CHAPTER - II

# VARIATIONAL PRINCIPLES

## Unit 1: Euler-Lagranges's Differential Equations:

- ***Introduction:***

We have seen that co-ordinates are the tools in the hands of a mathematician. With the help of these co-ordinates the motion of a particle and also the path followed by the particle can be discussed. The piece wise information of the path $y = f(x)$, whether it is minimum or maximum at a point can be obtained from differential calculus by putting $\dot{y} = 0$. The function is either maximum or minimum at the point depends upon the value of second derivative of the function at that point. The function is maximum at a point if its second derivative is negative at the point, and is minimum at the point if its second derivative is positive at that point.

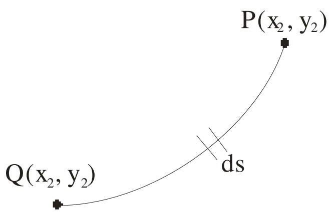
However, if we want to know the information about the whole path, we use integral calculus. i.e., the techniques of calculus of variation and are called variational principles. Thus the calculus of variation has its origin in the generalization of the elementary theory of maxima and minima of function of a single variable or more variables. The history of calculus of variations can be traced back to the year 1696, when John Bernoulli advanced the problem of the brachistochrone. In this problem one has to find the curve connecting two given points A and B that do not lie on a vertical line, such that a particle sliding down this curve under gravity from A reaches point B in the shortest time.

Apart from the problem of brachistochrone, there are three other problems exerted great influence on the development of the subject and are:

1. the problem of geodesic,
2. the problem of minimum surface of revolution and

3. the isoperimetric problem.

Thus in calculus of variation we consider the motion of a particle or system of particles along a curve y = f(x) joining two points $P(x_1, y_1)$ and $Q(x_2, y_2)$. The infinitesimal distance between two points on the curve is given by




$$ds = \left(dx^2 + dy^2\right)^{\frac{1}{2}}.$$

Hence the total distance between two point P and Q along the curve is given by

$$I\left(y(x)\right) = \int_{x_1}^{x_2} \left(1 + y'^2\right)^{\frac{1}{2}} dx, \quad y' = \frac{dy}{dx}$$

In general the integrand is a function of the independent variable x, the dependent variable y and its derivative $y'$. Thus the most general form of the integral is given by

$$I\left(y(x)\right) = \int_{x_1}^{x_2} f\left(x, y, y'\right) dx. \qquad \ldots (1)$$

This integral may represents the total path between two given points, the surface area of revolution of a curve, the time for quickest decent etc. depending upon the situation of the problem. The functional $I$ in general depends upon the starting point $(x_1, y_1)$, the end point $(x_2, y_2)$ and the curve between two points. The question is what function y is of x so that the functional $I\left(y(x)\right)$ has stationary value. Thus in this chapter we first find the condition to be satisfied by y(x) such that the functional $I\left(y(x)\right)$ defined in (1) must have extremum value. The fascinating principle in calculus of variation paves the way to find the curve of extreme distance between two points. Its object is to extremize the values of the functional. This is one of the most fundamental and beautiful principles in applied mathematics. Because from this principle one can determine the

(a) Newton's equations of motion,

(b) Lagrange's equations of motion,

(c) Hamilton's equations of motion,

(d) Schrödinger's equations of motion,

(e) Einstein's field equations for gravitation,

(f) Hoyle-Narlikar's equations for gravitation and so on and so forth by slightly modifying the integrand.

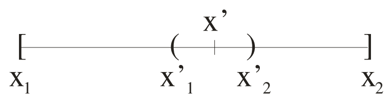
**Note :** A functional means a quantity whose values are determined by one or several functions. i.e., domain of a functional is a set of all admissible functions.

e.g. The length of the path $l$ between two points is a function of curves y(x), which it self is a function of x. Such functions are called functional.

- ***Basic Lemma :***

If $x_1$ and $x_2$ $(> x_1)$ are fixed constants and $G(x)$ is a particular continuous function for $x_1 \le x \le x_2$ and if $\int_{x_1}^{x_2} G(x)\eta(x)\,dx = 0$ for every choice of continuous differentiable function $\eta(x)$ such that $\eta(x_1) = 0 = \eta(x_2)$, then $G(x) = 0$ identically in $x_1 \le x \le x_2$.

**Proof** : Let the lemma be not true. Let us assume that there is a particular value $x'$ of x in the interval such that $G(x') \ne 0$. Let us assume that $G(x') > 0$.




Since $G(x)$ is continuous function in $x_1 \le x \le x_2$ and in particular it is continuous at $x = x'$. Hence there must exist an interval surrounding $x'$ say $x_1' \le x \le x_2'$ in which $G(x) > 0$ everywhere.

Let us now see whether the integral $\int_{x_1}^{x_2} G(x)\eta(x)\,dx = 0 \quad \forall$ permissible choice of $\eta(x)$.

We choose $\eta(x)$ such that

$$\begin{aligned} \eta(x) &= 0 && for \quad x_1 \le x \le x_1' \\ &= (x - x_1')^2 (x - x_2')^2 && for \quad x_1' \le x \le x_2' \\ &= 0 && for \quad x_2' \le x \le x_2 \end{aligned} \qquad \text{. . . (1)}$$

For this choice of $\eta(x)$ which also satisfies

$$\eta(x_1') = \eta(x_2') = 0,$$

the integral $\int_{x_1}^{x_2} G(x)\eta(x)\,dx$ becomes

$$\int_{x_1}^{x_2} G(x)\eta(x)\,dx = \int_{x_1}^{x_1'} G(x)\eta(x)\,dx + \int_{x_1'}^{x_2'} G(x)\eta(x)\,dx + \int_{x_2'}^{x_2} G(x)\eta(x)\,dx$$

$$\Rightarrow \quad \int_{x_1}^{x_2} G(x)\eta(x)\,dx = \int_{x_1'}^{x_2'} (x - x_1')^2 (x - x_2')^2 G(x)\,dx \qquad \text{. . . (2)}$$

Since

$$G(x) > 0 \text{ in } x_1' \le x \le x_2',$$

$\Rightarrow$ R. H. S. of equation (2) is definitely positive.

$$\Rightarrow \quad \int_{x_1}^{x_2} \eta(x) G(x)\,dx > 0,$$

This is contradiction to the hypothesis

$$\int_{x_1}^{x_2} G(x)\eta(x)\,dx = 0.$$

If $\quad G(x') < 0,$

we obtain the similar contradiction. This contradiction arises because of our assumption that $G(x) \neq 0$ for x in $x_1 \leq x \leq x_2$.

This implies that $G(x) = 0$ identically in $x_1 \leq x \leq x_2$.

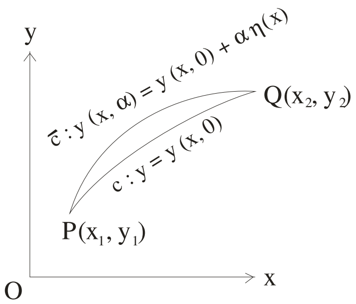
This completes the proof.

**Theorem 1 :** Find the Euler- Lagrange differential equation satisfied by twice differentiable function y(x) which extremizes the functional

$$I(y(x)) = \int_{x_1}^{x_2} f(x, y, y')dx$$

where y is prescribed at the end points.

**Proof**: Let $P(x_1, y_1)$ and $Q(x_2, y_2)$ be two fixed points in $xy$ plane. The points P and Q can be joined by infinitely many curves. Accordingly the value of the integral $I$ will be different for different paths. We shall look for a curve along which the functional $I$ has an extremum value. Let c be a curve between P and Q whose equation is given by $y = y(x,0)$.




Let also the value of the functional along the curve c be extremum and is given by

$$I(y(x)) = \int_{x_1}^{x_2} f(x, y, y')dx \qquad \ldots (1)$$

We can label all possible paths starting from P and ending at Q by the family of equations

$$y(x, \alpha) = y(x, 0) + \alpha\eta(x), \qquad \ldots (2)$$

where $\alpha$ is a parameter and $\eta(x)$ is any differentiable function of x.

For different values of $\alpha$ we get different curves. Accordingly the value of the integral $I$ will be different for different paths. Since y is prescribed at the end points, this implies that there is no variation in y at the end points. i.e., all the curves of the family must be identical at fixed points P and Q.

$$\Rightarrow \quad \eta(x_1) = 0 = \eta(x_2) \qquad \ldots (3)$$

Conversely, the condition (3) ensures us that the curves of the family that all pass through the points P and Q. Let the value of the functional along the neighboring curve be given by

$$I\big(y(x,\alpha)\big) = \int_{x_1}^{x_2} f\big(x, y(x,\alpha), y'(x,\alpha)\big)\,dx \qquad \ldots (4)$$

From differential calculus, we know the integral $I$ is extremum if $\left(\frac{\partial I}{\partial \alpha}\right)_{\alpha=0} = 0$, since for $\alpha = 0$ the neighboring curve coincides with the curve which gives extremum values of $I$.

