n} C_k a_k(t) \right] dt \right\} = 0$$

But the functions $a_i(s)$ are linearly independent, therefore

$$\therefore C_i - \lambda \int_a^b b_i(t) \left[\sum_{k=1}^{n} C_k a_k(t) \right] dt = 0$$

$$\therefore C_i - \lambda \sum_{k=1}^{n} C_k \int_a^b b_i(t) a_k(t)\, dt = 0$$

$$\text{Denote } \int_a^b b_i(t) a_k(t) dt = a_{ik}$$

$$\therefore C_i - \lambda \sum_{k=1}^{n} C_k a_{ik} = 0; (i = 1, 2, ...n)$$

which is system of homogeneous linear equations for the unknown $C_1, ..., C_2$ as given by

$$C_1(1-\lambda a_{11})-C_2\lambda a_{12}-\cdots-\lambda C_n a_{1n}=0$$
$$-C_1\lambda a_{21}+C_2(1-\lambda a_{22})-\cdots-\lambda C_n a_{2n}=0$$
$$\cdots \quad \cdots \quad \cdots\cdots \quad \cdots \quad \cdots$$
$$-C_1\lambda a_{n1}-C_2\lambda a_{n2}-\cdots+\lambda(1-\lambda a_{nn})=0$$

which can be written as

$$\begin{bmatrix} 1-\lambda a_{11} & -\lambda a_{12} & -\lambda a_{1n} \\ -\lambda a_{21} & (1-\lambda a_{22}) & -\lambda a_{2n} \\ & & \\ & & \\ -\lambda a_{n1} & -\lambda a_{n2} & 1-\lambda a_{nn} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ . \\ . \\ c_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ . \\ . \\ 0 \end{bmatrix}$$

$$\therefore [I-\lambda A]c=0 \qquad \text{-----(5)}$$

Where, I is the unit matrix of order n and A is the matrix (a_{ij}).

Let $D(\lambda)=|I-\lambda A|=0$

$\Rightarrow$ The algebraic system (5) has infinite number of solution.

$\therefore$ For each non trivial solution of algebraic system (5) there corroponds a non trivial solution of homogeneous integeral equation (1). Let λ coincides with a certain eigen value λ_0 and assume that $D(\lambda_0)=|I-\lambda_0 A|$ has the rank p, $1\le P\le n$ then there are r = n - p linearly independent solution of algebraic system (5). (r is called index of eigen value λ_0)

$\Rightarrow$ The integral equation (1) has r linearly independent solution ($\because$ 4). Let us denote these r linearly independ out solution as

$$g_{01}(s)\ ,g_{02}(s),\ldots\ldots g_{0r}(s)$$

$\Rightarrow$ For each values λ_0 of index r = n-p, there corrosponds a solution

$$g_0(s) = \sum_{k=1}^{r} \alpha_k \ g_{0k}(s)$$

Where α_k are the arbitrary constants. Let m is the multiplicity of the eigen value λ_0

i.e. $D(\lambda) = 0$ has m equal roots λ_0

$\Rightarrow D(\lambda) = |I - \lambda A|$ has m+1 identical rows. (See theory of linear algebra)

Rank p of the determinant $D(\lambda) = 0$ is greater than or equal to n-m

$\therefore r = n - p = n - (n - m) = m$

If $a_{ij} = a_{ji}$ then the rank is n - m

$-(n - m) = m$

If aij = aji then equation (1) has m linearly independent solutions.

♦♦♦

Unit – 4

METHOD OF SUCCESSIVE APPROXIMATION'S

4.1 Successive Approximations for Fredholm Integral Equation :

Consider the Fredholm integral equation of second kind.

$$g(s) = f(s) + \lambda \int_a^b k(s, t)\, g(t)\, dt \qquad \text{-----(1)}$$

Where f(s) and k(s,t) are L_2 functions. Taking zero order approximation as $g_0(s) = f(s)$,

to the required solution g(s) we get the first order approximation.

$$g_1(s) = f(s) + \lambda \int_a^b k(s, t)\, g_0(t)\, dt$$

Putting $g_1(s)$ in (1) we get second order approximation.

$$g_2(s) = f(s) + \lambda \int_a^b k(s, t)\, g_1(t)\, dt$$

Continuing the process, the $(n + 1)^{th}$ approximation is given by

$$g_{n+1}(s) = f(s) + \lambda \int_a^b k(s, t)\, g_n(t)\, dt \qquad \text{-----(2)}$$

It $g_n(s)$ tends uniformly to a limit as $n \to \infty$ then this limit is the required solution.

To find such a limit we proceed in detail.

The first and second order approximations are

$$g_1(s) = f(s) + \lambda\int_a^b k(s, t)\, f(t)\, dt \quad [\because\ g_0(s) = f(s)] \qquad \text{-----(3)}$$

$$g_2(s) = f(s) + \lambda\int_a^b k(s, t)\, g_1(t)\, dt \qquad \text{-----(4)}$$

Using (3) in (4) we get

$$g_2(s) = f(s) + \lambda\int_a^b k(s,t)\left\{ f(t) + \lambda\int_a^b k(t,x) f(x) dx \right\} dt$$

$$= f(s) + \lambda\int_a^b k(s,t) f(t) dt + \lambda^2\int_a^b k(s,t)\left[\int_a^b k(t,x) f(x)\, dx\right] dt$$

Changing the order of integration

$$g_2(s) = f(s) + \lambda\int_a^b k(s,t) f(t) dt + \lambda^2\int_a^b f(x)\left[\int_a^b k(s,t) k(t,x) dt\right] dx$$

Interchanging the variable x & t

$$= f(s) + \lambda\int_a^b k(s,t) f(t) dt + \lambda^2\int_a^b f(t)\left[\int_a^b k(s,x) k(x,t) dx\right] \mathrm{dt}$$

Putting $k_2(s,t) = \int_a^b k(s,x)\ k(x,t)\ dx$

we get

$$g_2(s) = f(s) + \lambda\int_a^b k(s,t) f(t) dt + \lambda^2\int_a^b k_2(s,t) f(t) dt$$

Similarly

$$g_3(s) = f(s) + \lambda\int_a^b k(s,t)f(t)dt + \lambda^2\int_a^b k_2(s,t)f(t)dt$$

$$+\lambda^3\int_a^b k_3(s,t)f(t)dt \qquad \text{-----(5)}$$

where $k_3(s,t) = \int_a^b k(s,x)k_2(x,t)\ dx$

Equation (5) can be written as

$$g_3(s) = f(s) + \sum_{m=1}^{3}\lambda^m\int_a^b k_m(s,t)f(t)dt$$

By continuing same procedure, we have

$$g_n(s) = f(s) + \sum_{m=1}^{n}\lambda^m\int_a^b k_m(s,t)f(t)dt \qquad \text{-----(6)}$$

Where $k_m(s,t) = \int_a^b k(s,x)k_{m-1}(x,t)dt$ -----(7)

We call the expression $k_m(s, t)$ the m^{th} iterate or m^{th} iterated kernel.

The solution of equation (1) is given by

$$= f(s) + \sum_{m=1}^{\infty}\lambda^m\int_a^b k_m(s,t)f(t)dt \qquad \text{-----(8)}$$

$$g(s) = \lim_{n\to\infty} g_n(s)$$

Condition for convergence of equation (8) :

Applying schwartz inequality for any s to the partial sum of the series in (8)

$$\left|\int_a^b k_m(s,t)f(t)dt\right|^2 = \mathrm{K}\left|k_m,t\right|^2 \le \left\|k_m\right\|^2 . \left\|f\right\|^2 \le \left[\int_a^b \left|k_m(s,t)\right|^2 dt\right]\left[\int \left|\mathrm{f(t)}\right|^2 dt\right]$$

$\left[\because \text{ schwartz inequality : } \left|(\phi,\psi)\right| \le \|\phi\| \ \|\psi\|\right]$

Let D be the norm of f then

$$D = \|f\| = \left[\int_a^b \left|\mathrm{f(t)}\right|^2 dt\right]^{\frac{1}{2}}$$

$$\therefore \ \mathrm{D}^2 = \int_a^b \left|\mathrm{f(t)}\right|^2 dt$$

$\therefore$ Equation (9) becomes

$$\left|\int_a^b k_m(s,t)f(t)dt\right|^2 \le \mathrm{D}^2 \int_a^b \left|\mathrm{k_m}(s,t)\right|^2 dt \qquad \text{-----(10)}$$

Now from equation (7)

$$\left|k_m(s,t)\right|^2 = \left|\int_a^b k(x,t)\ \mathrm{k_{m-1}}(s,x)dx\right|^2 \le \int_a^b \left|k(x,t)\right|^2 dx \cdot \int_a^b \left|\mathrm{k_{m-1}}(s,x)\right|^2 dx$$

$[\ \because$ schwartz inequality$]$

Integrating with respect to t we get

$$\therefore \int_a^b |k_m(s,t)|^2 dt \le \int_a^b\int_a^b |k(x,t)|^2 dx\, dt \int_a^b |k_{m-1}(s,x)|^2 dx$$

$$\therefore \int_a^b |k_m(s,t)|^2 dt \le B^2 \int_a^b |k_{m-1}(s,x)|^2 dx\, dt \qquad \text{-----(11)}$$

where $B^2 = \int_a^b\int_a^b |k(x,t)|^2 dx\, dt$

Applying (11) repeatedly we have

$$\int_a^b |k_m(s,t)|^2 dt \le B^2\ B^2 \int_a^b |k_{m-2}(s,x)|^2\ dx$$

$$\le B^2 . B^2 . B^2 \int_a^b |k_{m-3}(s,x)|^2\ dx$$

.

.

.

$$\le \left(B^2\right)^{m-1} \int_a^b |k_1(s,x)|^2\ dx$$

$$\therefore \int_a^b |k_m(s,t)|^2\ dt \le (B^{m-1})^2\ C_1^2 \qquad \text{-----(12)}$$

Where C_1^2 is an upper bound of the integral $\int_a^b |k_1(s,x)|^2\ dx$

Using (12) in Equation (10) we get

$$\left|\int_a^b k_m(s,t) f(t)\, dt\right|^2 \le C_1^2\ D^2 (B^{m-1})^2\ ;\ m \ge 1$$

$$i.e. \left|\int_a^b k_m(s,t) f(t) dt\right| \leq C_1 \; D \, B^{m-1}; \; m \geq 1$$

$$\therefore \left|\lambda^m \int_a^b k_m(s,t) f(t) dt\right| \leq C_1 \; D \; |\lambda|^m \; B^{m-1}$$

$\therefore$ The series $\sum_{m=1}^{\infty} \lambda^m \int_a^b km(s,t) f(t) dt$ is dominated by the series $C_1 \; |\lambda| \; D \sum_{m=1}^{\infty} (|\lambda| \; B)^{m-1}$ which is geometric series and converges if $|\lambda| \; B < 1$. Hence, if $|\lambda| \; B < 1$ the Neumann series converges uniformly for all s; $a \leq s \leq b$.

To prove equation (1) has unique solution :

If possible g_1 (s) and g_2 (s) be two solutions of equation (1) then,

$$g_1(s) = f(s) + \lambda \int_a^b k(s,t) \; g_1(t) \; dt$$

$$g_2(s) = f(s) + \lambda \int_a^b k(s,t) \; g_2(t) \; dt$$

By substracting these equation and setting $g_1(s) \; - g_2(s) = \phi(s)$

We get, $\phi(s) = \lambda \int_a^b k(s,t) \; \phi(\mathrm{t}) \; dt$

Applying schwartz inequality

For any $\mathrm{S} \in [a,b] b$

$$|\phi(s)|^2 = |\lambda|^2 \; \left|\int_a^b \mathrm{k(s,t)} \phi t \; dt 2\right|^2$$

$$= |\lambda|^2 \, |\mathrm{k}(\mathrm{s},\phi)|^2$$

$$\leq |\lambda|^2 \, \|\mathrm{k}\|^2 \, \|\phi\|^2$$

$$|\phi(s)|^2 \;\leq\; |\lambda|^2 \int_a^b |\mathrm{k}(\mathrm{s,t})|^2 \, dt \int_a^b |\phi(\mathrm{t})|^2 dt$$

Integrating w. r. t. s. we get

$$\int_a^b |\phi(s)|^2 ds \;\leq\; |\lambda|^2 \int_a^b\int_a^b |k(s,t)|^2 \; ds\; dt \int_a^b |\phi(s)|^2 \, ds$$

$$= \; |\lambda|^2 \; \mathrm{B}^2 \int_a^b |\phi(s)|^2 ds$$

$$\therefore \; \left(1 - |\lambda|^2 B^2\right) \int_a^b \; |\phi(\mathrm{s})|^2 \, ds \;\leq\; 0$$

Since $|\lambda| B < 1 \;\Rightarrow\; 1 - |\lambda|^2 \; \mathrm{B}^2 \; > 0$

$$\Rightarrow \int_a^b |\phi(s)|^2 \, ds = 0 \quad \Rightarrow \phi(s) \equiv 0, \; \forall \; \mathrm{S} \; \in [\mathrm{a,b}]$$

$$\therefore \; \mathrm{g}_1(s) - g_2(s) \equiv 0 \quad \Rightarrow \; \mathrm{g}_1(s) \; \equiv \; \mathrm{g}_2(s)$$

Which proves the uniqueness of the solution.

Now changing the order of integration and summation in Neumann series (8) we get

$$g(s) = f(s) + \lambda \int_a^b \left[\sum_{m=1}^{\infty} \lambda^{m-1} k_m(s,t) \right] f(t) dt$$

Which we can be written as

$$g(s) = f(s) + \lambda \int_a^b \Gamma(s,t;\lambda) f(t) dt \qquad \text{-----(13)}$$

Where $\Gamma(s,t;\lambda) = \sum_{m=1}^{\infty} \lambda^{m-1} k_m(s,t)$ -----(14)

is the resolvent kernel which can be finded by determing iterated kernel k_m (s, t).

$\therefore$ Equation (13) is the solution of (1) which $\Gamma(s,t;\lambda)$ can be determined by using expression (14).

From above discussion we can infer that the series (14) is also convergent atleast for $|\lambda|$ B < 1. Hence, the resolvant kernel is analytic function of λ regular atleast inside the circle $|\lambda|$ < B^{-1}

Uniqueness of Γ $(s,t;\lambda)$:

From the uniqueness of the solution of (1) we can prove that the resolvent kernel Γ $(s,t;\lambda)$ is unique.

If possible for $\lambda = \lambda_0$ equation (1) have two resolvent kernels

$\Gamma_1(s,t;\lambda_0)$ & $\Gamma_2(s,t;\lambda_0)$

Since equation (1) has unique solution for an arbitrary function f(s) we get

$$f(s)+\lambda_0 \int_a^b \Gamma_1(s,t,\lambda_0) f(t)dt \equiv f(s)+\lambda_0 \int_a^b \Gamma_2(s,t;\lambda_0) f(t)dt$$

$$\therefore\ \lambda_0 \int_a^b \left[\Gamma_1(s,t;\lambda_0)-\Gamma_2(s,t;\lambda_0)\right] f(t)\ dt \equiv 0$$

Setting $\psi(s,t;\lambda_0)=\Gamma_1(s,t;\lambda_0)-\Gamma_2(s,t;\lambda_0)$

we get $\int_a^b \psi(s,t;\lambda_0) f(t)dt \equiv 0$

for an abitrary function f(t). Let us take $f(t)=\psi^*(s,t;\lambda_0)$ with fixed s.

This implies that

$$\int_a^b |\psi(s,t;\lambda_0)|^2\ dt \equiv 0$$

$$\Rightarrow \psi(s,t;\lambda_0) \equiv 0$$

$$\Rightarrow \Gamma_1(s,t;\lambda_0) \equiv \Gamma_2(s,t;\lambda_0)$$

The above analysis is summed up in the following theorem.

Theoram 1 : To each L_2 - kernel k(s,t), there corresponds a unique resolvent kernel $\Gamma(s,t;\lambda)$ which is an analytic function of λ regular, atleast inside the circle $|\lambda| < B^{-1}$, and represented by the power series.

$$\Gamma(s,t;\lambda)=\sum_{m=1}^{\infty} \lambda^{m-1}\, k_m(s,t)$$

Furthermore, if f(s) is also an L_2- function, then the unique L_2- solution of the fredholm integral equation.

$$g(s) = f(s) + \lambda \int_a^b k(s,t)\ g(t)dt$$

valid in the circle $|\lambda| < B^{-1}$ is given by the formula

$$g(s) = f(s) + \lambda \int_a^b \Gamma(s,t;\lambda) f(t)\ dt$$

4.2 Resolvent kernel :

The resolvent kernel $\Gamma(s,t;\lambda)$ of fredholm integral equation.

$g(s) = f(s) + \lambda \int_a^b k(s,t) g(t) dt$ is given by

$$\Gamma(s,t;\lambda) = \sum_{m=1}^{\infty} \lambda^{m-1} k_m(s,t)$$

Where $k_m(s,t)$ is m^{th} iterated kernel defined as $k_1(s,t) = k(s,t)$

and $k_m(s,t) = \int_a^b k(s,x) k_{m-1}(x,t) dx$

Theorem 2 : The resolvent kernel $\Gamma(s,t;\lambda)$ of a fredholm integral equation

$$g(s) = f(s) + \lambda \int_a^b k(s,t) g(t)\ dt$$

satisfied the integral equation

$$\Gamma(s,t;\lambda) = k(s,t) + \lambda \int_a^b \Gamma(s,t;\lambda)\ k(x,t)\ dx$$

Proof : The resolvent kernel is given by

$$\Gamma(s,t;\lambda) = \sum_{m=1}^{\infty} \lambda^{m-1} k_m(s,t) \qquad \text{-----(1)}$$

Where k_1 (s, t) = k(s, t)

and $k_m(s,t) = \int_a^b k_{m-1}(s,x)\ k(x,t)\ dx$

$\therefore$ Equation (1) becomes

$$\Gamma(s,t;\lambda) = \sum_{m=1}^{\infty} \lambda^{m-1} \int_a^b k_{m-1}(s,x)\ k(x,t)\ dx$$

$$= k_1(s,t) + \sum_{m=2}^{\infty} \lambda^{m-1} \int_a^b k_{m-1}(s,x)\ k(x,t)\ dx$$

$$= k\ (s,t) + \sum_{m=1}^{\infty} \lambda^{m} \int_a^b k_{m}(s,x)\ k(x,t)\ dx$$

$$= k\ (s,t) + \lambda \int_a^b \left[\sum_{m=1}^{\infty} \lambda^{m-1} k_m(s,x) \right] k(x,t)\ dx$$

$$\Gamma(s,t;\lambda) = k(s,t) + \lambda \int_a^b \Gamma(s,x;\lambda)\ k(x,t)\ dx$$

This completes the proof.

Some standard series useful in finding resolvent kernel.

1. $(1-x)^{-1} = 1 + x + x^2 + x^3 + \ldots = \sum_{n=0}^{\infty} x^n$

2. $(1+x)^{-1} = 1 - x + x^2 - x^3 + ... = \sum_{n=0}^{\infty} (-1)^n x^n$

3. $\sum_{n=1}^{\infty} x^n = \frac{x}{1-x}; \sum_{n=2}^{\infty} x^n = \frac{x^2}{1-x}$ and so on

4. $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$

5. $\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{3} + ...$

6. $-\log(1-x) = x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + ...$

7. $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + ...$

8. $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{5!} + ...$

9. $\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + ...$

10. $\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + ...$

Problem 1 : Find resolvent kernel associated with following kernels.

i) $k(s, t) = e^{s+t}$; in the intervel (0, 1)

ii) $k(s, t) = st$; in the intervel (0, 1)

iii) $k(s, t) = (1 + s)(1 - t)$; in the intervel (0, 1)

Solution : i) The interated kernel is

$k_1(s,t) = k(s,t)$

$$k_m(s,t) = \int_a^b k(s,x)\, k_{m-1}(x,t)\, dx; m \geq 2$$

Therefore $k_1(s,t) = e^{s+t}$

$$\therefore k_2(s,t) = \int_0^1 k(s,x)\, k_1(x,t)\, dx$$

$$= \int_0^1 e^{s+x}\; e^{x+t}\; dx$$

$$= e^{s+t}\int_0^1 e^{2x} dx \quad = e^{s+t}\left[\frac{e^{2x}}{2}\right]_0^1 \quad = e^{s+t}\left[\frac{e^2-1}{2}\right]$$

Now, $k_3(s,t) = \int_0^1 k(s,x)\, k_2(x,t)\, dx$

$$= \int_0^1 e^{s+x} e^{x+t}\left[\frac{e^2-1}{2}\right] dx$$

$$= e^{s+t}\left[\frac{e^2-1}{2}\right]\int_0^1 e^{2x} dx \quad = e^{s+t}\left[\frac{e^2-1}{2}\right]\left[\frac{e^{2x}}{2}\right]_0^1 \quad = e^{s+t}\left[\frac{e^2-1}{2}\right]^2$$

Similary $k_4(s,t) = e^{s+t}\left[\frac{e^2-1}{2}\right]^3$

In general, we have

$$k_m(s,t) = e^{s+t}\left[\frac{e^2-1}{2}\right]^{m-1} \qquad \text{-----(1)}$$

The resolvent kernel is given by

$$\Gamma(s,t;\lambda) = \sum_{m=1}^{\infty} \lambda^{m-1} k_m(s,t) \qquad \text{-----(2)}$$

Using (1) in (2) we get

$$\Gamma(s,t;\lambda) = e^{s+t}\sum_{m=1}^{\infty}\lambda^{m-1}\left[\frac{e^2-1}{2}\right]^{m-1}$$

$$= e^{s+t}\sum_{m=0}^{\infty}\lambda^{m}\left[\frac{e^2-1}{2}\right]^{m}$$

$$= e^{s+t}\frac{1}{1-\lambda\left(\frac{e^2-1}{2}\right)};\left|\lambda\left(\frac{e^2-1}{2}\right)\right|<1$$

$$\Gamma(s,t;\lambda) = \frac{2e^{s+t}}{2-\lambda(e^2-1)};|\lambda|<\frac{2}{e^2-1}$$

is the required resolvent kernel.

ii) The interated kernel is k_1 (s,t) = k(s,t) &

$$k_m(s,t) = \int_a^b k(s,x)\; k_{m-1}(x,t)\; dx$$

Thus, k_1 (s,t) = st

Now $k_2(s,t) = \int_0^1 k(s,x)\; k_1(x,t)\; dx$

$$= \int_0^1 (sx)\;(xt)\; dx$$

$$= st\int_0^1 x^2\; dx \qquad = st\left[\frac{x^3}{3}\right]_0^1 \qquad = st\left(\frac{1}{3}\right)$$

$$k_3(s,t) = \int_0^1 k(s,x)\; k_2(x,t)\; dx$$

$$= \int_0^1 (sx)\;(xt)\left(\frac{1}{3}\right)\; dx$$

$$= \frac{1}{3} st \int_0^1 x^2 dx \quad = \frac{1}{3} st \left[\frac{x^3}{3} \right]_0^1 \quad = \left(\frac{1}{3} \right)^2 st$$

Similarly $k_4(s,t) = \left(\frac{1}{3} \right)^3 st$

Ingenerse $k_m(s,t) = \left(\frac{1}{3} \right)^{m-1} (st)$

The resolvent kernel is given by

$$\Gamma(s,t;\lambda) = \sum_{m-1}^{\infty} \lambda^{m-1} k_m(s,t)$$

$$= \sum_{m-1}^{\infty} \lambda^{m-1} \left(\frac{1}{3} \right)^{m-1} st$$

$$= st \sum_{m=0}^{\infty} \left(\frac{\lambda}{3} \right)^m$$

$$= st \frac{1}{1 - \frac{\lambda}{3}}; \quad \left| \frac{\lambda}{3} \right| < 1$$

$$\Gamma(s,t;\lambda) = \frac{3st}{3-\lambda}; \quad |\lambda| < 3$$

is the required resolvent kernel.

iii) The interated kernel is given by k_1 (s,t) = k(s,t)

$$k_m(s,t) = \int_a^b k(s,x)\, k_{m-1}(x,t)\, dx$$

Thus $k_1(s,t) =$ (1+s)(1-t)

$$k_2(s,t) = \int_0^1 k(s,x) k_1(x,t) dx$$

$$= \int_0^1 (1+s)\ (1-x)\ (1+x)\ (1-t)\ dx$$

$$= (1+s)\ (1-t) \int_0^1 (1-x^2)\ dx$$

$$= (1+s)\ (1-t) \left[x - \frac{x^3}{3} \right]_0^1$$

$$= (1+s)(1-t)\left(1-\frac{1}{3}\right)$$

$$= \frac{2}{3}(1+s)(1-t)$$

Now $k_3(s,t) = \int_0^1 k(s,x)k_2(x,t)\,dx$

$$= \int_0^1 (1+s)(1-x)(1+x)(1-t)\left(\frac{2}{3}\right)dx$$

$$= \frac{2}{3}(1+s)(1-t)\int_0^1 \left(1-x^2\right)dx$$

$$= \left(\frac{2}{3}\right)^2 (1+s)\ (1-t)$$

Similarly $k_4(s,t) = \left(\frac{2}{3}\right)^3 (1+s)\ (1-t)$

Ingeneral $k_m(s,t) = \left(\frac{2}{3}\right)^{m-1} (1+s)\ (1-t)$

The resolvent kernel is

$$\Gamma(s,t;\lambda) = \sum_{m=0}^{\infty} \lambda^{m-1} k_m(s,t)$$

$$= \sum_{m=0}^{\infty} \lambda^{m-1} \left(\frac{2}{3}\right)^{m-1} (1+s)\ (1-t)$$

$$= (1+s)(1-t)\sum_{m=0}^{\infty}\left(\frac{2\lambda}{3}\right)^{m}$$

$$= (1+s)(1-t)\frac{1}{1-\frac{2\lambda}{3}};\left|\frac{2\lambda}{3}\right| < 1$$

$$\Gamma(s,t;\lambda) = \frac{3(1+s)(1-t)}{3-2\lambda};|\lambda| < \frac{3}{2}$$

is the required resolvent kernel.

4.3 Solution of Fredholm Integral Equation by Method of Resolvent Kernel :

Problem 1 : Solve the integral equation

$$g(s) = f(s) + \lambda\int_0^1 e^{s-t} g(t)\ dt$$

Solution : Here, $k(s,t) = e^{s-t}$

The interated kernel is given by $k_1(s,t) = k(s,t)$ and

$$k_m(s,t) = \int_a^b k(s,x)\ k_{m-1}(x,t)\ dx; m \geq 2$$

$\therefore\ k_1(s,t) = e^{s-t}$

Now, $k_2(s,t) = \int_0^1 k(s,x)\ k_1(x,t)\ dx$

$$= \int_0^1 e^{s-x} \ e^{x-t} \ dx$$

$$= e^{s-t} \int_0^1 \ dx = e^{s-t}$$

$$k_3(s,t) = \int_0^1 k(s,x) \ k_2(x,t) \ dx$$

$$= \int_0^1 e^{s-x} e^{x-t} \ dx \quad = e^{s-t} \int_0^1 dx \quad = e^{s-t}$$

Similarly $k_4(s,t) = e^{s-t}$

Ingeneral $k_m(s,t) = e^{s-t}$

The resolvent kernel is

$$\Gamma(s,t;\lambda) = \sum_{m=1}^{\infty} \lambda^{m-1} k_m(s,t)$$

$$= \sum_{m=1}^{\infty} \lambda^{m-1} e^{s-t}$$

$$= e^{s-t} \sum_{m=0}^{\infty} \lambda^m$$

$$= \frac{e^{s-t}}{1-\lambda}; |\lambda| < 1$$

$\therefore$ The required solution is

$$g(s) = f(s) + \lambda \int_0^1 \Gamma(s,t;\lambda) f(t) \ dt$$

$$= f(s) + \lambda \int_0^1 \frac{e^{s-t}}{1-\lambda} f(t)\, dt$$

$$= f(s) + \frac{\lambda}{1-\lambda} \int_0^1 e^{s-t} f(t)\, dt$$

Problem 2 : Solve the fredholm integral equation

$g(s) = 1 + \lambda \int_0^1 (1-3st)\, g(t)\, dt$ by determining resolvent kernel.

Solution : k(s, t) = 1 - 3st

The resolvent kernel is

$$k_1(s,t) = k(s,t)$$

$$k_m(s,t) = \int_0^1 k(s,x)\, k_{m-1}(x,t)\, dx; m \geq 2$$

$\therefore$ k_1(s, t) = 1 - 3st

$$k_2(s,t) = \int_0^1 k(s,x)\, k_1(x,t)\, dx$$

$$= \int_0^1 (1-3sx)\,(1-3xt)\, dx$$

$$= \int_0^1 \left[1 - 3xt - 3sx + 9st\, x^2\right] dx$$

$$= \int_0^1 \left[1 - 3(s+t)x + 9st\, x^2\right] dx$$

$$= \left[x - 3(s+t)\frac{x^2}{2} + 9st\, x^3\right]_0^1$$

$$=1-\frac{3}{2}(s+t)+9st$$

$$k_3(s,t)=\int_0^1 k(s,x)\,k_2(x,t)\,dx$$

$$=\int_0^1 (1-3sx)\left[1-\frac{3}{2}(x+t)+3xt\right]dx$$

$$=\int_0^1\left[1-\frac{3}{2}x-\frac{3t}{2}+3xt-3sx+\frac{9}{2}sx^2+\frac{9}{2}stx-9stx^2\right]dx$$

$$=\int_0^1\left[\left(1-\frac{3t}{2}\right)-3x\left(\frac{1}{2}-t+s-3st\right)+9sx^2\left(\frac{1}{2}-t\right)\right]dx$$

$$=\left[\left(1-\frac{3t}{2}\right)x-\frac{3}{2}x^2\left(\frac{1}{2}-t+s-3st\right)+3sx^3\left(\frac{1}{2}-t\right)\right]_0^1$$

$$=\frac{1}{4}(1-3st)$$

$$k_3(s,t)=\frac{1}{4}k_1(s,t)$$

$$k_4(s,t)=\int_0^1 k(s,x)\,k_3(x,t)\,dx$$

$$=\frac{1}{4}\int_0^1 (1-3sx)(1-3xt)\,dx$$

$$=\frac{1}{4}\left[1-\frac{3}{2}(s+t)+3st\right]$$

$$=\frac{1}{4}k_2(s,t)$$

Similarly $k_5(s,t)=\left(\frac{1}{4}\right)^2 k_1(s,t)$

$$k_6(s,t)=\left(\frac{1}{4}\right)^2 k_2(s,t)$$

$$k_7(s,t)=\left(\frac{1}{4}\right)^2 k_1(s,t)$$

$$k_8(s,t)=\left(\frac{1}{4}\right)^3 k_2(s,t) \quad \text{and so on}$$

The resolvent kernel is $\Gamma(s,t;\lambda)=\sum_{m=1}^{\infty}\lambda^{m-1}k_m(s,t)$

$$\therefore \Gamma(s,t;\lambda)=k_1(s,t)+\lambda k_2(s,t)+\lambda^2 k_3(s,t)+...$$

$$=k_1(s,t)+\lambda k_2(s,t)+\lambda^2\left(\frac{1}{4}\right)k_1(s,t)+\lambda^3\left(\frac{1}{4}\right)k_2(s,t)+\lambda^4\left(\frac{1}{4}\right)^2 k_1(s,t)$$

$$+\lambda^5\left(\frac{1}{4}\right)^2 k_2(s,t)+...$$

$$=k_1(s,t)\left[1+\left(\frac{1}{4}\right)\lambda^2+\left(\frac{1}{4}\right)^2\lambda^4+\left(\frac{1}{4}\right)^3\lambda^6+...\right]+\lambda k_2(s,t)\left[1+\frac{1}{4}\lambda^2+\left(\frac{1}{4}\right)^2\lambda^4+...\right]$$

$$=k_1(s,t)\left[1+\left(\frac{\lambda^2}{4}\right)+\left(\frac{\lambda^2}{4}\right)^2+...\right]+\lambda k_2(s,t)\left[1+\frac{\lambda^2}{4}+\left(\frac{\lambda^2}{4}\right)^2+...\right]$$

$$=\left[k_1(s,t)+\lambda k_2(s,t)\right]\left[1+\frac{\lambda^2}{4}+\left(\frac{\lambda^2}{4}\right)^2+...\right]$$

$$=\left[k_1(s,t)+\lambda k_2(s,t)\right]\frac{1}{1-\frac{\lambda^2}{4}};|\lambda|<2$$

$$\therefore \Gamma(s,t;\lambda)=\left[k_1(s,t)+\lambda k_2(s,t)\right]\frac{4}{4-\lambda^2};|\lambda|<2$$

$$=\frac{4}{4-\lambda^2}\left[(1-3st)+\lambda\left(1-\frac{3}{2}(s+t)+3st\right)\right]$$

The required solution is

$$g(s) = f(s) + \lambda \int_0^1 \Gamma(s,t;\lambda)\, f(t)\, dt$$

$$= 1 + \lambda \int_0^1 \frac{4}{4-\lambda^2}\left[1 - 3st + \lambda\left(1 - \frac{3}{2}(s+t) + 3st\right)\right] dt$$

$$= 1 + \frac{4\lambda}{4-\lambda^2}\int_0^1 \left[\left(1 + \lambda - \frac{3\lambda s}{2}\right) - 3t\left(s + \frac{\lambda}{2} - s\lambda\right)\right] dt$$

$$= 1 + \frac{4\lambda}{4-\lambda^2}\left[\left(1 + \lambda - \frac{3\lambda s}{2}\right)t - \frac{3t^2}{2}\left(s + \frac{\lambda}{2} - s\lambda\right)\right]_0^1$$

$$= 1 + \frac{4\lambda}{4-\lambda^2}\left[\left(1 + \lambda - \frac{3\lambda s}{2}\right) - \frac{3}{2}\left(s + \frac{\lambda}{2} - s\lambda\right)\right]$$

$$g(s) = \frac{4 + 2\lambda(2-3s)}{4-\lambda^2}; |\lambda| < 2$$

is the required solution.

Problem 3 : Solve the integral equation

$$g(s) = 1 + \lambda \int_0^{\pi} \sin(s+t)\, g(t)\, dt$$

Solution : The iterated kernel is

$$k_1(s,t) = k(s,t)$$

and $k_m(s,t) = \int_a^b k(s,x)\, k_{m-1}(x,t)\, dx,\ m \geq 2$

$$k_1(s,t) = \sin(s+t)$$

$$k_2(s,t)=\int_0^{\pi} k(s,x)\,k_1(x,t)\,dx$$

$$=\int_0^{\pi} \sin(s+x)\sin(x+t)\,dx$$

$$=\frac{1}{2}\int_0^{\pi}\left[\cos(s-t)-\cos(s+2x+t)\right]dx$$

$$\left[\because 2\sin A\sin B=\cos(A-B)-\cos(A+B)\right]$$

$$=\frac{1}{2}\left[x\ cos(s-t)-\frac{1}{2}\sin(s+2x+t)\right]_0^{\pi}$$

$$=\frac{1}{2}\left[\pi\ cos(s-t)-\frac{1}{2}\sin(2\pi+s+t)+\frac{1}{2}\sin(s+t)\right]$$

$$=\frac{1}{2}\left[\pi\ cos(s-t)-\frac{1}{2}\sin(s+t)+\frac{1}{2}\sin(s+t)\right]$$

$$=\frac{\pi}{2}\cos(s-t)$$

Now,

$$k_3(s,t)=\int_0^{\pi} k(s,x)k_2(x,t)\,dx$$

$$=\int_0^{\pi} \sin(s+x)\frac{\pi}{2}\cos(x-t)\,dx$$

$$=\frac{\pi}{4}\int_0^{\pi}\left[\sin(s+2x-t)+\sin(s+t)\right]dx$$

$$\left[\because 2\sin A\cos B=\sin(A+B)+\sin(A-B)\right]$$

$$=\frac{\pi}{4}\left[-\frac{1}{2}\cos(s+2x-t)+x\sin(s+t)\right]_0^{\pi}$$

$$=\frac{\pi}{4}\left[-\frac{1}{2}\cos(s-t)+\frac{1}{2}\cos(s-t)+\pi\sin(s+t)\right]$$

$$k_3(s,t)=\left(\frac{\pi}{2}\right)^2\sin(s+t)$$

Similarly, $k_4(s,t)=\left(\frac{\pi}{2}\right)^3\cos(s-t)$

$$k_5(s,t)=\left(\frac{\pi}{2}\right)^4\sin(s+t)$$

$$k_6(s,t)=\left(\frac{\pi}{2}\right)^5\cos(s-t)\ \textit{etc.}$$

The resolvent kernel is

$$\Gamma(s,t;\lambda)=\sum_{m=1}^{\infty}\lambda^{m-1}k_m(s,t)$$

$$\therefore\ \Gamma(s,t;\lambda)=k_1(s,t)+\lambda k_2(s,t)+\lambda^2k_3(s,t)+\lambda^3k_4(s,t)+\ldots$$

$$=\sin(s+t)+\lambda\left(\frac{\pi}{2}\right)\cos(s-t)+\lambda^2\left(\frac{\pi}{2}\right)^2\sin(s+t)+\lambda^3\left(\frac{\pi}{2}\right)^3\cos(s-t)+\lambda^4\left(\frac{\pi}{2}\right)^4\sin(s+t)+\ldots$$

$$=\sin(s+t)\left[1+\left(\frac{\pi\lambda}{2}\right)^2+\left(\frac{\pi\lambda}{2}\right)^4+\ldots\right]$$

$$+\frac{\pi\lambda}{2}\cos(s-t)\left[1+\left(\frac{\pi\lambda}{2}\right)^2+\left(\frac{\pi\lambda}{2}\right)^4+\ldots\right]$$

$$=\frac{1}{1-\left(\frac{\pi\lambda}{2}\right)^2}\left[\sin(s+t)+\frac{\pi\lambda}{2}\cos(s-t)\right]$$

$$=\frac{2}{4-\lambda^2\pi^2}\left[2\sin(s+t)+\pi\lambda\cos(s-t)\right]$$

The solution is given by

$$g(s)=f(s)+\lambda\int_0^{\pi}\Gamma(s,t;\lambda)g(t)\,dt$$

$$\therefore g(s)=1+\frac{2\lambda}{4-\lambda^2\pi^2}\int_0^{\pi}\left[2\sin(s+t)+\pi\lambda\cos(s-t)\right]dt$$

$$=1+\frac{2\lambda}{4-\lambda^2\pi^2}\left[-2\cos(s+t)-\pi\lambda\sin(s-t)\right]_0^{\pi}$$

$$g(s)=1+\frac{4\lambda}{4-\lambda^2\pi^2}\left[2\cos s+\pi\lambda\sin s\right]$$

is the required answer.

Problem 4 : Solve the fredholm integral equation.

$$g(s)=e^s-\frac{e}{2}+\frac{1}{2}+\frac{1}{2}\int_0^1 g(t)\,dt$$

by finding resolvent kernal

Solution : Here $k\ (s,t)=1$

The iterated kernal is

$k_1(s,t)=k(s,t)$ and

$$k_m(s,t)=\int_a^b k(s,x)\,k_{m-1}(x,t)\,dx;m\geq 2$$

$$k_1(s,t) = 1$$

$$k_2(s,t) = \int_0^1 k(s,x)\, k_1(x,t)\, dx$$

$$= \int_0^1 dx = 1$$

$$k_3(s,t) = \int_0^1 k(s,x)\, k_2(x,t)\, dx$$

$$= \int_0^1 dx = 1$$

Similarly $k_4(s,t) = 1$

Ingenerally $k_m(s,t) = 1$

The resolvent kernal is

$$\Gamma(s,t;\lambda) = \sum_{m=1}^{\infty} \lambda^{m-1} k_m(s,t)$$

$$= \sum_{m=0}^{\infty} \lambda^m \qquad (\because k_m(s,t) = 1)$$

$$= \sum_{m=0}^{\infty} \left(\frac{1}{2}\right)^m \qquad (\because \lambda = \frac{1}{2})$$

$$= \frac{1}{1 - \frac{1}{2}}$$

$$\therefore \Gamma(s,t;\lambda) = 2$$

The required solution is

$$g(s) = f(s) + \lambda \int_0^1 \Gamma(s,t;\lambda) f(t)\, dt$$

$$= e^s - \frac{e}{2} + \frac{1}{2} + \frac{1}{2}\int_0^1 2\left(e^t - \frac{e}{2} + \frac{1}{2}\right) dt$$

$$= e^s - \frac{e}{2} + \frac{1}{2} + \left(e^t - t\frac{e}{2} + t\frac{1}{2}\right)_0^1$$

$$= e^s - \frac{e}{2} + \frac{1}{2} + e - \frac{e}{2} + \frac{1}{2} - 1$$

$g(s) = e^s$ is the required solution.

4.4 Method of successive approximation :

1) If the sum of infinite series occurring in the formula of the resolvent kernal cannot be determined in such a case we use the method of successive approximations (Method of iteration)

2) The n^{th} approximation for the fredholm integral equation.

$$g(s) = f(s) + \lambda \int_a^b k(s,t)\, g(t)\, dt$$

is given by

$$g_n(s) = f(s) + \lambda \int_a^b k(s,t)\, g_{n-1}(t)\, dt; \quad n \geq 1.$$

in which zero order approximation is ingenerally we are taking as $g_0(s) = f(s)$

3) One can take zero order approximation $g_0(s)$ different from f(s) illustrated in problem 6.

Problem 1 : Solve the inhomogeneous fredholm integral equation of the second kind.

$$g(s) = 2s + \lambda \int_0^1 (s+t)\, g(t)\, dt, g_o(s) = 1$$

by method of succcessive apprximations to the third order.

Solution : Let g_0 (s) = 1 Then

The n^{th} order approximation is

$$g_n(s) = 2s + \lambda \int_0^1 (s,t)\, g_{n-1}(t)\, dt; \; n \geq 1.$$

Now $g_1(s) = 2s + \lambda \int_0^1 (s+t)\, g_0(t)\, dt$

$$= 2s + \lambda \int_0^1 (s+t)\, dt$$

$$= 2s + \lambda \left[st + \frac{t^2}{2} \right]_0^1$$

$$\therefore g_1(s) = 2s + \lambda \left(s + \frac{1}{2} \right)$$

$$g_2(s) = 2s + \lambda \int_0^1 (s+t)\, g_1(t)\, dt$$

$$= 2s + \lambda \int_0^1 (s+t) \left[2t + \lambda \left(t + \frac{1}{2} \right) \right] dt$$

$$= 2s + \lambda \int_0^1 \left[t^2(2+\lambda) + t\left(\frac{\lambda}{2} + 2s + \lambda \right) + \frac{s\lambda}{2} \right] dt$$

$$= 2s + \lambda \left[\frac{t^3}{3}(2+\lambda) + \frac{t^2}{2}\left(\frac{\lambda}{2} + 2s + \lambda \right) + \frac{ts\lambda}{2} \right]_0^1$$

$$= 2s + \lambda\left(s + \frac{2}{3} \right) + \lambda^2\left(s + \frac{7}{12} \right)$$

$$g_3(s) = 2s + \lambda \int_0^1 (s+t) g_2(t)\, dt$$

$$= 2s + \lambda \int_0^1 (s+t)\left[2s + \lambda\left(s + \frac{2}{3} \right) + \lambda^2\left(s + \frac{7}{12} \right) \right] dt$$

$$= 2s + \lambda \int_0^1 \left[t^2(2+\lambda+\lambda^2) + t\left(\frac{2\lambda}{3} + \frac{7\lambda^2}{12} + 2s + s\lambda + s\lambda^2 \right) + s\left(\frac{2\lambda}{3} + \frac{7\lambda^2}{12} \right) \right] dt$$

$$= 2s + \lambda\left[s + \frac{2}{3} \right] + \lambda^2\left[\left(\frac{7}{6} \right)s + \frac{2}{3} \right] + \lambda^3\left[\left(\frac{13}{12} \right)s + \frac{5}{8} \right]$$

Problem 2 : Solve the fredholm integral equation

$$g(s) = 1 + \lambda \int_0^1 (1 - 3st)\, g(t)\, dt$$

by method of successive approximation starting with g_0 (s) = 1

Solution : The zero - order approximation is given that g_0 (s) = 1

The n^{th} order approximation is given by

$$g_n(s) = f(s) + \lambda \int_a^b k(s,t) g_{n-1}(t)\, dt; \quad n \ge 1$$

$$\therefore g_1(s) = f(s) + \lambda \int_a^b k(s,t) g_0(t)\, dt$$

Here k(s,t) = 1 - 3st, $g_0(t) = 1$, & $f(s) = 1$

$$\therefore g_1(s) = 1 + \lambda \int_0^1 (1-3st)\, dt$$

$$= 1 + \lambda \left[t - 3s\frac{t^2}{2} \right]_0^1$$

$$= 1 + \lambda \left(1 - \frac{3}{2}s \right)$$

$$g_2(s) = 1 + \lambda \int_0^1 k(s,t)\, g_1(t)\, dt$$

$$= 1 + \lambda \int_0^1 (1-3st) \left[1 + \lambda \left(1 - \frac{3}{2}t \right) \right] dt$$

$$= 1 + \lambda \int_0^1 \left\{ (1-3st) + \lambda \left[1 - 3t\left(\frac{1}{2} + s \right) + \frac{9s}{2}t^2 \right] \right\} dt$$

$$= 1 + \lambda \left[\left(t - \frac{3}{2}st^2 \right) + \lambda \left(t - \frac{3t^2}{2} \right)\left(\frac{1}{2} + s \right) + \frac{3st^3}{2} \right]_0^1$$

$$= 1 + \lambda \left[\left(1 - \frac{3s}{2} \right) + \lambda \left(1 - \frac{3}{2}\left(\frac{1}{2} + s \right) + \frac{3s}{2} \right) \right]$$

$$g_2(s) = 1 + \lambda \left(1 - \frac{3s}{2} \right) + \frac{\lambda^2}{4}$$

The third order approximation is

$$g_2(s) = 1 + \lambda \int_0^1 (1-3st)\, g_2(t)\, dt$$

$$=1+\lambda\int_0^1 (1-3st)\left(1+\lambda\left(1-\frac{3t}{2}\right)+\frac{\lambda^2}{4}\right)dt$$

$$=1+\lambda\int_0^1\left[(1-3st)\left(1+\frac{\lambda^2}{4}\right)+\lambda\left(1-3t\left(\frac{1}{2}+s\right)\right)+\frac{9s}{2}t^2\right]dt$$

$$=1+\lambda\left[\left(t-\frac{3s}{2}t^2\right)\left(1+\frac{\lambda^2}{4}\right)+\lambda\left(1-\frac{3t^2}{2}\left(\frac{1}{2}+s\right)+\frac{3st^3}{2}\right)\right]_0^1$$

$$=1+\lambda\left(1-\frac{3s}{2}\right)\left(1+\frac{\lambda^2}{4}\right)+\frac{\lambda^2}{4}$$

$$=1+\lambda\left(1-\frac{3s}{2}\right)+\frac{\lambda^3}{4}\left(1-\frac{3s}{2}\right)+\frac{\lambda^2}{4}$$

$$=1+\lambda\left(1-\frac{3s}{2}\right)\left[1+\frac{\lambda^2}{4}\right]$$

Similarly $g_4(s)=1+\lambda\left(1+\frac{3s}{2}\right)\left[1+\frac{\lambda^2}{4}+\left(\frac{\lambda^2}{4}\right)^2\right]$

$$\therefore g(s)=\lim_{n\to\infty} g_n(s)$$

$$=\left[1+\lambda\left(1-\frac{3s}{2}\right)\right]\left[1+\left(\frac{\lambda^2}{4}\right)+\left(\frac{\lambda^2}{4}\right)^2+\left(\frac{\lambda^2}{4}\right)^3+\dots\right]$$

$$=\left[1+\lambda\left(1-\frac{3s}{2}\right)\right]\frac{1}{1-\frac{\lambda^2}{4}};\left|\frac{\lambda^2}{4}\right|<1$$

$$g(s) = \frac{4}{4-\lambda^2}\left[1+\lambda\left(1-\frac{3s}{2}\right)\right]; |\lambda| < 2$$

is the required solution.

Exercise :

1) Find the resolvent kernel associated with the following kernel.

(a) $e^{|x|+t}$ in the interval (-1, 1)

(b) $|s-t|$ in the interval (0, 1)

(c) $s^2 t^2$ in the interval (-1, 1)

(d) $\exp\left[-|s-t|\right]$ in the interval (0, 1)

(e) $st + s^2 t^2$ in the interval (-1, 1)

(f) cos (s + t) in the interval (0, 2π)

(g) $(s - t)^2$ in the interval (-1, 1)

2) Solve the fredholm integral equation by finding resolvent kernel.

(a) $g(s) = \sin s - \frac{s}{4} + \frac{1}{4}\int_0^{\frac{\pi}{2}} st\ g(t)\ dt$

(b) $g(s) = \frac{5s}{6} + \frac{1}{2}\int_0^1 st\ g(t)\ dt$

(c) $g(s) = s + \int_0^{\frac{1}{2}} g(t)\ dt$

(d) $g(s)=\frac{5s}{6}-\frac{1}{9}+\frac{1}{3}\int_0^1 (s+t)\, g(t)\, dt$

3) Solve the following fredholm integral equation by method successive approximation.

(a) $g(s)=e^x+\frac{1}{c}\int_0^1 u(t)\, dt; \quad g_0(s)=0$

(b) $g(s)=s+\lambda\int_0^1 s\, g(t)\, dt; \quad g_0(s)=0$

(c) $g(s)=1+\int_0^1 s\, g(t)\, dt$

(d) $g(s)=\frac{11}{12}s+\frac{1}{4}\int_0^1 st\, g(t)\, dt$

(e) $g(s)=\sin s+\int_0^\pi \sin s \cos t\, g(t)\, dt$

(f) $g(s)=\frac{5}{4}+\sin s-\frac{1}{4}\int_0^\pi s\, g(t)\, dt$

(g) $g(s)=\frac{6}{7}s^3+\frac{5}{7}\int_0^1 s^3\, t\, g(t)\, dt$

4.4 Volterra integral equation :

Applying the same procedure in fredholm integral equation to the volterra integral equation of the second kind.

$$g(s)=f(s)+\lambda\int_a^s k\,(s,t)\, g(t)\, dt$$

we get its solution as

$$g(s) = f(s) + \lambda\int_a^s \Gamma\,(s,t;\,\lambda)\; f(t)\; dt \qquad \text{-----(1)}$$

where $\Gamma(s,t;\lambda) = \sum_{m=1}^{\infty} \lambda^{m-1} k_m(s,t)$

Which is called resolvent kernel, in which the interated kernal $k_m(s,t)$ is given by

$k_1(s,t) = k(s,t)$

and

$$k_m(s,t) = \int_t^s k(s,x)\; k_{m-1}(x,t)\; dx; \quad m \geq 1$$

we see this in detail in unit V

Note : Expression (1) also can be written as

$$g(s) = f(s) + \sum_{m=1}^{\infty} \lambda^m \int_a^s k_m(s,t)\; f(t)\; dt$$

which is called Neumann series.

♦♦♦

Unit – 5

VOLTERRA INTEGRAL EQUATION

5.1 Volterra Integral Equation Definition and some Properties

The linear integral equation of the form

$$h(s)g(s) = f(s) + \lambda \int_a^s k(s,t)g(t)\,dt \qquad \text{-----(1)}$$

Where the upper limit 's' is the variable is called volterra integral equation.

Special types of volterra integral equation :

i) Put h(s) = 0 in equation (1) we get

$$0 = f(s) + \lambda \int_a^s k(s,t)g(t)\,dt$$

it is called volterra integral equation of first kind.

ii) Putting h(s) = 1 in equation (1) we get

$$g(s) = f(s) + \lambda \int_a^s k(s,t)g(t)\,dt \ldots\ldots\ldots\ldots(2)$$

it is called volterra integral equation of second kind.

a) Putting f (s) = 0 in equation (2) we get

$$g(s) = \lambda \int_a^s k(s,t)g(t)\,dt$$

it is called homogeneous volterra integral equation of second kind.

b) If $f(s) \neq 0$ in equation (2) it is called nonhomogeneous volterra integral equation of second kind.

Property 1 : If k(s, t) is continuous then the only possible continuous solution to the homogeneous volterra integral equation of the second kind is trivial zero solution.

Solution : Consider the homogeneous volterra integral equation of the second kind.

$$g(s) = \lambda \int_a^s k(s,t)\, g(t)\, dt \qquad \text{-----(1)}$$

We know the fact that, if k is continuous and ψ is bounded, then

$$\int_a^b k(s,t)\psi(t)\, dt$$

is continuous.

Thus g(s) defined by equation (1) is continuous if it is bounded.

Let $|k(s,t)| \leq M$

Let b be a number such that,

$(b-a)|\lambda| M < 1$. Let $a \leq s \leq b$

Then, $(s-a)|\lambda| M < 1$

Now from (1)

$$|g(s)| = |\lambda| \int_a^s k(s,t)\, g(t)\, dt$$

$$\leq |\lambda| \int_a^s |k(s,t)| |g(t)|\, dt$$

$$\leq |\lambda||g(s)|M\int_a^s dt$$

$$=(s-a)|\lambda|M|g(s)|$$

$$<|g(s)|$$

Thus we have proved

$$|g(s)|<|g(s)|$$

Thus is only possible if g(s) is identically zero in $a \leq s \leq b.$

Properties 2 : The volterra integral equation of first kind $f(s)+\lambda\int_a^s k(s,t)\, g(t)\, dt = 0$ is consistent if f (a) = 0

Property 2 : Any solution to the volterra integral equation of second kind.

$$g(s) = f(s)+\lambda\int_a^s k(s,t)\, g(t)\, dt$$

cannot be correct unless g(o) = f(a).

Property 3 :

If k (s, s) is non zero then the volterra integral equation of the first kind can be reduced to a volterra integral equation of second kind.

Proof :

Consider the volterra integral equation of first kind

$f(s)+\lambda\int_0^s k(s,t)\, g(t)\, dt = 0$, where the kernal satisfy $k(s,s) \neq 0, \ \forall$ S

Differentiating w.r.t.s., we obtain

$$f'(s) + \lambda \int_0^s \frac{\partial k(s,t)}{\partial s} g(t)\, dt + k(s,s)\, g(s) = 0 \qquad \text{-----(1)}$$

Since $k(s,s) \neq 0$ it is possible to divide through by it.

$$\therefore \frac{f'(s)}{k(s,s)} + \lambda \int_0^s \left[\frac{\partial k(s,t)}{\partial s} \Big/ k(s,s) \right] g(t)\, dt + g(s) = 0$$

Which can be written as

$$g(s) = \phi(s) + \lambda \int_0^s k^*(s,t)\, g(t)\, dt \qquad \text{-----(2)}$$

Where $\phi(s) = \dfrac{-f'(s)}{k(s,s)}$

and $k^*(s,t) = \dfrac{-\partial k(s,t)}{\partial s} \Big/ k(s,s)$

Equation (2) is volterra integral of second kind.

Note that $g(0) = \phi(0)$

It gives the correct solution ($\because$ Property 2)

If k (s, s) = 0 then (1) becomes

$$f'(s) + \lambda \int_0^s \frac{\partial k(s,t)}{\partial s} g(t)\, dt = 0$$

which can be written as

$$\phi(s) + \lambda \int_0^s k^*(s,t)\, g(t)\, dt = 0$$

where $\phi(s) = f'(s)$

and $k^*(s,t) = \dfrac{\partial k(s,t)}{\partial s}$

which is volterra integral equation of first kind.

Property 4 : If k(x, y) is continuous on [o, a] x [o, a] then prove that, there are no eigen values and eigen function associated with the homogeneous integral equation

$$\phi(x) = \lambda \int_0^x k(x,y)\,\phi(y)\,dy \qquad \text{-----(1)}$$

Solution :

Let k(x, y) be continuous on [o, a] x [o, a] and hence bounded.

Let $|k(x,y)| \le M; \quad \forall\, x, y$

$\therefore$ From equation (1)

$$|\phi(x)| \le |\lambda| \int_0^x |k(x,y)||\phi(y)|\,dy \le |\lambda|\, M \int_0^x |\phi(y)|\,dy$$

$$\therefore |\phi(x)| \le |\lambda| MP \qquad \text{-----(2)}$$

Where $p = \int_0^x |\phi(y)|\,dy$

From (1) and (2)

$$|\phi(x)| \le |\lambda| \int_0^x |k(x,y)|\,|\phi(y)|\,dy$$

$$\therefore |\phi(x)| \le |\lambda| M \int_0^x |\lambda|\, MP\,dy$$

$$= |\lambda|^2 M^2 P [y]_0^x$$

$$\therefore |\phi(x)| \le |\lambda|^2 M^2 Px \qquad \text{-----(3)}$$

From (1) and (3)

$$|\phi(x)| \le |\lambda| \int_0^x |k(x,y)| \, |\phi(y)| \, dy$$

$$\le |\lambda| M \int_0^x |\lambda|^2 M^2 py \, dy$$

$$= |\lambda|^3 M^3 P \left[\frac{y^2}{2} \right]_0^x$$

$$\therefore |\phi(x)| \le |\lambda|^3 M^3 P \frac{x^2}{2!}$$

Continuing in this way we get

$$|\phi(x)| \le |\lambda|^n M^n P \frac{x^{n-1}}{(n-1)!} \qquad \text{-----(4)}$$

$$\le \frac{|\lambda|^n M^n P}{(n-1)!} a^{n-1}$$

Now $P|\lambda| M \sum_{n=1}^{\infty} \frac{(|\lambda| Ma)^{n-1}}{(n-1)!} < \infty$

$$\Rightarrow \frac{(|\lambda| Ma)^{n-1}}{(n-1)!} \to 0 \text{ as } n \to \infty$$

Letting $n \to \infty$ in equation (4) we have $|\phi(x)| \to 0 \Rightarrow \phi(x) = 0$

$\therefore$ The homogeneous equation have trivial solution and hence the equation has no eigen value and eigen function.

5.2 Solution of Volterra integral equation by method of differentiation :

Problem 1 : Solve the integral equation

$$x^2 = \int_0^x \sin a(x-y)\, \phi(y)\, dy; a \neq 0$$

Solution : Given integral equation is

$$x^2 = \int_0^x \sin a(x-y)\, \phi(y)\, dy \qquad \text{-----(1)}$$

Differentiating w. r. t. *x* we get

$$2x = a\int_0^x \cos a(x-y)\, \phi(y)\, dy + \sin a(x-x)\, \phi(x)$$

$$= a\int_0^x \cos a(x-y)\, \phi(y)\, dy$$

Differentiating again w.r.t.x, we get

$$2 = a^2\int_0^x -\sin a(x-y)\, \phi(y)\, dy + a\cos a(x-x)\, \phi(x)$$

$$\therefore 2 = -a^2\int_0^x \sin a(x-y)\, \phi(y)\, dy + a\, \phi(x)$$

$$\therefore 2 = -a^2x^2 + a\, \phi(x) \qquad (\because \textit{equation } (1))$$

$$\therefore \phi(x) = \frac{2+a^2x^2}{a}$$

is the required solution.

Problem 2 : Solve the integral equation

$$\phi(x) = (x+1) + \int_0^x \left[1+2(x-y)\right] \phi(y)\, dy$$

and verify your solution

Solution : Given integral equation is

$$\phi(x) = (x+1) + \int_0^x [1+2(x-y)]\,\phi(y)\,dy$$

Differentiating w. r. t. *x* we get

$$\phi'(x) = 1 + 2\int_0^x \phi(y)\,dy + [1+2(x-x)]\,\phi(x)$$

$$\therefore \phi'(x) = 1 + 2\int_0^x \phi(y)\,dy + \phi(x)$$

Differentiating w. r. t. *x*, again

$$\phi''(x) = 2\,\phi(x) + \phi'(x)$$

$$\therefore \phi''(x) - \phi'(x) - 2\,\phi(x) = 0 \qquad \text{-----(1)}$$

Also, $\phi(0) = 1,$

$\phi'(0) = 2$

The auxilliarly equation for (1) is

$$D^2 - D - 2 = 0$$

$$(D+1)\,(D-2) = 0$$

$$\Rightarrow D = -1,\, 2$$

$\therefore$ The solution is

$$\phi(x) = C_1\,e^{-x} + C_2\,e^{2x}$$

$$\Rightarrow \phi'(x) = -\,C_1\,e^{-x} + 2C_2\,e^{2x}$$

Now, $1 = \phi(0) = C_1 + C_2$ -----(2)

and $2 = \phi'(0) = -C_1 + 2C_2$ -----(3)

Adding (2) and (3), we get

$3C_2 = 3 \Rightarrow C_2 = 1$

$\therefore C_1 = 0$

Thus required solution is

$\phi(x) = e^{2x}$

Verification :

Consider

$$\int_0^x [1 + 2(x-y)]\,\phi(y)\,dy$$

$$= \int_0^x [1 + 2(x-y)]\,e^{2y}\,dy$$

$$= \int_0^x e^{2y}\,dy + 2x\int_0^y e^{2y}dy - 2\int_0^x ye^{2y}dy$$

$$= (1+2x)\int_0^x e^{2y}\,dy - \int_0^x y\,e^{2y}dy$$

$$= (1+2x)\left[\frac{e^{2y}}{2}\right]_0^x - 2\left[y\frac{e^{2y}}{2} - \frac{e^{2y}}{4}\right]_0^x$$

$$= (1+2x)\left(\frac{e^{2x}-1}{2}\right) - xe^{2x} + \left(\frac{e^{2x}-1}{2}\right)$$

$$= \frac{e^{2x}}{2} - \frac{1}{2} + x\,e^{2x} - x - x\,e^{2x} + \frac{e^{2x}}{2} - \frac{1}{2}$$

$= e^{2x} - x - 1$

$= \phi(x) - (x+1)$

$$\therefore \phi(x) = (x+1) + \int_0^x [1 + 2(x-y)] \phi(y)\, dy$$

$\Rightarrow \phi(x) = e^{2x}$ satisfies the given volterra integral equation.

Exercise :

1. Solve the integral equation $\phi(x) = 3\int_0^x \cos(x-y)\, \phi(y)\, dy + e^x$ by method of differentiation.

2. Solve the following integral equation by method of differentiation.

(a) $\phi(x) = 3 + \int_0^x (5x - 3t)\, \phi(t)\, dt$

(b) $\phi(x) = -x - \int_0^x (x-t)\, \phi(t)\, dt$

(c) $\phi(x) = 1 + \int_0^x \phi(t)\, dt$

(d) $\phi(x) = \cos x - x - \int_0^x (x-t)\, \phi(t)\, dt$

(e) $\phi(x) = \cos x - \int_0^x (x-t)\, \phi(t)\, dt$

(f) $\phi(x) = 1 - x - 4\sin x + \int_0^x [3 - 2(x-t)]\, \phi(t)\, dt$

(g) $\phi(x) = (6x-5) + \int_0^x (5 - 6x + 6t)\, \phi(t)\, dt$

5.3 Successive approximation for Volterra Integral Equations :

Consider volterra integral equation of the second kind.

$$g(s) = f(s) + \lambda \int_a^s k(s,t)\, g(t)\, dt \qquad \text{-----(1)}$$

Let $g_0(s) = f(s)$ be the zero-order approximation to the desired solution g(s) substituting this in equation (1) to get the first order approximation.

$$g_1(s) = f(s) + \lambda \int_a^s k(s,t)\, g_0(t)\, dt \qquad \text{-----(2)}$$

This function when substituded into (2) yields second approximation.

$$g_2(s) = f(s) + \lambda \int_a^s k(s,t)\, g_1(t)\, dt$$

Further, if $g_n(s)$ and $g_{n-1}(s)$ are n^{th} and $(n-1)^{th}$ order approximation respectively then,

$$g_n(s) = f(s) + \lambda \int_a^s k(s,t)\, g_{n-1}(t)\, dt \qquad \text{-----(3)}$$

If $g_n(s)$ tends uniformly to a limit as $n \to \infty$, then this limit is the required solution. To find this limit we proceed in detail.

The first and second order approximations are

$$g_1(s) = f(s) + \lambda \int_a^s k(s,t)\, f(t)\, dt \qquad \text{-----(4)}$$

and $g_2(s) = f(s) + \lambda \int_a^s k(s,t)\, g_1(t)\, dt$

$$= f(s) + \lambda \int_a^s k(s,t)\, g_1(x)\, dx \qquad \text{-----(5)}$$

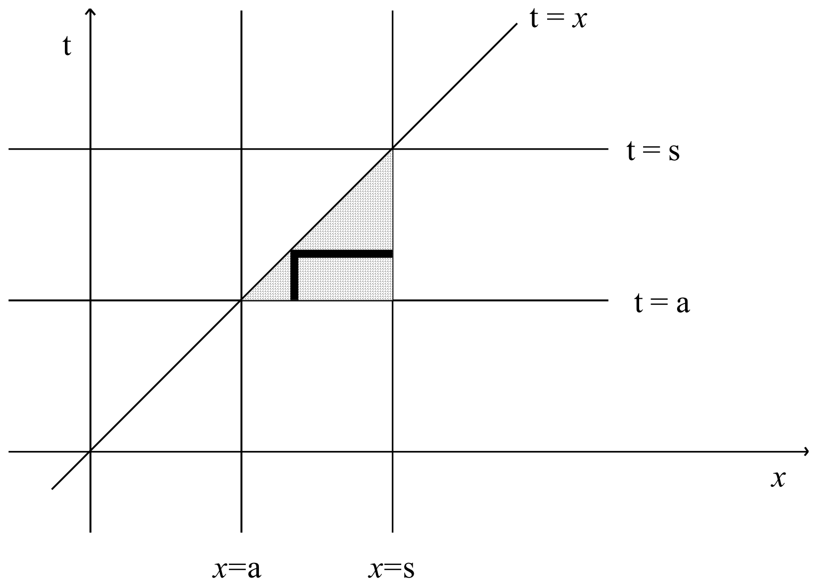
Using (4) in (5) we get

$$g_2(s) = f(s) + \lambda \int_a^s k(s,x) \left[f(x) + \lambda \int_a^x k(x,t)\, f(t)\, dt \right] dx$$

$$= f(s) + \lambda \int_a^s k(s,x)\, f(x)\, dx$$

$$+\lambda^2 \int_a^s k(s,x) \left[\int_a^x k(x,t)\, f(t)\, dt \right] dx \qquad \text{-----(6)}$$

Changing the order of integration. Here, given strip is parallel to t-axis therefore to change the order of integration consider the strip parallel to x - axis as shown in following figure.



The equation (6) becomes

$$g_2(s) = f(s) + \lambda \int_a^s k(s,x)\, f(x)\, dx$$

$$+\lambda^2 \int_a^s \left[\int_t^s k(s,x)\, k(x,t)\, dx \right] f(t)\, dt$$

$$= f(s) + \lambda \int_a^s k(s,t) f(t)\, dt + \lambda^2 \int_a^s k_2(s,t) f(t)\, dt$$

Where k_1 (s, t) = k (s, t)

$$k_2(s,t) = \int_t^s k(s,x)\, k_1(x,t)\, dx$$

$$g_2(s) = f(s) + \sum_{m=1}^{2} \lambda^m \int_a^s k_m(s,t)\, f(t)\, dt$$

Similarly $g_3(s) = f(s) + \sum_{m=1}^{3} \lambda^m \int_a^s k_m(s,t)\, f(t)\, dt$

Where $k_3(s,t) = \int_t^s k(s,x)\, k_2(x,t)\, dx$

By continuing this process we get

$$g_n(s) = f(s) + \sum_{m=1}^{n} \lambda^m \int_a^s k_m(s,t)\, f(t)\, dt \qquad \text{-----(7)}$$

Where, $k_m(s,t) = \int_t^s k(s,x)\, k_{m-1}(x,t)\, dx \,;\; m \geq 2$

We call the expression k_m(s, t) the m^{th} iterated kernel (or m^{th} iterate)

where k_1 (s, t) = k (s, t)

Letting $n \to \infty$ the expression (7) become

$$g(s) = \lim_{n \to \infty} g_n(s)$$

$$= f(s) + \sum_{m=1}^{\infty} \lambda^m \int_a^s k_m(s,t)\, f(t)\, dt \qquad \text{-----(8)}$$

Provided the series on R.H.S. of (8) is converges uniformly.

Expression (8) is called Neumann series which can be written as,

$$g(s) = f(s) + \int_a^s \left[\sum_{m=1}^{\infty} \lambda^m k_m(s,t) \right] f(t)\, dt$$

$$= f(s) + \lambda \int_a^s \left[\sum_{m=1}^{\infty} \lambda^{m-1} k_m(s,t) \right] f(t)\, dt$$

$$g(s) = f(s) + \lambda \int_a^s \Gamma(s,t;\lambda)\, f(t)\, dt \qquad \text{-----(9)}$$

is the required solution of equation (1)

Where $\Gamma(s,t;\lambda) = \sum_{m=1}^{\infty} \lambda^{m-1}\, k_m(s,t)$

which called resolvent kernel.

Now solution (9) exists if $\sum_{m=1}^{\infty} \lambda^m\, k_m(s,t)$ converges uniformly.

Aim : $\sum_{m=1}^{\infty} \lambda^m\, k_m(s,t)$ converges uniformly.

Suppose $0 \le s,t \le a$, and $|k(s,t)| \le M$

Then

$$|k_2(s,t)| = \left| \int_t^s k(s,x)\, k_1(x,t)\, dx \right|$$

$$\le \int_t^s |k(s,x)|\,|k(x,t)|\,dx$$

$$\le M^2 \int_t^s dx$$

$$\therefore |k_2(s,t)| \le M^2 (s-t) \; if \quad s \ge t$$

also $k_2(s,t) = 0 \quad if \quad s \le t$

Similarly

$$|k_3(s,t)| = \left| \int_t^s k(s,x)\, k_2(x,t)\, dx \right|$$

$$\le \int_t^s |k(s,x)|\,|k_2(x,t)|\,dx$$

$$\le M^3 \int_t^s (x-t)\, dx$$

$$= M^3 \left[\frac{(x-t)^2}{2} \right]_t^s$$

$$= M^3 \frac{(s-t)^2}{2!} \; if \quad s \ge t$$

and $k_3(s,t) = 0 \quad if \quad s \le t$

Therefore ingeneral

$$|k_m(s,t)| \le M^m \frac{(s-t)^{m-1}}{(m-1)!} \quad if \quad s \ge t$$

and $k_m(s,t) = 0 \quad if \quad s \le t$

$$\therefore \left|\lambda^m k_m(s,t)\right| \le |\lambda|^m M^m \frac{|s-t|^{m-1}}{(m-1)!}$$

Now $|s-t| \le 2a$

$$\therefore \left|\lambda^m k_m(s,t)\right| \le |\lambda|^m M^m \frac{(2a)^{m-1}}{(m-1)!}$$

$$i.e. \left|\lambda^m k_m(s,t)\right| \le |\lambda| M \frac{(2a|\lambda|M)^{m-1}}{(m-1)!}$$

$\Rightarrow \sum_{m=1}^{\infty} \lambda^m k_m(s,t)$ is dominated by the series

$|\lambda| M \sum_{m=1}^{\infty} \frac{(2a|\lambda|M)^{m-1}}{(m-1)!}$ which is convergent

$\therefore \sum_{m=1}^{\infty} \lambda^m k_m(s,t)$ converges uniformly.

Uniqueness :

If g_A and g_B are two solutions of equation (1) then

$$g_A(s) = f(s) + \lambda \int_a^s k(s,t)\, g_A(t)\, dt$$

and $g_B(s) = f(s) + \lambda \int_a^s k(s,t)\, g_B(t)\, dt$

$$\therefore g_A(s) - g_B(s) = \lambda \int_a^s k(s,t)\left[g_A(t) - g_B(t)\right] dt$$

$\Rightarrow g_A(s) - g_B(s)$ is the solution of the homogeneous volterra integral equation. But we know that if k(s,t) is continous then the homogeneous integral equation has only trivial solution.

$\therefore g_A(s) - g_B(s) \equiv 0$

$\Rightarrow g_A(s) \equiv g_B(s)$

5.4 Resolvent Kernel of Volterra Integral Equation :

The resolvent kernel $\Gamma(s;t;\lambda)$ of volterra integral equation

$g(s) = f(s) + \lambda \int_a^s k(s,t) g(t) dt$ is given by

$\Gamma(s,t;\lambda) = \sum_{m=1}^{\infty} \lambda^{m-1} km(s,t)$ where $km(s,t)$ is mth iterated kernel given by

$k_1(s,t) = k(s,t)$ and $k_m(s,t) = \int_t^s k_{m-1}(s,x) k(x,t) dx; m \geq 2$

Theoram 1 : The resolvent kernel $\Gamma(s,t;\lambda)$ of the volterra integral equation

$$g(s) = f(s) + \lambda \int_a^s k(s,t) g(t) dt$$

satisfies the integral equation

$$\Gamma(s,t;\lambda) = k(s,t) + \lambda \int_t^s (s,x;\lambda) k(x,t) dx$$

Proof : The resolvent kernel is

$$\Gamma(s,t;\lambda) = \sum_{m=1}^{\infty} \lambda^{m-1} k_m(s,t) \qquad \text{-----(1)}$$

Where the iterated kernel k_m (s,t) is given by

$$k_1(s,t) = k(s,t)$$

$$k_m(s,t) = \int_t^s k_{m-1}(s,x) k(x,t) dx; m \geq 2$$

$\therefore$ Equation (1) becomes

$$\Gamma(s,t;\lambda)=\sum_{m=1}^{\infty}\lambda^{m-1}\int_t^s k_{m-1}(s,x)k(x,t)\,dx$$

$$=k_1(s,t)+\sum_{m=2}^{\infty}\lambda^{m-1}\int_t^s k_{m-1}(s,x)k(x,t)\,dx$$

$$=k(s,t)+\sum_{m=1}^{\infty}\lambda^{m}\int_t^s k_{m}(s,x)k(x,t)\,dx$$

Changing order of integration

$$=k(s,t)+\lambda\int_t^s\left[\sum_{m=1}^{\infty}\lambda^{m-1}k_m(s,x)\right]k(x,t)\,dx$$

$$\Gamma(s,t;\lambda)=k(s,t)+\lambda\int_t^s\Gamma(s,x;\lambda)k(x,t)\,dx \qquad (\because (1))$$

Problem 1 : Find the resolvent kernel of the volterra integral equation with the kernel.

$i) k(s,t)=1$

$ii) k(s,t)=e^{s-t}$

$iii) k(s,t)=\dfrac{1+s^2}{1+t^2}$

Solution : $i) k(s,t)=1$

The iterated kernel is

$$k_1(s,t)=k(s,t)$$

$$k_m(s,t)=\int_t^s k(s,x)k_{m-1}(x,t)\,dx\ ;\ m\geq 2$$

Now,

$$\therefore k_1(s,t) = 1$$

$$k_2(s,t) = \int_t^s k(s,x) k_1(x,t)\, dx$$

$$= \int_t^s dx = (s-t)$$

$$k_3(s,t) = \int_t^s k(s,x)\; k_2(x,t)\; dt$$

$$= \int_t^s (x-t)\; dt$$

$$= \left[\frac{(x-t)^2}{2} \right]_t^s = \frac{(s-t)^2}{2}$$

$$k_4(s,t) = \int_t^s k(s,x)\; k_3(x,t)\; dx$$

$$= \int_t^s \frac{(x-t)^2}{2} dx = \left[\frac{(x-t)^3}{3.2} \right]_t^s = \frac{(s-t)^3}{3!}$$

Similarly $k_4(s,t) = \dfrac{(s-t)^3}{3!}$

Ingenerally $k_m(s,t) = \dfrac{(s-t)^{m-1}}{(m-1)!}$

The resolvent kernel is

$$\Gamma(s,t;\lambda) = \sum_{m=1}^{\infty} \lambda^{m-1} k_m(s,t)$$

$$= \sum_{m=1}^{\infty} \lambda^{m-1} \frac{(s-t)^{m-1}}{(m-1)!}$$

$$= \sum_{m=0}^{\infty} \frac{\left[\lambda(s-t)\right]^{m}}{m!}$$

$$= e^{\lambda(s-t)}$$

ii) $k(s,t) = e^{s-t}$

The iterated kernel is defined by

$$k_1(s,t) = k(s,t)$$

and $k_m(s,t) = \int_t^s k(s,x)\, k_{m-1}(x,t)\, dx\,; \quad m \geq 2$

$$\therefore k_1(s,t) = e^{s-t}$$

$$k_2(s,t) = \int_t^s k(s,x)\, k_1(x,t)\, dx$$

$$= \int_t^s e^{s-x} e^{x-t}\, dx$$

$$= e^{s-t} \int_t^s dx$$

$$= e^{s-t}(s-t)$$

$$k_3(s,t) = \int_t^s k(s,x)\, k_2(x,t)\, dx$$

$$= \int_t^s e^{s-x}\, e^{x-t} (x-t)\, dx$$

$$= e^{s-t}\int_t^s (x-t)\,dx$$

$$= e^{s-t}\left[\frac{(x-t)^2}{2}\right]_t^s$$

$$= e^{s-t}\,\frac{(x-t)^2}{2}$$

$$k_4(s,t) = \int_t^s k(s,x)\,k_3(x,t)\,dx$$

$$= \int_t^s e^{s-x}\,e^{x-t}\,\frac{(x-t)^2}{2}\,dx$$

$$= e^{s-t}\int_t^s \frac{(x-t)^2}{2}\,dx$$

$$= e^{s-t}\,\frac{(s-t)^3}{3!}$$

Similarly $k_5(s,t) = e^{s-t}\,\dfrac{(s-t)^4}{4!}$

Ingeneral , $k_m(s,t) = e^{s-t}\,\dfrac{(s-t)^{m-1}}{(m-1)!}$

$\therefore$ Resolvent kernel is

$$\Gamma(s,t;\lambda) = \sum_{m=1}^{\infty} \lambda^{m-1} k_m(s,t)$$

$$= \sum_{m=1}^{\infty} \lambda^{m-1}\,\frac{(s-t)^{m-1}}{(m-1)!}\,e^{(s-t)}$$

$$= e^{s-t} \sum_{m=0}^{\infty} \frac{[\lambda(s-t)]^m}{m!}$$

$$= e^{s-t} e^{\lambda(s-t)} = e^{(1+\lambda)(s-t)}$$

iii) $k(s,t) = \dfrac{1+s^2}{1+t^2}$

The iterated kernel is given by

$$k_1(s,t) = k(s,t)$$

$$k_m(s,t) = \int_t^s k(s,x)\, k_{m-1}(x,t)\, dx\,; \quad m \ge 2$$

$$\therefore k_1(s,t) = \frac{1+s^2}{1+t^2}$$

$$k_2(s,t) = \int_t^s k(s,x)\, k_1(x,t)\, dx$$

$$= \int_t^s \frac{1+s^2}{1+x^2}\, \frac{1+x^2}{1+t^2}\, dx$$

$$= \frac{1+s^2}{1+t^2} \int_t^s dx$$

$$= \frac{1+s^2}{1+t^2}(s-t)$$

$$k_3(s,t) = \int_t^s k(s,x)\, k_2(x,t)\, dx$$

$$= \int_t^s \frac{1+s^2}{1+x^2}\, \frac{1+x^2}{1+t^2}(x-t)\, dx$$

$$= \frac{1+s^2}{1+t^2} \int_t^s (x-t)\, dx$$

$$= \frac{1+s^2}{1+t^2} \frac{(s-t)^2}{2}$$

$$k_4(s,t) = \int_t^s k(s,x)\, k_3(x,t)\, dx$$

$$= \int_t^s \frac{1+s^2}{1+x^2} \frac{1+x^2}{1+t^2} \frac{(x-t)^2}{2}\, dx$$

$$= \frac{1+s^2}{1+t^2} \int_t^s \frac{(x-t)^2}{2}\, dx = \frac{1+s^2}{1+t^2} \frac{(s-t)^3}{3!}$$

Similarly $k_5(s,t) = \dfrac{1+s^2}{1+t^2} \dfrac{(s-t)^4}{4!}$

Ingeneral $k_m(s,t) = \dfrac{1+s^2}{1+t^2} \dfrac{(s-t)^{m-1}}{(m-1)!}$

$\therefore$ The resolvent kernel $\Gamma(s,t;\lambda)$ becomes

$$\Gamma(s,t;\lambda) = \sum_{m=1}^{\infty} \lambda^{m-1} k_m(s,t)$$

$$= \sum_{m=1}^{\infty} \lambda^{m-1} \frac{1+s^2}{1+t^2} \frac{(s-t)^{m-1}}{(m-1)!}$$

$$= \frac{1+s^2}{1+t^2} \sum_{m=0}^{\infty} \frac{\left[\lambda(s-t)\right]^m}{m!}$$

$$= \frac{1+s^2}{1+t^2} e^{\lambda(s-t)}$$

5.5 Solution of Volterra Integral Equation by Method of Resolvent Kernel :

Problem 1 : Solve the integral equation

$$g(s) = f(s) + \lambda \int_0^s e^{s-t} g(t)\, dt$$

Solution : $k(s,t) = e^{s-t}$

The resolvent kernel for $k(s,t) = e^{s-t}$ is

$\Gamma(s,t;\lambda) = e^{(1+\lambda)(s-t)}$ (see : Problem 1 (ii))

$\therefore$ The solution is given by

$$g(s) = f(s) + \lambda \int_0^s \Gamma(s,t;\lambda)\, f(t)\, dt$$

$$= f(s) + \lambda \int_0^s e^{(1+\lambda)(s-t)}\, f(t)\, dt$$

Problem 2 : Solve $g(s) = 1 + s^2 + \int_0^s \frac{1+s^2}{1+t^2} g(t)\, dt$

Solution : $k(s,t) = \frac{1+s^2}{1+t^2}$

$\therefore$ The resolvent kernel is

$\Gamma(s,t;\lambda) = \frac{1+s^2}{1+t^2} e^{s-t}$ (see : Problem 1 (ii))

The required solution is

$$g(s) = f(s) + \lambda \int_0^s \Gamma(s,t;\lambda)\, f(t)\, dt$$

Here $f(s) = 1 + s^2,\ \lambda = 1$

$$\therefore g(s) = 1 + s^2 + \int_0^s \frac{1+s^2}{1+t^2} e^{s-t} (1+t^2)\, dt$$

$$= (1 + s^2) + (1 + s^2)\int_0^s e^{s-t}\, dt$$

$$= (1+s^2) + e^s (1+s^2)\left[-e^{-t}\right]_0^s$$

$$= (1+s^2) + e^s (1+s^2)\left[1-e^{-s}\right]$$

$g(s) = e^s (1+s^2)$ is the required solution.