Thus $$\left(\frac{\partial I}{\partial \alpha}\right)_{\alpha=0} = 0, \Rightarrow \int_{x_1}^{x_2} \left(\frac{\partial f}{\partial y}\eta(x) + \frac{\partial f}{\partial y'}\eta'(x)\right)dx = 0.$$

Integrating the second integration by parts, we get

$$\int_{x}^{x_2} \frac{\partial f}{\partial y}\eta(x)\,dx + \left(\frac{\partial f}{\partial y'}\eta(x)\right)_{x_1}^{x_2} - \int_{x_1}^{x_2} \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)\eta(x)\,dx = 0 \qquad \ldots (5)$$

As y is prescribed at the end points, hence on using equations (3) we obtain

$$\int_{x_1}^{x_2} \left(\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)\right)\eta(x)\,dx = 0.$$

By using the basic lemma of calculus of variation we get

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0. \qquad \ldots (6)$$

This is required Euler- Lagrange differential equation to be satisfied by y(x) for which the functional $I$ has extremum value.

- ***Important Note :***

If however, y is not prescribed at the end points then there is a difference in y even at the end points and hence $(\eta(x))_{x_1}^{x_2} \neq 0$. As the value of the functional $I$ is taken only on the extremal between two points and hence we must have the Euler-Lagrange equation is true. Consequently, in this case we must have from equation (5) that

$$\left(\frac{\partial f}{\partial y'}\right)_{x_1}^{x_2} = 0 \;\Rightarrow\; \left(\frac{\partial f}{\partial y'}\right)_{x_1} = 0 \quad and \quad \left(\frac{\partial f}{\partial y'}\right)_{x_2} = 0. \qquad \text{. . . (7)}$$

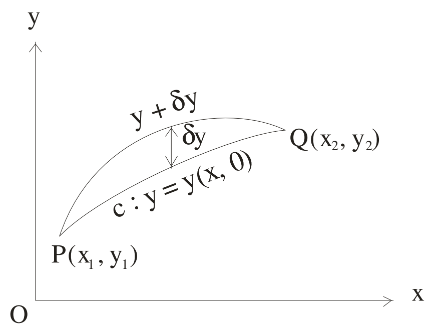
We will prove this result a little latter in Theorem No. 2.

**<u>Aliter :</u>** (Proof of the above Theorem (1)):

Let $P(x_1, y_1)$ and $Q(x_2, y_2)$ be two fixed points in xy plane. Let c be the curve between P and Q whose equation is given by y = y(x).Let the extremum value of the functional along the curve c be given by

$$I(y(x)) = \int_{x_1}^{x_2} f(x, y, y')dx. \qquad \text{. . . (1)}$$

To find the condition to be satisfied by y(x), let the curve c be slightly deformed from the original position such that any point $y$ on the curve c is displaced to $y+\delta y$, where $\delta y$ is the variation in the path for an arbitrary choice of $\alpha$, at any point except at the end points, as $y$ prescribed there. Mathematically this means that




$$y(x_1) = y_1, y(x_2) = y_2$$

$$\Rightarrow \quad (\delta y)_{x_1}^{x_2} = 0. \qquad \text{. . . (2)}$$

Thus the value of the functional along the varied path is given by

$$I' = \int_{x_1}^{x_2} f\left(x, y + \delta y, y' + \delta y'\right) dx . \qquad \text{. . . (3)}$$

Hence the change in the value of the functional due to change in the path is given by

$$I' - I = \int_{x_1}^{x_2} \left[ f\left(x,\, y + \delta y,\, y' + \delta y'\right) - f\left(x,\, y,\, y'\right) \right] dx ,$$

Let $$I' - I = \delta I = \int_{x_1}^{x_2} \left[ f\left(x,\, y + \delta y,\, y' + \delta y'\right) - f\left(x,\, y,\, y'\right) \right] dx . \qquad \text{. . . (4)}$$

We recall the Taylor's series expansion for the function of two variables'

$$f\left(x,\, y + \delta y,\, y' + \delta y'\right) = f\left(x,\, y,\, y'\right) + \left( \frac{\partial f}{\partial y} \delta y + \frac{\partial f}{\partial y'} \delta y' \right) + \ldots$$

Since $\delta y$ is very small, therefore by neglecting the higher order terms in $\delta y$ and $\delta y'$ we have

$$f\left(x,\, y + \delta y,\, y' + \delta y'\right) - f\left(x,\, y,\, y'\right) = \left( \frac{\partial f}{\partial y} \delta y + \frac{\partial f}{\partial y'} \delta y' \right).$$

Substituting this in the equation (4) we get

$$\delta I = \int_{x_1}^{x_2} \left( \frac{\partial f}{\partial y} \delta y + \frac{\partial f}{\partial y'} \delta y' \right) dx$$

We know $$\delta \frac{dy}{dx} = \frac{d}{dx} \delta y ,$$

hence we have

$$\delta I = \int_{x_1}^{x_2} \left( \frac{\partial f}{\partial y} \delta y + \frac{\partial f}{\partial y'} \frac{d}{dx} (\delta y) \right) dx .$$

Integrating the second integral by parts we get

$$\delta I = \int_{x_1}^{x_2} \frac{\partial f}{\partial y} \delta y \, dx + \left( \frac{\partial f}{\partial y'} \delta y \right)_{x_1}^{x_2} - \int_{x_1}^{x_2} \frac{d}{dx} \left( \frac{\partial f}{\partial y'} \right) \delta y \, dx .$$

On using equation (2) we get

$$\delta I = \int_{x_1}^{x_2} \left[ \frac{\partial f}{\partial y} - \frac{d}{dx}\left( \frac{\partial f}{\partial y'} \right) \right] \delta y \, dx . \qquad \text{. . . (5)}$$

If $I$ is extremum along the curve y = y(x) then change in $I$ is zero. $\left( i.e., \quad \delta I = 0 \right)$. This resembles very closely with a similar condition of extremum of a function in differential calculus.

$$\Rightarrow \qquad \int_{x_1}^{x_2} \left[ \frac{\partial f}{\partial y} - \frac{d}{dx}\left( \frac{\partial f}{\partial y'} \right) \right] \delta y \, dx = 0 .$$

Since $\delta y$ is arbitrary, we have

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\left( \frac{\partial f}{\partial y'} \right) = 0 .$$

This is the Euler-Lagrange's differential equation to be satisfied by y(x) for the extremum of the functional between two points.

- ***Generalization of Theorem (1) : Euler-Lagrange's equations for several dependent variables.***

**Theorem 1a :** Derive the Euler-Lagrange's equations that are to be satisfied by twice differential functions $y_1, y_2, ..., y_n$ that extremize the integral

$$I = \int_{x_1}^{x_2} f\left( x, y_1, y_2, ..., y_n, y_1', y_2', ..., y_n' \right) dx$$

with respect to those functions $y_1, y_2, ..., y_n$ which achieve prescribed values at the fixed points $x_1, x_2$.

**Proof:** The functional which is to be extremized can be written as

$$I = \int_{x_1}^{x_2} f\left( x, y_i, y_i' \right) dx, \quad i = 1, 2, ..., n .$$

Choose the family of neighboring curves as

$$y_i\left( x, \alpha \right) = y_i\left( x, 0 \right) + \alpha \eta_i\left( x \right)$$

and repeating the procedure delineated either in the Theorem (1) or in the alternate proof we arrive the following set of Euler-Lagrange's equations

$$\frac{\partial f}{\partial y_i}-\frac{d}{dx}\left(\frac{\partial f}{\partial y_i'}\right)=0,\quad i=1,2,\ldots,n\,.$$

**Geodesic** : Geodesic is defined as the curve of stationary (extremum) length between two points.

## Worked Examples

**Example 1 :** Show that the geodesic (shortest distance between two points) in a Euclidian plane is a straight line.

**Solution**: Take $P(x_1, y_1)$ and $Q(x_2, y_2)$ be two fixed points in a Euclidean plane. Let $y=f(x)$ be the curve between P and Q. Then the element of distance between two neighboring points on the curve $y=f(x)$ joining P and Q is given by

$$ds^2=dx^2+dy^2$$

Hence the total distance between the point P and Q along the curve is given by

$$I=\int_P^Q ds$$

$$\Rightarrow \quad I=\int_{x_1}^{x_2}\left(1+y'^2\right)^{\frac{1}{2}}dx,\quad y'=\frac{dy}{dx} \qquad \ldots(1)$$

Here the functional $I$ is extremum if the integrand

$$f=\left(1+y'^2\right)^{\frac{1}{2}} \qquad \ldots(2)$$

must satisfy the Euler-Lagrange's differential equation

$$\frac{\partial f}{\partial y}-\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)=0 \qquad \ldots(3)$$

Now from equation (2) we find that

$$\frac{\partial f}{\partial y} = 0 \quad and \quad \frac{\partial f}{\partial y'} = \frac{y'}{\sqrt{1+y'^2}}$$

$$\Rightarrow \quad \frac{d}{dx}\left(\frac{y'}{\sqrt{1+y'^2}}\right) = 0.$$

Integrating we get

$$y' = c\sqrt{1+y'^2}.$$

Squaring we get

$$y' = c_1, \quad where \quad c_1 = \frac{c}{\sqrt{1+c^2}}.$$

Integrating we get

$$y = c_1 x + c_2. \qquad \ldots (4)$$

This is the required straight line. Thus the shortest distance between two points in a Euclidean plane is a straight line.