Problem 3 : Find the Neumann series for the solution of the integral equation

$$g(s) = (1+s) + \lambda \int_0^s (s-t)\, g(t)\, dt$$

Hence find the solution when $\lambda = 1$

Solution : Here k(s,t) = s- t

The iterated kernel is given by $k_1(s, t) = k(s, t)$

and $k_m(s,t) = \int_t^s k(s,x)\, k_{m-1}(x,t)\, dx$

$$\therefore k_1(s,t) = s - t$$

$$k_2(s,t) = \int_t^s k(s,x)\, k_1(x,t)\, dx$$

$$\therefore k_2(s,t) = \int_t^s (s-x)\,(x-t)\, dx$$

$$= \left[(s-x)\frac{(x-t)^2}{2} - (-1)\frac{(x-t)^3}{3!} \right]_t^s$$

$$= \frac{(s-t)^3}{3!}$$

$$k_3(s,t) = \int_t^s k(s,x)\, k_2(x,t)\, dt$$

$$= \int_t^s (s-x)\, \frac{(x-t)^3}{3!}\, dt$$

$$= \left[(s-x)\frac{(x-t)^4}{4!} - (-1)\frac{(x-t)^5}{5!} \right]_t^s$$

$$= \frac{(s-t)^5}{5!}$$

Similarly $k_4(s,t) = \frac{(s-t)^7}{7!}$

Ingeneral $k_m(s,t) = \frac{(s-t)^{2m-1}}{(2m-1)!}; m \geq 1$

$\therefore$ The reolvent kernel is given by

$$\Gamma(s,t;\lambda) = \sum_{m=1}^{\infty} \lambda^{m-1} k_m(s,t)$$

$$= \sum_{m=1}^{\infty} \lambda^{m-1} \frac{(s-t)^{2m-1}}{(2m-1)!}$$

The solution of integral equation is

$$g(s) = f(s) + \lambda \int_0^s \Gamma(s,t;\lambda)\, f(t)\, dt$$

Here f(s) = 1 + s

$$\therefore g(s) = 1 + s + \lambda \int_0^s \sum_{m=1}^{\infty} \lambda^{m-1} \frac{(s-t)^{2m-1}}{(2m-1)!} (1+t)\, dt$$

Changing the order of summation and integration.

$$\therefore g(s) = 1 + s + \sum_{m=1}^{\infty} \frac{\lambda^m}{(2m-1)!} \int_0^s (s-t)^{2m-1} (1+t)\, dt$$

$$= (1+s) + \sum_{m=1}^{\infty} \frac{\lambda^m}{(2m-1)!} \left[(1+t) \frac{(s-t)^{2m}}{2m} - \frac{(s-t)^{2m+1}}{(2m)(2m+1)} \right]_0^s$$

$$= (1+s) + \sum_{m=1}^{\infty} \frac{\lambda^m}{(2m-1)!} \left[\frac{s^{2m}}{2m} + \frac{s^{2m+1}}{(2m)(2m+1)} \right]$$

$$g(s) = (1+s) + \sum_{m=1}^{\infty} \lambda^m \left[\frac{s^{2m}}{(2m)!} + \frac{s^{2m+1}}{(2m+1)!} \right]$$

is the required solution.

Now for $\lambda = 1$

$$g(s) = (1+s) + \sum_{m=1}^{\infty} \left[\frac{s^{2m}}{(2m)!} + \frac{s^{2m+1}}{(2m+1)!} \right]$$

$$= 1 + s + \frac{s^2}{2!} + \frac{s^3}{3!} + \frac{s^4}{4!} + \frac{s^5}{5!} + \dots$$

$$= \sum_{m=0}^{\infty} \frac{s^m}{m!}$$

$g(s) = e^s$ is the required solution.

5.6 Solution of Volterra Integral Equation by Method of iteration :

1) If the sum of infinite series occuring in the formula of the resolvent kernel cannot be determined in such a case we use the method of successive approximations.

(Method of iteration)

2) The n^{th} approximation for the fredhlom integral equation.

$$g(s) = f(s) + \lambda \int_a^s k(s,t)\, g(t)\, dt$$

is given by $g_n(s) = f(s) + \lambda \int_a^s k(s,t)\, g_{n-1}(t)\, dt;\ \ n \geq 1$

in which zero order approximation is ingenerally we are taking as $g_0(s) = f(s)$

3) One can take zero order approximation $g_0(s)$ different from f(s).

Problem 1 : Solve the integral equation by the method of successive approximation.

$$g(s) = 1 - \int_0^s (s-t)\, g(t)\, dt;\ \ g_0(s) = 0$$

Solution : The n^{th} order approximation is given by

$$g_n(s) = f(s) + \lambda \int_a^s k(s,t)\, g_{n-1}(t)\, dt;\ \ n \geq 1$$

$\therefore$ Taking $g_0(s) = 0$

Then $g_1(s) = 1 - \int_0^s (s-t)\, g_0(t)\, dt$

$$= 1\ (\because g_0(s) = 0)$$

$$g_2(s) = 1 - \int_0^s (s-t)\, g_1(t)\, dt$$

$$=1-\int_0^s (s-t)\,dt$$

$$=1-\left[\frac{(s-t)^2}{-2}\right]_0^s$$

$$=1-\frac{s^2}{2!}$$

$$g_3(s)=1-\int_0^s (s-t)\,g_2(t)\,dt$$

$$=1-\int_0^s (s-t)\left(1-\frac{t^2}{2}\right)dt$$

$$=1-s\int_0^s \left(1-\frac{t^2}{2}\right)dt+\int_0^s\left(t-\frac{t^3}{2}\right)dt$$

$$=1-s\left[t-\frac{t^3}{6}\right]_0^s+\left[\frac{t^2}{2}-\frac{t^4}{8}\right]_0^s$$

$$=1-s\left(s-\frac{s^3}{6}\right)+\frac{s^2}{2}-\frac{s^4}{8}$$

$$=1-s^2+\frac{s^4}{6}+\frac{s^2}{2}-\frac{s^4}{8}$$

$$=1-\frac{s^4}{2}+\frac{s^4}{24}$$

$$g_3(s)=1-\frac{s^2}{2!}+\frac{s^4}{4!}$$

Similarly $g_4(s) = 1 - \frac{s^2}{2!} + \frac{s^4}{4!} - \frac{s^6}{6!}$

Ingeneral $g_n(s) = 1 - \frac{s^2}{2!} + \frac{s^4}{4!} - \frac{s^6}{6!} + \ldots (-1)^{n+1} \frac{x^{2(n-1)}}{(2n-2)!}$

$\therefore$ The required solution is

$$g(s) = \lim_{n \to \infty} g_n(s)$$

$$= 1 - \frac{s^2}{2!} + \frac{s^4}{4!} - \frac{s^6}{6!} + \ldots$$

g(s) = cos s

Problem 2 : Find an approximate solution of

$$g(s) = s - \int_0^s (s-t)\, g(t)\, dt;\ g_0(s) = 0$$

Solution : The n[th] order approximation is

$$g_n(s) = f(s) + \lambda \int_0^s k(s,t)\, g_{n-1}(t)\, dt;\ \ n \geq 1$$

Here $f(s) = s,\ \lambda = -1,\ k(s,t) = (s-t)$

$$\therefore g_n(s) = s - \int_0^s (s-t)\, g_{n-1}(t)\, dt;\ \ n \geq 1$$

$$\therefore g_1(s) = s - \int_0^s (s-t)\, g_0(t)\, dt$$

$$= s \quad (\because g_0(s) = 0)$$

Now $g_2(s) = s - \int_0^s (s-t)\, g_1(t)\, dt$

$$= s - \int_0^s (s-t)\, t\, dt$$

$$= s - \left[s\frac{t^2}{2} - \frac{t^3}{3} \right]_0^s$$

$$= s - \frac{s^3}{2} + \frac{s^3}{3}$$

$$= s - \frac{s^3}{6}$$

$$= s - \frac{s^3}{3!}$$

$$g_3(s) = s - \int_0^s (s-t)\, g_2(t)\, dt$$

$$= s - \int_0^s (s-t) \left(t - \frac{t^3}{3!} \right) dt$$

$$= s - s\left[\frac{t^2}{2} - \frac{t^4}{24} \right]_0^s + \left[\frac{t^3}{3} - \frac{t^5}{30} \right]_0^s$$

$$= s - \frac{s^3}{2} + \frac{s^5}{24} + \frac{s^3}{3} - \frac{s^5}{30}$$

$$= s - \frac{s^3}{6} + \frac{s^5}{120}$$

$$= s - \frac{s^3}{3!} + \frac{s^5}{5!}$$

Similarly $g_4(s) = s - \frac{s^3}{3!} + \frac{s^5}{5!} - \frac{s^7}{7!}$

Ingeneral $g_n(s) = s - \frac{s^3}{3!} + \frac{s^5}{5!} - + ... + (-1)^{n-1}\frac{s^{2n-1}}{(2n-1)!}$

$$g(s) = \lim_{n \to \infty} g_n(s)$$

$$= s - \frac{s^3}{3!} + \frac{s^5}{5!} - \frac{s^7}{7!} + ...$$

g(s) = sin s is the required solution.

Problem 3 :

Solve $g(s) = 1 + s + \int_0^s (s-t)\, g(t)\, dt$

$g_0(s) = 1$ by method of successive approximations.

Solution : The n^th order approximation is given by

$$g_n(s) = f(s) + \lambda \int_0^s k(s,t)\, g_{n-1}(t)\, dt; \quad n \geq 1$$

Here $f(s) = 1 + s, \lambda = 1, k(s,t) = s - t$

$$\therefore g_n(s) = 1 + s + \int_0^s (s-t)\, g_{n-1}(t)\, dt$$

Now $g_1(s) = 1 + s + \int_0^s (s-t)\, g_0(t)\, dt$

Given $g_0(s) = 1$

$$\therefore g_1(s) = 1 + s + \int_0^s (s-t)\, dt$$

$$= 1 + s + \left[st - \frac{t^2}{2} \right]_0^s$$

$$= 1 + s + s^2 - \frac{s^2}{2}$$

$$= 1 + s + \frac{s^2}{2}$$

Now $g_2(s) = 1 + s + \int_0^s (s - t)\ g_1(t)\ dt$

$$= 1 + s + \int_0^s (s - t) \left(1 + t + \frac{t^2}{2} \right) dt$$

$$= 1 + s + s\left[t + \frac{t^2}{2} + \frac{t^3}{6} \right]_0^s$$

$$- \left[\frac{t^2}{2} + \frac{t^3}{3} + \frac{t^4}{8} \right]_0^s$$

$$= 1 + s + s^2 + \frac{s^3}{2} + \frac{s^4}{6} - \frac{s^2}{2} - \frac{s^3}{3} - \frac{s^4}{8}$$

$$= 1 + s + \frac{s^2}{2} + \frac{s^3}{6} + \frac{s^4}{24}$$

$$g_2(s) = 1 + s + \frac{s^2}{2!} + \frac{s^3}{3!} + \frac{s^4}{4!}$$

Similarly,

$$g_3(s) = 1 + s + \frac{s^2}{2!} + \frac{s^3}{3!} + \frac{s^4}{4!} + \frac{s^5}{5!} + \frac{s^6}{6!}$$

Ingeneral $g_n(s) = 1 + s + \frac{s^2}{2!} + ... + \frac{s^n}{n!} = \sum_{m=0}^{n} \frac{s^m}{m!}$

$$g(s) = \lim_{n \to \infty} g_n(s)$$

$$= \sum_{m=0}^{\infty} \frac{s^m}{m!}$$

$$= e^s$$

is the required solution.

5.5 Exercise :

1. Find the resolvent kernel of the volterra integral equation with the following kernel.

(a) $k(s,t) = e^{s^2 - t^2}$

b) $k(s,t) = \frac{\cosh s}{\cosh t}$

c) $k(s,t) = \frac{2 + \cos s}{2 + \cos t}$

d) k(s,t) = t-s

e) $k(s,t) = e^{s-t};\ a > 0$

2. Solve the following volterra integral by finding resolvent kernel (or Neumann series solution)

a) $g(s) = 1 + \int_0^s st\, \phi(t)\, dt$

b) $g(s) = 1 + s - \int_0^s \phi(t)\, dt$

c) $g(s) = \frac{1}{1+s^2} + \int_0^s \sin(s-t)\, g(t)\, dt$

d) $g(s)=1+\int_0^s (s-t)\ g(t)\ dt$

e) $g(s)=s3^s-\int_0^s 3^{s-t}\ g(t)\ dt$

f) $g(s)=e^{s^2+2s}+2\int_0^s e^{s^2-t^2}\ g(t)\ dt$

g) $g(s)=e^s+\int_0^s e^{s-t}\ g(t)\ dt$

h) $g(s)=e^{-s}+\int_0^s e^{-(s-t)}\ \sin(s-t)g(t)\ dt$

i) $g(s)=se^{\frac{s^2}{2}}+\int_0^s e^{-(s-t)}\ g(t)\ dt$

3. Solve the following volterra integral equation by method of successive approximation (Method of iteration).

a) $g(s)=1+\int_0^s g(t)\ dt\ ;\ g_0(s)=0$

b) $g(s)=1+\int_0^s (s-t)g(t)\ dt\ ;\ g_0(s)=0$

c) $g(s)=\frac{s^2}{2}+s-\int_0^s g(t)\ dt$

i) $g_0(s)=\frac{s^2}{2}+s$

ii) $g_0(s)=1$

iii) $g_0(s)=s$

d) $g(s) = s + 1 - \int_0^s g(t)\,dt;\ \ g_0(s) = s + 1$

e) $g(s) = \dfrac{s^3}{2} - 2s - \int_0^s g(t)\,dt;\ \ g_0(t) = s^2$

f) $g(s) = 2s + 2 - \int_0^s g(t)\,dt$

i) $g_0(s) = 1$

ii) $g_0(s) = 2$

g) $g(s) = 2s^2 + 2 - \int_0^s sg(t)\,dt$

i) $g_0(s) = 2$

ii) $g_0(s) = 2s$

♦♦♦

Unit – 6

SYMMETRIC KERNELS

6.1 Preliminaries :

1. Normed Linear Space :

A normed linear space is a linear space N in which to each vector x there corrosponds a real number denoted by $\|x\|$ called norm of x such that

i) $\|x\| > 0$ and $\|x\| = 0 \Leftrightarrow x = 0$

ii) $\|x + y\| \leq \|x\| + \|y\|$

iii) $\|\alpha x\| = |\alpha| \|x\|; \alpha$ is scalar

(*i.e.* If there is a function $\| \ \| : N \to R^+$ satisfying (i), (ii) and (iii) then N is called normed linear space).

2. Normed linear space : is a metric space with respect to the metric d defined by

$d(x, y) = \|x - y\|$

3. Complete metric space :

A complete metric space is a metric space in which every cauchy sequence is convergent.

4. Banch space:

A banach space is a complete normed linear space.

5. Hilbert space :

A Hilbert space H is complax banach space in which there is defined a function

$(\ ) : H \times H \to$ C with the following properties.

$i)\overline{(x,y)} = (y,x)$

$ii)(\alpha x, +By, z) = \alpha(x,z) + \beta(y,z)$

$iii)(x,x) = \|x\|^2$

for any vecters x, y, z, ε H and α, β be a scaler.

6. Square integrable function : A function ϕ (t) is called square integrable if

$$\int_a^b |\phi(t)|^2 \, dt < \infty$$

7. L2 - function : A square integrable function ϕ (t) is called an L_2 function.

8. L2 - kernel : The kernel K (s, t) is called an L_2 kernel (L_2 function) if.

a) For each set of values of s, t in the square $a \le s \le b, a \le t \le b$

$$\int_a^b \int_a^b |k(s,t)|^2 \, ds dt < \infty$$

b) For each value of s in $a \le s \le b$

$$\int_a^b |k(s,t)|^2 \, dt < \infty$$

c) For each value of t in $a \le t \le b$

$$\int_a^b |k(s,t)|^2 \, ds < \infty$$

9. The inner produce (scalar product) of two

L_2 - function : The inner or scalar product (ϕ, ψ) of two complex L_2 functions ϕ *and* ψ of real variable is $a \le s \le b$ is defined as $(\phi, \psi) = \int_a^b \phi(t) \psi^*(t) dt$

Where * denotes the complex conjugete.

10. Orthogonal functions :

Two L_2 functions are called orthogonal if their inner product is zero.

i.e. Two L_2 - functions ϕ *and* ψ of real variable t; $a \le t \le b$ are orthogonal if.

$$(\phi,\psi) = 0 \quad \int_a^b \phi(t)\psi^*(t)\,dt = 0$$

11. Norm of L_2 - function :

The norm of L_2 - function ϕ (t) is given by the relation.

$$\|\phi\| = \left[\int_a^b \phi(t)\phi^*(t)\,dt\right]^{\frac{1}{2}} \quad = \left[\int_a^b \left|\phi(t)^2\right| dt\right]^{\frac{1}{2}}$$

12. Normalized function :

A function ϕ is called normalized if $\|\phi\| = 1$

Note : A function ϕ (t) with $\|\phi\| \ne 0$ (i.e. a nonnull function) can always be normalised by dividing it by its norm.

13. Orthogonal set : (System of orthogonal functions) A finite or an infinite set $\{\phi_k\}$ is said to be an orthogonal set if

$$(\phi_{i,}\phi_j) = 0; \quad i \ne j$$

14. Orthonormal set :

A set $\{\phi_k\}$ is orthornormal if

$$(\phi_{i,}\phi_j) = \begin{cases} 0; & i \ne j \\ 1; & i = j \end{cases}$$

15. Compex Hilbert space L_2 (a, b) or L_2 - Space :

Let H be the set of complex valued function ϕ (t) defined on an interval (a, b) such that-

$$\int_a^b |\phi(t)|^2 < \infty$$

i) Define inner product by $(\phi, \psi) = \int_a^b \phi(t)\psi^*(t)\,dt$

ii) Define norm $\|\phi\|$ by

$$\|\phi\| = \left[\int |\phi(t)|^2\,dt\right]^{\frac{1}{2}}$$

Then the function d defined by

$$d(\phi, \psi) = \|\phi - \psi\|$$

generates the metric on H

with information (i) and (ii)

H is Hilbert space denoted by L_2 (a, b) are L_2-space.

15. Following both in equality hold sin L_2- space

i) Scharz inequality : $|(\phi, \psi)| \le \|\phi\|\,\|\psi\|$

ii) Minkowski inequality : $\|\phi + \psi\| \le \|\phi\| + \|\psi\|$

16. Bessels inequality :

If f(s) is real continuous and square integrable function and $\phi_i(s); i = 1, 2, 3, ...$ is real and continuous consisting normalized orthogonal set then

$$\sum_{i=1}^{\infty} |(f\phi_i)|^2 \le \|f\|^2$$

6.2 Symmetric Kernel and Properties :

A kernel k (s,t) is symmetric or complex symmetric or Hermitian if

$k(s,s) \neq 0, \ \forall s,t \in [a,b]$

Where the asterisk * denote the complex conjugate.

Note : i) A real kernel is symmetric if k(s,t) = k (t,s), $\forall$ s,t $\in [a,b]$

ii) In the place of asterisk (*) we may use bare (-) to denote the complex conjugate.

Iterated kernel :

The iterated kernel k_n (s, t) are defined as follows.

$k_1(s,t) = k(t,s)$

and $k_{n+1}(s,t) = \int k(s,x) k_n(x,t) dx \quad ; n \geq 1$

or $k_{n+1}(s,t) = \int k_n(s,x) k(x,t) dx \quad ; n \geq 1$

Theoram 1: If a kernel is symmetric then all its iterated kernels are also symmetric.

Proof : The iterated kernel k_n (s,t) is defined as $k_1(s,t) = k(s,t)$ and

$k_{n+1}(s,t) = \int k(s,x) k_n(x,t) dx \ ; n \geq 1$

Given k(s,t) is symmetric $\Rightarrow k(s,t) = k^*(t,s)$

Now $k_1(s,t) = k(s,t) = k^*(t,s) = k_1^*(t,s)$

Now $k_2(s,t) = \int k(s,x) k_1(x,t) dx$ -----(1)

$$= \int k(s,x) k(x,t) dx$$

$$= \int k^*(t,x) k^*(x,s) dx$$

$$= \left[\int k(t,x) k(x,s) \, dx \right]^*$$

$$=\left[\int k(t,x)k,(x,s)\,dx\right]^*$$

$$=k_2^*(t,s)$$

Which shows the result is true for n = 1, 2.

Assume that result is true for n = m

i.e. Let k_m (s, t) is symmetric

$$\Rightarrow k_m(s,t)=k_m^*(t,s)$$

Now we prove result for n = m + 1

$$k_{m+1}(s,t)=\int k(s,x)k_m(x,t)dx$$

$$=\int k_m^*(t,x)k^*(x,s)dx$$

$$=\left[\int k_m(t,x)k(x,s)\,dx\right]^*$$

$$=k^*{}_{m+1}(t,s)$$

$\Rightarrow$ Result is true for n = m + 1

$\therefore$ By method of induction, the kernel is symmetric.

Fredholm operator :

The fredholm operator k is

$$\mathrm{K}\phi=\int k(s,t)\phi(t)\,dt$$

and the operator K* defined by

$\mathrm{K}^*\psi=\int k^*(t,s)\psi(t)\,dt$ is the adjoint operator of k.

Problem 1 : Let the operator K and K* are defined by

$$K\phi=\int k(s,t)\phi(t)\,dt$$

and $K^*\psi = \int k^*(t,s)\psi(t)\,dt$

where ϕ *and* ψ are L_2 function, then prove that :

a) K* is an adjoint operator of k i.e. $(K\phi,\psi) = (\phi, K^*\psi)$

b) If K is symmetric kernel then

i) K is self adjoint operator

ii) Inner product $(K\phi,\phi)$ is always real.

Solution : Given

$$K\phi(s) = \int k(s,t)\phi(t)\,dt$$

$$K^*\psi(s) = \int K^*(t,s)\psi(t)\,dt$$

a) $(K\phi,\psi) = \int K\phi(t)\psi^*(t)\,dt$

$$= \int\left[\int k(t,y)\phi(y)\,dy\right]\psi^*(t)\,dt$$

$$= \int \phi(y)\left[\int k(t,y)\psi^*(t)\,dt\right]dy$$

$$= \int \phi(y)\left[\int k^*(t,y)\psi\ \ (t)\,dt\right]^* dy$$

$$= \int \phi(y)\left[k^*\psi(y)\right]^* dy$$

$\therefore (K\phi,\psi) = (\phi, K^*\psi).$

$\Rightarrow$ K^* is an adjoint operator of K

b) (i) Let k (s, t) be symmetric kernel then

$$k(s,t) = k^*(t,s)$$

For all s we have

$$\left(K^*\psi\right)(s)=\int K^*(t,s)\psi(t)\,dt$$

$$=\int k(s,t)\psi(t)\,dt$$

$$=K\psi(s)$$

$$K^*\psi=K\psi$$

$\Rightarrow$ $K^* = K$

$\Rightarrow$ K is self adjoint operator.

Aliter :

From equation (1)

$$\left(K\phi,\psi\right)=\int\phi(y)\left[\int k^*(t,y)\psi(t)\,dt\right]^* dy$$

since k is symmetric we have

$$k^*(t,y)=k(y,t)$$

$$\left(K\phi,\psi\right)=\int\phi(y)\left[\int K(y,t)\psi(t)\,dt\right]^* dy$$

$$=\int\phi(y)\left[K\psi(y)\right]^* dy$$

$$\left(K\phi,\psi\right)=\left(\phi,K\psi\right)$$

$\Rightarrow$ K is self adjoint operator

ii) $\left(K\phi,\phi\right)^* = \left[\int K\phi(t)\phi^*(t)\,dt\right]^*$

$$=\int\left[K\phi(t)\right]^*\phi(t)\,dt$$

$$=\int\left[\int k(t,y)\phi(y)\,dy\right]^*\phi(t)\,dt$$

$$=\int\left[\int k^*(t,y)\phi^*(y)\,dy\right]\phi(t)\,dt$$

$$=\int\left[\int k(y,t)\phi^*(y)\,dy\right]\phi(t)\,dt$$

($\because$ k is symmetric)

$$=\int\left[\int k(y,t)\phi(t)\,dt\right]\phi^*(y)\,dy$$

($\because$ Changing order of integration)

$$=\int \mathrm{K}\phi(y)\phi^*(y)\,dy$$

$$=(\mathrm{K}\phi,\phi)$$

$\Rightarrow (\mathrm{K}\phi,\phi)$ is real

Exercise :

1) If $(k\ \phi,\phi)$ is real for all ϕ then prove that k is symmetric fredholm operator.

Orthogonal Set : A finite or an infinite set $\{\phi_k\}$ of function's is said to be an orthogonal set if

$$(\phi_i,\phi_j)=0;\quad i\neq j$$

If none of the elements of this set is a zero then it is said to be a proper orthogonal set.

6.3 Orthonormal set :

A set of functions is said to be an orthonormal set if,

$$(\phi_i,\phi_j)=\begin{cases}0\ ;\ i\neq j\\ 1\ ;\ i=j\end{cases}$$

Normalized function : A function ϕ for which $\|\phi\|=1$ is called normalized function.

Gram-Schmidt procedure to construct an orthonormal set :

Given a finite or infinite denumerable independent set of functions

$\{\psi_1, \psi_2, \psi_3, ..., \psi_k, ...\}$

We can construct an orthonormal set $\{\phi_1, \phi_2, ...\phi_k ...\}$ by Gram-Schmidt procedure as follows.

Let $\phi_1 = \dfrac{\psi_1}{\|\psi_1\|}$

To obtain ϕ_2 define

$$w_2(s) = \psi_2 - (\psi_2 \phi_1)\phi_1$$

Then, w_2 is orthogonal to ϕ_1

ϕ_2 is obtained by setting $\phi_2 = \dfrac{\omega_2}{\|\omega_2\|}$

Continuing this process we have

$$w_k(s) = \psi_k(s) - \sum_{i=1}^{k-1} (\psi_k, \phi_i)\phi_i, \ \mathrm{K} \geq 2$$

Then, $\phi_k = \dfrac{\omega_k}{\|\omega_k\|}$

Which given the orthonormal set.

Note :

1. A set of orthogonal functions, we can convert it into an orthonormal set simply by dividing each function by its norm.

2. If $\{\phi_1, \phi_2, ...\phi_k ...\}$ be an orthonormal set then $\{\phi_1^*, \phi_2^*, ...\phi_k^* ...\}$ also form an orthonormal set.

6.4 Fundamental properties of eigen values and eigen functions for symmetric kernel :

Property 1 : The eigen values of non zero symmetric kernel are real.

Proof : Let λ and $\phi(s)$ be an eigen value and a corrosponding eigen function of the symmetric kernel k(s,t)

$$\phi(s) = \lambda\int_a^b k(s,t)\phi(t)\,dt \qquad \text{-----(1)}$$

Multiply both side by $\overline{\phi}(s)$ and integrate w.r.t.s. from a to b.

$$\int_a^b \overline{\phi}(s)\phi(s)\,ds = \lambda\int_a^b \overline{\phi}(s)\left[\int_a^b k(s,t)\phi(t)\,dt\right]ds$$

$$= \lambda\int_a^b\int_a^b \overline{\phi}(s)k(s,t)\phi(t)\,dtds \qquad \text{-----(2)}$$

Now from (1)

$$\overline{\phi}(s) = \overline{\lambda}\left[\int_a^b k(s,t)\phi(t)\,dt\right]$$

$$= \overline{\lambda}\int_a^b \overline{k}(s,t)\overline{\phi}(t)\,dt$$

multiply both side by $\phi(s)$ and integrating w.r.t.s from a to b

$$\int_a^b \phi(s)\overline{\phi}(s)\,ds = \overline{\lambda}\int_a^b \phi(s)\left[\int_a^b \overline{k}(s,t)\overline{\phi}(t)\,dt\right]ds$$

$$= \overline{\lambda}\int_a^b\int_a^b \phi(s)\overline{k}(s,t)\overline{\phi}(t)\,dtds$$

$$= \overline{\lambda}\int_a^b\int_a^b \phi(s)k(t,s)\overline{\phi}(t)\,dtds \quad (\because k \text{ is symmetric})$$

$$= \bar{\lambda} \int_a^b \int_a^b \bar{\phi}(s)\, k(s,t)\, \phi(t)\, dtds$$

From equation (2) and (3) we have

$$= \lambda \int_a^b \int_a^b \bar{\phi}(s) k(s,t) \phi(t)\, dtds = \bar{\lambda} \int_a^b \int_a^b \bar{\phi}(s) k(s,t) \phi(t)\, dtds$$

$$\Rightarrow \lambda = \bar{\lambda}$$

$$\Rightarrow \lambda \textit{ is real}$$

This completes the proof.

Aliter : Let λ and $\phi(s)$ be an eigen value and corroponding eigen function of the kernel k (s, t)

$$\therefore \phi(s) = \lambda \int_a^b k(s,t) \phi(t)\, dt$$

$$\therefore \phi(s) = \lambda k \phi(s)$$

Multiply both side by $\bar{\phi}(s)$ and itegrate w.r.t.s. from a to b

$$\int_a^b \bar{\phi}(s) \phi(s)\, ds = \lambda \int_a^b k \phi(s) \bar{\phi}(s)\, ds$$

$$\|\phi(s)\|^2 = \lambda (k\phi, \phi)$$

Note that, $\langle kd, \phi \rangle \neq \phi$, when ϕ is an eigntunurm, Threfor, we can write,

$$\therefore \lambda = \frac{\|\phi(s)\|^2}{(k\phi, \phi)}$$

Since k is symmetric therefore $(k\phi, \phi)$ is real, also $\|\phi(s)\|$ is real.

$\therefore \lambda$ is real.

Property 2 : The eigen functions of a symmetric kernel, corresponding to different eigen values, are orthoginal

Proof : Let ϕ_1 *and* ϕ_2 are eigen functions corresponding, to the eigen values λ_1 *and* λ_2 of symmetric kernel k(s,t).

$$\phi_1(s) = \lambda_1 \int_a^b k(s,t)\phi_1(t)\,dt \qquad \text{-----(1)}$$

and

$$\phi_2(s) = \lambda_2 \int_a^b k(s,t)\phi_2(t)\,dt \qquad \text{-----(2)}$$

from (2)

$$\bar{\phi}_2(s) = \bar{\lambda}_2 \int_a^b \bar{k}(s,t)\bar{\phi}_2(t)\,dt$$

$$= \lambda_2 \int_a^b k(s,t)\bar{\phi}_2(t)\,dt$$

($\because$ eigen values of symmetric kernel are real)

$$(\phi_1, \phi_2) = \int_a^b \phi_1(s)\bar{\phi}_2(s)\,ds$$

$$= \int_a^b \phi_1(s)\left[\lambda_2 \int_a^b k(t,s)\bar{\phi}_2(t)\,dt\right]ds$$

$$= \lambda_2 \int_a^b \bar{\phi}_2(t)\left[\int_a^b k(t,s)\phi_1(s)\,ds\right]dt \qquad \text{-----(3)}$$

From (1), $\phi_1(t) = \lambda_1 \int_a^b k(t,s)\phi_1(s)\,ds$

Using this in equation (3) we get

$$(\phi_1,\phi_2)=\lambda_2\int_a^b \bar{\phi}_2(t)\frac{\phi_1(t)}{\lambda_1}dt$$

$$=\frac{\lambda_2}{\lambda_1}\int_a^b \phi_1(t)\phi_2(t)\,dt$$

$$=\frac{\lambda_2}{\lambda_1}(\phi_1,\phi_2)$$

$$\therefore \lambda_1(\phi_1,\phi_2)=\lambda_2(\phi_1,\phi_2)$$

$$\therefore (\lambda_1-\lambda_2)(\phi_1,\phi_2)=0$$

$$\Rightarrow (\phi_1,\phi_2)=0 \quad (\because \lambda_1\neq\lambda_2)$$

$\Rightarrow \phi_1$ *and* ϕ_2 are orthogonal

Theorem : Riesz - Fischer Theorem :

If $\{\phi(s)\}$ is a given orthonormal system of functions in L_2 and $\{\alpha_i\}$ is a given sequence of complex numbers such that the series $\sum_{k=1}^{\infty}|\alpha_k^2|$ converges, then there exists a unique function *f*(s) for which α_i are the Fourier cofficients with respect to the orthonormal system $\{\phi_i\}$ and to which the Fourier series converges in the mean that is

$$\left\| f(s)-\sum_{i=1}^{n}\alpha_i\phi_i \right\| \to 0 \quad as \quad n\to\infty$$

Property 3 : The normalized eigen functions associated with symmetric kernel form an orthonormal set.

Proof : Let k (s, t) be symmetric kernel and $\{\phi_k\}$ be the set of normalized eigen functions associated with k(s,t)

For each k we have

$$\|\phi_k\| = 1$$

$$\Rightarrow \|\phi_k\|^2 = 1$$

$$\Rightarrow \int_a^b |\phi_k(t)|^2 \, dt = 1$$

$$\Rightarrow \int_a^b \phi_k(t)\phi^*_k(t)\, dt = 1$$

Thus, $\langle \phi_k , \phi_k \rangle = 1$ for each k ----- (1)

Also we know that, "Eigen function of symmetric kernel corrosponding to different eigen values are orthogonal".

$$\Rightarrow \int_a^b \phi_i(t)\phi^*_j(t)\, dt = 0 \quad i \neq j$$

i.e. $(\phi_i, \phi_j) = 0 \; for \quad i \neq j$ ----- (2)

From (1) and (2), the collection $\{\phi_k\}$ is such that

$$(\phi_i, \phi_j) = \begin{cases} 1; & i = j \\ 0; & i \neq j \end{cases}$$

$\Rightarrow$ The collection $\{\phi_k\}$ form orthonormal set.

Property 4 : The multiplicity of any non zero eigen value is finite for every symmetric kernel for which $\iint |k(s,t)|^2 \, dsdt$ is finite.

Proof : Let k(s,t) be a symmetric such that $\iint |k(s,t)|^2 \, dsdt$ is finite.

Let the functions $\phi_{1\lambda}(s), \phi_{2\lambda}(s), ..., \phi_{n\lambda}(s), ...$ be the linearly independent eigen functions which corespond to a non zero eigen value λ. By the Gram - Schmidth Procedure, we can find linear combinations of these functions which form an orthonormal system $\{u_{k\lambda}(s)\}$. Then, the coresponding complex conjugate system $\{u^*_{k\lambda}\}$ also forms an orthonormal system.

By Reisz -Fischer Theoram, for fixed s, the fourier series associated with k(s,t) in terms of an orthonormal set $\{u^*_{k\lambda}\}$ is given by

$$k(s,t) = \sum_i a_i u^*_{i\lambda}(t)$$

Where $a_i = \int k(s,t)\left(u^*_{i\lambda}(t)\right)^* dt = \left(k, u^*_{i\lambda}\right)$

Also $a_i = \lambda^{-1} u_{i\lambda}(s) \quad \because u_{i\lambda}$ is an eigen function.

Using Bessels in equality we have

$$\sum_i |a_i|^2 = \sum_i \left|\left(k, u_{i\lambda}^*\right)\right|^2 \le \|k\|^2$$

$$\therefore \sum_i |a_i|^2 \le \int \left|k(s,t)\right|^2 dt \qquad \text{----- (1)}$$

But $\sum_i |a_i|^2 = \sum_i \left|\lambda^{-1} u_{i\lambda}(s)\right|^2$

$$= \sum_i \left(\lambda^{-1}\right)^2 \left|u_{i\lambda}(s)\right|^2$$

$$= \sum_i \left(\lambda^{-1}\right)^2 \left|u_{i\lambda}(s)\right|^2$$

Using this in equation (1) we have

$$\therefore \sum_i \left(\lambda^{-1}\right)^2 \int \left|u_{i\lambda}(s)\right|^2 \le \left|k(s,t)\right|^2 dtds$$

Integrating w.r.t.s.

$$\therefore \sum_i \left(\lambda^{-1}\right)^2 \int \left|u_{i\lambda}(s)\right|^2 \le \iint \left|k(s,t)\right|^2 dtds$$

i.e. $\sum_i \left(\lambda^{-1}\right)^2 \left\|u_{i\lambda}(s)\right\| \le \iint \left|k(s,t)\right|^2 dtds$

$$\Rightarrow \sum_i \left(\lambda^{-1}\right)^2 \le \iint \left|k(s,t)\right|^2 dtds$$

Let m is the multiplicity of λ then,

$$m \le \lambda^2 \iint \left|k(s,t)\right|^2 dtds$$

since $\iint \left|k(s,t)\right|^2 dtds$ is finite

$\Rightarrow$ m is finite.

Propery 5 : The eigen values of a symmetric L_2- kernel form a finite or an infinite sequence $\{\lambda_n\}$ with no finite limit point.

Proof : Let k(s,t) be symmetric L_2-kernel

$$\therefore \iint \left|k(s,t)\right|^2 dsdt < \infty$$

Let $u_{ik}(s)$ be the orthonormal eigen functions corrosponding to different eigen values λ_i.

Then proceding as in the proof of the property 4, we have

$$\sum_i \left(\lambda_i^{-1}\right)^2 \le \iint \left|k(s,t)\right|^2 dsdt < \infty$$

If there exists an enumerable infinity of λ_i then we must have

$$\sum_{n=1}^{\infty} \left(\frac{1}{\lambda_n}\right)^2 < \infty$$

Then by Cauchy Theorem

$$\Rightarrow \frac{1}{\lambda_n} \to 0 \quad as \quad n \to \infty$$

$$\Rightarrow \lambda_n \to \infty \quad as \quad n \to \infty$$

$\therefore \infty$ is the only limit point of the eigen values. This complets the proof.

Problem 2 : Prove that $K^2\phi(s) = K_2\phi(s)$ where k_2 is second itercated kerenel of k.

Solution :

$$k^2\phi(s) = kk\phi(s)$$

$$= \int_a^b k(s,x) k\phi(x)\, dx$$

$$= \int_a^b k(s,x) \left[\int_a^b k(x,t) \phi(t)\, dt \right] dx$$

$$= \int_a^b \left[\int_a^b k(s,x) k(x,t) dx \right] \phi(t)\, dt$$

($\because$ Changing the order of integration)

$$= \int_a^b k_2(s,t) \phi(t)\, dt$$

($\because$ Definition of second iterated kernel)

$$= k_2\phi(s), \ \forall \text{ S}$$

$$\Rightarrow k^2\phi = k_2\phi$$

Property 6 : The set of eigen values of the second iterated kernel coincide with the set of squares of the eigen values of the given kernel.

[(i.e. λ is an eigen value of k (s,t) iff λ^2 is an eigen value of k_2 (s,t)]

Proof : Let λ be an eigen value of k corrosponding to eigen function $\phi(s)$

$$\therefore \phi(s) = \lambda \int_a^b k(s,t)\phi(t)\,dt$$

i.e. $\phi(s) = \lambda K\phi(s)$ *where* $K\phi(s) = \int_a^b k(s,t)\phi(t)\,dt$

$$\Rightarrow \phi = \lambda K\phi$$

$$\therefore (I - \lambda K)\phi = 0$$

Where, I is an identity operator

Now, operate on both side $I + \lambda K$

$$\therefore (I + \lambda K)(I - \lambda K)\phi = 0$$

$$\therefore (I - \lambda^2 K^2)\phi = 0$$

$$\Rightarrow \phi(s) = \lambda^2 K^2 \phi(s)$$

$$= \lambda^2 \int_a^b k_2(s,t)\phi(t)\,dt$$

$\Rightarrow \lambda^2$ is an eigen value of kernel k_2 (s,t)

Conversely, let λ^2 is an eigen value of kernel k_2 (s,t)

$$\therefore \phi(s) = \lambda^2 \int_a^b k_2(s,t)\phi(t)\,dt$$

$$\Rightarrow (I - \lambda^2 k^2)\phi = 0$$

$$\therefore (I - \lambda k)(I + \lambda k)\phi = 0$$

If λ is an eigen value then proof is complete

If λ is not an eigen value

$$\therefore \phi(s) \neq \lambda \int_a^b \mathrm{K}(s,t)\phi(t)\,dt$$

$$\therefore \phi(s) \neq \lambda \mathrm{K}\phi(s)$$

$$i.e.\ \ \phi \neq \lambda \mathrm{K}\phi \Rightarrow I \neq \lambda \mathrm{K}$$

$$\Rightarrow (I - \lambda \mathrm{K}) \neq 0$$

(i.e. $I - \lambda \mathrm{K}$ is not null operator)

from (1) $(I + \lambda \mathrm{K})\phi = 0$

$$i.e.\ \ \phi(s) = -\lambda k \phi(s)$$

$$\therefore \phi(s) = -\lambda \int_a^b k(s,t)\phi(t)\,dt$$

i.e. $-\lambda$ is an eigen value of k(s,t). This complete the proof.

Aliter : Let λ is eigen value and $\phi(s)$ is corrosponding eigen function.

$$\therefore \phi(s) = \lambda \int_a^b k(s,t)\phi(t)\,dt \qquad \text{----- (1)}$$

Now $\therefore \phi(s) - \lambda^2 \int_a^b k_2(s,t)\phi(t)\,dt$

$$= \phi(s) - \lambda^2 \int_a^b \left[\int_a^b k(s,x)\,k(x,t)\,dx \right] \phi(t)\,dt$$

Changing order of integration

$$= \phi(s) - \lambda \int_a^b k(s,x) \left[\lambda \int_a^b k(x,t)\phi(t)\,dt \right] dx$$

$$= \phi(s) - \lambda \int_a^b k(s,x)\phi(x)\,dx \qquad (\because (1))$$

$$\therefore \phi(s) - \lambda^2 \int_a^b k_2(s,t)\phi(t)\,dx = \phi(s) - \lambda \int_a^b k(s,t)\phi(t)\,dt$$

$\Rightarrow \lambda^2$ is eigen value of $k_2(s,t)$ if and only if λ is an eigen value of k(s,t)

Property 7 : A symmetric kernel possesses at least one eigen value.

Definition : The spectrum of the kernel k(s,t) is the set of all eigen values.

In this terminology the above property 7 may be stated as, the spectrum of a symmetric kernel is never empty.

Property 8 : If λ_1 is the smallest eigen value of the kernel k, then

$$\frac{1}{|\lambda_1|} = \max\left[\frac{|(\mathrm{K}\phi,\phi)|}{\|\phi\|^2}\right]$$

or equivalently

$$\frac{1}{|\lambda_1|} = \max|(\mathrm{K}\phi,\phi)| \quad ; \|\phi\| = 1$$

Proof : If λ *and* $\phi(s)$ be an eigen value and coresponding eigen function then we have

$$\phi(s) = \lambda \int k(s,t)\,\phi(t)\,dt$$

$$= \lambda\,\mathrm{K}\phi(s)$$

$$i.e. \phi = \lambda\,\mathrm{K}\phi \Rightarrow \mathrm{K}\phi = \lambda^{-1}\phi$$

Now $(\mathrm{K}\phi,\phi) = (\lambda^{-1}\phi,\phi)$

$$= \lambda^{-1}(\phi,\phi)$$

$$= \lambda^{-1} \|\phi\|^2$$

$$\therefore \lambda^{-1} = \frac{(K\phi, \phi)}{\|\phi\|^2}$$

$$i.e. \frac{1}{\lambda} = \frac{(K\phi, \phi)}{\|\phi\|^2}$$

$$\frac{1}{|\lambda|} = \frac{|(K\phi, \phi)|}{\|\phi\|^2}$$

$$\therefore \frac{1}{|\lambda|} = \max\left[\frac{|(K\phi, \phi)|}{\|\phi\|^2}\right] if\, \lambda \;\; is \text{ minimum}$$

Thus, It λ, is the smallest eigen value then

$$\frac{1}{|\lambda_1|} = \max\left[\frac{|(K\phi, \phi)|}{\|\phi\|^2}\right]$$

If $\|\phi\| = 1$ then $\frac{1}{|\lambda_1|} = \max|(K\phi, \phi)|$

6.5 Expansion of symmetric kernel in eigen function.

Let k (s, t) be a non null, symmetric kernel which has finite or an infinite number of eigen values (always real and non zero). We order them in sequence.

$\lambda_1, \lambda_2, ..., \lambda_n, ...$ ----- (1)

in such a way that each eigen value is repeated as many times as its multiplicity.

We order them as follows.

$$0 < |\lambda_1| \le |\lambda_2| \le ... |\lambda_n| \le |\lambda_{n+1}| \le ...$$

Let $\phi_1(s), \phi_2(s) ... \phi_n(s), ...$ ----- (2)

be the sequence of eigen functions corrosponding to the eigen values given by the sequence (1) and arranged in such a way that they are no longer repeated and are linearly independent in each group corrosponding to the same eigen value.

Thus, to each eigen value λ_k in (1) there corrosponds just one eigen function $\phi_k(s)$ in (2).

Suppose that these eigen functions have been orthonormalized (property3)

we know that symmetric L_2- kernel k(s,t) has atleast one eigen value, say λ_1, then $\phi_1(s)$ is the corrosponding eigen function. Define, second truncated kernel, by

$$k^{(2)}(s,t) = k(s,t) - \frac{\phi_1(s)\phi_1^*(t)}{\lambda_1} \qquad \text{----- (3)}$$

Then $k^{(2)}(s,t)$ is nonnull and symmetric kernel.

Indeed, since K(s,t) is non null, K(2) (s,t) is non-null further, from (3)

$$\left[k^{(2)}(s,t)\right]^* = k^*(s,t) - \frac{\phi_1^*(s)^* \phi_1(t)^*}{\lambda_1^*}$$

($\because \lambda_1^* = \lambda_1$ as eigen ($\therefore$ k(s,t) is symmetric) value of symmetric kernel are real)

$$= k(t,s) - \frac{\phi^*{}_1(s)\phi_1(t)}{\lambda_1}$$

$$= k^{(2)}(t,s)$$

$= k^{(2)}(s,t)$ is symmetric kernel.

$\because$ As the second truncated kernel is symmetric and non null, it has atleast one eigen value, say λ_2 (Property 7). (We choose smallest if there are more.)

Let ϕ_2(s) be the corresponding normalized eigen function of $k^{(2)}$(s,t)

Then $\phi_1(s) \neq \phi_2(s)$ even if $\lambda_1 = \lambda_2$ (i.e. $\phi_1(s)$ is not an eigen function of kernel $k^{(2)}$(s,t) Which can be shown as below)

$$= \int K^{(2)}(s,t)\ \phi_1(t)\ dt$$

$$= \int \left\{ K(s,t) - = \frac{\phi_1(s)\phi_1(t)}{\lambda_1} \right\} \phi_1(t)\,dt$$

$$= \int \left\{ K(s,t)\ \phi_1(t)\,dt - \frac{\phi_1(s)}{\lambda_1} \right\} \phi_1^*(t)\phi_1(t)\,dt$$

$$= \int \left\{ K(s,t)\ \phi_1(t)\,dt - \frac{\phi_1(s)}{\lambda_1} \right\} (\phi_1, \phi_1) \qquad \text{----- (4)}$$

$\Rightarrow$ $\phi_1(s)$ can not be an eigen function corresponding to λ_2 $K^{(2)}$ (s,t).

Thus, we have proved $\phi_1(s)$ cannot be an eigen function of $k^{(2)}$(s,t) similarly we can prove that the third truncated kernel.

$$k^{(3)}(s,t) = k^{(2)}(s,t) - \frac{\phi_2(s)\phi^*{}_2(t)}{\lambda_2}$$

$$= k(s,t) - \frac{\phi_1(s)\phi^*{}_1(t)}{\lambda_1} - \frac{\phi_2(s)\phi^*{}_2(t)}{\lambda_2}$$

$$k^{(3)}(s,t) = k(s,t) - \sum_{k=1}^{2} \frac{\phi_k(s)\phi^*{}_k(t)}{\lambda_k}$$

is also non null and symmetric. Hence given third eigen value λ_3 and eigen function $\phi_3(s)$

As discussed above the function $\phi_3(s)$ cannot be same with $\phi_1(s)$ and $\phi_2(s)$

Proceeding in this way the n^{th} truncated kernel is given by

$$k^{(n)}(s,t) = k(s,t) - \sum_{k=1}^{n-1} \frac{\phi_k(s)\phi^*{}_k(t)}{\lambda_k}$$

Repeating this procedure countably many times we have two possibillities.

a) This processes terminates after n steps that is, $k^{(n+1)}(s,t) \equiv 0$

$$\Rightarrow k(s,t) = \sum_{k=1}^{n} \frac{\phi_k(s)\phi^*{}_k(t)}{\lambda_k}$$

Thus, k (s, t) is (degenerated kernel) separable kernal

b) The process can be continued in finitely and there are an infinite number of eigen values and eigen functions.

Theorom 3 : Let the sequence $\{\phi_k(s)\}$ be all the eigen functions of a symmetric L_2-kernel k (s,t), with $\{\lambda_k\}$ as the corroponding eigen values. Then, the series $\sum_{n=1}^{\infty} \frac{|\phi_n(s)|^2}{\lambda_n^2}$ converges and its sum is bounded by C_1^2 which is an upper bound of the integral.

$$\int |k(s,t)|^2\, dt$$

Proof: Let $\{\phi_K(s)\}$ be the all eigen functions of a symmetric kernel k(s,t) with $\{\lambda_k\}$ as the corresponding eigen values. We know that sequence of eigen functions of a symmetric kernel can be made orthonormal. Thus we suppose that the sequence $\{\phi_k(s)\}$ is orthonormalized.

The nth truncated kernel $k^{(n)}(s,t)$ is given by

$$k^{(n)}(s,t) = k(s,t) - \sum_{i=1}^{n-1} \frac{\phi_i(s)\phi_i^*(t)}{\lambda_i}$$

Now $\int \left|k^{(n)}(s,t)\right|^2 dt = \int k^{(n)}(s,t)\left[k^{(n)}(s,t)\right]^* dt$

$$= \int \left[k(s,t) - \sum_{i=1}^{n-1} \frac{\phi_i(s)\phi_i^*(t)}{\lambda_i}\right]\left[k^*(s,t) - \sum_{i=1}^{n-1} \frac{\phi_i^*(s)\phi_i(t)}{\lambda_i}\right] dt$$

($\because \lambda_i^* = \lambda_i \quad k^n(s,t)$ is symmetric)

$$= \int k(s,t) k^*(s,t)\, dt - \int k(s,t) \sum_{i=1}^{n-1} \frac{\phi^*_i(s)\phi_i(t)}{\lambda_i} - \int \left[k^*(s,t) \sum_{i=1}^{n-1} \frac{\phi_i(s)\phi_i(t)}{\lambda} \right] dt$$

$$+\int \left[\sum_{i=1}^{n-1} \frac{\phi_i(s)\phi_i^*(t)}{\lambda_i} \sum_{j=1}^{n-1} \frac{\phi^*_j(s)\phi_j(t)}{\lambda_j} \right] dt$$

$$= \int |k(s,t)|^2\, dt - \sum_{i=1}^{n-1} \frac{\phi_i^*(s)}{\lambda_i} \int k(s,t)\phi_i(t)\, dt - \sum_{i=1}^{n-1} \frac{\phi_i(s)}{\lambda_i} \int k^*(s,t)\phi_i^*(t)\, dt$$

$$+\sum_{i=1}^{n-1} \frac{\phi_i(s)\phi_i^*(s)}{\lambda_i^2} \int \phi_i^*(t)\phi_i(t)\, dt$$

$$\left(\therefore \int \phi_i^*(t)\phi_j(t)\, dt = 0; i \neq j \right)$$

Using $\phi_i(s) = \lambda_i \int k(s,t)\phi_i(t)\, dt$

$$\Rightarrow \int k(s,t)\phi_i(t)\, dt = \frac{\phi_i(s)}{\lambda_i}$$

$$\Rightarrow \int k^*(s,t)\phi_i^*(t)\, dt = \frac{\phi_i^*(s)}{\lambda_i}$$

and $\int \phi_i(t)\phi_i^*(t)\, dt = 1$

we get

$$\int \left| k^{(n)}(s,t) \right|^2 dt = |k(s,t)|^2\, dt - \sum_{i=1}^{n-1} \frac{\phi_i^*(s)\phi_i(s)}{\lambda_i^2} - \sum_{i=1}^{n-1} \frac{\phi_i(s)\phi^*{}_i(s)}{\lambda_i^2} + \sum_{i=1}^{n-1} \frac{\phi_i(s)\phi^*{}_i(s)}{\lambda_i^2}$$

$$\int \left| k^{(n)}(s,t) \right|^2 dt = \int |k(s,t)|^2\, dt - \sum_{i=1}^{n-1} \frac{\phi_i(s)\phi^*{}_i(s)}{\lambda_i^2}$$

Since $k^{(n)}(s,t)$ is non null, for all n

$\left|k^{(n)}(s,t)\right| dt \neq 0$ for all n

$$\therefore 0 \leq \int \left|k(s,t)\right|^2 dt - \sum_{i=1}^{n-1} \frac{\phi_i(s)\phi^*_i(s)}{\lambda_i^2}$$

$$\therefore \sum_{i=1}^{n} \frac{\phi_i(s)\phi^*_i(s)}{\lambda_i^2} \leq \int \left|k(s,t)\right|^2 dt$$

Since C_1^2 is an upper bound of the integral $\int \left|k^2(s,t)\right| dt$

We have $\sum_{i=1}^{n} \frac{\phi_i(s)\phi^*_i(s)}{\lambda_i^2} \leq c_1^2$

Taking limit $n \to \infty$ we get

$$\therefore \sum_{i=1}^{\infty} \frac{\phi_i(s)\phi^*_i(s)}{\lambda_i^2} \leq c_1^2$$

i.e. $\sum_{n=1}^{\infty} \frac{\left|\phi_n(s)\right|^2}{\lambda_n^2} \leq c_1^2$

$\therefore \sum_{n=1}^{\infty} \frac{\left|\phi_n(s)\right|^2}{\lambda_n^2}$ Converges and its sum is bounded by C_1^2

Theorem 4 : Let the sequence $\{\phi_n(s)\}$ be all the eigen functions of a symmetric kernel k(s,t) with $\{\lambda_n\}$ as the corrosponding eigen values.

Then, the truncated kernel.

$$k^{(n+1)}(s,t) = k(s,t) - \sum_{m=1}^{n} \frac{\phi_m(s)\phi^*_m(t)}{\lambda_m}$$

has the eigen values $\lambda_{n+1}, \lambda_{n+2}, \ldots$ to which corrospond the eigen functions

$\phi_{n+1}(s), \phi_{n+2}(s), \ldots$ The kernel $k^{(n+1)}(s,t)$ has no other eigen values or eigen functions.

Proof: Let $\{\lambda_n\}$ be the all eigen values of symmetric kernel k(s,t) with corrosponding eigen function $\{\phi_n(s)\}$

(a) For $j \geq n+1$ $\quad (i.e\ j = n+1, n+2, \ldots)$

$$\lambda_j \int k^{(n+1)}(s,t)\phi_j(t)\,dt$$

$$= \lambda_j \int \left[k(s,t) - \sum_{m=1}^{n} \frac{\phi_m(s)\phi^*{}_m(t)}{\lambda_m} \right] \phi_j(t)\,dt$$

$$= \lambda_j \int k(s,t)\phi_j(t)\,dt - \lambda_j \sum_{m=1}^{n} \frac{\phi_m(s)}{\lambda_m} \int \phi^*{}_m(t)\,\phi_j(t)\,dt$$

$$= \lambda_j \int k(s,t)\phi_j(t)\,dt - \lambda_j \sum_{m=1}^{n} \frac{\phi_m(s)}{\lambda_m} (\phi_j \phi_m)$$

$$= \lambda_j \int k(s,t)\phi_j(t)\,dt$$

$$\left[\begin{array}{l} \because \text{since } 1 \leq m \leq n; \quad j \geq n+1 \ldots \\ \Rightarrow j \neq m \Rightarrow (\phi_j \phi_m) = 0 \end{array} \right]$$

$$= \phi_j(s)$$

$i.e\ \phi_j(s) = \lambda_j \int k^{(n+1)}(s,t)\phi_j(t)\,dt;$ for $j \geq n+1$

$\Rightarrow \lambda_j$ and ϕ_j are eigen values and eigen function of $k^{(n+1)}(s,t)$ for $j \geq n+1$

b) Let λ and $\phi(s)$ be an eigen value and eigen function of the kernel $k^{(n+1)}(s,t)$

$$\phi(s) = \lambda \int k^{(n+1)}(s,t)\phi(t)\,dt$$

$$\phi(s) = \lambda\left[\int k(s,t) - \sum_{m=1}^{n} \frac{\phi_m(s)\phi^*{}_m(t)}{\lambda_m}\phi(t)\right]dt$$

$$= \lambda\int k(s,t)\phi(t)\,dt - \lambda\sum_{m=1}^{n}\frac{\phi_m(s)}{\lambda_m}\int \phi_m{}^*(t)\phi(t)\,dt$$

$$\phi(s) = \lambda K\phi(s) - \lambda\sum_{m=1}^{n}\frac{\phi_m(s)}{\lambda_m}(\phi,\phi_m)$$

Taking inner product with $\phi_j; j \le n$ on both side we get.

$$(\phi,\phi_j) = \lambda(K\phi,\phi_j) - \lambda\sum_{m=1}^{n}\frac{(\phi_m\phi_j)(\phi,\phi_m)}{\lambda_m} \qquad \text{(i.e. j= 1, 2, ...n)}$$

Using $(\phi_m,\phi_j) = \begin{cases} 1 & \text{for} \quad j = m \\ 0 & \text{for} \quad j \ne m \end{cases}$

for fixed $j; j \le n$ we have

$$(\phi,\phi_j) = \lambda(K\phi,\phi_j) - \frac{\lambda(\phi,\phi_j)}{\lambda_j} \qquad \text{----- (1)}$$

Now, $(K\phi,\phi_j) = \int K\phi(s)\phi_i^*(s)\,ds$

$$= \int\left[\int k(s,t)\phi(t)\,dt\right]\phi_j^*(s)\,ds$$

Changing order of integration

$$= \int \phi(t)\left[\int k(s,t)\phi^*_j(s)\,ds\right]dt$$

interchanging t and s

$$= \int \phi(s)\left[k(t,s)\phi_j^*(t)\,dt\right]ds$$

$$=\int \phi(s)\left[\int k^*(s,t)\phi_j^*(t)\,dt\right]ds$$

$$=\int \phi(s)\left[\int k(s,t)\phi_j(t)\,dt\right]^* ds$$

$$=\int \phi(s)\left[K\phi_j(s)\right]^* ds$$

$$=\left(\phi, K\phi_j\right)$$

$$=\left(\phi, \lambda_j^{-1}\phi_j\right) \qquad (\because \phi_j = \lambda k\phi_j;\text{as } \phi_j \text{ is an eigen function for } k(s,t))$$

$$\left(K\phi,\phi_j\right)=\frac{1}{\lambda_j}\left(\phi,\phi_j\right)$$

Thus equation (1) becomes

$$\left(\phi,\phi_j\right)=\frac{\lambda}{\lambda_j}\left(\phi,\phi_j\right)-\frac{\lambda}{\lambda_j}\left(\phi,\phi_j\right)=0$$

$$\left(\phi,\phi_j\right)=0$$

$\Rightarrow \phi$ and ϕ_j are orthoginal for all $j \le n$

$\Rightarrow \phi \ne \phi_j; j \le n$

If $\phi(s)$ is an eigen function of $k^{(n+1)}(s,t)$

then $\phi(s) \ne \phi_j(s); j = 1,2,3,...n$

$\Rightarrow \phi(s)$ must be from the set

$\left\{\phi_{n+1}(s), \phi_{n+2}(s)...\right\}$

and λ must be from the set

$\left\{\lambda_{n+1}, \lambda_{n+2}...\right\}$

This completes the proof

Theorem 5 : A necessary and sufficient condition for a symmetric L_2- kernel to be degenerate is that it have a finite number of eigen values.

Proof : Let k (s,t) be symmetric L_2- kernel.

If k(s,t) is degenerate

i.e. $k(s,t) = \sum_{i=1}^{n} a_i(s) b_i(t)$,

Where $\{a_1(s), a_2(s), ..., a_n(s),\}$, $\{b_1(t), b_2(t), ..., b_n(t),\}$ are linearly independent. Let D (λ) denote the Fredholm determinant of the homogeneous Fredholm integral equation corresponding to the kernel K(s,t). Then D (λ) = 0 is a polynomial in λ of degree at most n. This implies that k (s,t) gives at the most n given values.

If $\Rightarrow k(s,t)$ is degenerate then it has a finite number of eigen values.

Conversly let k(s,t) has finite number of eigen values say 'n', $\lambda_1, \lambda_2, ...\lambda_n$ &

Corresponding orthonormalized eigen functions are $\phi_1(s), \phi_2(s), ..., \phi_n(s)$

Now, the $(n+1)^{th}$ truncated kernel is

$$k^{(k+1)}(s,t) = k(s,t) - \sum_{i=1}^{n} \frac{\phi_i(s)\phi^*_i(t)}{\lambda_i} \qquad \text{----- (1)}$$

since k(s,t) has 'n' eigen value which finite & λ_j, ϕ_j, (j =1, 2, ...,n) are has not eigen values & eigen functions of $k^{(n+1)}(s,t)$

$\therefore\ k^{(k+1)}(s,t) = 0, \forall$, s, t,

using in (1) we get

$$k(s,t) = \sum_{i=1}^{n} \frac{\phi_i(s)\phi^*_i(t)}{\lambda_i}$$ which is (degenerate) separable

Theorem 6 : Let $\{\phi_r\}$ be all the eigen functions of symmetric L_2 kernel k(*x*,y) with $\{\lambda_r\}$ as eigen values. If the series

$$\sum_{r=1}^{\infty} \frac{\phi_r(x)\overline{\phi_r(y)}}{\lambda_r} \qquad \text{-----(1)}$$

converges uniformly then,

$$k(x,y) = \sum_{r=1}^{\infty} \frac{\phi_r(x)\overline{\phi_r(y)}}{\lambda_r} \qquad \text{(Bilinear form)}$$

Proof : Suppose series in (1) converges uniformly to L(x, y). Let,

$$R(x,y) = k(x,y) - L(x,y)$$

We show that, $R(x,y) = 0$

Since R(*x*, y) is also symmetric and hence it has atleast one eigen value say λ_R

Let ϕ_R be the corresponding eigen function.

$$\phi_R(x) = \lambda_R \int R(x,y)\,\phi_R(y)\,dy \qquad \text{-----(2)}$$

For each eigen function ϕ_m(m =1, 2,.......) of k (s,t)

Corresponding to λ_m, we have

$$\int \phi_R(x)\overline{\phi_m(x)}dx$$

$$= \lambda_R \int \overline{\phi_m(x)} \int R(x,y)\,\phi_R(y)\,dydx$$

$$= \lambda_R \int \phi_R(y) \int R(x,y)\,\overline{\phi_m(x)}\,dxdy$$

$$= \lambda_R \int \phi_R(y)[\int k(x,y)\,\overline{\phi_m(x)}\,dx - \int L(x,y)\,\overline{\phi_m(x)}\,dx]dy$$

$$(\because R(x,y) = k(x,y) - L(x,y))$$

$$= \lambda_R \int \phi_R(y)[\int k(x,y)\,\overline{\phi_m(x)}\,dx - \int \sum_{r=1}^{\infty} \frac{\phi_r(x)\overline{\phi_r(y)}}{\lambda_r}\overline{\phi_m(x)}\,dx]$$

$$= \lambda_R \int \phi_R(y)[\int k(x,y)\,\overline{\phi_m(x)}\,dx - \int \sum_{r=1}^{\infty} \phi_r(x)\left(\int \overline{k(y,z)}\,\overline{\phi_r(z)}dz\right).\overline{\phi_m(x)}dx\,dy$$

$$\begin{bmatrix} \therefore \overline{\phi_r(y)} = \lambda_r \int \overline{k(y,z)}\,\overline{\phi_r(z)}\,dz \\ \Rightarrow \dfrac{\overline{\phi_r(y)}}{\lambda_r} = \int \overline{k(y,z)}\,\overline{\phi_r(z)}\,dz \end{bmatrix}$$

$$= \lambda_R \int \phi_R(y)[\int k(x,y)\,\overline{\phi_m(x)}\,dx - \sum_{r=1}^{\infty} \int \phi_r(x)\,\overline{\phi_m(x)}\,dx. \int \overline{k(y,z)}\overline{\phi_r(z)}\,dz]dy$$

$$= \lambda_R \int \phi_R(y)[\int k(x,y)\,\overline{\phi_m(x)}\,dx - \int \overline{k}(y,z)\overline{\phi_m(z)}\,dz]dy$$

$(\because \{\phi_r\}$ are orthogonal)

$$= \lambda_R \int \phi_R(y)[\int k(x,y)\,\overline{\phi_m(x)}\,dx - \int k(z,y)\overline{\phi_m(z)}\,dz]dy$$

$$= \lambda_R \int \phi_R(y)[\int k(x,y)\,\overline{\phi_m(x)}\,dx - \int k(x,y)\overline{\phi_m(x)}\,dx]\,dy$$

$$= 0$$

$$\therefore \int \phi_R(x)\overline{\phi_m(x)}\,dx = 0 \Rightarrow \langle \phi_R, \phi_m \rangle = 0, \text{ (m=1, 2, ...)}$$

ϕ_R is orthogonal to ϕ_m -----(3)

$$\therefore \int L(x,y)\,\phi_R(y)dy = \int \left(\sum_{r=1}^{\infty} \frac{\phi_r(x)\overline{\phi_r(y)}}{\lambda_r} \right) \phi_R(y)dy$$

$$= \sum_{r=1}^{\infty} \frac{\phi_r(x)}{\lambda_r} \int \phi_R(y)$$

$$= \sum_{r=1}^{\infty} \int \phi_R(y)\,\overline{\phi_r(y)}\,dy \frac{\phi_r(x)}{\lambda_r}$$

$$= \sum_{r=1}^{\infty} \frac{\phi_r(x)}{\lambda_r} \langle \phi_R, \phi_r \rangle = 0$$

$$= \sum_{r=1}^{\infty} \int \phi_R(y)\,\overline{\phi_r(y)}\,dy \frac{\phi_r(x)}{\lambda_r}$$

$$= \sum_{r=1}^{\infty} \frac{\phi_r(x)}{\lambda_r} \langle \phi_R, \phi_r \rangle = 0 \quad (\therefore \text{from (3)})$$

($\therefore$ the series converges uniformly)

From equation (2)

$$\phi_R(x) = \lambda_R \int [k(x,y) - L(x,y)]\phi_R(y)dy$$

$$\phi_R(x) = \lambda_R \int k(x, y)\, \phi_R(y) dy$$

ϕ_R is eigen function of the kernel k (x, y) w.r.t. λ_R as eigen value.

By equation (3) we get

$$\int \phi_R(x)\, \overline{\phi_R(x)} dx = 0$$

$$\therefore \int |\phi_R(x)|^2\, dx = 0 \Rightarrow \|\phi_1\|^2 = 0$$

$\Rightarrow \phi_R$ is a zero function.

Since ϕ_R is the eigen function of the symmetric kernel R (x, y).

$\therefore R(x, y) = 0, \quad \forall$, x, y

$\therefore L(x, y) = k(x, y)$

$$i.e. \quad k(x, y) = \sum_{i=1}^{\infty} \frac{\phi_r(x)\phi_r(y)}{\lambda_r}$$

♦♦♦

Unit – 7

HILBERT THEOREM AND ITS CONSEQUENCES

7.1 Hilbert Schmidt Theorem :

Theorem 1 : (Hilbert - Schmidt Theorem) :

If *f*(s) can be written in the form, $f(s) = \int k(s,t)h(t)dt$ -----(1)

Where k (s, t) is a symmetric L_2 - kernel and h (t) is a L_2 - function then *f*(s) can be expanded in an absolutely and uniformly convergent Fourier series with respect to the orthonormal system of eigen functions $\{\phi_n(s)\}$ of the kernel k(s,t) ;

$$f(s) = \sum_{n=1}^{\infty} f_n \phi_n(s), f_n = (f,\phi_n) \quad \text{-----(2)}$$

The fourier coefficients of the function f(s) are related to the fourier coefficients h_n of the function h (s) by the relations.

$$f_n = \frac{h_n}{\lambda_n}, \text{h}_\text{n} = (h,\phi_n), \quad \text{-----(3)}$$

Where $\{\lambda_n\}$ are the eigen values of the kernel k(s,t).

Proof : Let k(s,t) be symmetric, L_2-kernel and h(t) is an L_2 - Function

Let $f(s) = \int k(s_1 t)\, h\, (f)dt = (Kh)(s)$,

Where K is the fredholm operator cooresponding to k (s,t)

Let $\{\lambda_n\}$ be the sequence of eigen values of k (s,t) that is arranged as follows :

$o < |\lambda_1| \le |\lambda_2| \le |\lambda_3| \le .... \le |\lambda_n| \le |\lambda_{n+1}| \le ....$ and the corresponding eigen functions are $\phi_1(s), \phi_2(s), \phi_3(s), ... \phi_n(s), \phi_{n+1}(s), ...$ We suppose that the sequence $\{\phi_n(s)\}$ is orthonormalized.

Let the fourier coefficients f_n of the function f (s) corresponding to orthonormal system $\{\phi_n(s)\}$ are

$$fn = \langle f, \phi_n \rangle$$

$$= \langle Kh, \phi_n \rangle$$

$$= \langle h, K\phi_n \rangle \qquad (\because k(s,t) \text{ is symmetric k is self adjoint})$$

$$= (h, \lambda_n^{-1}\phi_n) \qquad (\because \phi_n \text{ is an eigen function} \Rightarrow \phi_n = \lambda_n K\phi_n)$$

$$= \lambda_n^{-1}(h, \phi_n)$$

$$= \frac{h_n}{\lambda_n}$$

$(\because h_n = h, \phi_n)$ are fourier coef. of fucntion h relative to rothonormal set $\{\phi_n\}$)

Thus, the fourier series for *f*(s) is

$$f(s) = \sum_{n=1}^{\infty} f_n\phi_n(s) = \sum_{n=1}^{\infty} \frac{h_n}{\lambda_n}\phi_n(s) \qquad \text{-----(4)}$$

Aim : The series (4) converges absolutely and uniformly. For this we show that, the remainder term

$\sum_{k=n+1}^{n+p} \frac{h_k}{\lambda_k}\phi_k(s)$ of the series (4) can be made arbitrarily small for any n and p.

Using, Cauchy schwartz inequality,

$$\left|\sum_{k=n+1}^{n+p} \frac{h_k}{\lambda_k}\phi_k(s)\right|^2 \le \sum_{k=n+1}^{n+p} \left|h_k^2\right| \sum_{k=n+1}^{n+p} \frac{|\phi_k(s)|^2}{\lambda_k^2}$$

using theorems 3 of unit VI

$\sum_{n=1}^{\infty} \frac{|\phi_n(s)|^2}{\lambda_n^2}$ Converges and its sum is bounded by C_1^2 which is an upper bound of the integral $\int |k(s,t)|^2 \, dt$

$$\therefore \left|\sum_{k=n+1}^{n+p} \frac{h_k}{\lambda_k}\phi_k(s)\right|^2 \le C_1^2 \sum_{k=n+1}^{n+p} \left|h_k^2\right| \qquad \text{-----(5)}$$

Also, the Bessel's inequality gives

$$\sum_{n=1}^{\infty} |h_n|^2 = \sum_{n=1}^{\infty} \left|(h, \phi_n)\right|^2 \le \|h\|^2 = \int |h(t)|^2 dt \prec \infty$$

($\because$ h is a L_2 function)

$\therefore \sum_{n=1}^{\infty} |h_n|^2$ is convergent, hence by Cauchy cirterion $\sum_{k=n+1}^{n+p} h_k^2$ can be made arbitrarily small.

thus, for every $\varepsilon > 0, \sum_{k=n+1}^{n+p} h_k^2 < \varepsilon^2$

Equation (5) becomes

$$\therefore \left|\sum_{k=n+1}^{n+p} \frac{h_k}{\lambda_k}\phi_k(s)\right|^2 \le C_1^2 \varepsilon^2$$

$$\therefore \left| \sum_{k=n+1}^{n+p} \frac{h_k}{\lambda_k} \phi_k(s) \right| \le C_1 \varepsilon$$

Since ε was an arbitrary, we can made, the remainder term under modulus sign arbitrarily small (Taking $\varepsilon \to 0$)

$\therefore$ The series (4) converges absolutely and uniformly.

Aim : The series (4) converges to *f*(s) consider, the partial sum

$$\psi_n(s) = \sum_{m=1}^{n} \frac{h_m}{\lambda_m} \phi_m(s)$$

our aim is to show that

$$\| f(s) - \psi_n(s) \| \to 0 \ \ as \ \ n \to \infty$$

Now,

$$f(s) - \psi_n(s) = f(s) - \sum_{m=1}^{n} \frac{h_m}{\lambda_m} \phi_m(s)$$

$$= \int k(s,t)\ h(t)\ dt - \sum_{k=1}^{n} \frac{(h, \phi_m)}{\lambda_m} \phi_m(s)$$

$$= \int k(s,t)\ h(t)\ dt - \sum_{k=1}^{n} \frac{\phi_m(s)}{\lambda_m} \int h(t)\ \phi_m^*(t)\ dt$$

$$= \int \left[k(s,t) - \sum_{k=1}^{n} \frac{\phi_m(s)\phi_m^*(t)}{\lambda_m} \right] h(t)\ dt$$

$$= \int k^{(n+1)}(s,t)\ h(t)\ dt$$

$$= \mathrm{K}^{(n+1)} h$$

Where $k^{(n+1)}$ is the truncated kernel of k(s,t)

$\therefore \|f(s)-\psi_n(s)\|^2 = \|\mathrm{K}^{(n+1)}h\|^2$

$= (\mathrm{K}^{(n+1)}h, \quad \mathrm{K}^{(n+1)}h)$

$= (h, (\mathrm{K}^{(n+1)})^* \quad \mathrm{K}^{(n+1)}h)$ -----(6)

[$\because \mathrm{K}^{n+1}$ is an self adjoint operator]

$= (h, \mathrm{K}^{(n+1)}\mathrm{K}^{(n+1)}h)$

($\because \mathrm{K}^{(n+1)}$ is an self adjoint operator.)

$= (h, \mathrm{K}_2^{(n+1)}h)$

($\because k^2\phi = k_2\phi$ for any kernel K)

* By theorem 4 of Unit VI the eigenvalues of the truncated kernel $k^{(n+1)}$ (s,t) are

$\lambda_{n+1,}\lambda_{n+2,}\lambda_{n+3},...with\,|\lambda_{n+1}| \leq |\lambda_{n+2}| \leq ...$

* By property (6) of chapter VI, the eigenvalues of

$K_2^{n+1}(s,t) are \lambda_{n+1}^2, \lambda_{n+2}^2, ...$

The least eigenvalue of $k_2^{(n+1)}(s,t)$ *is* λ_{n+1}^2 now we know the property, if λ_1, is the smallest eigen value of kernel k, then

$$\frac{1}{|\lambda_1|} = \max\left[\frac{|(K\phi,\phi)|}{\|\phi\|^2}\right]$$

Since λ_{n+1}^2 is the smallest eignevalue for $k_2^{(n+1)}(s,t)$ we have

$$\frac{1}{\lambda_{n+1}^2} = \max\left\{\frac{(\mathrm{K}_2^{(n+1)}\phi,\phi)}{\|\phi\|^2}\right\}$$

$$= \max\left\{\frac{\left\langle \phi, \mathrm{K}_2^{(n+1)}\right\rangle}{\|\phi\|^2}\right\}$$

$$\frac{1}{\lambda_{n+1}^2} = \max\left[\frac{\left|(h, K_2^{(n+1)}h)\right|}{\|h\|^2}\right]$$

We have omitted the modulus sign from the scalar product $(h, k_2^{(n+1)}h)$, as it is a positive quantity (Sec 6)

Thus $\dfrac{(h, K_2^{n+1}h)}{\|h\|^2} \le \dfrac{1}{\lambda_{n+1}^2}$

$$\Rightarrow (h, K_2^{n+1}h) \le \frac{\|h\|}{\lambda_{n+1}^2} \qquad \text{-----(7)}$$

Combining (6) and (7) we get

$$\|f(s) - \psi_n(s)\| \le \frac{\|h\|^2}{\lambda_{n+1}^2}$$

But we know the eigen values of symmetric kernel has no finite limit point (Sec Property (5))

$\therefore \lambda_{n+1} \to \infty \; as \; n \to \infty$

Therefore $n \to \infty \Rightarrow \|f(s) - \psi_n(s)\| \to 0$

$\Rightarrow \psi_n(s) \to f(s)$

Hence, $\sum_{n=1}^{\infty} f_n \; \phi_n(s) = f(s)$

Uniqueness : Let ψ be the limit of the series with partial sum ψ_η

Then $\quad \|f - \psi\| = \|f - \psi_n + \psi_n - \psi\|$

$\le \|f - \psi_n\| + \|\psi_n - \psi\| \to 0 \; as \; n \to \infty$

$\therefore f = \psi$

7.2 Application of Hilbert - Schmidt Theorem :

Let us apply the Hilbert - Schmidt theorem to find solution of the inhomogeneous Fredholm integral equation of the second kind

$$g(s) = f(s) + \lambda \int k(s,t) g(t) dt \qquad \text{-----(1)}$$

With a L_2 - Symmetric kernel.

Let $\lambda_1, \lambda_2, \cdots, \lambda_n, \cdots$ be the sequence of eigen values repeated as many times as its multiplicity, which are arranged as

$$o < |\lambda_1| \le |\lambda_2| .... |\lambda_n| \le |\lambda_{n+1}| \le ....$$

With corresponding normalized eigen functions $\{\phi_n,(s)\}$ of homogeneous integral equation.

Case I : λ is not an eigen value i.e. $\lambda \ne \lambda_n ; \forall n$

By (1) we have

$$g(s) - f(s) = \int k(s,t) [\lambda g(t)] dt \qquad \text{-----(2)}$$

Which has the same form as the integral equation in Hilbert - Schmidt theorem

$\therefore$ By Hilbert - schmidt theorem

$$g(s) - f(s) = \sum_{n=1}^{\infty} C_n \phi_n(s) \qquad \text{-----(3)}$$

Where $C_n = (g - f, \phi_n)$ & $C_n = \dfrac{\lambda_g, \phi_n}{\lambda_n}$

$$= \int (g - f)(s) \phi_n^*(s) ds$$

$$= \int g(s) \phi_n^*(s) ds - \int f(s) \phi_n^*(s) ds$$

$$C_n = g_n - f_n \qquad \text{-----(4)}$$

Where $g_n = \int g(s)\phi_n^*(s)ds = (9,\phi_n)$ -----(5)

$f_n = \int f(s)\phi_n^*(s)ds = \left(f,\phi_n\right)$ -----(6)

Also $C_n = \frac{(\lambda_9,\phi_n)}{\lambda_n}$

$$C_n = \frac{1}{\lambda_n}\int \lambda\, g(s)\, \phi_n^*(s)\, ds$$

$$= \frac{\lambda}{\lambda_n} g_n \qquad (\because (5))$$

$$\Rightarrow g_n = \frac{\lambda_n C_n}{\lambda} \qquad \text{-----(7)}$$

Using (7) in (4) we get

$$C_n = \frac{\lambda_n C_n}{\lambda} - f_n$$

$$C_n\left(1 - \frac{\lambda_n}{\lambda}\right) = -f_n$$

$$C_n = \frac{-f_n}{\left(\frac{\lambda-\lambda_n}{\lambda}\right)} = \left(\frac{\lambda}{\lambda_n-\lambda}\right)f_n \qquad \text{-----(8)}$$

Using (8) in (3) we get

$$g(s) - f(s) = \sum_{n=1}^{\infty}\left(\frac{\lambda}{\lambda_n-\lambda}\right)f_n\phi_n(s)$$

$$\therefore g(s) = f(s) + \lambda\sum_{n=1}^{\infty}\frac{f_n}{\lambda_n-\lambda}\phi_n(s) \qquad \text{-----(9)}$$

OR $$g(s) = f(s) + \lambda \sum_{n=1}^{\infty} \frac{\phi_n(s)}{\lambda_n - \lambda} \int f(t) \phi_n^*(t) dt$$

$$= f(s) + \lambda \int \sum_{n=1}^{\infty} \left[\frac{\phi_n(s) \phi_n^*(t)}{\lambda_n - \lambda} \right] f(t) dt$$

$$g(s) = f(s) + \lambda \int \Gamma(s,t,\lambda) f(t) dt \qquad \text{-----(10)}$$

where $\Gamma(s,t;\lambda) = \sum_{n=1}^{\infty} \frac{\phi_n(s)\phi_n^*(t)}{\lambda_n - \lambda}$

Which is the resolvent kernel

$\therefore$ The solution of equation (1) is given by the equation (9) or (10) in terms of absolutely and uniformly convergent series.