**Example 2 :** Show that the shortest distance between two polar points in a plane is a straight line.

**Solution**: Define a curve in a plane. If A $(x, y)$ and B$(x+dx, y+dy)$ are infinitesimal points on the curve, then an element of distance between A and B is given by

$$ds^2 = dx^2 + dy^2. \qquad \ldots (1)$$

Let $\theta = \theta(r)$ be the polar equation of the curve and $P(r_1, \theta_1)$ *and* $Q(r_2, \theta_2)$ be two polar points on it. Recall the relations

$$x = r\cos\theta,$$
$$y = r\sin\theta.$$

Hence equation (1) becomes

$$ds^2 = dr^2 + r^2 d\theta^2. \qquad \ldots (2)$$

Thus the total distance between the points P and Q becomes

$$I = \int_{r_1}^{r_2} \left(1 + r^2\theta'^2\right)^{\frac{1}{2}} dr, \quad \theta' = \frac{d\theta}{dr}. \qquad \text{. . . (3)}$$

The functional $I$ is shortest if the integrand

$$f = \left(1 + r^2\theta'^2\right)^{\frac{1}{2}} \qquad \text{. . . (4)}$$

must satisfy the Euler-Lagrange's differential equation

$$\frac{\partial f}{\partial \theta} - \frac{d}{dr}\left(\frac{\partial f}{\partial \theta'}\right) = 0, \qquad (5)$$

$$\Rightarrow \quad \frac{d}{dr}\left(\frac{r^2\theta'}{\sqrt{1 + r^2\theta'^2}}\right) = 0,$$

$$\Rightarrow \quad r^2\theta' = h\sqrt{1 + r^2\theta'^2}.$$

Squaring and solving for $\theta'$ we get

$$\frac{d\theta}{dr} = \pm \frac{h}{r\left(r^2 - h^2\right)^{\frac{1}{2}}}.$$

On integrating we get

$$\theta = \pm \cos^{-1}\left(\frac{h}{r}\right) + \theta_0,$$

where $\theta_0$ is a constant of integration. We write this as

$$h = r\cos(\theta - \theta_0). \qquad \text{. . . (6)}$$

This is the polar form of the equation of straight line. Hence the shortest distance between two polar points is a straight line.

**Note :** If $r = r(\theta)$ is the polar equation of the curve, then the length of the curve is given by

$$I = \int_{\theta_0}^{\theta_1} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta.$$

Since the integrand $f = \sqrt{r^2 + r'^2}$ does not contain $\theta$, we therefore have

$$f - r'\frac{\partial f}{\partial r'} = h.$$

Solving this equation we readily obtain the same polar equation of straight line as the geodesic.

**Example 3 :** Show that the geodesic $\phi = \phi(\theta)$ on the surface of a sphere is an arc of the great circle.

**Solution :** Consider a sphere of radius r described by the equations

$$\begin{aligned} x &= r\sin\theta\cos\phi, \\ y &= r\sin\theta\sin\phi, \\ z &= r\cos\theta. \end{aligned} \qquad \ldots (1)$$

If $A(x, y, z)$ and $B(x+dx, y+dy, z+dz)$ be two neighboring points on the curve joining the points P and Q. Then the infinitesimal distance between A and B along the curve is given by

$$ds^2 = dx^2 + dy^2 + dz^2, \qquad \ldots (2)$$

where from equation (1) we find

$$\begin{aligned} dx &= r\cos\theta\cos\phi d\theta - r\sin\theta\sin\phi d\phi, \\ dy &= r\cos\theta\sin\phi d\theta + r\sin\theta\cos\phi d\phi, \\ dz &= -r\sin\theta d\theta. \end{aligned} \qquad \ldots (3)$$

Squaring and adding these equations we readily obtain

$$ds^2 = r^2 d\theta^2 + r^2\sin^2\theta d\phi^2. \qquad \ldots (4)$$

Hence the total distance between the points P and Q along the curve $\phi = \phi(\theta)$ is given by

$$I = \int_{\theta_1}^{\theta_2} r\left(1 + \sin^2\theta\phi'^2\right)^{\frac{1}{2}} d\theta, \quad \phi' = \frac{d\phi}{d\theta} \qquad \ldots (5)$$

where

$$f = r\left(1 + \sin^2\theta\phi'^2\right)^{\frac{1}{2}}. \qquad \ldots (6)$$

The curve is geodesic if the functional $I$ is stationary. This is true if the function f must satisfy the Euler-Lagrange's equations.

$$\frac{\partial f}{\partial \phi} - \frac{d}{d\theta}\left(\frac{\partial f}{\partial \phi'}\right) = 0 \qquad \ldots (7)$$

$$\Rightarrow \quad \frac{d}{d\theta}\left(\frac{r\sin^2\theta\,\phi'}{\sqrt{1+\sin^2\theta\,\phi'^2}}\right) = 0\,.$$

Integrating we get

$$\Rightarrow \quad \frac{\sin^2\theta\phi'^2}{\sqrt{1+\sin^2\theta\phi'^2}} = c,$$

Solving for $\phi'$ we get

$$\phi' = \frac{c\cos ec^2\theta}{\left(1-c^2\cos ec^2\theta\right)^{\frac{1}{2}}}\,.$$

On simplifying we get

$$\frac{d\phi}{d\theta} = \frac{k\cos ec^2\theta}{\left(1-k^2\cot^2\theta\right)^{\frac{1}{2}}}\,. \qquad \ldots (8)$$

Put $\qquad k\cot\theta = t \Rightarrow \quad -\cos ec^2\theta\,d\theta = dt\,,$

Therefore we have

$$d\phi = -\frac{dt}{\sqrt{1-t^2}}\,.$$

Integrating we get

$$\phi = \alpha - \sin^{-1} t\,,$$

or

$$\phi = \alpha - \sin^{-1}\left(k\cot\theta\right),$$

$$\Rightarrow \quad k\cot\theta = \sin\left(\alpha-\phi\right),$$

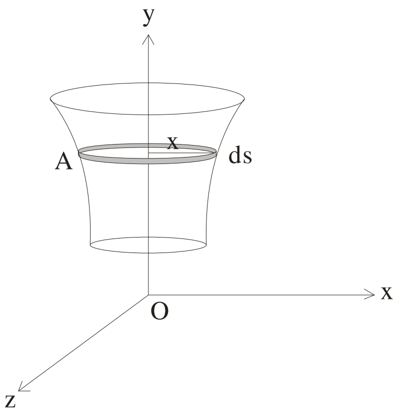
$$\Rightarrow \quad k\cos\theta = \sin\alpha\sin\theta\cos\phi - \cos\alpha\sin\theta\sin\phi,$$

$$\Rightarrow \quad kz = x\sin\alpha - y\cos\alpha\,. \qquad \ldots (9)$$

This is the first-degree equation in x, y, z, which represents a plane. This plane passes through the origin, hence cutting the sphere in a great circle. Hence the geodesic on the surface of a sphere is an arc of a great circle.

**Example 4 :** Show that the curve is a catenary for which the area of surface of revolution is minimum when revolved about y-axis.

**Solution**: Consider a curve between two points $(x_1, y_1)$ and $(x_2, y_2)$ in the $xy$ plane whose equation is $y = y(x)$. We form a surface by revolving the curve about y-axis. Our claim is to find the nature of the curve for which the surface area is minimum. Consider a small strip at a point A formed by revolving the arc length $ds$ about y –axis. If the distance of the point A on the curve from y-axis is x, then the surface area of the strip is equal to $2\pi x\, ds$.