Case II : λ is an eigen value of k (s,t)

If λ is an eigen value, then it necessarily occurs in the sequence $\{\lambda_n\}$ and perhaps it is repeated several times.

Let $\lambda = \lambda_m = \lambda_{m+1} = \lambda_{m'}$

and $\phi_m, \phi_{m+1,} \ldots \phi_{m'}$ are the corresponding eigen functions.

Then for k = m, m + 1, ... m'

$\lambda - \lambda_k = o$

Also from relation (8)

$$f_k = \left(\frac{\lambda_k - \lambda}{\lambda} \right) C_k = O$$

$$\Rightarrow \frac{f_k}{\lambda - \lambda_k} = \frac{0}{0}; \qquad \text{-----(11)}$$

a indeterminate form which has some constant value.

$\therefore$ The Solution of (1) is given by the formula (9) where the coefficients with the indeterminate form $\frac{0}{0}$ have to be taken as arbitrary numbers.

Thus in this case solution is given by

$$g(s) = f(s) + \lambda \sum_{n=1}^{\infty} \frac{f_n}{\lambda_n - \lambda} \phi_n(s) + C_m \phi_m(s) + C_{m+1} \phi_{m+1}(s) + \ldots + C_{m'} \phi_{m'}(s)$$

$$(n \neq m, m+1, \ldots m')$$

Where $C_m, C_{m+1}, \ldots C_{m'}$ are arbitrary constants,

From (11) we see that integral equation (1) is soluble if and only if

$$f_k = o; k = m, m+1, \ldots\ldots m'$$

$$\Rightarrow \int f(s) \phi_k^*(s) ds = 0 \quad k = m, m+1, \ldots m'$$

i.e. equation (1) is soluble iff f(s) is orthogonal with $\phi_k(s); k = m, m+1, \ldots m'$

7.3 The solution of Fredholm integral equation of the second kind by Hilbert Schmidt theorem :

Consider, $g(s) = f(s) + \lambda \int k(s,t)\, g(t)\, dt$

With symmetric L_2 - kernel is

Its Solution is given by

$$g(s) = f(s) + \lambda \sum_{k=1}^{\infty} \frac{f_k}{\lambda_k - \lambda} \phi_k(s)$$

Where, $\lambda_k; k = 1, 2, 3\ldots$ are the eigenvalues of homogeneous integral equation. $\phi_k(s); k = 1, 2, 3\ldots$ are the corresponding normalized eigen functions of homogeneous integral equation.

i.e. if g_k are the eigen functions of homogeneous integral equation then.

$$\phi_k = \frac{g_k}{\|g_k\|}$$

and $f_k = (f, \phi_k) = \int f(s)\, \phi_k^*(s) ds$

Problem 1 : Solve the symmetric integral equation

$$g(s) = (s+1)^2 + \int_{-1}^{1} (st + s^2t^2)\, g(t)\, dt$$

by using Hilbert Schmidt theorem.

Solution : $g(s) = (s+1)^2 + \int_{-1}^{1} (st + s^2t^2) g(t) dt$

Comparing with standard equation

$f(s) = (s+1)^2, \lambda = 1$

Firstly we find the eigen values and corresponding normalized eigen function of

$$g(s) = \lambda \int_{-1}^{1} (st + s^2t^2) g(t) dt$$

Above equation can be written as

$$g(s) = \lambda s \int_{-1}^{1} tg(t) dt + \lambda s^2 \int_{-1}^{1} t^{2g(t)dt}$$

$$\therefore g(s) = \lambda s c_1 + \lambda s^2 c_2 \qquad \text{-----(1)}$$

Where $c_1 = \int_{-1}^{1} tg(t) dt \quad and \quad c_2 = \int_{-1}^{1} t^2 g(t)$

Using (1) in (2) we get

$$c_1 (1 - \frac{2\lambda}{3}) + 0.c_2 = 0 \qquad \text{-----(4)}$$

Using (1) in (3) we get

$$0.c_1 + \left(1 - \frac{2\lambda}{5}\right)c_2 = 0 \qquad \text{-----(5)}$$

For non trivial solution, we must have

$$\begin{vmatrix} 1 - \frac{2\lambda}{3} & 0 \\ 0 & 1 - \frac{2\lambda}{5} \end{vmatrix} = 0$$

$$\therefore (1 - \frac{2\lambda}{3})(1 - \frac{2\lambda}{5}) = o = \lambda = \frac{3}{2}, \frac{5}{2}$$

Case I : For $\lambda_1 = \frac{3}{2}$ equation (4) and (5) becomes

$$\left.\begin{array}{l} o.c_1 + o.c_2 = o \\ o.c_1 + \frac{2}{5}.c_2 = o \end{array}\right\} \Rightarrow c_2 = o \text{ and } C_1 \text{ is an arbitrary.}$$

$$\therefore g_1(s) = c_1 \frac{3}{2} s$$

Eigen function corresponding to $\lambda_1 = \frac{3}{2}$ is $g_1(s) = s$

Corresponding normalized eigen function is

$$\phi_1(s) = \frac{g_1(s)}{\|g_1(s)\|} = \frac{g_1(s)}{\left[\int_{-1}^{1} |g_1(s)|^2\right]^{\frac{1}{2}}}$$

$$= \frac{(s)}{\left[\int_{-1}^{1} s^2\right]^{\frac{1}{2}}}$$

$$= \frac{(s)}{\left\{\left[\frac{s^3}{3}\right]_{-1}^{1}\right\}^{\frac{1}{2}}}$$

$$= \frac{s}{\left(\frac{2}{3}\right)^{\frac{1}{2}}} = \sqrt{\frac{3}{2}}\, s$$

Case 2 : $\lambda_2 = \frac{5}{2}$

Equation (4) and (5) becomes

$$\left.\begin{aligned} -\frac{2}{3}c_1 + o.c_2 = o \\ o - c_1 + o.c_2 = o \end{aligned}\right\} \Rightarrow c_1 = 0 \quad and$$ C_2 is an arbitrary

$$\therefore g_1(s) = \frac{5}{2} c_2 s^2$$

Eigen function corrosponding to $\lambda_2 = \frac{5}{2}$ is $g_1(s) = s^2$

Corrosponding normalized eigen function is

$$\phi_2(s) = \frac{\phi_2(s)}{\|\phi_2(s)\|} = \frac{\phi_2(s)}{\left[\int_{-1}^{1} |\phi_2(s)|^2 \, ds\right]^{\frac{1}{2}}}$$

$$= \frac{s^2}{\left[\int_{-1}^{1} s^4\right]^{\frac{1}{2}}} = \frac{s^2}{\left[\left[\frac{s^5}{5}\right]\right]^{\frac{1}{2}}} = \frac{s^2}{\sqrt{\frac{2}{5}}}$$

$$\phi_2(s) = \frac{\sqrt{10}}{2} s^2$$

$$\therefore f_1 = (f, \phi_1) = \int_{-1}^{1} f(s)\, \phi_1^*(s) ds$$

$$= \int_{-1}^{1} (s+1)^2 \frac{\sqrt{6}}{2} s ds$$

$$= \frac{\sqrt{6}}{2} \int_{-1}^{1} (s^3 + 2s^2 + s) ds$$

$$= \frac{\sqrt{6}}{2} \left[2(2) \int_{o}^{1} s^2 ds \right]$$

$$= \frac{\sqrt{6}}{2} 4 \left[\frac{s^3}{3} \right]_o^1$$

$$= \frac{2\sqrt{6}}{3}$$

$$f_2 = (f, \phi_2) = \int_{-1}^{1} f(s)\, \phi_2^*(s) ds$$

$$= \int_{-1}^{1} (s+1)^2 \frac{\sqrt{10}}{2} s^2 ds$$

$$= \frac{\sqrt{10}}{2} \int_{-1}^{1} \left(s^4 + 2s^3 + s^2 \right) ds$$

$$= \frac{\sqrt{10}}{1} \left[2 \int_{o}^{1} (s^4 + s^2) ds \right]$$

$$= \frac{\sqrt{10}}{1} \left[\frac{s^5}{5} + \frac{s^3}{3} \right]_0^1$$

$$= \sqrt{10}\left[\frac{1}{5}+\frac{1}{3}\right]$$

$$= \frac{8\sqrt{10}}{!5}$$

Since $(\lambda \neq \lambda_1 and \lambda \neq \lambda_2)\, \lambda = 1$ is not an eigen value, then we have unique solution given by

$$g(s) = f(s) + \lambda \sum_{K=1}^{2} \frac{f_k}{\lambda_k - \lambda} \phi_k(s)$$

$$= f(s) + \frac{\lambda f_1}{\lambda_1 - \lambda} \phi_1(s) + \frac{\lambda f_2}{\lambda_2 - \lambda} \phi_2(s)$$

$$= (s+1)^2 + \frac{\frac{2\sqrt{6}}{3}}{\frac{3}{2}-1} \frac{\sqrt{6}}{2} 5 + \frac{8\sqrt{10}}{\frac{5}{2}-1} \frac{\sqrt{10}}{2} s^2$$

$$g(s) = \frac{25}{9} s^2 + 6s + 1$$

Problem 2 : Solve the symmetric integral equation

$$g(s) = s^2 + 1 + \frac{3}{2} \int_{-1}^{1} (st + s^2 t^2)\, g(t)\, dt$$

by using Hilbert - Schmidt theorem

Solution : $g(s) = s^2 + 1 + \frac{3}{2} \int_{-1}^{1} (st + s^2 t^2)\, g(t)\, dt$

Comparing with standard equations.

$$f(s) = s^2 + 1, \lambda = \frac{3}{2}$$

Case I : $\lambda_1 = \dfrac{3}{2}; \phi_1(s) = \dfrac{\sqrt{6}}{2} s$

Case II : $\lambda_2 = \dfrac{5}{2}; \phi_2(s) = \dfrac{\sqrt{10}}{2} s^2$

$$f_1 = (f, \phi_1) = \int_{-1}^{1} f(s)\, \phi_1^*(s) ds = \int_{-1}^{1} (s^2+1) \frac{\sqrt{6}}{2} s\, ds$$

$$= \frac{\sqrt{6}}{2} \int_{-1}^{1} (s^2 + s) ds = o \quad (\therefore \text{ odd function})$$

$$f_2 = (f, \phi_2) = \int_{-1}^{1} f(s) \phi_2^*(s) ds = \int_{-1}^{1} (s^2+1) \frac{\sqrt{10}}{2} s^2 ds$$

$$= \frac{\sqrt{10}}{2} \int_{-1}^{1} (s^4 + s^2) ds$$

$$= \frac{\sqrt{10}}{2} 2 \int_{-1}^{1} (s^4 + s^2)\, ds$$

$$= \sqrt{10} \left[\frac{s^5}{5} + \frac{s^3}{3} \right]_0^1 = \sqrt{10} \left[\frac{1}{5} + \frac{1}{3} \right]$$

$$= \frac{8\sqrt{10}}{15}$$

$$\therefore g(s) = f(s) + \lambda \sum_{k=1}^{2} \frac{f_k}{\lambda_x - \lambda} \phi_k(s)$$

$$= f(s) + \lambda \frac{f_1}{\lambda_1 - \lambda} \phi_1(s) + \lambda \frac{f_2}{\lambda_2 - \lambda} \phi_2(s)$$

Since $\lambda = \lambda_1 = \frac{3}{2}$ *and* $f_1 = 0$

$\frac{f_1}{\lambda_1 - \lambda} = \frac{o}{o}$ a indeterminate form hence it should be taken as arbitrary constent, Say C

$$\therefore\ g(s) = s^2 + 1 + \frac{3}{2} C \frac{\sqrt{6}}{2} s + \frac{3}{2} \frac{\frac{8\sqrt{10}}{15}}{\frac{5}{2} - \frac{3}{2}} \sqrt{\frac{10}{2}} s^2$$

$$= 5s^2 + Cs + 1$$

is the required solution.

Problem 3 : Solve the following symmetric integral equation with the help of Hilbert - Schmidt theorem.

$$g(s) = 1 + \lambda \int_0^{\pi} \cos\,(s+t)\; g(t)\; dt$$

Solution : $f(s) = 1, \lambda = \lambda$ firstly finding eigen value and corresponding normal eigen function of

$$\therefore\ g(s) = \lambda \int_0^{\pi} \cos(s+t)\; g(t)\; dt$$

$$= \lambda \cos s c_1 - \lambda \sin s c_2 \qquad \text{-----(1)}$$

Where $c_1 = \int_0^{\pi} \cos t\; g(t) dt$ -----(2)

$$c_2 = \int_0^{\pi} \sin t g(t) dt \qquad \text{-----(3)}$$

Using (1) in (2)

$$c_1(2 - \lambda\pi) + o.c_2 = o \qquad \text{-----(4)}$$

Using (3) in (2)

$$o.c_1 + (2+\lambda\pi)c_2 = o \qquad \text{-----(5)}$$

for non trivial solution

$$\begin{vmatrix} 2-\lambda\pi & o \\ o & 2+\lambda\pi \end{vmatrix} = o \Rightarrow \lambda = \frac{2}{\pi}, -\frac{2}{\pi}$$

Case 1 : $\lambda_1 = \frac{2}{\pi}$ (4) and (5) becomes

$$\left.\begin{aligned} o.c_1 + o.c_2 &= o \\ o.c_1 + 4.c_2 &= o \end{aligned}\right\} \Rightarrow \text{ and } c_1 \text{ is an arbitrary.}$$

Equation (1) gives

$$g(s) = \frac{2}{\pi} C_1 \cos s$$

$\therefore$ Eigen function corresponding to $\lambda_1 = \frac{2}{\pi}$ is $g_1(s) = \cos s$ and

corresponding normalized eigen function is

$$\phi_1(s) = \frac{g(s)}{\|g(s)\|} = \frac{g(s)}{\left[\int_o^\pi |g(s)|^2\right]^{\frac{1}{2}}}$$

$$= \frac{\cos s}{\left[\int_o^\pi \cos^2 ds\right]^{1/2}} = \sqrt{\frac{2}{\pi}} \cos s$$

Case 2 : $\lambda_1 = -\frac{2}{\pi}$ equation (4) and (5) gives

$$\left.\begin{aligned} 4c_1 + o.c_2 &= o \\ o.c_1 + o.c_2 &= o \end{aligned}\right\} \Rightarrow \text{ and } c_2 \text{ is an arbitrary}$$

$$\therefore g(s) = -\lambda \sin s \, c_2 = \frac{2}{\pi} c_2 \sin s$$

$\therefore g(s) = \sin s$ is an eigenfunctions corresponding to $\lambda_2 = -\dfrac{2}{\pi}$

$$\phi_2(s) = \frac{g_2(s)}{\|g_2(s)\|} = \frac{g_2(s)}{\left[\int_0^{\pi} |g_2(s)|^2 \, ds\right]^{\frac{1}{2}}}$$

$$= \frac{\sin s}{\left[\int_0^{\pi} \sin^2 s ds\right]^{\frac{1}{2}}} = \sqrt{\frac{2}{\pi}} \sin s$$

Now $f_1 = (f, \phi_1) = \int_0^{\pi} f(s) \, \phi_1^*(s) \, dt$

$$= \int_0^{\pi} (1) \cos s \left(\sqrt{\frac{2}{\pi}}\right) ds = \sqrt{\frac{2}{\pi}} \left[\sin s\right]_0^{\pi} = 0$$

$$f_2 = (f, \phi_2) = \int_0^{\pi} f(s) \, \phi_2^*(s) \, ds = \int_0^{\pi} \sqrt{\frac{2}{\pi}} \sin s \, ds = 2\sqrt{\frac{2}{\pi}}$$

Now the required solution is

$$g(s) = f(s) + \lambda \sum_{k=1}^{2} \frac{f_k}{\lambda_k - \lambda} \phi_k(s)$$

$$= f(s) + \lambda \frac{f_1}{\lambda_1 - \lambda}\phi_1 + \lambda \frac{f_2}{\lambda_2 - \lambda}\phi_2 \qquad \text{-----(6)}$$

Case 1 : If $\lambda \neq \lambda_1, \quad \lambda \neq \lambda_2$ then equation (6)

becomes

$$g(s) = 1 + \frac{\lambda\sqrt{\frac{2}{\pi}\cos s(0)}}{\frac{2}{\pi} - \lambda} + \frac{\lambda 2\sqrt{\frac{2}{\pi}}}{-\frac{2}{\pi} - \lambda}\sqrt{\frac{2}{\pi}}\sin s$$

$$= 1 - \frac{4\lambda \sin s}{2 + \pi\lambda}$$

Case 2 : If $\lambda = \lambda_2 = \frac{2}{\pi}$ Since $f_2 \neq 0$ there fore given integral equation has no solution

Case 3 : $\lambda = \lambda_1 = \frac{2}{\pi}$ Since $f_1 = 0$

equation (6) becomes with $\frac{f_1}{\lambda - \lambda_1}$ is an arbitrary.

$$g(s) = 1 + C\cos s + \frac{2}{\pi}\frac{2\sqrt{\frac{2}{\pi}}}{-\frac{2}{\pi} - \frac{2}{\pi}}\sqrt{\frac{2}{\pi}}\sin s$$

$$= 1 + C\cos s - \frac{2}{\pi}\sin s$$

Where C is an arbitrary constant is the required solution.

Problem 4 : Solve the symmetric integral equation

$$g(s) = f(s) + \lambda \int k(s)k(t)g(t)dt \qquad \text{-----(1)}$$

Solution : Firstly, we find the eigenvalues and corresponding normalized eigen functions of the homogeneous equation.

$$g(s) = \lambda \int k(s)k(t)g(t)dt$$

$$= \lambda k(s) \int k(t)g(t)dt$$
$$= \lambda k(s)c$$

Where $c = \int k(t)g(t)dt$

using (2) in (3) we get

$$c = \int k(t)\lambda k(t)c \ \ dt$$

$$= \lambda c \int [k(t)]^2 dt$$

$$c\left[1 - \lambda \int [k(t)]^2 \, dt\right] = o$$

for non zero solution we must have $c \neq 0$

$$1 - \lambda \int [k(t)]^2 dt = o$$

The eigenvalue is $\lambda_1 = \dfrac{1}{\int [k(t)]^2 \, dt}$

$$\therefore \ g(s) = \frac{ck(s)}{\int [k(t)]^2 \, dt}$$

The corresponding eigen function is

$$g_1(s) = k(s)\left[\therefore \frac{c}{\int [k(t)]^2 \, dt} = const\right]$$

Normalized eigen function is

$$\phi_1(s) = \frac{g(s)}{\|g_1(s)\|} = \frac{k(s)}{\left[\int [k(s)]^2\, ds\right]^{\frac{1}{2}}}$$

and

$$f_1 = (f, \phi_1) = \int f(s)\phi_1^*(s)ds$$

$$= \int \frac{f(s)k(s)ds}{\left[\int [k(s)]^2\, ds\right]^{\frac{1}{2}}}$$

$$= \left[\int [k(s)]^2\, ds\right]^{-1/2} \int f(s)k(s)ds$$

Case 1 : $\lambda \neq \lambda_1$

$$g(s) = f(s) + \lambda \frac{f_1}{\lambda_1 - \lambda}\phi_1(s)$$

$$= f(s) + \frac{\lambda \left[\int [k(s)]^2\, ds\right]^{-\frac{1}{2}} \int f(s)k(s)ds}{\left\{\left[\int [k(t)]^2\, dt\right]^{-1} \lambda\right\}} \cdot \frac{k(s)}{\left[\int [k(s)]^2\, ds\right]^{\frac{1}{2}}}$$

is the required solution

Case 2 : Let $\lambda = \lambda_1$ and $f_1 = 0$ ie and f and ϕ_1 is orthogonal

Then $\frac{f_1}{\lambda_1 - \lambda}$ be an arbitrary const.

Required solution is

$g(s) = f(s) + C\phi_1(s)$; where C is const.

$$= f(s) + c\frac{k(s)}{\left[\int [k(s)]^2 ds\right]^{1/2}}$$

Case 3 : $\lambda = \lambda_1$ and $f_1 \neq 0$

Then the solution does not exists.

Problem 5 : Solve the symmetric fredholm integral equation of the first kind.

$$\int_o^1 k(s,t)g(t)dt = f(s)$$

Where

$$K(s,t) = \begin{cases} s(1-t); s < t \\ (1-s)t; s > t \end{cases}$$

Solution : Firstly we find eigen values and Normalized eigen function of

$$g(s) = \lambda \int_0^1 k(s,t)g(t)dt$$

$$= \int_0^s \lambda k(s,t)\, g(t)\, dt + \int_s^1 \lambda k(s,t)\, g(t)\, dt$$

$$= \int_o^s \lambda t(1-s)g(t)dt + \int_s^I \lambda(1-t)sg(t)dt$$

$$g'(s) = \int_o^s -\lambda t g(t)dt + \lambda s(1-s)g(s) + \int_s^1 \lambda(1-t)g(t)\, dt + \lambda(1-s)s\, g(s)$$

$$\therefore g^{11}(s) = -\lambda s g(s) - \lambda(1-s)g(s)$$

$$\therefore g^{11}(s) + \left[\lambda s + \lambda(1-s)\right]g(s) = o$$

$$\left.\begin{aligned}&\therefore g^{11}(s)+\lambda g(s)=0\\ &\text{and } g(o)=g(1)=0\end{aligned}\right\}\quad \text{-----(2)}$$

Case 1 : $\lambda = o \Rightarrow g^{11}(s) = o \Rightarrow g(s) = Ax + B$

$$\left.\begin{aligned}&o = g(o) = B\\ &\& o = g(1) = A + B\end{aligned}\right\}\Rightarrow A = B = O$$

$\therefore g(s) = o$ Not an eigen function

Case 2 : $\lambda = -k^2; k \neq o$

$\therefore g^{11} - K^2 g = o$

Auxillary equation $\therefore D^2 - k^2 = o \Rightarrow D = \pm k$

Solution is

$g(s) = Ae^{-ks} + Be^{ks}$

$\therefore o = g(o) = A + B \Rightarrow A = -B$

$o = g(1) = Ae^{-K} + Be^{K}$

$\therefore B(e^k - e^{-k}) = B = o \; (\because e^k - e^{-k} \neq o)$

$\therefore A = B = o \therefore g(s) = o$ not an eigen functions.

Case 3 : $\therefore \lambda = k^2; k \neq o$

$\therefore g^{11} + K^2 g = o$

A. E. is $\therefore D^2 = -K^2 \Rightarrow D = \pm ik$

$g(s) = A\cos Ks + B\sin Ks$

$\therefore o = g(o) = A$

$\therefore g(s) = B\sin ks$

$0 = g(1) = B\sin k \;\Rightarrow \sin k = 0$

$\left(\because B = o \text{ gives g(s) = o; not eigen function}\right)$

$\therefore k = n\pi$

$\therefore \lambda = K^2 = n^2\pi^2$

$\therefore g(s) = B\sin n\pi s$

Eigen values are $\lambda_n = n^2\pi^2$ and corresponding eigen function are

$g_n(s) = \sin n\pi s; \quad n = 1, 2, 3...$

Corrosponding normalized eigen function are

$$\phi_n(s) = \frac{g_n(s)}{\|g_n(s)\|}$$

$$= \frac{\sin n\pi s}{\left[\int_o^1 [\sin n\pi s]^2\, ds\right]^{\frac{1}{2}}}$$

$$= \frac{\sin n\pi s}{\left[\int_o^1 \frac{1-\cos 2n\pi s}{2}\, ds\right]^{\frac{1}{2}}}$$

$$= \frac{\sin n\pi s}{\sqrt{\frac{1}{2}\left[s - \frac{\sin 2n\pi s}{2n\pi}\right]_0^1}}$$

$$= \frac{\sin n\pi^{s}}{\sqrt{\frac{1}{2}}} = \sqrt{2}\sin n\pi s.$$

$\phi_n(s) = \sqrt{2}\sin n\pi s; n = 1, 2, 3...$

$$\therefore fn = (f, \phi_n(s)) = \int_0^1 f(s)\phi_n^*(s)ds$$

$$= \sqrt{2}\int_0^1 f(s)\sin n\pi s\; ds$$

Required solution is

$$g(s) = f(s) + \lambda \sum_{n=1}^{\infty} \frac{f_n}{\lambda_n - \lambda} \phi_n(s)$$

$$= f(s) + \sum_{n=1}^{\infty} \frac{\sqrt{2}\left[\int_0^1 f(s)\sin n\pi s ds\right]}{n^2\pi^2 - 1}\sqrt{2}\sin n\pi s \qquad (\because \lambda = 1)$$

$$\therefore g(s) = f(s) + 2\sum_{n=1}^{\infty} \frac{\sin n\pi s\left[\int_0^1 f(s)\sin n\pi s ds\right]}{n^2\pi^2 - 1}$$

and solution exists if the series on R.H.S is convergent

Exercise :

Using Hilbert Schmidt theorem solve the following integral equations.

1) $g(s) = f(s) + \lambda \int_o^{2\pi} \sin(s+t)\; g(t)\; dt$

2) $g(s) = e^s + \lambda \int_0^1 k(s,t)\; g(t)\; dt$

$$K(s,t) = \begin{cases} \dfrac{\sinh s \sinh(t-1)}{\sinh 1}; o \le s \le t \\ \dfrac{\sinh t \sinh(s-1)}{\sinh 1}; t \le s \le 1 \end{cases}$$

3) $g(s) = \cos \pi s + \lambda \int_0^1 k(s,t) g(t) dt$

$$k(s,t) = \begin{cases} (s+1)t; o \le s \le t \\ (t+1)s; t \le s \le 1 \end{cases}$$

4) $g(s) = s + \lambda \int_0^1 k(s,t) g(t) dt$

$$k(s,t) = \begin{cases} s(t-1); o \le s \le t \\ t(s-1); t \le s \le 1 \end{cases}$$

5) $g(s) = s + \lambda \int_0^1 (s+t) g(t)\, dt$

6) $g(s) = s + \lambda \int_0^1 g(t) dt$

7) $g(s) = s + \lambda \int_0^1 tsg(t) dt$

8) $g(s) = \frac{1}{2} - s + \int_0^1 g(t) dt$

♦♦♦

Unit – 8

INTEGRAL TRANSFORM METHOD

8.1 Laplace Transform :

If f(t) be a function of t defined for all values of t, then laplace transform of f(t) denoted by L[f(t)] or $\overline{f}(p)$ is defined by

$$L[f(t)] = \int_o^\infty e^{-pt} f(t)dt$$

Provided the integral exists for some values of p.

1. The Laplace transformation is a linear transformation.

$$L[c_1 f(t) + c_2 g(t)] = C_1 L[f(t)] + C_2 L[g(t)]$$

Where C_1 and C_2 are constants.

2. Laplace Transform of elementary functions.

Table : 1

f(t)	L[f(t)]	f(t)	L[f(t)]
1	$\frac{1}{p}; p > o$	sin h at	$\frac{a}{p^2 - a^2}; p^2 > a^2$
t^n ; n=0, 1, 2, ...	$\frac{n!}{p^{n+1}}; p > 0$	cos h at	$\frac{p}{p^2 - a^2}; p^2 > a^2$
t^a ; a> - 1	$\frac{\Gamma(a+1)}{P^{a+1}}, P > o$	Jo (at) Bessels function	$\frac{1}{\sqrt{p^2 - a^2}}$

e^{at}	$\frac{1}{p-a}; p > a$	s(t-a) Impulse function	e^{-ap}
sin at	$\frac{a}{p^2+a^2}; p > o$		
cos at	$\frac{p}{p^2+a^2}; p > o$		

3. First shifting theorem :

If $L\left[f(t)\right] = \overline{f}(p)$ then

$$L\left[e^{at}f(t)\right] = \overline{f}(p-a)$$

4. Second shifting theorem :

If $L\left[f(t)\right] = \overline{f}(p)$ and $\overline{g}(t)$ is

function defined by

$$g(t) = \begin{cases} f(t-a); t \geq a \\ o \quad ; t < a \end{cases}$$

then

$$L[g(t)] = e^{-ap}\,\overline{f}(p)$$

5. Change of scale property:

If $L\left[f(t)\right] = \overline{f}(p)$ then

$$L\left[f(at)\right] = \frac{1}{a}\overline{f}(\frac{p}{a})$$

6. Laplace transform of t^n f(t)

If $L\left[f(t)\right]=\overline{f}(p)$

then $L\left[t^n f(t)\right]=(-1)^n \frac{d^n}{dp^n}\overline{f}(p)$

7. Laplace transform of $\frac{f(t)}{t}$:

If $L\left[f(t)\right]=\overline{f}(p)$

then $L\left[\frac{f(t)}{t}\right]=\int_p^\infty \overline{f}(p)dp$

Note: $L\left[\frac{f(t)}{t^n}\right]=\int_p^\infty \overline{f}(p)dp^n$

8. Laplace transform of derivative:

$L\left[f'(t)\right]=PL\left[f(t)\right]-f(o)$

$\Rightarrow$ where $f(0)=\lim_{t\to 0} f(t)$

$L\left[f''(t)\right]=P^2L\left[f(t)\right]-Pf(o)-f^1(o)$

$L\left[f'''(t)\right]=P^3L\left[f(t)\right]-P^2f(o)-Pf'(o)-f^{11}(o)$

9. Laplace transform of Integral :

$L\left[\int_o^t f(x)dx\right]=\frac{1}{P}L\left[f(t)\right]=\frac{1}{p}\overline{f}(p)$

10. Laplace transform of periodic function :

Let f(t) Is periodic function with period T > 0, i.e. f(t+ T) = f(t)

Then $L[f(t)] = \frac{1}{1-e^{-PT}} \int_{o}^{T} e^{-pt} f(t)dt$

11. Convolution theorem :

Let $L[f(t)] = \overline{f}(p)$ *and* $L[g(t)] = \overline{g}(p)$

then $L\left[\int_{o}^{t} f(x)g(t-x)dx\right] = L[f(x) * g(x)] = \overline{f}(p).\overline{g}(p)$

OR

If $L^{-1}\left[\overline{f}(p)\right] = f(t)$ and $L^{-1}\left[\overline{g}(p)\right] = g(t)$

Then $L^{-1}\left[\overline{f}(p)\overline{g}(p)\right] = \int_{o}^{t} f(x)g(t-x)dx$

12. Inverse Laplace Transform:

If $L[f(t)] = \overline{f}(p)$ then $L^{-1}\left[\overline{f}(p)\right] = f(t)$

13. L^{-1} is an Linear operator:

For any constant C_1 and C_2

$$L^{-1}[C_1 f(t) + C_2 g(t)] = C_1 L^{-1}[f(t)] + C_2 L^{-1}[g(t)]$$

Some elementary Inverse Laplace Transform :

Table : 2

f(p)	$L^{-1}\left[\overline{f}(p)\right] = f(t)$
$\frac{1}{p}$	1
$\frac{1}{p^n}; n = 1, 2, 3...$	$\frac{t^{n-1}}{(n-1)!}$

$\frac{1}{p-a}$	e^{at}
$\frac{1}{p^{\alpha}}$	$\frac{t^{\alpha-1}}{\Gamma^{(\alpha)}}$
$\frac{1}{p^2+a^2}$	$\frac{\sin at}{a}$
$\frac{P}{p^2+a^2}$	cos at
$\frac{1}{p^2-a^2}$	$\frac{\sinh at}{a}$
$\frac{P}{p^2-a^2}$	cos h at
$\frac{1}{\sqrt{P^2+a^2}}$	Jo (at)
$\overline{e}^{ap}$	δ (t-a)

Note : $L^{-1}\left[\frac{\overline{f}(p)}{p^n}\right] = \int_o^t f(t)dt^n$

If $L^{-1}\left[\overline{f}(p)\right] = f(t)$

Method of finding Inverse L.T.

1. By property of L.T.

2. By method of partial fraction

3. By convolution theorem

8.2 Solution of volterra integral equation with convolution type kernel by method of Laplace transform.

Consider the volterra integral equation of first kind with convolution type kernel.

$$g(s) = f(s) + \int_o^s K(s-t)g(t)dt$$

$$= f(s) + k(s) * g(s)$$

Taking Laplace transform of both sides assuming

$L[g(s)] = \overline{g}(p), L[f(s)] = \overline{f}(p)$ *and* $L[k(s)] = \overline{k}(p)$ and using convolution theorem.

$$\overline{g}(p) = \overline{f}(p) + \overline{k}(p)\overline{g}(p)$$

$$\therefore \overline{g}(p) = \frac{\overline{f}(p)}{1-\overline{k}(p)}$$

Taking the inverse laplace transform of both side we get the required solution.

$$g(t) = L^{-1}\left[\frac{\overline{f}(p)}{1-\overline{k}(p)}\right]$$

Problem 1 : Solve the integral equation.

$$s = \int_o^s e^{s-t} g(t)dt$$

Solution : $s = \int_o^s e^{s-t} g(t)dt$

$$= e^s * g(s)$$

Taking Laplace transform on both side

$$L[s] = L\left[e^s * g(s)\right] = L\left[e^s\right] L\left[g(s)\right]$$

$$\frac{1}{P^2} = \frac{1}{p-1}\overline{g}(p)$$

$$\therefore \overline{g}(p) = \frac{p-1}{p^2} = \frac{1}{p} - \frac{1}{p^2}$$

Taking inverse laplace transform we get g(s) = 1 - s

Problem : 2 Solve $\mathbf{g(s) = s + 2\int_0^s \cos(s - t)g(t)dt}$

Solution : $g(s) = s + 2\int_o^s \cos(s-t)g(t)dt$

$$= s + 2\cos s * g(s)$$

Taking Laplace transform on both side we get

$$\bar{g}(p) = \frac{1}{p^2} + 2\frac{p}{p^2+1}\bar{g}(p)$$

$$\therefore \bar{g}(p)\left[1 - \frac{2p}{p^2+1}\right] = \frac{1}{p^2}$$

$$\therefore \bar{g}(p) = \frac{p^2+1}{p^2(p-1)^2}$$

By partial fraction

$$\bar{g}(p) = \frac{2}{p} + \frac{1}{p^2} - \frac{2}{p-1} + \frac{2}{(p-1)^2}$$

on inversion, we get.

g(s) = 2 + s - 2 e^s + 2s e^s

Problem 3 : Solve sins = $\mathbf{\int_0^s J_o(s - t)g(t)dt}$

$$= Jo\ (s) * g(s)$$

Taking laplace transform on both side

L [sin s] = L [Jo(s) * g(s)] = L [Jo(s)] L [g(s)]

$$\frac{1}{p^2+1}=\frac{1}{\sqrt{p^2+1}}\bar{g}(p)$$

$$\therefore \bar{g}(p)=\frac{\sqrt{p^2+1}}{p^2+1}=\frac{1}{\sqrt{p^2+1}}$$

$$\therefore g(s)=L^{-1}\left[\frac{1}{\sqrt{p^2+1}}\right]=Jo(s)$$

Problem 4 : Solve the integral equation

$$g(s)=1-\int_o^s (s-t)g(t)dt$$

Solution : $g(s)=1-\int_o^s (s-t)g(t)dt$

$$=1-s*g(s)$$

$$\therefore L[g(s)]=L[1]-L[s]L[g(s)]$$

$$\bar{g}(p)=\frac{1}{p}-\frac{1}{p^2}\bar{g}(p)$$

$$\therefore \bar{g}(p)\left[1+\frac{1}{p^2}\right]=\frac{1}{p}$$

$$\therefore \bar{g}(p)=\frac{1}{p}\frac{p^2}{p^2+1}=\frac{p}{p^2+1}$$

Taking inverse laplce transform g(s) = Coss.

Problem 5 : Solve $\mathbf{g(s) = s^2 + \int_0^s \sin(s - t)g(t)dt}$

Solution : $g(s) = s^2 + \int_o^s \sin(s-t)g(t)dt$

$$= s^2 + \sin s * g(s)$$

Taking Laplace transform, we get

$$L[g(s)] = L[s^2] + L[\sin s]L[g(s)]$$

$$\overline{g}(p) = \frac{2}{p^3} + \frac{1}{p^2+1}\overline{g}(p)$$

$$\therefore \overline{g}(p)\left[1 - \frac{1}{p^2+1}\right] = \frac{2}{p^3}$$

$$\therefore \overline{g}(p) = \frac{2(p^2+1)}{p^5} = \frac{2}{p^3} + \frac{2}{p^5}$$

$$\therefore g(s) = L^{-1}\left[\frac{2}{p^3}\right] + L^{-1}\left[\frac{2}{p^5}\right]$$

$$= 2\frac{s^2}{2!} + 2\frac{s^4}{4!}$$

$$g(s) = s^2 + \frac{s^4}{12}$$

Problem 6 : Solve $\int_0^s \mathbf{g(t)g(s-t)dt = 4\sin gs}$

Solution : $\int_o^s g(t)g(s-t)dt = 4\sin gs$

$$\Rightarrow g(s) * g(s) = 4\sin gs$$

Taking Laplace transform on both side

L[g(s)] L [g(s)] = 4 L [sin gs]

$$\therefore \left[\overline{g}(p)\right]^2 = 4\left[\frac{g}{p^2+81}\right]$$

$$\therefore \overline{g}(p) = \sqrt{\frac{36}{p^2+81}} = \pm\frac{6}{\sqrt{p^2+81}}$$

$$\therefore g(s) = \pm 6L^{-1}\left[\frac{1}{\sqrt{p^2+81}}\right]$$

$$g(s) = \pm 6Jo(gs)$$

is the required solution.

8.3 Solution of Integro - differential equation by Laplace transform method.

Problem 1 : Solve $\mathbf{g^1(s) + 3g(s) + 2\int_0^s g(t)dt = s}$

given g(0)=1

Solution : $g'(s) + 3g(s) + 2\int_0^s g(t)dt = s$

Taking Laplace transform an both side we get

$$L\left[g'(s)\right] + 3L\left[g(s)\right] + 2L\left[\int_o^s g(t)dt\right] = L\left[s\right]$$

$$\left[p\overline{g}(p) - g(o)\right] + 3\overline{g}(p) + 2\frac{1}{p}\overline{g}(p) = \frac{1}{p^2}$$

$$\left[p + 3 + \frac{2}{p}\right]\overline{g}(p) = 1 + \frac{1}{p^2}\left[\because g(o) = 1\right]$$

$$\therefore \overline{g}(p) = \frac{p^2+1}{p^2}\frac{p}{p^2+3p+2}$$

$$=\frac{(p^2+1)}{P(p+1)(p+2)}$$

$$=\frac{1}{2}\frac{1}{p}-\frac{2}{p+1}+\frac{5}{2}\frac{1}{p+2}$$

Taking inverse Laplace transform we get

$$\therefore g(s)=\frac{1}{2}L^{-1}\left[\frac{1}{p}\right]-2L^{-1}\left[\frac{1}{p+1}\right]+\frac{5}{2}L^{-1}\left[\frac{1}{p+2}\right]$$

$$=\frac{1}{2}-2e^{-s}+\frac{5}{2}e^{-2s}$$

is the required solution.