But we know the element of arc $ds$ is given by

$$ds = \sqrt{1+y'^2}\,dx.$$

Thus the surface area of the strip $ds$ is equal to

$$2\pi x\sqrt{1+y'^2}\,dx.$$

Hence the total area of the surface of revolution of the curve $y = y(x)$ about y- axis is given by

$$I = \int_{x_1}^{x_2} 2\pi x\sqrt{1+y'^2}\,dx. \qquad \ldots (1)$$

This surface area will be minimum if the integrand

$$f = 2\pi x\sqrt{1+y'^2} \qquad \ldots (2)$$

must satisfy Euler-Lagrange's equation

$$\frac{\partial f}{\partial y}-\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)=0, \qquad \text{. . . (3)}$$

$$\Rightarrow \quad \frac{d}{dx}\left(\frac{2\pi\, x\, y'}{\sqrt{1+y'^2}}\right)=0,$$

$$\Rightarrow \quad \frac{d}{dx}\left(\frac{x\, y'}{\sqrt{1+y'^2}}\right)=0.$$

Integrating we get

$$xy' = a\sqrt{1+y'^2}\ .$$

Solving for $y'$ we get

$$\frac{dy}{dx}=\frac{a}{\sqrt{x^2-a^2}}\,.$$

Integrating we get

$$y = a\cosh^{-1}\left(\frac{x}{a}\right)+b\,.$$

Or $$x = a\cosh\left(\frac{y-b}{a}\right). \qquad \text{. . . (4)}$$

This shows that the curve is the catenary.

- ***The Brachistochrone Problem :***

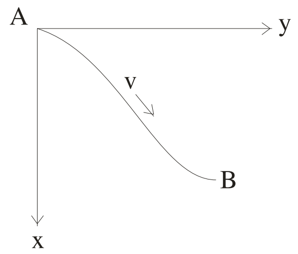
The Brachistochrone is the curve joining two points not lie on the vertical line, such that the particle falling from rest under the influence of gravity from higher point to the lower point in minimum time. The curve is called the cycloid.

**Example 5:** Find the curve of quickest decent.

**Or**

A particle slides down a curve in the vertical plane under gravity. Find the curve such that it reaches the lowest point in shortest time.

**Solution**: Let A and B be two points on the curve not lie on the vertical line. Let $v = \frac{ds}{dt}$ be the speed of the particle along the curve. Then the time required to fall an arc length $ds$ is given by




$$dt = \frac{ds}{v}$$

$$\Rightarrow \quad dt = \frac{\sqrt{1+y'^2}}{v}dx.$$

Therefore the total time required for the particle to go from A to B is given by

$$t_{AB} = \int_A^B \frac{\sqrt{1+y'^2}}{v}dx \qquad \ldots (1)$$

Since the particle falls freely under gravity, therefore its potential energy goes on decreasing and is given by

$$V = -mgx\,,$$

and the kinetic energy is given by

$$T = \frac{1}{2}mv^2\,.$$

Now from the principle of conservation of energy we have

$$T + V = \text{constant}.$$

Initially at point A, we have $x = 0$ and $v = 0$. Hence the constant is zero.

$$\Rightarrow \quad \frac{1}{2}mv^2 = mgx\,,$$

$$\Rightarrow \quad v = \sqrt{2gx}\,. \qquad \ldots (2)$$

Hence equation (1) becomes

$$t_{AB} = \int_{x_1}^{x_2} \frac{\sqrt{1+y'^2}}{\sqrt{2gx}}dx\,. \qquad \ldots (3)$$

Thus $t_{AB}$ is minimum if the integrand

$$f = \frac{\sqrt{1+y'^2}}{\sqrt{2gx}}, \qquad \ldots (4)$$

must satisfy Euler-Lagrange's equation

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0 \qquad \ldots (5)$$

$$\Rightarrow \quad \frac{d}{dx}\left(\frac{y'}{\sqrt{2gx(1+y'^2)}}\right) = 0$$

$$\Rightarrow \quad \frac{d}{dx}\left(\frac{y'}{\sqrt{x(1+y'^2)}}\right) = 0$$

Integrating we get

$$y' = c\sqrt{x(1+y'^2)}\ .$$

Solving it for $y'$ we get

$$\frac{dy}{dx} = \frac{\sqrt{x}}{\sqrt{a-x}}$$

Integrating we get

$$y = \int \frac{\sqrt{x}}{\sqrt{a-x}}\, dx + b \qquad \ldots (6)$$

Put

$$\begin{aligned} x &= a\sin^2(\theta/2) \\ \Rightarrow \quad dx &= 2a\sin(\theta/2)\cos(\theta/2)d\theta \end{aligned} \qquad \ldots (7)$$

Hence

$$y = a\int \sin^2(\theta/2)d\theta + b\,.$$

$$\Rightarrow \quad y = \frac{a}{2}(\theta - \sin\theta) + b\,,$$

If $\qquad y = 0, \theta = 0 \Rightarrow b = 0\,,$

hence

$$y = \frac{a}{2}(\theta - \sin\theta). \qquad \ldots (8)$$

Thus from equations (7) and (8) we have

$$x = b(1 - \cos\theta),$$

$$y = b(\theta - \sin\theta), \quad for \quad b = \frac{a}{2}$$

This is a cycloid. Thus the curve is a cycloid for which the time of decent is minimum.

**Example 6 :** Find the extremal of the functional

$$\int_0^{\frac{\pi}{2}} \left(y'^2 - y^2 + 2xy\right) dx$$

subject to the conditions that

$$y(0) = 0, \quad y\left(\frac{\pi}{2}\right) = 0.$$

**Solution**: Let the functional be denoted by

$$I = \int_0^{\frac{\pi}{2}} \left(y'^2 - y^2 + 2xy\right) dx. \qquad \ldots (1)$$

The functional is extremum if the integrand

$$f = y'^2 - y^2 + 2xy \qquad \ldots (2)$$

must satisfy the Euler-Lagrange's equation

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0, \qquad \ldots (3)$$

$$\Rightarrow \quad 2(x - y) - \frac{d}{dx}(2y') = 0,$$

$$\Rightarrow \quad y'' + y = x. \qquad \ldots (4)$$

This is second order differential equation, whose complementary function (C.F.) is given by

$$y = c_1 \cos x + c_2 \sin x \qquad \ldots (5)$$

where $c_1$ *and* $c_2$ are arbitrary constants.

The particular integral (P.I.) is

$$y = \left(1 + D^2\right)^{-1} x$$

$$y = x.$$

Hence the general solution is given by

$$y = c_1 \cos x + c_2 \sin x + x. \qquad \ldots (6)$$

This shows that the extremals of the functional are the two-parameter family of curves. On using the boundary conditions we obtain

$$y(0) = 0 \Rightarrow \quad c_1 = 0,$$

$$y\left(\frac{\pi}{2}\right) = 0 \Rightarrow \quad c_2 = -\frac{\pi}{2}.$$

Hence the required extremal is

$$y = x - \frac{\pi}{2} \sin x. \qquad \ldots (7)$$

**Example 7 :** Find the extremal of the functional

$$\int_1^2 \left(\frac{x^3}{y'^2}\right) dx$$

subject to the conditions that

$$y(1) = 0, \quad y(2) = 3.$$

**Solution:** Let the functional be denoted by

$$I = \int_1^2 \left(\frac{x^3}{y'^2}\right) dx \qquad \ldots (1)$$

The functional is extremum if the integrand

$$f = \frac{x^3}{y'^2} \qquad \ldots (2)$$

must satisfy the Euler-Lagrange's equation

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0. \qquad (3)$$

$$\Rightarrow \qquad \frac{d}{dx}\left(\frac{x^3}{y'^3}\right) = 0.$$

Integrating we get

$$x^3 = cy'^3$$

or $\qquad y' = ax.$

Integrating we get

$$y = \frac{a}{2}x^2 + b. \qquad \ldots (4)$$

Now using the boundary conditions we get

$$y(1) = 0 \Rightarrow \frac{a}{2} + b = 0,$$
$$y(2) = 3 \Rightarrow 2a + b = 3.$$

Solving these two equations we obtain

$$a = 2, \quad b = -1.$$

Hence the required functional becomes

$$y = x^2 - 1. \qquad \ldots (5)$$

**Example 8 :** Show that the time taken by a particle moving along a curve $y = y(x)$ with velocity $\frac{ds}{dt} = x,$ from the point (0,0) to the point (1,1) is minimum if the curve is a circle having its center on the x-axis.