Problem 2 : Solve the integro differential equation

$$g''(s)+\int_o^s e^{2(s-t)}\phi'(t)dt=1$$ subject g(0) = 0, g'(0) = 0

Solution : $g''(s)+\int_o^s e^{2(s-t)}9'(t)dt=1$

Which can be written as

$$g''(s)+e^{2s}*9'(s)=1$$

Taking Laplace transform on both side we have

$$L\left[g''(s)\right]+L\left[e^{2s}\right]L\left[g'(s)\right]=L[1]$$

$$\left[p^2\overline{g}(p)-pg(o)-g'(o)\right]+\frac{1}{p-2}\left[p\overline{g}(p)-g(o)\right]=\frac{1}{p}$$

$$p^2\overline{g}(p)+\frac{p}{p-2}\overline{g}(p)=\frac{1}{p}$$

$\left[\because g(o) = g'(o) = o\right]$

$$\left[p^2 + \frac{p}{p-2}\right]\bar{g}(p) = \frac{1}{p}$$

$$\left[\frac{p^3 - 2p^2 + p}{p-2}\right]\bar{g}(p) = \frac{1}{p}$$

$$\therefore \bar{g}(p) = \frac{p-2}{p^2(p^2-2p+1)} = \frac{p-2}{p^2(p-1)^2}$$

$$\therefore \bar{g}(p) = \frac{-3}{p} - \frac{2}{p^2} + \frac{3}{p-1} - \frac{1}{(p-1)^2} \qquad [\because \text{ partial fraction}]$$

Taking inverse laplace transform on both side

g(s) = 3 - 2s + 3 e^s - s e^s

= 3 - 2s + (3 - s) e^s

is the required solution.

Problem 3 : Solve $y'(x) = 2 + 3\int_0^x \cos 2(x-t)y(t)dt$ with $y(0) = 1$

Solution : The given equation can be written as $\Rightarrow y'(x) = 2 + 3\cos 2x * y(x)$

Taking Laplace transform on bs

$$L[y'(x)] = \frac{2}{s} + 3\,L[\cos 2x]L[y(x)]$$

$$s\bar{y}(s) - y(o) = \frac{2}{s} + \frac{3s}{s^2+4}\bar{y}(s)$$

Putting y(0) = 1 and sloving for $\bar{y}(s)$

$$\bar{y}(s) = \frac{(s+2)(s^4+4)}{s^2(s^2+1)}$$

$$=\frac{4}{s}+\frac{8}{s^2}-3\frac{s}{s^2+1}-6\frac{1}{s^2+1}$$

on inversion we obtain

$$\Rightarrow y(x)=4+8x-3\cos x-6\sin x$$

is the required solution

Problem 4 : Solve $\mathbf{g'(s) = s + \int_o^s costg(s - t)dt, g(o) = 4}$

Solution : $g'(s)=s+\int_o^s \cos t g(s-t)dt$

$$=s+\cos s * g(s)$$

Taking Laplace transform on both side

$$\therefore L[g'(s)]=L[s]+L[\cos s]L[g(s)]$$

$$\left[p\overline{g}(p)-g(0)\right]=\frac{1}{p^2}+\frac{p}{p^2+1}\overline{g}(p)$$

$$\therefore \overline{g}(p)\left[p-\frac{p}{p^2+1}\right]=\frac{1}{p^2}+4\left[\because g(o)=4\right]$$

$$\therefore \overline{g}(p)=\frac{p^2+1}{p^3}\left[\frac{1}{p^2}+4\right]$$

$$=\left(\frac{1}{p}+\frac{1}{p^3}\right)\left(\frac{1}{p^2}+4\right)$$

$$=\frac{1}{p^5}+\frac{5}{p^3}+\frac{4}{p}$$

$$\therefore g(s) = L^{-1}\left[\frac{1}{p^5}\right] + 5L^{-1}\left[\frac{1}{p^3}\right] + 4L^{-1}\left[\frac{1}{p}\right]$$

$$= \frac{s^4}{24} + \frac{5}{2}s^2 + 4$$

is the required solution.

Problem 5 : Solve $\mathbf{g'(s) = \int_o^s g(t)\cos(s - t)dt}$ Subject to g(0) =1

Solution : $g'(s) = \int_o^s g(t)\cos(s-t)dt = g(s) * \cos s$

$$\therefore L\left[g'(s)\right] = L\left[g(s)\right]L\left[\cos s\right]$$

$$p\overline{g}(p) - g(o) = \overline{g}(p)\frac{p}{p^2+1}$$

$$\overline{g}(p)\left[p - \frac{p}{p^2+1}\right] = 1\left[\because g(o) = 1\right.$$

$$\overline{g}(p)\left[\frac{p^3+p-p}{p^2+1}\right] = 1$$

$$\overline{g}(p) = \frac{p^2+1}{p^3} = \frac{1}{p} + \frac{1}{p^3}$$

$$\therefore g(s) = L^{-1}\left[\frac{1}{p}\right] + L^{-1}\left[\frac{1}{p^3}\right]$$

$$= 1 + \frac{s^2}{2!}$$

$$= 1 + \frac{s^2}{2}$$

is the required solution

Remark : i) The gamma function is definded as

i) $\Gamma(n) = \int_{o}^{\infty} e^{-x} x^{n-1} dx; (n \succ o)$

ii) $\Gamma(n+1) = n\Gamma(n)$

iii) $\Gamma(n+1) = n! \quad ; n \in N$

iv) $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$

v) $\Gamma(\alpha)\Gamma(1-\alpha) = \frac{\pi}{\sin \alpha\pi}$

8.4 Solution of Able integral equation by laplace transform method.

Problem 1 : Solve the Abel integral equation

$$f(s) = \int_{o}^{s} \frac{g(t)}{(s-t)^{\alpha}} dt; o < \alpha < 1$$

Solution : $f(s) = \int_{o}^{s} (s-t)^{-\alpha} g(t)\, dt; o < \alpha < 1$

$= s^{-\alpha} * g(s)$

Taking Laplace transform of both side

$$L[f(s)] = L\left[s^{-\alpha} * g(s)\right] = L\left[s^{-\alpha}\right] L[g(s)]$$

$$\overline{f}(p) = \frac{\Gamma(-\alpha+1)}{p^{-\alpha+1}} . \overline{g}(p)$$

$$\therefore \overline{g}(p) = \frac{p^{1-\alpha}}{\Gamma(1-\alpha)} \overline{f}(p)$$

$$= \frac{p}{\Gamma(\alpha)\Gamma(1-\alpha)} \left[P^{-\alpha} \Gamma(\alpha) \overline{f}(p) \right]$$

Using

$$\left[(\alpha)\Gamma(1-\alpha) = \frac{\pi}{\sin \alpha\pi} \right]$$

$$\therefore \overline{g}(p) = \frac{p}{\left[\dfrac{\pi}{\sin \alpha\pi} \right]} \left[p^{-\alpha} \Gamma(\alpha) \overline{f}(p) \right]$$

$$= \frac{\sin \alpha\pi}{\pi} p \left[\frac{\Gamma(\alpha)}{p^{\alpha}} . \overline{f}(p) \right] \qquad \text{-----(1)}$$

Now $L^{-1} \left[\overline{f}(p) \right] = f(t)$

and $L^{-1} \left[\dfrac{\Gamma(\alpha)}{p^{\alpha}} \right] = t^{\alpha-1}$

By convolution transform

$$L^{-1} \left[\frac{\Gamma(\alpha)}{p^{\alpha}} \overline{f}(p) \right] = \int_{o}^{s} (s-t)^{\alpha-1} f(t) dt$$

$$i.e. \left[\frac{\Gamma(\alpha)}{p^{\alpha}} \overline{f}(p) \right] = L \left[\int_{o}^{s} (s-t)^{\alpha-1} f(t) dt \right]$$

thus equation (1) becomes

$$\overline{g}(p) = \frac{\sin \alpha\pi}{\pi} PL \left[\int_{o}^{s} (s-t)^{\alpha-1} f(t) dt \right]$$

$$\Rightarrow g(s) = \frac{\sin \alpha\pi}{\pi} L^{-1} \left\{ PL \left[\int_{o}^{s} (s-t)^{\alpha-1} f(t) dt \right] \right.$$

We know,

$$L^{-1}\left[PL\left[f(s)\right]\right]=f'(s)=\frac{d}{ds}f(s) \text{ if f(0) =0}$$

Provided $f(0)=0$

$$g(s)=\frac{\sin\alpha\pi}{\pi}\frac{d}{ds}\left[\int_o^s (s-t)^{\alpha-1}f(t)dt\right]$$

is the required solution.

Problem 2 : Solve the Abel's Integral Equation

$$\int_o^5 \frac{g(t)}{(s-t)^{1/2}}dt=1+s+s^2$$

Solution : $1+s+s^2=s^{-1/2}*g(s)$

Taking Laplace transform of both side

$$L\left[1+s+s^2\right]=L\left[s^{-1/2}*g(s)\right]=L\left[s^{-1/2}\right]L\left[g(s)\right]$$

$$\frac{1}{p}+\frac{1}{p^2}+\frac{2}{p^3}=\frac{\Gamma(\frac{1}{2})}{p^{1/2}}\overline{g}(p)$$

$$\therefore \overline{g}(p)=\left[\frac{1}{p^{1/2}}+\frac{1}{p^{3/2}}+\frac{2}{p^{5/2}}\right]\frac{1}{\sqrt{\pi}}$$

$$g(s)=L^{-1}\left[\overline{g}(p)\right]$$

$$=\frac{1}{\sqrt{\pi}}\left[\frac{s^{-1/2}}{\sqrt{(1/2)}}+\frac{s^{1/2}}{\sqrt{(3/2)}}+2\frac{s^{3/2}}{\sqrt{(5/2)}}\right]$$

$$\left[\therefore L^{-1}\left[\frac{1}{p^{\alpha}}\right]=\frac{t^{\alpha-1}}{\overline{|(\alpha)}}\right] \qquad \text{-----(1)}$$

Using $\Gamma(n+1)=n\Gamma(n) and \Gamma\left(\frac{1}{2}\right)=\sqrt{\pi}$

We have $\Gamma\left(\frac{3}{2}\right)=\frac{1}{2}\Gamma\left(\frac{1}{2}\right)=\frac{\sqrt{\pi}}{2}$

$$\Gamma\left(\frac{5}{2}\right)=\frac{3}{2}\Gamma\left(\frac{3}{2}\right)=\frac{3}{2}\frac{\sqrt{\pi}}{2}=\frac{3\sqrt{\pi}}{4}$$

(1) becomes

$$g(s)=\frac{1}{\sqrt{\pi}}\left[\frac{s^{-1/2}}{\sqrt{\pi}}+\frac{2s^{1/2}}{\sqrt{\pi}}+\frac{2(4)}{3\sqrt{\pi}}s^{3/2}\right]$$

$$=\frac{1}{\pi}\left[s^{-1/2}+2s^{1/2}+\frac{8}{3}s^{3/2}\right]$$

is the required answer.

Problem 3 : Solve the Abel's integral

$$\int_{o}^{s}\frac{g(s)}{(s-t)^{1/3}}dt=s(1+s)$$

Solution : The given integral equation can be written as

$$\Rightarrow g(s)*s^{-1/3}=s+s^{2}$$

Taking Laplace transform on both side

$$\bar{g}(p)\frac{\Gamma\left(\frac{2}{3}\right)}{p^{2/3}}=\frac{1}{p^{2}}+\frac{2}{p^{3}}$$

$$\therefore \bar{g}(p) = \frac{1}{\Gamma\left(\frac{2}{3}\right)}\left[\frac{p^{2/3}}{p^2} + 2\frac{p^{2/3}}{p^3}\right]$$

$$= \frac{1}{\Gamma\left(\frac{2}{3}\right)}\left[\frac{1}{p^{4/3}} + \frac{2}{p^{7/3}}\right]$$

Taking inverse Laplace transform we get

$$\therefore g(s) = \frac{1}{\Gamma\left(\frac{2}{3}\right)}\left[\frac{s^{4/3-1}}{\Gamma\left(\frac{4}{3}\right)} + \frac{s^{7/3-1}}{p^{7/3}}\right]$$

$$= \frac{1}{\Gamma^{\left(2/3\right)}}\left[\frac{s^{1/3}}{\Gamma^{\left(4/3\right)}} + \frac{s^{4/3}}{\Gamma^{\left(7/3\right)}}\right]$$

is the required solution.

Exercise :

Q.1 Solve the following integral equation by method of Laplace transform.

a) $g(s) = 1 + \int_o^s \sin(s-t)g(t)dt$

b) $g(s) = e^{-s} - 2\int_o^1 \cos\,(s-t)g(t)dt$

c) $g(s) = 1 - \int_o^s (s-t)g(t)dt$

d) $g(s) = a\sin s - 2\int_o^1 \cos(s-t)g(t)dt$

e) $g(s) = e^s - \int_o^s e^{s-t} g(t)dt$

f) $g(s) = \cos s - \int_o^s (s-t)\cos(s-t) g(t)dt$

g) $g(s) = \sin s + \int_o^s (s-t) g(t)dt$

Q. 2 Solve the following integro - differential equation by method of Laplace transform.

a) $g'(s) = \sin s + \int_0^s \cos t \; g(s-t)dt,$ subject to $g(0) = 0$

b) $g''(s) + \int_0^s \mathrm{e}^{2(s-t)} g'(s)\mathrm{dt} = \mathrm{e}^{2s}$ subject to $g'(0) = 0 = g(0)$

c) $g''(s) + 2g'(s) - 2\int_0^s \sin\,(s\text{-}t)\; g'(s)\mathrm{dt} = \cos s$ subject to $g'(0) = 0 = g(0)$

d) $g'(s) = \sin s + \int_0^s \cos\, t\; u(s\text{-}t)\mathrm{dt},\; g(0) = 0$

Q. 3 Solve $g(s) = f(s) + \lambda \int_0^s \frac{g(t)}{(s-t)^\alpha} dt; 0 < \alpha < 1.$

8.5 Determination of Resolvent Kernel :

To find resolvent kernel of the volterra integral equation with a convolution type kernel by intergral transform method:

1. If original kernel k(s, t) is difference kernel k (s - t) then the resolvent is also a function of (s-t) i.e. of the form $\Gamma(s-t)$

2. The solution of volterra integral equation.

$$g(s) = f(s) + \int_0^s k(s-t)(t)dt \qquad \text{-----(1)}$$

is given by

$$g(s) = f(s) + \int_0^s \Gamma(s-t) f(t)dt \qquad \text{-----(2)}$$

Note : Equation (2) is called resolvent of integral equation (1)

Procedure to determine resolvent kernel :

Equation (1) can be written as

$$g(s) = f(s) + k(s) * g(s)$$

Taking laplace transform on both side and using convolution theorem we get

$$\therefore \overline{g}(p) = \overline{f}(p) + \overline{k}(p)\overline{g}(p)$$

$$\therefore \overline{g}(p)\left[1 - \overline{k}(p)\right] = \overline{f}(p)$$

$$\therefore \overline{g}(p) = \frac{\overline{f}(p)}{1 - \overline{k}(p)} \qquad \text{-----(3)}$$

Also, equation (2) can be written as

$$g(s) = f(s) + \Gamma(s) * f(s)$$

Taking laplace transform and using convolution theorem we have

$$\overline{g}(p) = \overline{f}(p) + L\left[\Gamma(s)\right]\overline{f}(p)$$

Using equation (3) we get

$$\frac{\overline{f}(p)}{1 - \overline{k}(p)} = \left\{1 + L\left[\Gamma(s)\right]\right\}\overline{f}(p)$$

$$\therefore \frac{1}{1 - \overline{k}(p)} = 1 + L\left[\Gamma(s)\right]$$

$$\therefore L[\Gamma(s)] = \frac{1}{1-\bar{k}(p)} - 1$$

$$\frac{\bar{k}(p)}{1-\bar{k}(p)}$$

$$\therefore \Gamma(s) = L^{-1}\left[\frac{\bar{k}(p)}{1-\bar{k}(p)}\right]$$

Which gives $\Gamma(s-t)$ the required resolvent kernel.

Problem 1 : Find the reslovent kernel for the following volterra type kernel by using method of Laplace transform.

a) k (s,t) = sin (s-t)

b) k (s,t) = s - t

Solution : a) k (s, t) = sin (s - t) $\Rightarrow$ k (s) = sin s

$$k(s,t) = \sin(s-t) \Rightarrow k(s) = \sin s$$

$$\therefore \bar{k}(p) = L[k(s)] = L[\sin s] = \frac{1}{p^2+1}$$

$$\therefore \Gamma(s) = L^{-1}\left[\frac{\bar{k}(p)}{1-\bar{k}(p)}\right]$$

$$= L^{-1}\left[\frac{\frac{1}{p^2+1}}{1-\frac{1}{p^2+1}}\right]$$

$$= L^{-1}\left[\frac{1}{p^2+1-1}\right] = L^{-1}\left[\frac{1}{p^2}\right]$$

= S

The resolvent kernel is

$\Gamma(s-t) = (s-t)$

b) k (s, t) = s - t $\Rightarrow$ k(s) = s

$$\bar{k}(p) = L[s] = \frac{1}{p^2}$$

$$\therefore \Gamma(s) = L^{-1}\left[\frac{\bar{k}(p)}{1-\bar{k}(p)}\right]$$

$$= L^{-1}\left[\frac{\frac{1}{p^2}}{1-\frac{1}{p^2}}\right]$$

$$= L^{-1}\left[\frac{1}{p^2-1}\right]$$

$$= \sin\text{h } s$$

$\Rightarrow \Gamma(s,t) = \sinh(s-t)$ is the resolvent kernel

Problem 2 : Find the resolvent of the integral equation

$$g(s) = f(s) + \int_o^s e^{s-t} g(t)dt$$

Solution : Given integral equation can be written as

$$g(s) = f(s) + \int_o^s e^{s-t} g(t)dt \qquad \text{-----(1)}$$

can be written as; $g(s) = f(s) + e^s * g(s)$

Taking Laplace transform using convolution theorem we get

$$\bar{g}(p) = \bar{f}(p) + \frac{1}{p-1} - \bar{g}(p)$$

$$\therefore \left[1 - \frac{1}{p-1}\right]\bar{g}(p) = \bar{f}(p)$$

$$\therefore \bar{g}(p) = \left(\frac{p-1}{p-2}\right)\bar{f}(p) \qquad \text{-----(2)}$$

The solution of equation (1) is given by

$$g(s) = f(s) + \int_o^s \Gamma(s-t) f(t)dt \qquad \text{-----(3)}$$

Where $\Gamma(s-t)$ is the resolvent kernel of $k(s,t) = e^{s-t}$

Now, equation (3) can be written as

$$g(s) = f(s) + \Gamma(s) * f(s)$$

Taking Laplace transform & using convolution theorem we get

$$\bar{g}(p) = \bar{f}(p) + L[\Gamma(s)]\bar{f}(p)$$

Using (2) we get

$$\left(\frac{p-1}{p-2}\right)\bar{f}(p) = \bar{f}(p)[1 + L[\Gamma(s)]]$$

$$\frac{p-1}{p-2} = 1 + L[\Gamma(s)]$$

$$\Rightarrow L[\Gamma(s)] = \frac{p-1}{p-2} - 1 = \frac{p-1-p+2}{p-2}$$

$$= \frac{1}{p-2}$$

$$\therefore \Gamma(s) = L^{-1}\left[\frac{1}{p-2}\right] = e^{2s}$$

$$\therefore \Gamma(s-t) = e^{2(s-t)}$$

Required resolvent kernel

Equation (3) becomes

$$g(s) = f(s) + \int_o^s e^{2(s-t)} f(t)dt$$

is the required solution.

Problem 3 : Find the resolvent of the intergral equation

$$g(s) = f(s) + \int_o^s (s-t)g(t)dt$$

Solution : The given IE is

$$g(s) = f(s) + \int_o^s (s-t)g(t)dt \qquad \text{-----(1)}$$

Which can be written as

g(s) = f(s) + s* g(s)

Taking Laplace transform we get

$$\overline{g}(p) = \overline{f}(p) + \frac{1}{p^2}\overline{g}(p)$$

$$\therefore \left(1 - \frac{1}{p^2}\right)\overline{g}(p) = \overline{f}(p)$$

$$\therefore \overline{g}(p) = \frac{p^2}{p^2-1}\overline{f}(p) \qquad \text{-----(2)}$$

The Solution of equation (1) is

$$g(s) = f(s) + \int_{o}^{s} \Gamma(s-t) f(t) dt \qquad \text{-----(3)}$$

Where $\Gamma(s-t)$ is the resolvent kernel of k(s,t) = s-t

Now, equation (3) can be written as

$$g(s) = -f(s) + \Gamma(s) * f(s)$$

Taking Laplace transform, we get

$$\bar{g}(p) = \bar{f}(p) + L[\Gamma(s)]\bar{f}(p)$$

Using (2) we have

$$\frac{p^2}{p^2-1} = 1 + L[\Gamma(s)]$$

$$\therefore L[\Gamma(s)] = \frac{p^2}{p^2-1} - 1$$

$$= \frac{p^2 - p^2 + 1}{p^2-1} = \frac{1}{p^2-1}$$

$$\therefore \Gamma(s) = L^{-1}\left[\frac{1}{p^2-1}\right] = \sinh s$$

$\therefore$ The resolvent kernel is

$$\Gamma(s-t) = \sinh(s-t)$$

Equation (3) becomes

$$g(s) = f(s) + \int_{0}^{s} \sinh(s-t) f(t) dt$$

is the required solution.

Problem 4 : Find the resolvent of integral equation

$$g(s) = f(s) + \lambda \int_0^s J_o(s-t)\, g(t)dt \qquad \text{-----(1)}$$

Solution : The given integral equation can be written as

$$g(s) = f(s) + \lambda Jo(s) * g(s)$$

Taking Laplace Transform on bothside, we get

$$\therefore L[g(s)] = L[f(s)] + \lambda L[J_0(s)] L[g(s)]$$

$$\overline{g}(p) = \overline{f}(p) + \lambda \left(\frac{1}{\sqrt{p^2+1}} \right) \overline{g}(p)$$

$$\therefore \overline{g}(p) \left(1 - \frac{\lambda}{\sqrt{p^2+1}} \right) = \overline{f}(p) \qquad \text{-----(2)}$$

Solution of IE (1) is given by

$$g(s) = f(s) + \lambda \int_o^s \Gamma(s-t) f(t)dt \qquad \text{-----(3)}$$

Where $\Gamma(s-t)$ is resolvent kernel

Now, IE (3) written as

$$g(s) = f(s) + \lambda \Gamma(s) * f(s)$$

Taxing L.T.

$$\therefore \overline{g}(p) = \overline{f}(p) + \lambda \overline{f}(p) L[\Gamma(s)]$$

$$= \overline{f}(p)[1 + \lambda L[\Gamma(s)]]$$

Using this is (II) we get

$$\overline{f}(p)\left[1+\lambda L\left[\Gamma(s)\right]\left(1-\frac{\lambda}{\sqrt{P^2+1}}\right)\right]=\overline{f}(p)$$

$$\left\{1+\lambda L\left[\Gamma(S)\right]\right\}\left(\frac{\sqrt{P^2+1}-\lambda}{\sqrt{P^2+1}}\right)=1$$

$$\therefore 1+\lambda L\left[\Gamma(s)\right]=\frac{\sqrt{p^2+1}}{\sqrt{p^2+1}-\lambda}$$

$$\therefore \lambda L\left[\Gamma(s)\right]=\frac{\sqrt{p+1}}{\sqrt{p^2+1}-\lambda}-1$$

$$=\frac{\sqrt{p+1}-\sqrt{p+1}+\lambda}{\sqrt{p+1}-\lambda}$$

$$=\frac{\lambda}{\sqrt{p+1}-\lambda}$$

$$\therefore L\left[\Gamma(s)\right]=\frac{1}{\sqrt{p+1}-\lambda}$$

$$\therefore \Gamma(s)=L^{-1}\left[\frac{1}{\sqrt{p+1}-\lambda}\right]$$

$=\Omega(s)$ ...(say)

$\therefore Then\Gamma(s-t)=\Omega(s-t)$

Required solution is

$$g(s)=f(s)+\lambda\int_o^s\Omega(s-t)f(t)dt$$

Exercise :

Find the resolvent of the following integral equation.

1) $g(s) = f(s) + \int_o^s (s^2 - t^2) g(t) dt$

2) $g(s) = f(s) + \int_o^s \frac{(s-t)^{m-t}}{(m-1)!} g(t)\, dt$

8.4 Fourier Transform :

Notations : We will denote the Fourier transform by F[f (t)] or F(P) Fourier cosine transform by F_c [f(t)] or Fc (p) and Fourier sine transform by Fs [f(t)]. All these transform are defined in the **Table 3**

We Recall some properties of Fourier transform.

1. Linearity property

F[a f(t) + b g(t)] = a F [f(t)] + b f [g(t)]

same is hold for Fourier sine and cosine transform.

2. Change of scale property :

$$F\left[f(at)\right] = \frac{1}{|a|} F\left[f(t)\right]_{s \to s/a}$$

same is true for Fourier sine and cosine transform.

3. Convolution :

The Convolution of two functions f(t) and g(t) over the interval $(-\infty, \infty)$ is defined as

$$f(t) * g(t) = \int_{-\infty}^{\infty} f(x) g(t-x) dx$$

4. Convolution theorem for Fourier Transform

F [f(t)* g(t)] = F [f(t)] F [g(t)]

Table of Fourier Transform (F.T.)

Name of F. T.	Interval	Fourier Transform	Inverse fourier Tranform
Complex Form of F.T.	$-\infty < t < \infty$	$F(p)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-ipt} f(t)dt$	$f(t)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{ipt} F(p)dp$
F. T. for even function	$-\infty < t < \infty$	$F(p)=\sqrt{\frac{2}{\pi}}\int_{0}^{\infty} f(t)\cos ptdt$	$f(t)=\sqrt{\frac{2}{\pi}}\int_{0}^{\infty} F(p)\cos ptdp$
F. T. for odd function	$-\infty < t < \infty$	$F(p)=\sqrt{\frac{2}{\pi}}\int_{o}^{\infty} f(t)\sin ptdt$	$f(t)=\sqrt{\frac{2}{\pi}}\int_{o}^{\infty} F(p)\sin ptdp$
Fourier cosine Troms from	$0 < t < \infty$	$F_c(p)=\sqrt{\frac{2}{\pi}}\int_{o}^{\infty} f(t)\cos ptdt$	$f(t)=\sqrt{\frac{2}{\pi}}\int_{o}^{\infty} F_c(p)\cos ptdp$
Fourier sine Trems form	$0 < t < \infty$	$F_s(p)=\sqrt{\frac{2}{\pi}}\int_{0}^{\infty} f(t)\sin ptdt$	$f(t)=\sqrt{\frac{2}{\pi}}\int_{o}^{\infty} F_s(p)\sin ptdp$

5. $F\left[f(t-a)\right]=e^{-iap}F\left[f(t)\right]$

6. $F\left[f'(t)\right]=ipF\left[f(t)\right]$

7. $F\left[f^{(k)}(t)\right]=(ip)^k F\left[f(t)\right]$

8. $if \quad h(t)=\int_{o}^{t} f(x)dx$

then $F\left[h(t)\right]=\frac{1}{ip}\mathrm{T}\left[f(t)\right]$

8.5 Solution by fourier transfrom method :

Problem 1 : Solve the integral equation by method of Fourier transform

$$\int_o^\infty f(x)\cos pxdx = e^{-p} \qquad \text{-----(1)}$$

Solution : We know Fourier cosine transform is

$$Fc(p) = \sqrt{\frac{2}{\pi}}\int_o^\infty f(x)\cos pxdx$$

$$= \sqrt{\frac{2}{\pi}}e^{-p}\left[\because ea^n(1)\right]$$

using corresponding inverse transform.

$$f(x) = \sqrt{\frac{2}{\pi}}\int_o^\infty F_c(p)\cos px\ dp$$

$$= \sqrt{\frac{2}{\pi}}\int_o^\infty \sqrt{\frac{2}{\pi}}e^{-p} \quad \cos px\ dp$$

$$= \frac{2}{\pi}\int_o^\infty e^{-p} \quad \cos px\ dp$$

$$= \frac{2}{\pi}\left[\frac{e^{-p}}{1+x^2}(-\cos px + x\sin px\right]_o^\infty$$

$$\left[\because \int e^{ap}\cos bpdp = \frac{e^{ap}}{a^{2+}b^2}\left[a\cos bp + b\sin bp\right]\right]$$

$$= \frac{2}{\pi}\left[o - \left(\frac{-1}{1+x^2}\right)\right]$$

$f(x)=\frac{2}{\pi}\frac{1}{1+x^2}$ is the required solution.

Problem 2 : Solve the integral equation

$$\int_0^\infty f(x)\cos pxdx=\begin{cases} I-p; 0\le p\le 1 \\ 0; \quad p>1 \end{cases} \qquad \text{-----(1)}$$

Solution : We know Fourier cosine transform is

$$F_c(p)=\sqrt{\frac{2}{\pi}}\int_o^\infty f(x)\cos pxdx$$

$$=\sqrt{\frac{2}{\pi}}\begin{cases} 1-p; 0\le p\le 1 \\ \;0\;;\lambda>1 \end{cases}$$

Using corresponding inverse transform

$$f(x)=\sqrt{\frac{2}{\pi}}\int_o^\infty Fc(P)\cos pxdp$$

$$=\sqrt{\frac{2}{\pi}}\left[\int_o^1 Fc(p)\cos px\,dp+\int_1^\infty Fc(p)\cos px\,dp\right]$$

$$=\sqrt{\frac{2}{\pi}}\int_o^1\sqrt{\frac{2}{\pi}}(1-p)\cos px\,dp$$

$$=\sqrt{\frac{2}{\pi}}\int_o^1 (1-p)\cos px\,dp$$

$$=\frac{2}{\pi}\left[(1-p)\frac{\sin px}{x}-(-1)\left(\frac{\cos px}{x^2}\right)\right]_0^1$$

$$=\frac{-2}{\pi}\left[\frac{\cos px}{x^2}\right]_o^1$$

$$= \frac{-2}{\pi}\left[\frac{\cos x - 1}{x^2}\right] = \frac{2}{\pi}\left[\frac{1-\cos x}{x^2}\right]$$

is the required solution.

Problem 3 : Solve the integral equation

$$\int_o^\infty f(x)\sin px\,dx = \begin{cases} 1\,;\,0 \le p < 1 \\ 2\,;\,1 \le p < 2 \\ 0\,;\,p \ge 2 \end{cases} \qquad \text{-----(1)}$$

Solution : Using Fourier sine transform

$$F_s(p) = \sqrt{\frac{2}{\pi}}\int_o^\infty f(x)\sin pxdx$$

$$= \sqrt{\frac{2}{\pi}}\begin{cases} 1\,;\,0 \le p < 1 \\ 2\,;\,1 \le p < 2 \\ 0\,;\,p \ge 2 \end{cases}$$

Using corresponding Inverse transform

$$f(x) = \sqrt{\frac{2}{\pi}}\int_o^\infty F_s(p)\sin px\,dp$$

$$= \sqrt{\frac{2}{\pi}}\left[\int_o^1 F_s(p)\sin pxdp + \int_1^2 F_s(p)\sin pxdp\right]$$

$$+\int_2^\infty F_s(p)\sin pxdp]$$

$$= \frac{2}{\pi}\left[\int_o^1 \sin px\,dp + \int_1^2 2\sin pxdp\right]$$

$$= \frac{2}{\pi}\left[-\left[\frac{\cos px}{x}\right]_o^1 + 2\left[\frac{-\cos px}{p}\right]_1^2\right]$$

$$= \frac{2}{\pi}\left[-\left[\frac{\cos x - 1}{x}\right] - 2\left[\frac{\cos 2x - \cos x}{x}\right]_1^2\right]$$

$$= \frac{2}{\pi}\left[\frac{-\cos x + 1 - 2\cos 2x + 2\cos x}{x}\right]$$

$f(x) = \frac{2}{\pi}\left[\frac{1 + \cos x - 2\cos 2x}{x}\right]$ is required solution.

Exercise :

1. Solve $\int_o^\infty f(t)\sin pt dt = \frac{p}{p^2 + a^2}$

2. Solve $\int_o^\infty f(t)\cos pt dt = \frac{1}{p^2 + a^2}$

♦♦♦

Unit – 9

GREEN'S FUNCTION

9.1 Introduction :

Since their introduction in 1828, Green's Function are very powerful mathematical tool for solving linear inhomogeneous differential equations. It is a basic solution of linear differential equation which can be used as building block to construct many useful solutions of the equation. The form of Green's function depends on the differential equation, and the boundary conditions to be satisfied by a solution of the equation. These functions are named in the honor of English mathematician and physicist George Green (1793 - 1841). The purpose of this unit is to introduce Green's Function and the method of constructing Green's Function corresponding to homogeneous differential equation with boundary conditions. We also discuss the use of Green's Function in solving inhomogeneous boundary value problems.

9.2 Motivation :

To understand the motivation behind Green's Function, we consider the B.V.P.

$$\frac{d^2\psi}{dx^2} + k^2\psi = -f(x), 0 \le x \le a \qquad \text{-----(1)}$$

$$\psi(0) = \psi(a) = 0$$

This B.V.P. arise in the study of a forced vibration of a string with fixed ends. We assume a solution of the form.

$$\psi(x) = A(x)\sin kx + B(x)\cos kx \qquad \text{-----(2)}$$

Where A(x) and B(x) to be determined with suitable restrictions.

$$\therefore \psi'(x) = A'(x)\sin kx + B'\cos kx + kA(x)\cos kx - kB(x)\sin kx$$

We assume that

$$A'(x)\sin kx + B'\cos kx = 0 \qquad \text{-----(3)}$$

$$\therefore \psi'(x) = kA(x)\cos kx - kB(x)\sin kx$$

Hence $\psi''(x) = kA'(x)\cos kx - kB'(x)\sin kx - k^2 A\sin kx - k^2 B\cos kx$

$\therefore By(1),$

$$kA'(x)\cos kx - kB'(x)\sin kx = -f(x) \qquad \text{-----(4)}$$

Thus $A'(x)$ and $B'(x)$ satify two linear equations (3) and (4)

$\therefore$ Solving these equations for $A'(x)$ and $B'(x)$ we get

$$A'(x) = -\frac{1}{k} f(x)\cos kx \quad \text{and} \quad B'(x) = \frac{1}{k} f(x)\sin kx$$

$$\therefore A(x) = -\frac{1}{k}\int_{c_1}^{x} f(y)\cos ky\, dy \quad \text{and} \quad B(x) = -\frac{1}{k}\int_{c_2}^{x} f(y)\sin ky\, dy$$

where C_1 and C_2 are constants to be determined

The solution of the BVP is of the form

$$\psi(x) = -\frac{1}{k}\sin kx \int_{c_1}^{x} f(y)\cos ky\, dy + \frac{1}{k}\cos kx \int_{c_2}^{x} f(y)\sin ky\, dy \qquad \text{-----(5)}$$

This ψ has to satisfy the boundary condition

$$\psi(0) = \psi(a) = 0$$

Hence we must choose C_1 and C_2 such that

$$0 = \psi(0) = \frac{1}{k}\int_{c_2}^{0} f(y)\sin ky\, dy \Rightarrow C_2 = 0$$

and $0 = \psi(a) = \frac{1}{k}\sin ka \int_{c_1}^{a} f(y)\cos ky\, dy + \frac{1}{k}\cos ka \int_{0}^{a} f(y)\sin ky\, dy$

$$\therefore -\frac{1}{k}\sin ka\int_{c_1}^{0} f(y)\cos ky\, dy + \frac{1}{k}\int_0^a (-\sin ka\cos ky + \cos ka\sin ky) f(y)\, dy = 0$$

or $$-\frac{1}{k}\int_{c_1}^{0} f(y)\cos ky\, dy = -\frac{1}{k}\int_0^a \frac{\sin k(y-a)}{\sin ka} f(y)\, dy$$

$\therefore$ *By* (5) *and* $C_2 = 0,$

$$\psi(x) = -\frac{1}{k}\sin kx\int_0^a \frac{\sin k(y-a)}{\sin ka} f(y)\, dy$$

$$-\frac{1}{k}\sin kx\int_0^x f(y)\cos ky\, dy + \frac{1}{k}\cos k\int_0^x f(y)\sin ky\, dy$$

$$\therefore \psi(x) = -\frac{1}{k}\frac{\sin kx}{\sin ka}\int_0^a \sin k(y-a) f(y) dy + \frac{1}{k}\int_0^x \sin k(y-x) f(y)\, dy$$

$$\therefore \psi(x) = \frac{1}{k}\int_0^x \left[\frac{\sin k(y-x)\sin ka - \sin kx\sin k(y-a)}{\sin ka}\right] f(y)\, dy$$

$$-\frac{1}{k}\int_x^a \frac{\sin kx\sin k(y-a)}{\sin ka} f(y) dy$$

$$or\ \ \psi(x) = \frac{1}{k}\int_0^x \left[\frac{\sin ky\sin k(a-x)}{\sin ka}\right] f(y)\, dy + \frac{1}{k}\int_x^a \left[\frac{\sin kx\sin k(a-y)}{\sin ka}\right] f(y) dy$$

Thus $\psi(x) = \int_0^a G(x,y) f(y)\, dy,$ Where

$$G(x,y) = \begin{cases} \dfrac{1}{k}\dfrac{\sin ky\sin k(a-x)}{\sin ka} & if\ 0 \le y < x \\[2ex] \dfrac{1}{k}\dfrac{\sin kx\sin k(a-y)}{\sin ka} & f\ \ x \le y \le a \end{cases}$$

is the solution of the B.V.P. (1) and (2)

The function G (x, y) defined above is called Green's Function for the B.V.P. (1) and (2) and (or Influence)

It exists if sin ka # 0

Thus we have reduced the problem of solving the inhomogeneous differential equation (1) with boundary conditions (2) to a simple formula.

$$\psi(x) = \int_0^a f(y)\, G(x, y) dy$$

The advantage of this formulation of the problem is that Green's function is independent of the forcing function f. That is it depends only on the form of the differential equation, k and the boundary conditions. Hence the solutions to all possible such problems with different forcing functions *f* are known, provided the integral $\int_0^a f(y)\, G(x, y) dy$ exists.

We also note that the function G(x, y) satisfies the properties :

1) G(x, y) is continuous at y = x

2) $\frac{\partial}{\partial x} G(x, y)$ is discontinuous at y = x

3) G (*x*, y) as a function of x satisfies the homogeneous differential equation $y'' + k^2 y = 0$ in each internal $0 \le x < y, y < x \le a$.

4) $G(x,0) = G(x,a) = 0$ that G satisfies the boundary conditions (2)

5) G is symmetric that is G(x, y) = G(y, x). Hence we take some of the above conditions (1-4) as defining conditions for Green's function for general BVP's.

9.3 Definition (Green's function) :

Consider the differential equation

$$L(y) = p_0(x) y^{(n)}(x) + p_1(x) y^{(n-1)}(x) + ... + p_n(x) y(x) = 0 \qquad \text{-----(1)}$$

where $p_o, p_1, p_2 ..., p_n$ are continuous functions on internal [a, b], $P_o(x) \neq 0$ on [a, b] with the boundary conditions :

$$V_k(y) = \alpha_k y(a) + \alpha_k' \; y'(a) + ... + \alpha_k^{n-1} y^{(n-1)}(a) + \beta k y(b) + \beta_k' y'(b) + ... + \beta_k^{n-1} y^{(n-1)}(b) = 0$$

$$k = 1, 2, ..., n \qquad \text{-----(2)}$$

Where the linear form, $V_1, V_2, ... V_n$ in y(a), y'(a), ...,$y^{(n-1)}$ (a), y(b), y' (b), ..., $y^{(n-1)}$(b) are linearly independent.

Green's function for the B.V.P. (1) and (2) is the function G(x,t) constructed for any point t, a < t < b such that,

(G_1) G(x, t) is continuous and has continuous derivatives w.r.t. x of order (n-2) on $a \leq x \leq b$

(G_2) The (n-1)th derivative of G(x, t) w.r.t. x at t = x has discontinuity of first kind and

$$G^{(n-1)}(t^+,t) - G^{(n-1)}(t^-,t) = \frac{1}{P_0(t)}$$

(G_3) G(x, t) as a function of x, is a solution of L(y) = o in each interval $a \leq x < t, t < x \leq b$

(G_4) G(x, t) satisfies the boundary conditions (2), that is $V_k(G) = 0, k = 1, 2, ..., n$

Note : If we taken $G\,(t^+,t) - G\,(t^-,t) = \dfrac{-1}{P_0(t)}$

we get - G(x, t) instead of G(x, t)

9.4 Existence and Uniqueness Thearem :

Theorem : If the B.V.P. (1) and (2) has only trivial solution y(x) = 0, then the operator L or the B.V.P. has one and only one Green's function G(x, t).

Proof : Let the B.V.P. (1) and (2) has only trivial solution y(x) = 0. Let $y_1, y_2, ..., y_n$ be linearly independent solutions of the equation L(y) = o. From the definition of Green's function, we want a function G(x, t) of the form

$$G(x,t) = \begin{cases} a_1 y_1(x) + a_2 y_2(x) + \cdots + a_n y_n(x), & if \;\; a \leq x < t \\ b_1 y_1(x) + b_2 y_2(x) + \cdots + b_n y_n(x), & if \;\; t < x \leq b \end{cases}$$

Where $a_1, a_2, ..., a_n, b_1, b_2, ..., b_n$ are functions of t and are to be so determined that G(x,t) satisfies the defining properties of Green's function.

By the defining condition (G_1)

$$[b_1 y_1(t) + \cdots + b_n y_n(t)] - [a_1 y_1(t) + \cdots + a_n y_n(t)] = 0$$
$$[b_1 y_1^1(t) + \cdots + b_n y_n^1(t)] - [a_1 y_1^1(t) + \cdots + a_n y_n^1(t)] = 0$$
$$\cdots\cdots \qquad \cdots\cdots\cdots\cdots\cdots\cdots \qquad \cdots\cdots\cdots\cdots$$
$$[b_1 y_1^{(n-2)}(t) + \cdots + b_n y_n^{(n-2)}(t)] - [a_1 y_1^{(n-2)}(t) + \cdots + a_n y_n^{(n-2)}(t)] = 0$$

and by second defining condition

$$[b_1 y_1^{(n-1)}(t) + \cdots + b_n y_n^{(n-1)}(t)] - [a_1 y_1^{(n-1)}(t) + \cdots + a_n y_n^{(n-1)}(t)] = \frac{1}{P_0(t)}$$

Let $C_k = b_k - a_k \quad for \quad k = 1, 2, ..., n$

$\therefore C_k$ satisfy the linear equations

$$[C_1 y_1(t) + C_2 y_2(t) + \cdots + C_n y_n(t)] = 0$$
$$[C_1 y_1'(t) + C_2 y_2'(t) + \cdots + C_n y_n'(t)] = 0$$
$$\cdots\cdots \qquad \cdots\cdots\cdots\cdots \qquad \cdots\cdots\cdots\cdots$$
$$[C_1 y_1^{(n-2)}(t) + C_2 y_2^{(n-2)}(t) + \cdots + C_n y_n^{(n-2)}(t)] = 0 \qquad \text{-----(4)}$$
$$[C_1 y_1^{(n-1)}(t) + C_2 y_2^{(n-1)}(t) + \cdots + C_n y_n^{(n-1)}(t)] = \frac{1}{P_0(t)}$$

The determinant of the above system of equations (4) is

$$\begin{vmatrix} y_1(t) & y_2(t)\cdots & y_n(t) \\ y_1'(t) & y_2'(t)\cdots & y_n'(t) \\ \vdots & & \\ y_1^{(n-1)}(t) & y_2^{(n-1)}(t)\cdots & y_n^{(n-1)}(t) \end{vmatrix} \neq 0$$

because it is the wronskian of L(y) = o at x = t

Therefore, the system (4) has unique solutions $C_k(t), k = 1, 2, ..., n.$

Now by the definining condition $(G_4), U_k(G) = 0, k = 1, 2, ..., n.$

$$\therefore \alpha_k G(a,t) + \alpha_k' \, G'(a,t) + \cdots \alpha_k^{n-1} G^{(n-1)}(a,t)$$
$$+\beta_k G(b,t) + \beta_k' \, G'(b,t) + \cdots + \beta_k^{n-1} G^{(n-1)}(b,t) = 0$$

for k = 1, 2, ..., n

That is

$$\alpha_k \left[a_1 y_1(a) + a_2 y_2(a) + \cdots + a_n y_n(a)\right]$$
$$+\alpha_k' \left[a_1 y_1' \, (a) + \cdots + a_n y_n' \, (a)\right] + \cdots + \alpha_k^{(n-1)} a_1 y_1^{(n-1)}(a) + \cdots$$
$$+a_n y_n^{(n-1)}(a) + \beta_k \left[b_1 y_1(b) + \cdots + b_n y_n(b)\right] + \beta_k' \, (b, y_1' \, (b) + \cdots +$$
$$b_n y_n' \, (b) + \cdots + \beta_k^{n-1} \left[b_1, y_1^{(n-1)}(b) + \cdots + b_n y_n^{(n-1)}(b)\right] = 0$$

for k = 1, 2, 3, ... n.

That is

$$a_1 A_k(y_1) + a_2 A_k(y_2) + \cdots + a_n A_k(y_n)$$
$$+b_1 B_k(y_1) + \cdots + b_n B_k(y_n) = 0, \quad k = 1, 2, ..., n.$$

Where

$$A_k(y) = \alpha_k y(a) + \cdots + \alpha_k^{n-1} y^{(n-1)}(a) \quad and$$
$$B_k(y) = \beta_k y(b) + \cdots + \beta_k^{n-1} y^{(n-1)}(b)$$

Now since $a_k = b_k - c_k$, we get

$$(b_1 - c_1) A_k(y_1) + (b_2 - c_2) A_k(y_2) + \cdots + (b_n - c_n) A_k(y_n)$$
$$+b_1 B_k(y_1) + \cdots + b_n B_k(y_n) = 0, \quad k = 1, 2, ..., n.$$

Hence since $V_k(y) = A_k(y) + B_k(y)$, we get

$$b_1V_k(y_1)+b_2V_k(y_2)+\cdots+b_nV_k(y_n)=C_1A_k(y_1)+\cdots+C_nA_k(y_n)$$

$$k=1,2,...,n.$$

This is the system of n linear equations in $b_1,b_2,\cdots,b_n$ and its determinant is

$$\begin{vmatrix} V_1(y_1)\cdots V_1(y_n) \\ V_2(y_1)\cdots V_2(y_n) \\ \vdots \\ V_n(y_1)\cdots V_n(y_n) \end{vmatrix} \neq 0$$

Hence the system has unique solution for $b_1(t),b_2(t),...,b_n(t)$

Hence $a_k=b_k-c_k,\quad a_k(t)$ are determined uniquely.

Thus, $a_1,a_2,...,a_n,b_1,b_2,...,b_n$ are uniqualy determined.

$\therefore G(x,t)$ exists and unique.

Special case :

Consider the B.V.P.

1) $(p(x,y')'+q(x)y=0, p(x)\neq 0 \; on \; (a,b), p(x)\in C'(a,b)$

2) $y(a)=y(b)=0$

Suppose y_1 is a solution of the equation (1) with the initial condition $y_1(a)=0, y_1'(a)=\alpha\neq 0$

3) Assume that $y_1(b)\neq 0$

Then the function $C_1y_1(x)$ where C_1 is arbitrary, is a solution of the equation (1) satisfying the boundary condition $y_1(a)=0$. Similarly suppose $y_2(x)\neq 0$ to be a solution of the equation (1) satisfying the boundary condition $y_2(b)=0$

We shall construct to Green's function of the form

$$G(x,t)=\begin{cases} C_1y_1(x) \; if \; a\le x<t \\ C_2y_2(x) \; if \; t<x\le b \end{cases}$$

We choose C_1 and C_2 so that G(x,t) is Green's function of the BVP (1) and (2)

Since G(x, t) is to be continuous at x = t, $C_1 y_1(t) = C_2 y_2(t) \rightarrow (h)$

Since $\frac{\partial}{\partial x} G(x,t)$ has jump at x = t equal to $\frac{1}{p(t)}$

$$C_2 y_2'(t) - C_1 y_1'(t) = \frac{1}{p(t)} \qquad \text{-----(5)}$$

Solving the equations (4) and (5) for C_1 and C_2 we get.

$$C_1 = \frac{y_2(t)}{p(t)w(t)}, C_2 = \frac{y_1(t)}{p(t)w(t)}$$

Where $w(t) = y_1(t)y_2'(t) - y_1'(t)y_2(t)$

Green's function for the B.V.P. (1) and (2) is

$$G(x,t) = \begin{cases} \dfrac{y_1(x)y_2(t)}{p(t)w(t)} & if \quad a \le x < t \\ \dfrac{y_1(t)y_2(x)}{p(t)w(t)} & if \quad t < x \le b \end{cases}$$

9.5 Construction of Greens function

Problems 1 : Construct Green's function for the BVP

$$y'' - y = 0, y(0) = y'(0), y(l) + y'(l) = 0 \qquad \text{-----(1)}$$

Solution : The general solution of the equation $y'' - y = 0$ is $y(x) = C_1 e^x + C_2 e^{-x}$

If it is the solution of B.V.P. (1), y(x) must satisfy the conditions

$$y(0) = y'(0), y(l) + y'(l) = 0$$

We must choose C_1 and C_2 such that

$$C_1 + C_2 = y(0) = y^1(0) = C_1 - C_2 \Rightarrow C_2 = 0$$

and $C_1e^l + C_2e^{-l} + C_1e^l - C_2e^{-l} = y(l) + y'(l) = 0$

$2C_1e^l = 0 \Rightarrow C_1 = 0 \qquad C_1 = C_2 = 0 \Rightarrow y(x) = 0$

Hence the BVP (1) has only trivial solution and by the existence theorem, the B.V.P. has Green's function of the form :

$$G(x,t) = \begin{cases} a_1e^x + a_2e^{-x} & if \;\; 0 \le x < t \\ b_1e^x + b_2e^{-x} & if \;\; t < x \le l \end{cases}$$

Where $a_1, a_2, b_1, b_2,$ are to be determined.

By defining condition (G1)

$$(b_1 - a_1)e^t + (b_2 - a_2)e^{-t} = 0$$
$$(b_1 - a_1)e^t - (b_2 - a_2)e^{-t} = 1$$

(Note that $P_0(x) = 1$)

Solving these equation for $b_1 - a_1$ and $b_2 - a_2$ we get $b_1 - a_1 = \frac{1}{2}e^{-t}$ and $b_2 - a_2 = \frac{1}{2}e^t$

Since $G(0,t) = G'(0,t)$

$a_1 + a_2 = a_1 - a_2 \Rightarrow a_2 = 0$

Since $G(l,t) + \phi\, G'(l,t) = 0,$

$b_1e^l + b_2e^{-t} + b_1e^l - b_2e^{-l} = 0$

$\Rightarrow 2b_1e^l = 0$

$\Rightarrow b_1 = 0$

$\therefore b_1 - a_1 = \frac{1}{2}e^{-t} \Rightarrow a_1 = \frac{1}{2}e^{-t}$

$a_2 = 0, b_2 - a_2 = -\frac{1}{2}e^{+t} \Rightarrow b_2 = -\frac{1}{2}e^t$

Thus, $a_1 = \frac{1}{2}e^{-t}, a_2 = 0, b_1 = 0, b_2 = -\frac{1}{2}e^{t}$

Hence Green's function for the BVP (1) is

$$G(x,t) = \begin{cases} \frac{1}{2}e^{-t}e^{x} & if \quad 0 \le x < t \\ -\frac{1}{2}e^{t}e^{-x} & if \quad t < x \le l \end{cases}$$

i.e. $$G(x,t) = \begin{cases} \frac{1}{2}e^{x-t} & if \quad 0 \le x < t \\ -\frac{1}{2}e^{t-x} & if \quad t < x \le l \end{cases}$$

Problem 2 : Determine Green's function for the B.V.P.

$$\frac{d}{dx}x\frac{dy}{dx} - \frac{y}{x} = 0, \qquad 0 < x < 1$$

with boundary conditions $y(0) = 0 = y(1)$

Solution : Note that $\frac{d}{dx}x\frac{dy}{dx} - \frac{y}{x} = 0 \Leftrightarrow x\frac{d}{dx}x\frac{dy}{dx} - y = 0$

Now let $x = e^{z}$ then

$$\frac{dy}{dx} = \frac{dy}{dz}.\frac{dz}{dx} = \frac{dy}{dz}.\frac{1}{x} \quad or \quad x\frac{dy}{dx} = \frac{dy}{dz}$$

$$\therefore \frac{d}{dx}x\frac{dy}{dx} = \frac{d}{dx}\left(\frac{dy}{dz}\right) = \frac{d^2y}{dz^2}.\frac{dz}{dx} = \frac{d^2y}{dz^2}.\frac{1}{x}$$

$$\therefore x\frac{d}{dx}x\frac{dy}{dx} = \frac{d^2y}{dz^2}$$

$$\therefore x\frac{d}{dx}x\frac{dy}{dx} - y = 0 \Rightarrow \frac{d^2y}{dz^2} - y = 0$$

But the general solution of $\frac{d^2y}{dz^2} - y = 0$ is $y = C_1e^z + C_2e^{-z}$

$\therefore\ y(x) = C_1 x + C_2 \frac{1}{x}$ is general solution of the equation

$\frac{d}{dx} x \frac{d}{dx} y - \frac{y}{x} = 0 \quad an \quad 0 < x < 1$ of this solution has to the conditions $y(0) = 0 = y(1)$

we must have $0 = y(0)$ which implies that $C_2 = 0$ and $y(1) = 0$ implies that $C_1 = 0$

Hence the BVP has only trivial solution and hence its Green's function exists and is of the form

$$G(x,t) = \begin{cases} a_1 x + a_2 \frac{1}{x} & if \quad 0 < x < t \\ b_1 x + b_2 \frac{1}{x} & if \quad t < x < 1 \end{cases}$$

If this G(x,t) is to be Green's function of the B.V.P., we must have

$$(b_1 - a_1)t + (b_2 - a_2)\frac{1}{t} = 0$$

and $(b_1 - a_1) - (b_2 - a_2)\frac{1}{t^2} = \frac{1}{t}$

Solving these equations for $b_1 - a_1$ and $b_2 - a_2$ we have $b_1 - a_1 = \frac{1}{2t}$ and $b_2 - a_2 = -\frac{t}{2}$

Also $0 = G(0,t) \Rightarrow a_2 = 0 \Rightarrow b_2 = -\frac{t}{2}$

$$0 = G(0,t) \Rightarrow b_1 + b_2 = 0 \Rightarrow b_1 = -b_2 = \frac{t}{2}$$

$$\therefore b_1 - a_1 = \frac{1}{2t} \Rightarrow a_1 = \frac{1}{2}\left(t - \frac{1}{t}\right)$$

Hence G(x,t) Green's function for the BVP is

$$G(x,t)=\begin{cases}\frac{1}{2}x\left(t-\frac{1}{t}\right)if \quad 0<x<t\\ \frac{1}{2}t\left(x-\frac{1}{x}\right)if \quad t<x<1\end{cases}$$

Exercise in the following examples establish whether Green's function exists for the given BVP and if it does, construct it.

1) $y''=0, y(0)=y'(1), y'(0)=y(1)$

2) $y''=0, y(0)=y(1), y'(0)=y'(1)$

3) $y''+y=0, y(0)=y(\pi)=0$

4) $y'''=0, y(0)=y(1), y'(0)=y'(1)$

5) $y''+4y=0, y(0)=0, y\left(\frac{\pi}{2}\right)=0$

6) $y'''=0, y(0)=y(1), y'(0)+y'(1)=0$

7) $y''+k^2y=0, y(0)=y(1)=0$

8) $y''=0, y'(0)=Ly(0), y'(1)=Hy(1)$

9) $y''+y'=0, y(0)=y(1), y'(0)+y'(1)=0$

9.6 Solution or Convertion of BVP to integral equation by using Green's function:

In this section we shall discuss the problem of convertion of B.V.P.'s into integral equation by using Green's function. The Green's function G(x, t) of a linear differential operator L, at a print t is any solution of

$$L\left[G(x,t)\right]=\delta(x-t) \qquad \text{-----(1)}$$

Where s is the dirac delta function. This delta is not a function in usual sense. It is defined by $\delta(x) = 0$ for except at x = o where its value is infinitely large in such way that, $\int_{\infty}^{\infty} \delta(x)dx = 1$. Also for any conditions function ϕ it satisfies.

$$\int_{\infty}^{\infty} \delta(x-t)\phi(x)dx = \phi(t) \qquad \text{-----(2)}$$

To see now Green's function can be used to convert a BVP to an integral equation we consider the differential operator.

$$L(G(s) = \left(A(s)\frac{d^2}{dy^2} + B(s)\frac{d}{dy} + C(s) \right) u(s) = 0, a < s < b$$

Where A(s) continuously differentiable positive function.

Its, adjoint operator is

$$M(\vartheta(s)) = \frac{d^2}{dy^2}(A(s)\vartheta(s)) - \frac{d}{y}(B(s)\vartheta(s), a < s < b$$

By integration by parts,

$$\int_a^b (\vartheta LH - um\vartheta)ds = \left[A(s)[\vartheta u^1 - u\vartheta^1] + u\vartheta(BA^1) \right]_{s=a}^{s=b}$$

This formula is known as Green's formula for L. If we take $p(s) = exp\int \frac{B(s)}{A(s)}ds$ and $q(s) = p(s)\frac{C(s)}{A(s)}ds$ between the operator L in (3) can be converted to

$$L \equiv \frac{d}{ds}p(s)\frac{d}{ds} + q(s)$$

$\therefore$ For approximate u and v,

$$kL\vartheta - \vartheta Lu = u\left[\frac{d}{ds}p(s)\frac{d}{ds}\vartheta(s) + q(s)\vartheta(s)\right] - \vartheta\left[\frac{d}{ds}p(s)\frac{d}{ds}u(s) + q(s)u(s)\right]$$

$$= u\frac{d}{ds}p(s)\frac{d}{ds}\vartheta(s) + q(s)u(s)\vartheta(s) - \vartheta\frac{d}{ds}p(s)\frac{d}{ds}u(s) - q(s)u(s)\vartheta(s)$$

$$= u\frac{d}{ds}p(s)\frac{d}{ds}\vartheta(s) - \vartheta\frac{d}{ds}p(s)\frac{d}{ds}u(s)$$

$$= u\left[p(s)\vartheta''(s) + p'(s)\vartheta'(s)\right] - \vartheta\left[p(s)u''(s) + p'(s)u'(s)\right]$$

$$= p(s)\left[u\vartheta'' - \vartheta u''(s)\right] + p'(s)\left[u\vartheta' - \vartheta u'\right]$$

$$= p(s)\left[u\vartheta'' + u'\vartheta' - \vartheta u'' - u'\vartheta'\right] + p'(s)\left[u\vartheta' - \vartheta u'\right]$$

$$= p(s)\left[\left(u\vartheta' - \vartheta u'\right)'\right] + p'(s)\left[uv' - \vartheta u'\right]$$

$$= \left(p(s)(u'\vartheta' - \vartheta u')\right)'$$

$$= \frac{d}{ds}\left[p(s)u(s)\vartheta'(s) - \vartheta(s)u'(s)\right]$$

$$\therefore \int_a^b (uL\vartheta - \vartheta Lu)ds = p(s)\left(u(s)\vartheta'(s) - \vartheta(s)u'(s)\right) \Big|_{s=a}^{s=b} \qquad \text{-----(6)}$$

$\therefore$ If G(s,t) is the Green's function for L, then

$$\int_a^b \left[G(s,t)L\vartheta - \vartheta L\left(G(s,t)\right)ds\right] = p(s)\left(G(s,t)\vartheta'(s) - \vartheta(s)\frac{\partial}{ds}G(s,t)\right)\Big|_{s=a}^{s=b}$$

$\therefore$ by (1)

$$\int_a^b G(s,t)L\vartheta ds - \int_a^b \vartheta(s)\delta(s,t)ds = p(s)\left(G(s,t)\vartheta'(s) - \vartheta(s)\frac{\partial}{ds}G(s,t)\right)\Big|_{s=a}^{s=b}$$

$\therefore$ by (2)

$$\int_a^b G(s,t)L\vartheta\, ds - \vartheta(t) = p(s)\left(G(s,t)\vartheta'(s) - \vartheta(s)\frac{\partial}{ds}G(s,t) \right) \Big|_{s=a}^{s=b}$$

or $$\vartheta(t) = \int_a^b G(s,t)L(\vartheta)ds - p(s)\left(G(s,t)\vartheta'(s) - \vartheta(s)\frac{\partial}{ds}G(s,t) \right) \Big|_{s=a}^{s=b}$$

$\therefore$ If y is a solution of L(y) = f(s), then

$$y(t) = \int_a^b G(s,t)f(s)ds - p(s)\left(G(s,t)y'(s) - y(s)\frac{\partial}{ds}G(s,t) \right) \Big|_{s=a}^{s=b} \qquad \text{-----(7)}$$

The equation (7) can be used to convert or to solve a BVP of second order. We shall illustrate this technique by considering examples.

Problem 1 : Consider the BVP

$$(py')' + qy = F(s), \quad (a < s < b) \qquad \text{-----(8)}$$

$$\left.\begin{array}{l} -\Gamma_1 y'(a) + v_1 y(a) = 0 \\ \Gamma_2 y'(b) + v_2 y(b) = 0 \end{array}\right\} \qquad \text{-----(9)}$$

Where P is continuously differentable positive function and F(s), q(s) are continuous function on (a, b). Find the solution of the BVP by using Green's function.

Solution : First consider the BVP.

$$L(y) = (py')' + qy = 0 \qquad \text{-----(10)}$$

subject to the boundry conditions (2)

Let G(s,t) be the Green's function the BVP (10) amd (9)

If y is the solution of the BVP (8) and (9) by equation (7),

$$y(t) = \int_a^b G(s,t)F(s)ds - p(s)\left(G(s,t)y'(s) - y(s)\frac{\partial}{ds}G(s,t) \right) \Big|_{s=a}^{s=b} \qquad \text{-----(11)}$$

Now $p(s)\left(G(s,t)y'(s) - y(s)\frac{\partial}{ds}G(s,t) \right) \Big|_{s=a}^{s=b}$

$$p(b)\left(G(b,t)y'(b) - y(b)\frac{\partial}{ds}G(s,t)_{s=b} \right) - p(a)\left(G(a,t)y'(a) - y(a)\frac{\partial}{ds}G(s,t)_{s=a} \right)$$

-----(12)

Since G is the Green function for the BVP (10), (9) it satisfies the boundary conditions, (g)

$$\therefore -\Gamma_1 \frac{\partial}{\partial s}G(s,t)\Big|_{s=a} + v_1\, G(a,t) = 0$$

$$\Gamma_2 \frac{\partial}{ds}G(s,t)\Big|_{s=b} + v_2\, G(b,t) = 0$$

Hence $\frac{\partial}{ds}G(s,t)\Big|_{s=a} = \frac{v_1}{\Gamma_1}G(a,t)$ and

$$\frac{\partial}{ds}G(s,t)\Big|_{J=b} = \frac{v_2}{\Gamma_1}G(b,t)$$

Also, $y'(b) = -\frac{v_2}{\Gamma_2}y(b)$ and $y'(a) = -\frac{v_1}{\Gamma_1}y(a)$

$\therefore$ L.H.S. of (12) is

$$p(b)\left\{ G(b,t)\left(-\frac{v_2}{\Gamma_2}y(b) \right) - y(b)\left(-\frac{v_2}{\Gamma_2}G(s,t) \right) \right\}$$

$$-p(a)\left\{ G(a,t)\left(\frac{v_1}{\Gamma_1}y(a) \right) - y(a)\left(\frac{v_1}{\Gamma_1}G(a,t) \right) \right\}$$

$= p(b).0 + p(a).0$

$= 0$

Hence by (11) the solution y(t) of the BVP (8), (9) is given by

$$y(t)=\int_a^b G(s,t)F(s)ds$$

Thus if we known the Green function for the BVP (10), (9), we can determine the solution of the BVP (8), (9).

Problem 2 : Solve the BVP

$$(py')'+qy=f(s), \qquad a<s<b \qquad \text{-----(13)}$$

$$y(a)=\alpha, y(b)=\beta \qquad \text{-----(14)}$$

by using Green's function.

Solution : Let G(x, t) to the Green's function for the BVP

$$(py')'+qy=0, y(a)=0, y(b)=0$$

Hence $G(\phi,t)=0, G(b,t)=0$ for $a<t<b$

By (7) the solution y(t) of the B.V.P.(13), (14), is

$$y(t)=\int_a^b G(s,t)F(s)ds-p(s)\left(G(s,t)y'(s)-y(s)\frac{\partial}{ds}G(s,t)\right)\Big|_{s=a}^{s=b}$$

$$y(t)=\int_a^b G(s,t)F(s)ds-p(b)\left(G(b,t)y'(b)-y(b)\frac{\partial}{ds}G(s,t)\big|_{s=b}\right)$$

$$+p(a)\left(G(a,t)y'(a)-y(a)\frac{\partial}{ds}G(s,t)\big|_{s=a}\right)$$

$$=\int_a^b G(s,t)F(s)ds-p(b)\left\{-\beta\frac{\partial}{ds}G(s,t)\big|_{s=b}\right\}+p(a)\left[-\alpha\frac{\partial}{ds}G(s,t)\big|_{s=a}\right]$$

$$\therefore y(t)=\int_a^b G(st)F(s)ds+g(t) \text{ where}$$

$$g(f) = \beta p(b) \frac{\partial}{ds} G(s,t)\big|_{s=b} - \alpha p(a) \frac{\partial}{ds} G(s,t)\big|_{s=a}$$

Problem 3 : Solve the BVP

$$y'' - y = -ze^{x}, y(a) = y'(0), y(l) + y'(t) = 0$$

by using Green's function.

Solution : First consider the BVP $y'' - y = 0, y(0) = y'(0), y(l) + y'(l) = 0$ Let G(x,t) be the Green's function for the BVP $y'' - y = 0, y(0) = y'(0), y(l) + y'(l) = 0$ Then by the Problem 1 (V-3)

$$G(x,t) = \begin{cases} \frac{1}{2} e^{x-t} & if\ 0 \le x < t \\ -\frac{1}{2} e^{t-x} & if\ t < x \le l \end{cases}$$

The equation $y'' - y = -2e^{x}$ can be converted to

$$\frac{d}{dx} p(x) \frac{dy}{dx} + q(x) y(x) = -2e^{x}$$

Where p(x) = 1 and q(x) = -1

By problem 1 of this section with $\Gamma_1 = 1 = v_1$ and $\Gamma_2 = v_1 = 1$ the solution of the BVP $y'' - y = 0$, $y(0) = y'(0), y(l) + y'(l) = 0$ is

$$y(t) = \int_a^b G(x,t) F(x) dx, \quad 0 < t < l$$

That is $y(t) = \int_o^t G(x,t) F(x) dx, + \int_t^l G(a,t) F(x) dx$

or $y(t) = \frac{1}{2} \int_o^t e^{x-t} (-2e^{x}) dx - \frac{1}{2} \int_t^l e^{t-x} (-2e^{x}) dx$

$$y(t) = -\int_o^t e^{-t} e^{2x} dx + \int_t^l e^t dx$$

$$y(t) = -e^{-t}\left[\frac{e^{2x}}{2} 1_0^t\right] - e^t (l - t)$$

$$y(t) = -e^{-t}\left[\frac{e^{2t}}{2} - \frac{1}{2}\right] - e^t (l - t)$$

$$y(t) = -\left[\frac{e^t - e^{-t}}{2}\right] + e^t (l - t)$$

$$y(t) = -\sin \text{h}\, t + e^t (l - t)$$

Example : Verify really it is a solution of the B.V.P.

Problem 4 : Using Green's function find the solution of the BVP y"- y = x, y(0) = y(1) = 0

Solution : Consider the BVP y" - y = 0, y (0) = y (1) = 0

The Green's function for this BVP exist and

is given by

$$G(x,t) = \begin{cases} \dfrac{\sinh x \sinh(t-1)}{\sin 1} & if \quad 0 \le x < t \\ \dfrac{\sinh t \sinh(x-1)}{\sin 1} & if \quad t < x \le 1 \end{cases}$$

(show this)

As above problem 3, the solution of the BVP

$$y'' - y = x, y(0) = y(1) = 0\; is$$

$$y(t)=\int_0^1 G(x,t)F(x)dx=\int_0^1 G(x,t)x\,dt$$

$$y(t)=\frac{\sinh(t-1)}{\sin 1}\int_0^t x\sin\text{h}\,x\,dx+\frac{\sinh t}{\sin 1}\int_t^1 x\sinh(x-1)dx$$

$$\therefore y(t)=\frac{\sinh(t-1)}{\sin 1}\left[2\cosh x\Big|_0^t-\int_0^t\cosh x\,dx\right]$$

$$+\frac{\sinh t}{\sin 1}\left[x\cosh(x-1)\Big|_t^1-\int_t^1\cosh(x-1)\,dx\right]$$

$$=\frac{\sinh(t-1)}{\sin 1}\left[t\cosh t-\sinh t\right]+\frac{\sinh t}{\sin 1}\left[1-x\cosh(t-1)+\sinh(t-1)\right]$$

$$\therefore y(t)=\frac{1}{\sin 1}\left[t(\sinh(t-1)\cosh t-\sinh t\cosh(t-1)\right]$$

$$-\sin n(t-1)\sinh t+\sinh t\sinh(t-1)+\sinh t$$

$$\therefore y(t)=\frac{1}{\sin 1}\left\{-t\sin\text{h}(x-(x-1))\right\}+\sinh t$$

$$y(t)=-t+\frac{\sinh t}{\sin 1}$$

It is really solution of the B.V.P.? Prove

Problem 5 : (Sturm - Liouville problem) : Convert the strum Liouville problem.

$$(p,y')'+qy+\lambda vy=F(s),a<s<b \qquad \text{-----(i)}$$

$$\left.\begin{array}{l}\Gamma_1 y'(a)+v_1 y(a)=0\\ \Gamma_1 y'(b)+v_2 y(b)=0\end{array}\right\} \qquad \text{-----(ii)}$$

Where λ is a parameter and p, q, r are continuous functions an (a,b) and p has continuos derivative and (a, b) with p $\neq$ 0 m (a, b) to an integral equation by using Green's function .

Solution : First consider the BVP

$$(py') + qy = 0 \qquad \text{-----(iii)}$$

With the boundary condition (ii)

Let G(s,t) be Gveen's function for the BVP (iii), (ii) Note that G(s,t) is symmetric i.e. G(s,t)= G(t,s)

The equation (i) can be written as

$$(py')' + qy = f(s)$$

where $f(s) = F(s) - \lambda v(s)y(s)$

Now we know that by problem 1 of this section the solution of the BVP.

$$(py')' + qy = f(s), -\Gamma_1 y'(a) + v_1 y(a) = 0$$

$$\Gamma_2 y'(b) + v_2 y(b) = 0$$

$$y(s) = \int_a^b G(t,s) f(t) dt$$

$$\therefore y(s) = \int_a^b G(t,s)\left[F(t) - \lambda v(t) y(t)\right] dt$$

So $y(s) = \int_a^b G(t,s)F(t)dt - \lambda \int_a^b r(t)G(t,s)r(t)y(t)dt$

Let $f_1(s) = \int_a^b G(t,s)F(t)$ (Note that G and F are known)

K(t, s) = G(t, s) v(t)

Then $y(s) = f_1(s) - \lambda \int_a^b k(t,s) y(t) dt$

Which is the fredholm integral equation of second kind with the kernel K(t, s) = v(t) G(t, s)

Note : Note that the kernel K in (*) may not be symmetric However if r(t)>0 for all t, then the BVP can be converted to Fredholm integral equation of second kind with symmetric kernel

If $v(t) > 0$ for all t, multiply both side of (*)

by $\sqrt{v(s)}$

$$\therefore \sqrt{v(s)}\, y(s) = \sqrt{v(s)}\, f_1(s) - \lambda \int_a^b \sqrt{v(t)}\sqrt{v(s)}\, f_1(t,s) \sqrt{v(t)}\, y(t) dt$$

by taking $f(s) = \sqrt{v(s)}\, f_1(s)$ and $\phi(s) = \sqrt{v(s)}\, y(s)$

and $k(t,s) = \sqrt{v(t)}\ \sqrt{v(s)}\ G(t,s)$

we get

$$\phi(s) = f(s) - \lambda \int_a^b k''(t,s)\ \phi(t) dt$$

Since $G(s,t) = G(t,s)$ we see that $k^2(t,s) = k^2(s,t)$

Problem 6 : Reduce the BVP

i) $x^2 \dfrac{d^2 y}{dx^2} + x\dfrac{dy}{dx} + (\lambda x^2 - 1) y = 0, \quad 0 < x < 1$

ii) $y(0) = 0, y(1) = 0$

to an integral equation by using Green's function.

Solution : Since x > 0, the differential equation (i) can be written as

$$\left(\frac{d}{dx}x\frac{d}{dx}y\right)-\frac{1}{x}y+\lambda xy=0$$

or $Ly+\lambda xy=0$ where $L=\frac{d}{dx}x\frac{d}{dx}-\frac{1}{x}$

Have $p(x)=x, q(x)=-\frac{1}{x}$

Now consider the BVP

$Ly(0)=0, y(0)=0, y(1)=0$

By the problem 2 (v-3), we know that Green's function of this BVP exists and is of the form

$$G(x,t)=\begin{cases}\frac{1}{2}x\left(t-\frac{1}{t}\right) & if \quad 0<x<t\\ \frac{1}{2}x\left(x-\frac{1}{x}\right) & if \quad t<x<1\end{cases}$$

Which is symmetric i.e. $G(x,t)=G(t,x)$

Now consider the BVP

$Ly+\lambda xy=0, y(0)=y(1)=0$

or $Ly=-\lambda xy, y(0)=y(1)=0$

Hence by the problem 1 of this section, the solution y of the BVP satisfies.

$$y(t)=\int_0^1 G(x,t)(-\lambda xy(x),dx$$

i.e. $y(t)=-\lambda\int_0^1 xG(x,t)y(x)dx$

Which is the Fredholm integral of second kind.

Example : Convert the BVP of problem 6 into Fredholm integral equation of second kind with symmetric kernel

(Note that the x G(x, t) is not symmetric But if we take $k(x,t)=\sqrt{x}\sqrt{t}G(x,t),$ then (*) above can be written as

$$\sqrt{t}y(t)=-\lambda\int_0^1 \sqrt{t}\sqrt{x}G(x,t)\sqrt{x}y(x)dx$$

or $\phi(x)=-\lambda\int_0^1 k(x,t)\phi(x)dx$

Which is Fredholm integral equation of second kind with symmetric kernel $k(x,t)=\sqrt{t}\sqrt{x}G(x,t)$

Exercise : 1 - Solve or reduce the following BVP to integral equations by using Green Function if exit.

1) $y''+\lambda y=x, y(0)=y\left(\frac{\pi}{2}\right)=0$

2) $y''+y=x, y(0)=0, y'(1)=0$

3) $y''+xy=1, y(0)=y(1)=0$

4) $\frac{d}{dx}x\frac{d}{dx}y-\frac{n^2y}{x^2}=0, y(0)=y(1)=0$

5) $y'''+\lambda y=2x, y(0)=y(1)=0, y'(0)=y'(1)$

6) $y''+\frac{\pi^2}{y}y-\lambda y=\cos\frac{\pi}{2}x, y(-1)=y(1), y'(-1)=y'(1)$

7) $y''+y=x, y(0)=1, y'(1)=0$

Exercise : 2 - Consider the problem

$$y^{11}(x) - k^2 y(x) = F(x), \qquad 0 \le x \le 1 \qquad \text{-----(i)}$$

$$y(0) = y^1(1) = 0 \qquad \text{-----(ii)}$$

Where $f(x)$ is given function , k is positive real number.

i) Construct Green's function for the BVP

ii) With Green's function constructed, find the solution for $f(x)$= x. Make sure that your solution is correct by verifying that it satisfies (i) and (ii)

♦♦♦

Unit 10

THE ADOMAIN DECOMPOSITION METHOD

10.1 Solution of Fredham Integral Equations by Adomain Decomposition Method

Solution of Fredholm integral equation by Adomain decomposition method (or decomposition method) provides a reliable and effective way for finding solutions of linear and nonlinear differential & integral equation. The method provides the solution of the integral equations in the form of series. In decomposition method we usually express the solution *u(x)* of the integral equation in the form of series

$$u(x)=\sum_{n=0}^{\infty}u_n(x) \qquad \text{-------------------- (1)}$$

It is important to note that the series obtained for *u(x)*, frequently provides the exact solution in the closed form.

However for some problem where series in Eq. (1) cannot be evaluated, a truncated series $\sum_{n=0}^{k} u_n(x)$ in usually used to approximate the solution *u(x)* if *a* numerical solution is desirsd.

We point out that few terms of truncated series is usually provide the higher accurate level of the approximate solution if compared with the existing numerical techniques.

Consider the fredholm integral equation of the second kind of the form,

$$u(x) = f(x) + \lambda \int_a^b k(x,t)u(t)dt \qquad x \in [a,b] \qquad (1)$$

Let. $$u(x) = \sum_{n=0}^{\infty} u_n(x) \qquad (2)$$

is the solution of integral equation (1) Substituting (2) in (1), we obtain.

$$\sum_{n=0}^{\infty} u_n(x) = f(x) + \lambda \int_a^b k(x,t) \left\{ \sum_{n=0}^{\infty} u_n(t) \; dt \right\}$$

Then,

$$u_0(x) + u_1(x) + \dots\dots\dots = f(x) + \lambda \int_a^b k(x,t)\; u_0(t)dt + \lambda \int_a^b k(x,t)u_1(t)dt + \dots.$$

$$+\lambda \int_a^b k(x,t)\; u_2\;(t)dt + \dots\dots. \qquad (3)$$

The component $u_0(x), u_1(x), \dots$ of $u(x)$ are completely determined in a recurrent manner if we set

$$\begin{aligned} u_0(x) &= f(x) \\ u_1(x) &= \lambda \int_a^b k(x,t)u_0(t)dt \\ u_2(x) &= \lambda \int_a^b k(x,t)u_1(t)dt \\ u_3(x) &= \lambda \int_a^b k(x,t)u_2(t)dt \end{aligned}$$

The above discussed scheme for determination of components $u_0(x), u_1(x), u_2(x), \dots$ of the solution $u(x)$ of the integral equation (1) can be written in a recuresive manner by

$$u_o(x) = f(x),\ u_{n+1}(x) = \lambda \int_a^b k(x,t) u_n(t)\,dt,\ n \geqslant 0$$

The solution $u(x)$ of integral equation (1) can be determined from the decomposition (2)

Problem 1 : Find the solution of Fredholm integral equation

$$u(x) = \frac{9}{10}x^2 + \int_0^1 \frac{x^2t^2}{2} u(t)dt$$

by using adomain decomposition mothod.