**Solution**: Let a particle be moving along a curve y = y(x) from the point (0, 0) to the point (1, 1) with velocity

$$x = \frac{ds}{dt}$$

$$\Rightarrow \quad dt = \frac{ds}{x}.$$

Therefore the total time required for the particle to move from the point (0, 0) to the point (1, 1) is given by

$$t = \int_0^1 \frac{ds}{x} \qquad \text{. . . (1)}$$

where the infinitesimal distance between two points on the path is given by

$$ds = \sqrt{1 + y'^2}\, dx, \quad y' = \frac{dy}{dx}.$$

Hence the equation (1) becomes

$$t = \int_0^1 \frac{\sqrt{1 + y'^2}}{x} dx \qquad \text{. . . (2)}$$

Time t is minimum if the integrand

$$f = \frac{\sqrt{1 + y'^2}}{x} \qquad \text{. . . (3)}$$

must satisfy the Euler-Lagrange's equation

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0. \qquad \text{. . . (4)}$$

$$\Rightarrow \quad \frac{d}{dx}\left(\frac{y'}{x\sqrt{1 + y'^2}}\right) = 0,$$

$$\Rightarrow \quad y' = cx\sqrt{1 + y'^2}.$$

Solving for $y'$ we get

$$y' = \frac{x}{\sqrt{a^2 - x^2}}, \quad for \quad a = \frac{1}{c}.$$

Integrating we get

$$y = \int \frac{x}{\sqrt{a^2 - x^2}} dx + b$$

Put $\quad x = a\sin\theta \Rightarrow \quad dx = a\cos\theta d\theta \qquad \ldots (5)$

Therefore,

$$y = \int a\sin\theta d\theta + b,$$

$$y = -a\cos\theta + b. \qquad \ldots (6)$$

Squaring and adding equations (5) and (6) we get

$$x^2 + (y - b)^2 = a^2,$$

which is the circle having center on y –axis.

**Example 9 :** Show that the geodesic on a right circular cylinder is a helix.

**Solution**: We know the right circular cylinder is characterized by the equations

$$x^2 + y^2 = a^2, \quad z = z \qquad \ldots (1)$$

The parametric equations of the right circular cylinder are

$$x = a\cos\theta,$$
$$y = a\sin\theta,$$
$$z = z.$$

where a is a constant. The element of the distance (metric)

$$ds^2 = dx^2 + dy^2 + dz^2$$

on the surface of the cylinder becomes

$$ds^2 = a^2 d\theta^2 + dz^2,$$

Hence the total length of the curve on the surface of the cylinder is given by

$$s = \int_{\theta_0}^{\theta} \sqrt{a^2 + z'^2} d\theta, for \quad z' = \frac{dz}{d\theta}. \qquad \ldots (2)$$

For s to be extremum, the integrand

$$f = \sqrt{a^2 + z'^2}. \qquad \ldots (3)$$

must satisfy the Euler-Lagrange's equation

$$\frac{\partial f}{\partial z}-\frac{d}{d\theta}\left(\frac{\partial f}{\partial z'}\right)=0. \qquad \text{. . . (4)}$$

$$\Rightarrow \quad \frac{d}{d\theta}\left(\frac{z'}{f}\right)=0.$$

Integrating the equation and solving for $z'$ we get

$$z'=a \quad \text{(constant).}$$

Integrating we get

$$z=a\theta+b, \quad a\neq 0, \qquad \text{. . . (5)}$$

where a, b are constants. Equation (5) gives the required equation of helix. Thus the geodesic on the surface of a cylinder is a helix.

**Example 10 :** Find the differential equation of the geodesic on the surface of an inverted cone with semi-vertical angle $\theta$.

**Solution**: The surface of the cone is characterized by the equation

$$x^2+y^2=z^2\tan^2\theta, \quad \theta=const. \qquad \text{. . . (1)}$$

The parametric equations of the cone are given by

$$\begin{aligned} x&=ar\cos\phi,\\ y&=ar\sin\phi,\\ z&=br. \end{aligned} \qquad \text{. . . (2)}$$

where for $a=\sin\theta,\ \ b=\cos\theta$ are constant.

Thus the metric $ds^2=dx^2+dy^2+dz^2$

on the surface of the cone becomes

$$ds^2=dr^2+a^2r^2d\phi^2 \qquad \text{. . . (3)}$$

Hence the total length of the curve $\phi=\phi(r)$ on the surface of the cone is given by

$$s=\int\sqrt{1+a^2r^2\phi'^2}\,dr,\ \phi'=\frac{d\phi}{dr} \qquad \text{. . . (4)}$$

The length s is stationary if the integrand

$$f=\sqrt{1+a^2r^2\phi'^2} \qquad \text{. . . (5)}$$

must satisfy the Euler-Lagrange's equation

$$\frac{\partial f}{\partial \phi}-\frac{d}{dr}\left(\frac{\partial f}{\partial \phi'}\right)=0\,. \qquad \text{. . . (6)}$$

$$\Rightarrow \quad \frac{d}{dr}\left(\frac{a^2r^2\phi'}{f}\right)=0\,,$$

Solving for $\phi'$ we get

$$\frac{d\phi}{dr}=\frac{c_1}{ar\sqrt{a^2r^2-c_1^2}}\,, \qquad \text{. . . (7)}$$

where $c_1$ = constant.

This is the required differential equation of geodesic, and the geodesic on the surface of the cone is obtained by integrating equation (7). This gives

$$\phi=\int\frac{c_1}{ar\sqrt{a^2r^2-c_1^2}}dr+\alpha\,.$$

$$\phi=\frac{1}{a}sec^{-1}\left(\frac{ar}{c_1}\right)+\alpha,$$

$$\Rightarrow \quad r=\frac{c_1}{a}\sec\left[a(\phi-\alpha)\right].$$

**Example 11 :** Find the curve for which the functional

$$I\left[y(x)\right]=\int_0^{\frac{\pi}{4}}\left(y^2-y'^2\right)dx$$

can have extrema, given that y(0)=0, while the right –hand end point can vary along the line $x=\frac{\pi}{4}$.

**Solution**: To find the extremal curve of the functional

$$I\left[y(x)\right]=\int_0^{\frac{\pi}{4}}\left(y^2-y'^2\right)dx, \qquad \ldots (1)$$

we must solve Euler's equation

$$\frac{\partial f}{\partial y}-\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)=0, \qquad \ldots (2)$$

where $$f=y^2-y'^2 \qquad \ldots (3)$$

$$\Rightarrow \quad y''+y=0. \qquad \ldots (4)$$

This is the second order differential equation, whose solution is given by

$$y=a\cos x+b\sin x. \qquad \ldots (5)$$

The boundary condition $y(0)=0$ gives a = 0.

$$\Rightarrow \quad y=b\sin x. \qquad \ldots (6)$$

The second boundary point moves along the line $x=\frac{\pi}{4}$.

$$\Rightarrow \left(\frac{\partial f}{\partial y'}\right)_{x=\frac{\pi}{4}}=0$$

$$\Rightarrow (y')_{x=\frac{\pi}{4}}=0,$$

where from equation (6) we have $y'=b\cos x$. Thus $y'\ at\ x=\frac{\pi}{4}$ gives

b= 0. This implies the extremal is attained on the line y = 0.

**Example 12 :** If $f$ satisfies Euler-Lagrange's equation

$$\frac{\partial f}{\partial y}-\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)=0.$$

Then show that $f$ is the total derivative $\frac{dg}{dx}$ of some function of x and y and conversely.

**Solution**: Given that $f$ satisfies Euler-Lagrange's equation

$$\frac{\partial f}{\partial y}-\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)=0. \qquad \ldots (1)$$

We claim that $f=\dfrac{dg}{dx}$,

where $g=g(x,y)$.

As $f=f(x,y,y')$,

we write equation (1) explicitly as

$$\frac{\partial f}{\partial y}-\frac{\partial^2 f}{\partial x\partial y'}-\frac{\partial^2 f}{\partial y\partial y'}y'-\frac{\partial^2 f}{\partial^2 y'^2}y''=0. \qquad \ldots (2)$$

We see from equation (2) that the first three terms on the l. h. s. of (2) contain at the highest the first derivative of y. Therefore equation (2) is satisfied identically if the coefficient of $y''$ vanishes identically.

$$\Rightarrow \quad \frac{\partial^2 f}{\partial y'^2}=0.$$

Integrating w. r. t. $y'$ we get

$$\frac{\partial f}{\partial y'}=q(x,y).$$

Integrating once again we get

$$f=q(x,y)\,y'+p(x,y), \qquad \ldots (3)$$

where $p(x,y)$ and $q(x,y)$ are constants of integration and may be function of x and y only. Then the function $f$ so determined must satisfy the Euler –Lagrange's equation (1). From equation (3) we find

$$\frac{\partial f}{\partial y}=y'\frac{\partial q}{\partial y}+\frac{\partial p}{\partial y},$$

and $\dfrac{\partial f}{\partial y'}=q(x,y)$.

Therefore equation (1) becomes

$$y'\frac{\partial q}{\partial y}+\frac{\partial p}{\partial y}-\frac{d}{dx}\big(q(x,y)\big)=0\,.$$

$$\Rightarrow \quad y'\frac{\partial q}{\partial y}+\frac{\partial p}{\partial y}-\frac{\partial q}{\partial x}-\frac{\partial q}{\partial y}y'=0\,.$$

$$\Rightarrow \quad \frac{\partial p}{\partial y}=\frac{\partial q}{\partial x}. \qquad .(4)$$

This is the condition that the equation $pdx+qdy$ is an exact differential equation $dg$ .

$$\Rightarrow \quad dg = pdx+qdy,$$

$$\Rightarrow \quad \frac{dg}{dx}=p+qy'=f \quad by \quad (3)$$

Therefore,

$$f=\frac{dg}{dx}. \qquad ...(5)$$

This proves the necessary part.

Conversely, assume that $f=\dfrac{dg}{dx}$. We prove that $f$ satisfies the Euler-Lagrange's equation

$$\frac{\partial f}{\partial y}-\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)=0\,.$$

Since $\qquad f=\dfrac{dg}{dx}=\dfrac{\partial g}{\partial x}+\dfrac{\partial g}{\partial y}y'$ ,

Therefore, we find

$$\frac{\partial f}{\partial y}=\frac{\partial^2 g}{\partial x\partial y}+\frac{\partial^2 g}{\partial y^2}y',$$

$$\frac{\partial g}{\partial y'}=\frac{\partial g}{\partial y}.$$

Consider now

$$\frac{\partial f}{\partial y}-\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)=\frac{\partial^2 g}{\partial x\partial y}+\frac{\partial^2 g}{\partial y^2}y'-\frac{d}{dx}\left(\frac{\partial g}{\partial y}\right).$$

$$\frac{\partial f}{\partial y}-\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)=\frac{\partial^2 g}{\partial x\partial y}+\frac{\partial^2 g}{\partial y^2}y'-\frac{\partial^2 g}{\partial x\partial y}-\frac{\partial^2 g}{\partial y^2}y'.$$

$$\frac{\partial f}{\partial y}-\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)=0$$

$$\Rightarrow \quad f=\frac{dg}{dx}$$

satisfies Euler-Lagrange's equation.

**Example 13 :** Show that the Euler-Lagrange's equation of the functional

$$I\left(y(x)\right)=\int_{x_1}^{x_2} f\left(x, y, y'\right)dx$$

has the first integral $f-y'\dfrac{\partial f}{\partial y'}=const$ , if the integrand does not depend on x.

**Solution**: The Euler-Lagrange's equation of the functional

$$I\left(y(x)\right)=\int_{x_1}^{x_2} f\left(x, y, y'\right)dx$$

to be extremum is given by

$$\frac{\partial f}{\partial y}-\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)=0. \qquad \text{. . . (1)}$$

$$\Rightarrow \frac{\partial f}{\partial y}-\frac{\partial^2 f}{\partial x\partial y'}-\frac{\partial^{2f}}{\partial y\partial y'}y'-\frac{\partial^2 f}{\partial^2 y'^2}y''=0$$

If $f$ does not involve x explicitly, then $\dfrac{\partial f}{\partial x}=0$.

Therefore, we have

$$\frac{\partial f}{\partial y}-\frac{\partial^2 f}{\partial y\partial y'}y'-\frac{\partial^2 f}{\partial^2 y'^2}y''=0. \qquad \ldots (2)$$

Multiply equation (2) by $y'$ we get

$$\frac{\partial f}{\partial y}y'-\frac{\partial^2 f}{\partial y\partial y'}y'^2-\frac{\partial^2 f}{\partial^2 y'^2}y''y'=0 \qquad \ldots (3)$$

But we know that

$$\frac{d}{dx}\left(f-y'\frac{\partial f}{\partial y'}\right)=y'\frac{\partial f}{\partial y}+y''\frac{\partial f}{\partial y'}-y''\frac{\partial f}{\partial y'}-y'^2\frac{\partial^2 f}{\partial y'\partial y}-y'y''\frac{\partial^2 f}{\partial y'^2},$$

$$\frac{d}{dx}\left(f-y'\frac{\partial f}{\partial y'}\right)=y'\frac{\partial f}{\partial y}-y'^2\frac{\partial^2 f}{\partial y'\partial y}-y'y''\frac{\partial^2 f}{\partial y'^2}. \qquad \ldots (4)$$

From equations (3) and (4) we see that

$$\frac{d}{dx}\left(f-y'\frac{\partial f}{\partial y'}\right)=0,$$

$$\Rightarrow \quad f-y'\frac{\partial f}{\partial y'}=const. \qquad \ldots (5)$$

This is the first integral of Euler-Lagrange's equation, when the functional

$$f=f\left(y,y'\right).$$

**Worked Examples**

**Example 14 :** Find the minimum of the functional

$$I\left(y(x)\right)=\int_0^1\left(\frac{1}{2}y'^2+yy'+y'+y\right)dx$$

if the values at the end points are not given.

**Solution**: For the minimum of the functional

$$I\left(y(x)\right)=\int_0^1\left(\frac{1}{2}y'^2+yy'+y'+y\right)dx \qquad \ldots (1)$$

the integrand

$$f = \frac{1}{2} y'^2 + yy' + y' + y \qquad \ldots (2)$$

must satisfy the Euler-Lagrange's equation

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0. \qquad \ldots (3)$$

$$\Rightarrow \quad y' + 1 - \frac{d}{dx}(y' + y + 1) = 0,$$

$$\Rightarrow \quad y'' = 1. \qquad \ldots (4)$$

Integrating we get

$$y' = x + c_1, \qquad \ldots (5)$$

Further integrating we get

$$y = \frac{x^2}{2} + c_1 x + c_2, \qquad \ldots (6)$$

where $c_1, c_2$ are constants of integration and are to be determined.

However, note that the values of y at the end points are not prescribed. In this case the constants are determined from the conditions.

$$\left(\frac{\partial f}{\partial y'}\right)_{x=0} = 0, \quad and \quad \left(\frac{\partial f}{\partial y'}\right)_{x=1} = 0. \qquad \ldots (7)$$

These two conditions will determine the values of the constants.

$$\Rightarrow \quad (y' + y + 1)_{x=0} = 0, \, and \, (y' + y + 1)_{x=1} = 0, \qquad \ldots (8)$$

where from equation (5) and (6) we have

$$y'(0) = c_1 \; and \; y(0) = c_2,$$

similarly, $\quad y'(1) = 1 + c_1 \; and \; y(1) = \frac{1}{2} + c_1 + c_2.$

Thus the equations (8) become

$$c_1 + c_2 + 1 = 0, \, 2c_1 + c_2 + \frac{5}{2} = 0.$$

Solving these equations for $c_1$ and $c_2$ we obtain $c_1 = -\frac{3}{2}$ *and* $c_2 = \frac{1}{2}$

Hence the required curve for which the functional given in (1) becomes minimum is

$$y = \frac{1}{2}\left(x^2 - 3x + 1\right). \qquad \ldots (9)$$

**Theorem 3 :** Find the Euler- Lagrange differential equation satisfied by four times differentiable function y(x) which extremizes the functional

$$I\left(y(x)\right) = \int_{x_1}^{x_2} f\left(x, y, y', y''\right) dx$$

under the conditions that both y and $y'$ are prescribed at the end points.

**Proof** : Let $P\left(x_1, y_1\right)$ and $Q\left(x_2, y_2\right)$ be two fixed points in $xy$ plane. The points P and Q can be joined by infinitely many curves. Accordingly the value of the integral $I$ will be different for different paths. We shall look for a curve along which the functional $I$ has an extremum value. Let c be a curve between P and Q whose equation is given by y = y(x, 0). Let also the value of the functional along the curve c be extremum and is given by

$$I\left(y(x)\right) = \int_{x_1}^{x_2} f\left(x, y, y', y''\right) dx \qquad \ldots (1)$$

We can label all possible paths starting from P and ending at Q by the family of equations

$$y(x, \alpha) = y(x, 0) + \alpha\eta(x), \qquad \ldots (2)$$

where $\alpha$ is a parameter and $\eta(x)$ is any differentiable function of x.