Sol^n : Given integral equation is,

$$u(x) = \frac{9}{10}\ x^2 + \int_0^1 \frac{x^2t^2}{2} u\ (t)dt \qquad \dots\dots\dots\dots(1)$$

Let $\quad u(x) = \sum_{n=0}^{\infty} u_n(x) \qquad \dots\dots\dots..(2)$

is the solution of the integral equation (1)

Then,

$$\sum_{n=0}^{\infty} u_n(x) = \frac{9}{10}\ x^2 + \int_0^1 \frac{x^2t^2}{2} \left\{ \sum_{n=0}^{\infty} u_n(t) \right\} dt$$

Thus,

$$= \frac{9}{10}\ x^2 + \frac{x^2}{2}\int_0^1 t^2 \sum_{n=0}^{\infty} u_n(t)dt$$

Then,

$$u_0(x) + u_1(x) + u_2(x) + \ldots\ldots\ldots\ldots$$

$$= \frac{9}{10}\ x^2 + \frac{x^2}{2}\int_0^1 t^2 u_0(t)dt + \frac{x^2}{2}\int_0^1 t^2 u_1(t)dt + \ldots\ldots\ldots\ldots$$

Set, $u_o\left(x\right) = \frac{9}{10}x^2$

$$u_1\left(x\right) = \frac{x^2}{2}\int_0^1 \mathrm{t}^2 u_o\left(\mathrm{t}\right)dt$$

$$= \frac{x^2}{2}\int_0^1 \mathrm{t}^2\left(\frac{9}{10}\mathrm{t}^2\right)dt \ = \frac{9x^2}{100}$$

$$u_2\left(x\right) = \frac{x^2}{2}\int_0^1 \mathrm{t}^2\ \frac{9t^2}{100}dt$$

$$= \frac{9x^2}{200}\left[\frac{t^5}{5}\right]_0^1 = \frac{9x^2}{1000}$$

$$u_3\left(x\right) = \frac{x^2}{2}\int_0^1 t^2\ \frac{9\mathrm{t}^2}{1000}dt$$

$$= \frac{9x^2}{2000}\left[\frac{t^5}{5}\right]_0^1 = \frac{9x^2}{10000}$$

Continuing in this way, we obtain,

$$u_n(x) = \frac{9x^2}{10^{n+1}}$$

Thus (2) reduces to.

$$u(x) = \sum_{n=0} u_n(x) = u_0(x) + u_1(x) + u_2(x) + \cdots$$

$$= \frac{9}{10}x^2 + \frac{9}{10^2}x^2 + \frac{9}{10^3}x^2 + \cdots$$

$$= \frac{9}{10}x^2\left[1 + \frac{1}{10} + \frac{1}{10^2} + \cdots\right]$$

$$= \frac{9}{10}x^2 \sum_{n=0}^{\infty}\left(\frac{1}{10}\right)^n$$

$$= \frac{9x^2}{10}\frac{1}{1-\frac{1}{10}}$$

$$= \frac{9x^2}{10}\frac{10}{9}$$

$$= x^2$$

Therefore, $u(x) = x^2$ is the required sol of integral equation (1)

Problem 2 Find the solution of Fredholm integral equation

$$u(x) = cosx + 2x + \int_0^{\pi} xtu(t)dt$$

by using Adomain decomposition method

Sol^n : Given integral equation is

$$u(x) = cosx + 2x + \int_0^{\pi} xt\ u(t)dt \qquad \ldots\ldots\ldots\ldots\ldots\ldots (1)$$

$$\text{Let} \quad u(x) = \sum_{n=0}^{\infty} u_n(x) \qquad \ldots\ldots\ldots\ldots(2)$$

be the solution of integral equation (1)

Then,

$$\sum_{n=0}^{\infty} u_n(x) = \cos x + 2x + x\int_0^{\pi} \mathrm{t}\left\{\sum_{n=0}^{\infty} u_n(t)\right\} dt$$

This gives,

$$u_0(x) + u_1(x) + u_2(x) + \cdots$$

$$= \cos x + 2x + x\int_0^{\pi} tu_0(t)\,dt + x\int_0^{\pi} tu_1(t)\,dt$$

$$+\ x\int_0^{\pi} \mathrm{t}u_2(+)\,dt + \cdots$$

Set, $u_0(x) = \cos x + 2x$

$$u_1(x) = x\int_0^{\pi} tu_0(t)\,dt = x\int_0^{\pi} [\cos \mathrm{t} + 2t]\,dt$$

$$u_1(x) \ = x\left[\int_0^{\pi} \mathrm{t}\cos t dt + 2\int_0^{\pi} t^2 dt\right]$$

$$= x\left\{\left[\mathrm{t}\sin\mathrm{t} - (-\cos\mathrm{t})\right]_0^{\pi} + 2\left[\frac{t^3}{3}\right]_0^{\pi}\right\}$$

$$= x\cdot\left\{\left[0 + \cos\ \pi - \cos 0\right] + 2\frac{\pi^3}{3}\right\}$$

$$= x\left[-2 + 2\frac{\pi^3}{3}\right] = -2x + \frac{2}{3}\pi^3 x$$

$$u_2(x) = x\int_0^{\pi} \mathrm{t}u_1(t)\,dt$$

$$= x\int_0^{\pi} t\left[-2t + \frac{2}{3}\cdot\pi^3 t\right]dt$$

$$= -2x\int_0^{\pi}\mathrm{t}^2 dt + \frac{2\pi}{3}x\int_0^{\pi}\mathrm{t}^2 dt$$

$$= -2x\left[\frac{t^3}{3}\right]_0^{\pi} + \frac{2\pi^3}{3}x\left[\frac{t^3}{3}\right]_0^{\pi}$$

$$= -2x + \frac{\pi^3}{3} + \frac{\pi^3}{3}x.\frac{\pi^3}{3}$$

$$= \frac{-2\pi^3}{3}x + \frac{2\pi^6}{9}x$$

$$u_3(x\ ) = x\int_0^{\pi}\mathrm{t}\ u_2(\mathrm{t})\,dt$$

$$= x\int_0^{\pi} t\left[\frac{-2\pi^3}{3}t + \frac{2\pi^6}{9}t\right]dt$$

$$= x\int_0^{\pi}\frac{-2\pi^3}{3}t^2dt + x\int_0^{\pi}\frac{2\pi^6}{9}t^2dt$$

$$= -\frac{2\pi^3}{3}x\cdot\frac{\pi^3}{3} + x\frac{2\pi^6}{9}\frac{\pi^3}{3}$$

$$= \frac{-2}{9}\pi^6 x + \frac{2\pi^9}{27}x$$

and so on

Therefore equation (2) reducer to,

$$u_0(x) + u_1(x) + u_2(x) + \cdots$$

$$= \cos x + 2x - 2x + \frac{2}{3}\pi^3 x - \frac{2\pi^3}{3}x + \frac{2\pi^6}{9}x$$

$$= \frac{2\pi^6}{9}x + \frac{2\pi^9}{27}x$$

$$= \cos x$$

Exercise : Solve the following Fredholm integral equation by using the Adomai integral equation by using the Adomain decomposition method.

1. $u(x) = e^x + 1 - e + \int_0^1 u(t)dt$

2. $u(x)=\sin x - x + \int_0^{\pi/2} xt\ u(t)dt$

3. $u(x)=1+\frac{1}{2}\sin^2 x\int_0^{\pi/2} u(t)dt$

4. $u(x)=1-\frac{x^2}{15}+\int_{-1}^{1}\left(xt+x^2t^2\right) u(t)dt$

5. $u(x)=x+(1-x)e^x+\int_0^1 x^2e^{t(x-1)}\ u(t)dt$

6. $u(x)=\frac{3}{2}e^x-\frac{1}{2}e^{x+2}+\int_0^1 e^{x+t}\ u(t)dt$

7. $u(x)=1-\frac{19}{15}x^2+\int_{-1}^{1}\left(xt+x^2t^2\right) u(t)dt$

10.2 Solution of Volterra integral equation by Adomain decomposition method:

Solution of Volterra integral equation of 2nd kind of the form

$$u(x) = f(x)+\lambda\int_a^x k(x,t)u(t)dt, \qquad x\in[a,b] \qquad \text{--------------} (1)$$

by the Adomain decomposition method is given by,

$$u(x)=\sum_{n=0}^{\infty}u_n(x) \qquad \text{------------} (2)$$

in which $u_n(x), (n\geq 0)$ are given by,

$$u_0(x) = f(x)$$

$$u_{n+1}(x) = f(x) = \lambda \int_a^x k(x,t) u_n(t)\, dt, \quad n \geq 0$$

Problem 1 : Solve the Volterra integral equation

$$u(x) = 1 + \int_0^x u(t)\, dt$$

by using adomain decomposition method

Solution : Given integral equation is

$$u(x) = 1 + \int_0^x u(t) dt \qquad \text{-------------- (1)}$$

Let $u_n(x) = \sum_{n=0}^{\infty} u_n(x)$

Then

$$\sum_{n=0}^{\infty} u_n(x) = 1 + \int_0^x \sum_{n=0}^{\infty} u_n(t)\, dt$$

$$u_0(x) + u_1(x) + u_2(x) + \cdots = 1 + \int_0^x u_0(t)\, dt + \int_0^1 u_1(t)\, dt + \cdots$$

Let $u_0(x) = 1$,

$$u_1(x) = \int_0^x u_0(t)\, dt = x$$

$$u_2(x) = \int_0^x u_1(t)\, dt = \int_0^x \text{t} dt = \frac{x^2}{2}$$

$$u_3(x) = \int_0^x u_2(t)\,dt = \int_0^x \frac{t^2}{2}\,dt = \frac{x^3}{3!}$$

and so on

Thereofre Eq (2) reduces to

$$u(x) = \sum_{n=0}^{\infty} u_n(x) = u_o(x) + u_1(x) + u_2(x) + ...$$

$$= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

$$= e^x$$

Problem 2 : $u(x) = x + \int_0^x (t-x)\,u(t)\,dt$

Solution : Given integral equations is,

$$u(x) = x + \int_0^x (t-x)\,u(t)\,dt \qquad .... (1)$$

Let

$$u(x) = \sum_{n=0}^{\infty} u_n(x) \qquad .... (2)$$

is the solution of integral equations (1)

Then

$$\sum_{n=0}^{\infty} u_n(x) = x + \int_0^x (t-x)\left\{\sum_{n=0}^{\infty} u_n(t)\right\}\,dt$$

Thus,

$$u_0(x)+u_1(x)+u_2(x)+.................. = x+\int_0^x (t-x)\ u_0(t)\ dt$$

$$+\int_0^x (t-x)\ u_1(t)\ dt+\int_0^x (t-x)\ u_2(t)\ dt+.......$$

Set $u_0\left(x\right)=x$

$$u_1\left(x\right)=\int_0^x (t-x)\ u_0\left(t\right)dt$$

$$=\int_0^x (t-x)t\ dt=\int_0^2 (t^2-xt)\ dt$$

$$=\left[\frac{t^3}{3}-x\cdot\frac{t^2}{2}\right]_0^x=\frac{x^3}{3}-\frac{x^3}{2}=-\frac{x^3}{6}=\frac{-x^3}{3!}$$

$$u_2\left(x\right)=\int_0^x (t-x)\ u_1\left(\mathrm{t}\right)\mathrm{dt}$$

$$=\int_0^x (t-x)\left(\frac{-t^3}{3!}\right)dt=\frac{-1}{3!}\int_0^x\left(t^4-xt^3\right)dt$$

$$=\frac{-1}{3!}\left[\frac{t^5}{5}-\frac{xt^4}{4}\right]_0^x$$

$$=\frac{-1}{3!}\left[\frac{x^5}{5}-\frac{x^5}{4}\right]-\frac{1}{3!}\frac{\left(-x^5\right)}{4\times 5}=\frac{x^5}{5!}$$

$$=\frac{-1}{3!}\left[\frac{-x^5}{4\times 5}\right]=\frac{x^5}{5!}$$

$$u_3(x) = \int_0^x (t-x)\ u_2(t)dt$$

$$= \int_0^x (t-x)\ \left(\frac{t^5}{5!}\right)dt$$

$$= \frac{1}{5!}\int_0^x (t^6 - xt^5)\ dt$$

$$= \frac{1}{5!}\left[\frac{t^7}{7} - \frac{xt^6}{6}\right]_0^x$$

$$= \frac{1}{5!}\left(\frac{-x^7}{6\times 7}\right)$$

$$= \frac{-x^7}{7!}$$

and so on

Therefore equation (2) reduces to,

$$u(x) = \sum_{n=0}^{\infty} u_n(x)$$

$$= u_0(x) + u_1(x) + u_2(x) + \cdots$$

$$= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + ... = -\sin x$$

Exercise : Solve the following volterra integral equations by using Adomain decomposition method.

1. $u(x) = 1 - \frac{x^2}{2} + \int_0^x u(t)dt$

2. $u(x) = 1 - x + \int_0^x (x-t)\ u(t)dt$

3. $u(x) = x - \frac{2}{3}x^3 - 2\int_0^x u(t)dt$

4. $u(x) = 1 - \int_0^x u(t)dt$

5. $u(x) = 1 - x^2 - \int_0^x (x-t)\ u(t)dt$

6. $u(x) = 6x - 3x^2 + \int_0^x\ u(t)dt$

10.3 Modified decomposition method:

10.3.1 Modified decomposition method for Fredholm integral equation:

Consider the Fredholm integral equation of 2nd kind

$$u(x) = f(x) + \lambda\int_a^b k\ (x,t)u(t)\,dt, \quad a \le x \le b \qquad \text{...... (1)}$$

In modified decomposition method we simply split the given finction $f(x)$ into two parts. defined by

$$f(x) = f_1(x) + f_2(x), \qquad \text{......... (2)}$$

where $f_1(x)$ consist of only one term of $f(x)$ (or if needed two terms in fewer cases), and $f_2(x)$ includes the remaining terms of $f(x)$, In the view of (2), equation (1) becomes,

$$u(x) = f_1(x) + f_2(x) + \int_a^b k\ (x,t)u(t)\,dt, \quad a \le x \le b \quad \ldots\ldots\ldots (3)$$

We write the solution $u(x)$ of integral equation (1) in series form defined by

$$u(x) = \sum_{n=0}^{\infty} u_n(x) \qquad \ldots\ldots\ldots\ldots\ldots (4)$$

Substituting (4) in (1) we obtain,

$$\sum_{n=0}^{\infty} u_n(x) = f_1\ (x)\ + f_2\ (x) + \lambda \int_a^b k(x,t)\left(\sum_{n=0}^{\infty} u_n(t)\ dt\right)$$

This give

$$u_0(x) + u_1(x) + u_2(x) + ... = f_1\ (x)\ + f_2\ (x) +$$

$$\lambda \int_a^b k(x,t)\ u_0(t)dt + \lambda \int_a^b k(x,t)\ u_1(t)dt + \lambda \int_a^b k(x,t)\ u_2(t)dt + ...$$

In modified decomposition method the components

$u_0(x), u_1(x), u_2(x), ... of\ u(x),$ we set as

$$u_0(x) = f_1(x) \qquad \text{-------------- (5)}$$

$$u_1(x) = f_2(x) + \lambda \int_a^b k(x,t)\ u_0(t)dt \qquad \text{------------ (6)}$$

$$u_2(x) = \lambda \int_a^b k(x,t)\ u_1(t)dt$$

$$u_3(x) = \lambda \int_a^b k(x,t)\ u_2(t)dt$$

and so on.

In most of the problems we need to use (5) & (6) only

A necessary condition is required to apply modified decomposition method is that $f(x)$ should consist of more than one terms as shown in equation (2)

It is recommended to apply modified decomposition method for the cases where the nonhomogenous part $f(x)$ in (1) consist of polynomial that includes many terms or in case $f(x)$ contain a combination of polynomial and other trigrometric transcendental functions.

Problem 1 :

Solve the Fredholm integral equation $u(x) = e^{3x} - \frac{1}{9}(2e^3 + 1)x + \int_0^1 xt\ u(t)dt$ int by modified decomposition method.

Solution : Given integral equation is

$$u(x) = e^{3x} - \frac{1}{9}(2e^3 + 1)x + \int_0^1 xt\ u(t)\,dt \qquad \text{--------------- (1)}$$

Here $f(x) = e^{3x} - \frac{1}{9}(2e^3 + 1)x$

$$= f_1(x) + f_2(x)$$

where we take

$$f_1(x) = e^{3x} \text{ and } f_2(x) = -\frac{1}{9}\left(2e^3 + 1\right)x$$

Let, $\quad u(x) = \sum_{n=0}^{\infty} u_n(x)$ ---- (2)

is the solution of integral equation (1)

Then $\quad \sum_{n=0}^{\infty} u_n(x) = f_1(x) + f_2(x) + \int_0^1 x\text{t}\left(\sum_{n=0}^{\infty} u_n(t)\right)dt$

$$u_0(x) + u_1(x) + u_2(x) + \cdots$$

$$= f_1(x) + f_1(x) + x\int_0^1 tu_0(t)\,dt + x\int_0^1 tu_1(t)\,dt + x\int_0^1 tu_2(t)dr + \cdots$$

Set, $\quad u_0(x) = f_1(x) = e^{3x}$

$$u_1(x) = f_2(x) + x\int_0^1 tu_0(t)\,dt$$

$$= \frac{-1}{9}\left(2e^3 + 1\right)x + x\int_0^1 te^{3t}\,dt$$

$$= \frac{-x}{9}\left(2e^3 + 1\right) + x\left[t\frac{e^{3t}}{3} + \frac{e^{3t}}{3}\right]_0^1$$

$$= \frac{-x}{9}\left(2e^3 + 1\right) + x\left[\left(\frac{1}{3}e^3 - \frac{e^3}{9}\right) - 0 + \frac{1}{9}\right]$$

$$= -\frac{x}{9}\left(2e^2+1\right)+x\left(\frac{3e^3-e^3+1}{9}\right)$$

$$= -\frac{x}{9}\left(2e^3+1\right)+\frac{x}{9}\left(2e^3+1\right)$$

$$= 0$$

$$u_2(x) = x\int_0^1 tu_1(t)\,dt = 0$$

Note that $u_n(x) = 0, \forall n \geqslant 1$

Thus equation (2) becomes,

$u(x) = u_0(x) = e^{3x}$.

is the required solution.

Problem 2 : Solve the Fredholm integral equation by Modifed decomposition method.

$$u(x) = \sin^{-1} x + \left(\frac{\pi}{2}-1\right)x - \int_0^1 xu(t)\,dt$$

Solution : Given integral equation

$$u(x) = \sin^{-1} x + \left(\frac{\pi}{2}-1\right)x - \int_0^1 xu(t) - dt$$

Here, $f(x) = \sin^{-1} x + x\left(\frac{\pi}{2}-1\right)$

$$= f_1(x) + f_2(x),$$

where we take $f_1(x) = \sin^{-1} x$, and $f_2(x) = x\left(\frac{\pi}{2} - 1\right)$

Let, $u(x) = \sum_{n=0}^{\infty} u_n(x)$ ……….. (2)

is the solution of integral equation (1) then.

$$\sum_{n=0}^{\infty} u_n(x) = f_1(x) + f_2(x) - \int_0^1 x \left\{\sum_{n=0}^{\infty} u_n(t)\right\} dt$$

Thus, $u_0(x) + u_1(x) + u_2(x) + u_3(x) + \cdots$

$$= f_1(x) + f_2(x) - x\int_0^1 u_0(t)\,dt - x\int_0^1 u_1(t)\,dt + ...$$

Set,

$$u_0(x) = f_1(x) = \sin^{-1} x$$

$$u_1(x) = f_2(x) - x\int_0^1 u_0(t)\,dt$$

$$= f_2(x) - x\int_0^1 \sin^{-1} t\,dt$$

$$= f_2(x) - x\left[\left[\sin^{-1} t \cdot t\right]_0^1 - \int_0^1 \frac{t}{\sqrt{1-t^2}}\,dt\right]$$

$$= f_2(x) - x\left[\left(\sin^{-1} 1 - 0\right) - \frac{1}{1}\int_0^1 \frac{t\;dt}{\sqrt{1-t^2}}\right]$$

$$= f_2(x) - x\left[\frac{\pi}{2} - \int_0^1 \frac{t}{\sqrt{1-t^2}}dt\right]$$

Put $\quad t^2 = P,\ 2tdt = dp \Rightarrow tdt = \frac{dp}{2}$

If t = 0 then p = o and if t=1, then p =1

Therefore, $\quad u_1(x) = f_2(x) - x\left[\frac{\pi}{2} - \frac{1}{2}\int_0^1 \frac{t}{\sqrt{1-t^2}}\right]$

$$= x\left(\frac{\pi}{2} - 1\right) - x\left\{\frac{\pi}{2} - \frac{1}{2}\int_0^1 \frac{dp}{\sqrt{1-p}}\right\}$$

$$= x\left(\frac{\pi}{2} - 1\right) - x\left\{\frac{\pi}{2} - \left[\sqrt{1-p}\right]_0^1\right\}$$

$$= x\left(\frac{\pi}{2} - 1\right) - x\left\{\frac{\pi}{2} - \left[\sqrt{1-p}\right]_0^1\right\}$$

$$= x\left(\frac{\pi}{2} - 1\right) - x\left(\frac{\pi}{2} - 1\right)$$

$$= 0$$

$$\therefore u_n(x) = 0 \ \forall\ n \geq 1$$

Equation (2) becomes,

$u\ (x) = u_0\ (x) = \sin^{-1} x$

Exercise : Solve the following Fredholm integral equations by using modified decomposition method :

1. $u(x)=\sin x-x+x\int_0^{\pi/2} t1+1dt$

2. $u(x)=(\pi-2)\,x+\sin^{-1}\left(\frac{x+1}{2}\right)-\sin^{-1}\left(\frac{x-1}{2}\right)-\int_0^1 x\,u(t)\,dt$

3. $u(x)=\ x+e^x-2e^{x-1}+2e^x-\int_0^x\ e^{x-t}u(t)dt$

4. $u(x)=\ \frac{2}{15}+\frac{7}{12}x+x^2+x^3-\int_0^1\ (1+x-t)\ u(t)dt$

5. $u(x)=\ -6+14x^4+21x^2+x-\int_0^1\ (x^4-t^4)\ u(t)dt$

6. $u(x)=\ (\pi+2)x+\sin x-\cos x-x\int_0^{\pi}\ \mathrm{t}\ u(t)dt$

7. $u(x)=\ e^{x+1}+e^{x-1}+(e^{x-1})\ e^{x-1}-\int_0^1\ e^{x-t}\ u(t)dt$

10.3.2 : The modified Decomposition method for volterra integral equation.

Solution of Volterra integral equations by using the modified Adomain decomposition method can be obtained by following same procedure that have used for Fredholm integral equation.

Problem 1: Solve Volterra integral equation

$$u(x)=\sec x\tan x-\frac{1}{4}\left(e^{\sec x}-e\right)x+\frac{1}{4}\int_0^x xe^{\sec t}dx+\frac{\pi}{x}\,|$$

by modified decomposition method.

Solution : Given integral equation is

$$u(x) = \sec x \tan x - \frac{1}{4}\left(e^{\sec x} - e\right)x + \frac{1}{4}\int_0^x x.e^{\sec t}u(t)\,dt, \quad x < \frac{\pi}{2}$$

Hers, $f(x) = \sec \sec x \tan x - \frac{x}{4}\left(e^{\sec x} - e\right)$

$$= f_1(x) + f_2(x)$$

where we take $f_1(x) = \sec x \tan x$

and $f_2(x) = \frac{-x}{4}\left(e^{\sec x} - x\right)$

Let $u(x) + x = \sum_{n=0}^{\infty} u_n(x)$ is the solution of integral equation (1)

Thus, $\sum_{n=0}^{\infty} u_n(x) = f_1(x) + f_2(x) + \frac{x}{4}\int_0^x e^{\sec t}\left(\sum_{n=0}^{\infty} u_n(t)\right)dt$

Thus, $u_0(x) + u_1(x) + u_2(x) + \cdots = f_1(x) + f_2(x) + \frac{x}{4}\int_0^x e^{\sec t}u_0(t)\,dt$

$$+\frac{x}{4}\int_0^x e^{\sec t}u_1(t)\,dt + \ldots\ldots\ldots\ldots\ldots$$

Set,

$$u_1(x) = f_1(x) = \sec x \tan x$$

$$u_1(x) = f_2(x) + \frac{x}{4}\int_0^x e^{\text{sect}}\text{sect tant}\,dt$$

Put $\sec t = p$

Then, $\sec t \cdot \tan t \; dt = dp$

If t = 0 then p = 1 and if t= x then p = secx

Therefore, $u_1(x) = \frac{-1}{4}\left(e^{\sec x} - e\right)x + \frac{x}{4}\int_0^{\sec x} e^p.dp$

$$= -\frac{1}{4}\left(e^{\sec x} - e\right)x + \frac{x}{4}\left[e^p\right]_1^{\sec x}$$

$$= -\frac{1}{4}\left(e^{\sec x} - e\right)x + \frac{x}{4}\left[e^{\sec x} - e\right]$$

$$= 0$$

Thus, $u_2(x) = \frac{\pi}{4}\int_0^x e^{\sec t}u_1(t)\,dt = 0$

We note $u_n(x) = 0, \forall n \geqslant 1$

Therefore equation (2) becomes,

$$u(x) = u_0(x)$$

$$= \sec x \ \tan x$$

Problem 2 : Solve volterra integral equation

$$u(x) = x^3 - x^5 + 5\int_0^x tu\ (t)\,dt$$

by Modified decomposition method

Solution : Given integral equation is

$$u(x) = x^3 - x^5 + 5\int_0^x tu\,(t)\,dt$$

Here $\quad f(x) = x^3 - x^5$

$$= f_1(x) + f_2(x)$$

where we take deletes $f_1(x) = x^3$ and $f_2(x) = -x^5$

Let $u(x) = \sum_{n=0}^{\infty} u_n(x)$ --------------------- (2)

is the solution of integral equation (1)

Then, $\sum_{n=0}^{\infty} u_n(x) = f_1(x) + f_2(x) + 5\int_0^x t\left(\sum_{n=0}^{\infty} u_n(t)\right)dt$

This is gives,

$$u_0(x) + u_1(x) + u_2(x) + u_3(x) + \cdots = f_1(x) + f_2(x)$$

$$+5\int_0^x tu_0(t)\,dt + 5\int_0^x tu_1(t)\,dt + \cdots$$

Set,

$$u_0(x) = f_1(x) = x^3$$

$$u_1(x) = f_2(x) + 5\int_0^x tu_0(t)\,dt$$

$$= -x^5 + 5\int_0^x t.t^3 dt$$

$$= -x^5 + 5\left[\frac{t^6}{5}\right]_0^x$$

$$= -x^5 + 5\left[\frac{x^5}{5} - 0\right]$$

$$= -x^5 + x^5$$

$$= 0$$

Therefore, $u_2(x) = 5\int_0^x t \cdot u_1(t)\,dt = 0$

We note that, $u_n(x) = 0, \quad \forall n \geqslant 1$

Therefore, equation (3) becomes

$$u(x) = u_0(x) = x^3$$

Exercies : Solve following Volterra integral equation by using modified decomposition method.

1. $u(x) = e^x + x\, e^x - x - \int_0^{\pi} xu(t) + dt$

2. $u\left(x\right)=\cos x+\sin x-\int\limits_0^{\pi} u(t)+dt$

3. $u\left(x\right)=\cos x-\left(1-e^{\sin x}\right)x-x\int\limits_0^{\pi} e^{\sin x}u(t)+dt$

4. $u\left(x\right)=\sinh x+\cosh x-1-\int\limits_0^{\pi} u(t)+dt$

5. $u\left(x\right)=2x-\left(1-e^{-x^2}\right)+\int\limits_0^{\pi} e^{-x^2+t^2}u(t)+dt$

6. $u\left(x\right)=1+x+x^2+\dfrac{x^3}{2}+\cosh x+x\sinh x-\int\limits_0^{\pi} u(t)+dt$

7. $u\left(x\right)=1+\sin x+x+x^2-\cos x-\int\limits_0^{\pi} xu(t)+dt$

10.3 Adomain Decomposition Method for Integro –differential Equations

We will focus on integer-differential equations with seperable kernel of the form,

$$k(x,t)=\sum_{i=1}^{n} a_i(x)b_i(t)$$

Without loss of gemerality we will make our analysis on a one term kernel $k\left(x,t\right)$ of the form

$$k\left(x,t\right)=g\left(x\right)h\left(t\right)$$

Consider the Fredholm integroditerential equations of the form

$$u^{(n)}(x)=f(x)+\lambda\int_a^b k(x,t)u(t)\,dt, u_k(a)=\lambda_k, k=0,1,2,....... \quad (2)$$

Then for one term kernel equations (2) takes the form

$$u^{(n)}(x)=f(x)+\lambda\int_a^b g(x)h(t)u(t)\,dt$$

$$=f(x)+\lambda g(x)\int_a^b h(t)u(t)\,dt,\ u^k(a)=\lambda_k \text{ k=0, 1, 2,.....} \quad .(3)$$

Let $L=\dfrac{d^n}{dx^n}$

Then equation (3) can be witten as,

$$L(u(x))=f(x)+\lambda g(x)\int_a^b h(t)u(t)\,dt$$

Wet L^{-1} is n-fold integration operretor and it is considered as definite integral from a to *x* for each integral. Applying L^{-1} to both sides of equation (4) we obtain,

$$u(x)=\alpha_0+\alpha_1(x-a)+\alpha_2\frac{x-a)^2}{2!}+........+\alpha_{n-1}\frac{(x-a)^{n-1}}{(n-1)!}$$

$$+L^{-1}(f(x))+\left[\lambda\int_a^b h(t)u(t)\,dt\right]L^{-1}(g(x)) \ .... (5)$$

In other were we integrate (3) *n* times from a to *x* and we used initial condition at every step of the integration.

Note that egm (5) is standard Frodholm integral equation and can be solved by using decomposition method.

Let $u(x) = \sum_{n=0}^{\infty} u_n(x)$ ………………. (6)

be the series solution of (5)

Substituting (6) in (5) we obtain

$$\sum_{n=0}^{\infty} \mathrm{u_n}(x) = \sum_{k=0}^{n-1} \alpha_k \frac{(x-a)^k}{k!} + L^{-1}\left(f(x)\right) + \left[\lambda \int_a^b h(t)\left(\sum_{n=0}^{\infty} u_n(t)\, dt\right)\right] L^{-1} g(x)$$

Therefore,

$$u_0(x) + u_1(x) + u_2(x) + \cdots = \sum_{k=0}^{n-1} \alpha_k \frac{(x-a)^k}{k!} + L^{-1}\left(f(x)\right)$$

$$+\left[\lambda \int_a^b h(t) u_0(t)\, dt\right] L^{-1}\left(g(x)\right)$$

$$+\left[\lambda \int_a^b h(t) u_1(t)\, dt\right] L^{-1}\left(g(x)\right)$$

$$+\left[\lambda \int_a^b h(t) u_2(t)\, dt\right] L^{-1}\left(g(x)\right)$$

$$+\left[\lambda \int_a^b h(t) u_3(t)\, dt\right] L^{-1}\left(g(x)\right)$$

We set,

$$u_0(x) = \sum_{k=0}^{n-1} \lambda_k \frac{(x-a)^k}{k!} + L^{-1}\left(f(x)\right)$$

$$u_1(x)=\left(\lambda\int_a^b h(t)u_0(t)\,dt\right)L^{-1}\left(g(x)\right)$$

$$u_2(x)=\left(\lambda\int_a^b h(t)u_1(t)\,dt\right)L^{-1}\left(g(x)\right)$$

$$u_3(x)=\left(\lambda\int_a^b h(t)u_2(t)\,dt\right)L^{-1}\left(g(x)\right)$$

and so on...

The components $u_0(x), u_1(x), u_2(x), \ldots$ of $u(x)$

in (6) can be written in recursively relations by

$$u_0(x)=\sum_{k=0}^{n-1}\alpha_k\,\frac{(x-a)^k}{k!}+L^{-1}\left(f(x)\right)$$

$$u_{n+1}(x)=\left(\lambda\int_a^b h(t)u_n(t)\,dt\right)L^{-1}\left(g(x)\right), n \geqslant 0$$

The Noise Term Phenomena:-

Definition : The self cancelling terms between the components $u_0(x)$ and $u_1(x)$ are called the noise terms.

This phenomena suggest that instead of evaluating several components, it is useful to examine the first two component u_0 and u_1.

If we observe the appearance of like terms in both component with opposite sign then by cancelling these terms, the remaining non cancelled terms of u_o may in some cases provide the solution of the integral equation (2).

This can be justified through direct substitution However if the exact solution was not attenable by using this phenomena then we should continue at determing all other components of $u(x)$ to get closed form of solution or approximate solution.

Problem 1 : Solve the Fredholm integrodifferential equation

$$u^1(x) = \cos x + \frac{x}{4} - \frac{1}{4}\int_0^{\pi/2} xt\ u(t)\ dt,\ u(0) = 0$$

by using Adomain decomposition method.

Solution : Given integral equation is,

$$u^1(x) = cosx + \frac{x}{4} - \frac{1}{4}\int_0^{\pi/4} xt(t)\ dt,\ u(0) = 0$$

Integrating w.r.t. x from o to $x,$ we get.

$$u(x) - u(0) = \sin x + \frac{x^2}{8} - \left(\frac{1}{4}\int_0^{\pi/2} tu(t)dt\right)\frac{x^2}{2}$$

This gives

$$u(x) = \sin x + \frac{x^2}{8} - \frac{x^2}{8}\int_0^{\pi/2} tu(t)dt \qquad \ldots\ldots\ldots\ldots \qquad (2)$$

Let $$u(x) = \sum_{n=0}^{\infty} u_n(x) \qquad \ldots\ldots\ldots\ldots \qquad (3)$$

be the solution of integral equation (2)

Then,

$$\sum_{n=0}^{\infty} u_n(x) = \sin x + \frac{x^2}{8} - \frac{x^2}{8}\int_0^{\pi/2} t\left(\sum_{n=0}^{\infty} u_n(t)\right)dt$$

This gives,

$$u_0(x) + u_1(x) + u_2(x) + u_3(x) + \cdots =$$

$$\sin x + \frac{x^2}{8} - \frac{x^2}{8}\int_0^{\pi/2} tu_0(t)\,dt - \frac{x^2}{8}\int_0^{\pi/2} tu_1(t)\,dt - \frac{x^2}{8}\int_0^{\pi/2} tu_2(t)\,dt + \ldots\ldots\ldots\ldots$$

We set,

$$u_0(x) = \sin x + \frac{x^2}{8}$$

$$u_1(x) = \frac{-x^2}{8}\int_0^{\pi/2} u_0(t)\,dt$$

$$= -\frac{x^2}{8}\int_0^{\pi/2} t\left(\sin t + \frac{t^2}{8}\right)dt$$

$$= \frac{-x^2}{8}\int_0^{\pi/2}\left(t\sin t + \frac{t^3}{8}\right)dt$$

$$= -\frac{x^2}{8}\left\{\left[-t\cos t - (1)\cdot(-\sin t)\right]_0^{\pi/2} + \left[\frac{t^4}{32}\right]_0^{\pi/2}\right\}$$

$$= \frac{-x^2}{8}\left\{\left[0+\sin\frac{\pi}{2}+0-\sin 0\right]+\frac{(\pi/2)^4}{32}\right\}$$

$$= -\frac{x^2}{8}\left\{1+\frac{\pi^4}{16\times 32}\right\}$$

$$= \frac{-x^2}{8}-\frac{x^2\pi^4}{3\times 16\times 32}$$

$$= -\frac{x^2}{8}-\frac{x^2\pi^4}{16^3}$$

Considering the first two components $u_o(x)$ and $u_1(x)$, we observe that the two identitical terms $\frac{x^2}{8}$ appears in their component with apposite sign. Cancelling these terms and substituting the remaining cancelled term in $u_0(x)$ i.e. sin$x$, we see that equation (1) satisfied which is as shown below. Letting $u(x)$ = sin x we have,

$u(0)=0$ and $\int \cos x+\frac{x}{4}-\frac{x}{4}\int_0^{\pi/2} t\sin t dt$

$$= \cos x+\frac{x}{4}-\frac{x}{4}\left[t(-\cos t)-1\cdot(-\sin t)\right]_0^{\pi/2}$$

$$= \cos x+\frac{x}{4}-\frac{x}{4}\left[0+1-0+0\right]$$

$$= \cos x+\frac{x}{4}-\frac{x}{4}=\cos x$$

$$= \frac{d}{dx}(\sin x)=u'(x)$$

Hence u (x) = sin$x$ is the required solution of the integro-differential equation (1)

Problem 2 : Solve the following fredhoim integrodiff exn,

$$u^1(x)=\frac{1}{6}-\frac{x}{18}+\int_0^1 xt\ u(t)\,dt,\ u(0)=0 \qquad \text{..............}(1)$$

by adomain decomposition method

Solution : Given integral equation is,

$$u'(x)=\frac{1}{6}-\frac{x}{18}+\int_0^1 xtu(t)\,dt,\ \ u(0)=0$$

Integrating from o to x, we get,

$$u(x)-u(0)=\frac{x}{6}-\frac{x^2}{36}+\frac{x^2}{2}\int_0^1 tu(t)\,dt \qquad \text{................ (2)}$$

Let $u(x)=\sum_{n=0}^{\infty}u_n(x)$ .... (3)

is the solution of integral equation (2)

$$\sum_{n=0}^{\infty}u_n(x)=\frac{x}{6}-\frac{x^2}{36}+\frac{x^2}{2}\int_0^1 t\left(\sum_{n=0}^{\infty}u_n(t)\,dt\right)$$

Thus, $u_0(x)+u_1(x)+u_2(x)+u_3(x)+\cdots$

$$=\frac{x}{6}-\frac{x^2}{36}+\frac{x^2}{2}\int_0^1 tu_0(t)\,dt+\frac{x^2}{2}\int_0^2 tu_1(t)\,dt+\frac{x^2}{2}\int_0^2 tu_2(t)\,dt$$

we set,

$$u_0\left(x\right)=\frac{x}{6}-\frac{x^2}{36}$$

$$u_1\left(x\right)=\frac{x^2}{2}\int_0^1 tu_0\left(t\right)dt$$

$$=\frac{x^2}{2}\int_0^1 t\left(\frac{t}{6}-\frac{t^2}{36}\right)dt$$

$$=\frac{x^2}{2}\left[\frac{t^3}{18}-\frac{t^4}{4\times16}\right]_0^1$$

$$=\frac{x^2}{2}\left(\frac{1}{18}-\frac{1}{4\times36}\right)$$

$$=\frac{x^2}{2}\left(\frac{8-1}{144}\right)$$

$$=\frac{7x^2}{2\times144}=\frac{7x^2}{288}$$

$$u_2\left(x\right)=\frac{x^2}{2}\int_0^1 t\ u_1+\left(t\right)dt$$

$$=\frac{x^2}{2}\int_0^1 t.\frac{7t^2}{2\times144}dt$$

$$=\frac{7x^2}{4\times144}\left[\frac{t^4}{4}\right]_0^1$$

$$=\frac{7x^2}{8\times 288}(1-0)$$

$$=\frac{7x^2}{8(288)}$$

$$u_3(x)=\frac{x^2}{2}\int_0^1 tu_2(t)\,dt=\frac{x^2}{2}\int_0^1 t\frac{7t^2}{4^2x144}dt$$

$$=\frac{7x^2}{2\times 4^2\times 144}\left[\frac{t^4}{4}\right]_0^1$$

$$=\frac{7x^2}{8\times 4^2\times 144}$$

$$=\frac{7x^2}{64\ (288)}$$

Therefore equation (3) reduces to

$$u(x)=u_0(x)+u_1(x)+u_2(x)+u_3(x)+\cdots$$

$$=\frac{x}{6}-\frac{x^2}{36}+\frac{7x^2}{288}+\frac{7x^2}{8\times 288}+\frac{7x^2}{64\times 288}+........$$

$$=\frac{x}{6}-\frac{x^2}{36}+\frac{7x^2}{288}\left[1+\frac{1}{8}+\frac{1}{8^2}+\cdots\right]$$

$$=\frac{x}{6}-\frac{x^2}{36}+\frac{7x^2}{288}\sum_{n=0}^{\infty}\left(\frac{1}{8}\right)^n$$

$$=\frac{x}{6}-\frac{x^2}{36}+\frac{7x^2}{288}.\frac{1}{1-\frac{1}{8}}$$

$$=\frac{x}{6}-\frac{x^2}{36}+\frac{7x^2\times 8}{288\times 7}$$

$$=\frac{x}{6}-\frac{x^2}{36}+\frac{x^2}{36}=\frac{x}{6}$$

$$=\frac{7x^2}{64\times 288}\left(1^4-0^4\right)$$

Exercise : Save the following Fredholam integro-differential equation by using adomain decomposition method.

1. $u^1\left(x\right)=-1+24x+\int_0^1 u(t)dt,\ \ u\ (0)=0$

2. $u^1\left(x\right)=6+17x+\int_0^1 x\ u(t)dt,\ \ u\ (0)=0$

3. $u^{11}\left(x\right)=2x-\cos x\int_0^{\pi} x+u(t)dt,\ u(0)=1,\ u^1(0)=0$

4. $u^{11}=-2-\cos x\int_0^{\pi}\left(x-t\right)u_1+dt,\ u(0)=1,\ u^1(0)=0$

5. $u^{(iv)}(x)=-2x-\sin x+\cos x\int_{-\pi/2}^{\pi/2} xt\ u(\text{t})\ \text{dt},$

   $u(0)=1,\ u^1(0)=0, u^{11}(0)=-1,\ u^{111}(0)=-1$

Exercise 2 : Save the following Volterra integro-differential equation by using adomain decomposition method.

1. $u^{111}(x) = -1 + x - \int_0^{\pi} (x-t) u(\text{t}) dt, u(0) = 1,\ u^{1}(0) = -1,\ u^{11}(0) = 1$

2. $u^{1}(x) = 1 + \int_0^{\pi} u(t) dt,\ u(0) = 0$

3. $u^{11}(x) = 1 + \int_0^{\pi} (x-t) u(\text{t}) u(\text{t}) \text{dt},\ u(0) = 1,\ u^{1}(0) = -1$

4. $u^{11}(x) = -1 - x + \int_0^{\pi} (x-t) u(\text{t}) \text{dt},\ u(0) = 1,\ u^{1}(0) = 1$

5. $u^{11}(x) = 1 + x\frac{x^3}{31} + \int_0^{\pi} (x-t) u(\text{t})\ u(\text{t}) \text{dt},\ u(0) = 1,\ u^{1}(0) = 2$