For different values of $\alpha$ we get different curves. Accordingly the value of the integral $I$ will be different for different paths. Since y and $y'$ are prescribed at the end points, this implies that there is no variation in y and $y'$ at the end points.

i.e., all the curves of the family and their derivative must be identical at fixed points P and Q.

This implies that

$$y(x_1,\alpha)=y(x_1,0)=y_1,$$
$$y(x_2,\alpha)=y(x_2,0)=y_2,$$

and also

$$y'(x_1,\alpha)=y'(x_1,0)=y_1',$$
$$y'(x_2,\alpha)=y'(x_2,0)=y_2'$$

$$\Rightarrow \qquad \eta(x_1)=0=\eta(x_2), \qquad \ldots (3a)$$

and $$\eta'(x_1)=0=\eta'(x_2). \qquad \ldots (3b)$$

Conversely, the conditions (3) ensure us that the curves of the family that all pass through the points P and Q. Let the value of the functional along the neighboring curve be given by

$$I\left(y(x,\alpha)\right)=\int_{x_1}^{x_2} f\left(x, y(x,\alpha), y'(x,\alpha), y''(x,\alpha)\right)dx. \qquad \ldots (4)$$

From differential calculus, we know the integral $I$ is extremum if $\left(\frac{\partial I}{\partial\alpha}\right)_{\alpha=0}=0$, because for $\alpha=0$ the neighboring curve coincides with the curve which gives extremum values of $I$. Thus

$$\left(\frac{\partial I}{\partial\alpha}\right)_{\alpha=0}=0,$$

$$\Rightarrow \int_{x_1}^{x_2}\left(\frac{\partial f}{\partial y}\eta(x)+\frac{\partial f}{\partial y'}\eta'(x)+\frac{\partial f}{\partial y''}\eta''(x)\right)dx=0.$$

Integrating the second and the third integrations by parts, we get

$$\int_{x_1}^{x_2} \frac{\partial f}{\partial y}\eta(x)\,dx + \left(\frac{\partial f}{\partial y'}\eta(x)\right)_{x_1}^{x_2} - \int_{x_1}^{x_2} \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)\eta(x)\,dx + \left(\frac{\partial f}{\partial y''}\eta'(x)\right)_{x_1}^{x_2} -$$
$$-\left(\frac{d}{dx}\left(\frac{\partial f}{\partial y''}\right)\eta(x)\right)_{x_1}^{x_2} + \int_{x_1}^{x_2} \frac{d^2}{dx^2}\left(\frac{\partial f}{\partial y''}\right)\eta(x)dx = 0, \qquad \ldots (5)$$

$$\int_{x_1}^{x_2} \frac{\partial f}{\partial y}\eta(x)\,dx + \left(\left[\frac{\partial f}{\partial y'} - \frac{d}{dx}\left(\frac{\partial f}{\partial y''}\right)\right]\eta(x)\right)_{x_1}^{x_2} - \int_{x_1}^{x_2} \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)\eta(x)\,dx +$$
$$+\left(\frac{\partial f}{\partial y''}\eta'(x)\right)_{x_1}^{x_2} + \int_{x_1}^{x_2} \frac{d^2}{dx^2}\left(\frac{\partial f}{\partial y''}\right)\eta(x)\,dx \quad = 0. \qquad \ldots (6)$$

As y and $y'$ are both prescribed at the end points, hence on using equations (3) we obtain

$$\int_{x_1}^{x_2} \left(\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) + \frac{d^2}{dx^2}\left(\frac{\partial f}{\partial y''}\right)\right)\eta(x)\,dx = 0. \qquad \ldots (7)$$

By using the basic lemma of calculus of variation we get

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) + \frac{d^2}{dx^2}\left(\frac{\partial f}{\partial y''}\right) = 0. \qquad \ldots (8)$$

This is required Euler- Lagrange differential equation to be satisfied by y(x) for which the functional $I$ has extremum value.

**Note :** If the functions y and $y'$ are not prescribed at the end points then we must have unlike the fixed end point problem, $\eta(x)$ and $\eta'(x)$ need no longer vanish at the points $x_1$ and $x_2$. In order that the curve y =y(x) to be a solution of the variable end point problem, y must be an extremal. i.e., y must be a solution of Euler's equation (8). Thus for the extremal we have from equation (6)

$$\Rightarrow \quad \left[\frac{\partial f}{\partial y''}\right]_{x=a}^{x=b} = 0, \quad \left[\frac{\partial f}{\partial y'} - \frac{d}{dx}\left(\frac{\partial f}{\partial y''}\right)\right]_{x=a}^{x=b} = 0. \qquad \ldots (9)$$

Thus to solve the variable end point problem, we must first solve Euler's equation (8) and then use the conditions (9) to determine the values of the arbitrary constants.
The above result can be summarized in the following theorem (3a).

**Theorem (3a) :** Derive the differential equation satisfied by four times differentiable function $y(x)$, which extremizes the integral

$$I = \int_{x_0}^{x_1} f(x, y, y', y'')dx$$

under the condition that both $y, y'$ are prescribed at both the ends. Show that if neither $y$ nor $y'$ is prescribed at either end points then

$$\left(\frac{\partial f}{\partial y''}\right)_{x=x_0} = \left(\frac{\partial f}{\partial y''}\right)_{x=x_1} = 0$$

$$\left[\frac{\partial f}{\partial y'} - \frac{d}{dx}\left(\frac{\partial f}{\partial y''}\right)\right]_{x=x_0}^{x=x_1} = 0$$

**Remark:** (General case of Theorem (3)) If the integrand in equation (1) of the Theorem (3) is of the form $f = f\left(x, y, y', y'', ..., y^n\right)$ with the boundary conditions

$$y(x_1) = y_1, \quad y'(x_1) = y_1', ....., y^{n-1}(x_1) = y_1^{n-1},$$
$$y(x_2) = y_2, \quad y'(x_2) = y_2', ....., y^{n-1}(x_2) = y_2^{n-1},$$

then the Euler-Lagrange's differential equation is

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) + \frac{d^2}{dx^2}\left(\frac{\partial f}{\partial y''}\right) + ..... + (-1)^n \frac{d^n}{dx^n}\left(\frac{\partial f}{\partial y^n}\right) = 0.$$

## Worked Examples

**Example 15 :** Find the curve, which extremizes the functional

$$I\left(y(x)\right) = \int_0^{\frac{\pi}{4}} \left(y''^2 - y^2 + x^2\right)dx$$

under the conditions that

$$y(0)=0, \quad y'(0)=1,$$
$$y\left(\frac{\pi}{4}\right)=y'\left(\frac{\pi}{4}\right)=\frac{1}{\sqrt{2}}$$

**Solution :** For the extremization of the functional

$$I(y(x))=\int_0^{\frac{\pi}{4}}\left(y''^2-y^2+x^2\right)dx \qquad \text{. . . (1)}$$

the integrand

$$f=y''^2-y^2+x^2 \qquad \text{. . . (2)}$$

must satisfy the Euler-Lagrange's equation

$$\frac{\partial f}{\partial y}-\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)+\frac{d^2}{dx^2}\left(\frac{\partial f}{\partial y''}\right)=0. \qquad \text{. . . (3)}$$

$$\Rightarrow \quad -2y+\frac{d^2}{dx^2}(2y'')=0,$$

i.e.,
$$\frac{d^4y}{dx^4}-y=0. \qquad \text{. . . (4)}$$

The solution of equation (4) is given by

$$y=ae^x+be^{-x}+c\cos x+d\sin x \qquad \text{. . . (5)}$$

where a, b, c, d are constants of integration and are to be determined.

Thus

$$y(0)=0\Rightarrow a+b+c=0,$$
$$y\left(\frac{\pi}{4}\right)=\frac{1}{\sqrt{2}}\Rightarrow \quad ae^{\frac{\pi}{4}}+be^{-\frac{\pi}{4}}+\frac{1}{\sqrt{2}}c+\frac{1}{\sqrt{2}}d=\frac{1}{\sqrt{2}},$$
$$y'(0)=1\Rightarrow \quad a-b+d=1,$$
$$y'\left(\frac{\pi}{4}\right)=\frac{1}{\sqrt{2}}\Rightarrow \quad ae^{\frac{\pi}{4}}-be^{-\frac{\pi}{4}}-\frac{1}{\sqrt{2}}c+\frac{1}{\sqrt{2}}d=\frac{1}{\sqrt{2}}$$

Solving these equations we get $a = b = c = 0$ and $d = 1$.

Hence the required curve is $y = \sin x$.

**Example 16 :** Minimize the functional

$$I = \frac{1}{2}\int_0^2 (\ddot{x})^2\, dt,$$

satisfies

$$x(0) = 1,\ \dot{x}(0) = 1,\ x(2) = 1,\ \dot{x}(2) = 0.$$

**Solution :** To minimize the functional

$$I = \frac{1}{2}\int_0^2 (\ddot{x})^2\, dt, \qquad \ldots (1)$$

the integral

$$f = \frac{1}{2}\ddot{x}^2 \qquad \ldots (2)$$

must satisfy the Euler-Lagrange's equation

$$\frac{\partial f}{\partial x} - \frac{d}{dt}\left(\frac{\partial f}{\partial \dot{x}}\right) + \frac{d^2}{dt^2}\left(\frac{\partial f}{\partial \ddot{x}}\right) = 0. \qquad \ldots (3)$$

This implies $\dfrac{d^4 x}{dt^4} = 0$. $\qquad \ldots (4)$

Integrating we get

$$x = c_1\frac{t^3}{6} + c_2\frac{t^2}{2} + c_3 t + c_4. \qquad \ldots (5)$$

where $x$ given in (5) must satisfy the conditions

$$x(0) = 1 \Rightarrow c_4 = 1,$$
$$\dot{x}(0) = 1 \Rightarrow c_3 = 1,$$
$$x(2) = 1 \Rightarrow 4c_1 + 6c_2 = -3,$$
$$\dot{x}(2) = 0 \Rightarrow 2c_1 + 2c_2 = -1.$$

Solving for $c_1$ and $c_2$ we get the required functional is

$$x = -\frac{t^2}{4} + t + 1.$$

**Example 17 :** Find the function on which the functional

$$I(y(x)) = \int_0^1 \left(y''^2 - 2xy\right)dx,$$

can be extremized such that

$$y(0) = y'(0) = 0,\ y(1) = \frac{1}{120}$$

and $y'(1)$ is not prescribed.

**Solution :** For the extremization of the functional

$$I(y(x)) = \int_0^1 \left(y''^2 - 2xy\right)dx \qquad \text{. . . (1)}$$

the integrand

$$f = y''^2 - 2xy \qquad \text{. . . (2)}$$

must satisfy the Euler-Lagrange's equation

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) + \frac{d^2}{dx^2}\left(\frac{\partial f}{\partial y''}\right) = 0. \qquad \text{. . . (3)}$$

$$\Rightarrow \quad -2x + \frac{d^2}{dx^2}(2y'') = 0,$$

$$\Rightarrow \quad \frac{d^4y}{dx^4} = x.$$

Integrating we obtain

$$y''' = \frac{x^2}{2} + a,$$

$$\Rightarrow \quad y'' = \frac{x3}{6} + ax + b,$$

$$\Rightarrow \quad y' = \frac{x^4}{24} + \frac{a}{2}x^2 + bx + c\,,$$

$$\Rightarrow \quad y = \frac{x^5}{120} + \frac{a}{6}x^3 + \frac{b}{2}x^2 + cx + d\,, \qquad \ldots (4)$$

where a, b, c, d are constants to be determined. Given that

$$y(0) = 0 \Rightarrow \quad d = 0,$$
$$y'(0) = 0 \Rightarrow \quad c = 0,$$
$$y(1) = \frac{1}{120} \Rightarrow \; a + 3b = 0.$$

Since $y'(1)$ is not prescribed. i.e., $y'$ at x= 1 is not given, then we have the condition that

$$\left(\frac{\partial f}{\partial y''}\right)_{x=1} = 0 \Rightarrow \quad y''(1) = 0.$$

This gives from above equation that

$$6a + 6b = -1.$$

Solving the equations for a and b we get

$$a = -\frac{1}{4},\; and\; b = \frac{1}{12}.$$

Substituting these values in equation (4) we get

$$y = \frac{x^5}{120} + \frac{1}{24}\left(x^2 - x^3\right). \qquad \ldots (5)$$

This is the required curve.

- ***When integrand is a function more than two dependent variables :***

**Example 18:** Prove that the shortest distance between two points in a Euclidean 3-space is a straight line.

**Solution**: Define the curve $y = y(x)$, $z = z(x)$ in the 3-dimentional Euclidean space. Let P $(x, y, z)$ and Q $(x + dx, y + dy, z + dz)$ be two neighboring points on the curve joining the points $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$.Thus the infinitesimal distance between P and Q along the curve is given by

$$ds^2 = dx^2 + dy^2 + dz^2.$$

Hence the total distance between the points A and B along the curve is given by

$$I = \int_{x_1}^{x_2} \left(1 + \dot{y}^2 + \dot{z}^2\right)^{\frac{1}{2}} dx, \quad \dot{y} = \frac{dy}{dx} \qquad \text{. . . (1)}$$

Let $$f = \left(1 + \dot{y}^2 + \dot{z}^2\right)^{\frac{1}{2}}. \qquad \text{. . . (2)}$$

We know the functional $I$ is shortest if the function f must satisfy the Euler-Lagrange's equations.

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial \dot{y}}\right) = 0, \qquad \text{. . . (3)}$$

and $$\frac{\partial f}{\partial z} - \frac{d}{dx}\left(\frac{\partial f}{\partial \dot{z}}\right) = 0. \qquad \text{. . . (4)}$$

$$\Rightarrow \quad \frac{d}{dx}\left(\frac{\dot{y}}{f}\right) = o \Rightarrow \quad \dot{y} = af, \; a = \text{constant}$$

and

$$\Rightarrow \quad \dot{y}^2\left(1 - a^2\right) - a^2\dot{z}^2 = a^2. \qquad \text{. . . (5)}$$

Similarly, from equation (4) we obtain

$$\dot{z}^2\left(1 - b^2\right) - b^2\dot{y}^2 = b^2. \qquad \text{. . . (6)}$$

Solving equations (5) and (6) for $\dot{y}$ and $\dot{z}$ we get

$$\dot{y} = \pm\frac{a}{\sqrt{1 - a^2 - b^2}}, \text{ and } \quad \dot{z} = \pm\frac{a}{\sqrt{1 - a^2 - b^2}},$$

i.e., $$\dot{y} = \pm c_1, \text{ for } c_1 = a\left(1 - a^2 - b^2\right)^{-\frac{1}{2}} \qquad \text{. . . (7)}$$

and $$\dot{z} = \pm c_2 \quad \text{for } c_2 = b\left(1 - a^2 - b^2\right)^{-\frac{1}{2}}. \qquad \text{. . . (8)}$$

Integrating equations (7) and (8) we get

$$y = \pm c_1 x + \phi(z), \qquad \ldots (9)$$

$$z = \pm c_2 x + \psi(y), \qquad \ldots (10)$$

where $\phi(z)$ and $\psi(y)$ are constants of integration and may be functions of z and y respectively. Thus the required curve is given by equations (9) and (10). But these equations represent a pair of planes. The common point of intersection of these planes is the straight line. Hence the shortest distance between two points in Euclidean 3-space is a straight line.

**Example 19 :** Show that the geodesic defined in the 3-dimentional Euclidean space by the equations $x = x(t), \quad y = y(t), \quad z = z(t)$ is a straight line.

**Solution :** Let

$$x = x(t), \quad y = y(t), \quad z = z(t) \qquad \ldots (1)$$

be a curve in 3-dimentional Euclidean space, where t is a parameter of the curve. The infinitesimal distance between two neighboring points on the curve (1) is given by

$$ds^2 = dx^2 + dy^2 + dz^2,$$

where from equation (1) we have

$$dx = \dot{x}dt, \quad dy = \dot{y}dt, \quad dz = \dot{z}dt.$$

Thus

$$ds^2 = \left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right)dt^2.$$

Hence the total length of the curve between the points $P(t_0)$ and $P(t_1)$ is given by

$$I = \int_{t_0}^{t_1} \left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right)^{\frac{1}{2}} dt \qquad \ldots (2)$$

The curve is geodesic if the length of the curve $I$ is extremum. This is true if the integrand

$$f=\sqrt{\dot{x}^2+\dot{y}^2+\dot{z}^2} \qquad \ldots (3)$$

must satisfy the Euler-Lagrange's equations.

$$\frac{\partial f}{\partial x_i}-\frac{d}{dt}\left(\frac{\partial f}{\partial \dot{x}_i}\right)=0 \quad \forall\, i=1,2,3 \text{ with } x_i=(x,y,z), \qquad \ldots (4)$$

where

$$\frac{\partial f}{\partial x_i}=0 \text{ and } \frac{\partial f}{\partial \dot{x}_i}=\frac{\dot{x}_i}{f} \quad \forall\, i=1,2,3.$$

$$\Rightarrow \qquad \dot{x}=af, \quad \dot{y}=bf, \quad \dot{z}=cf.$$

Thus the Euler-Lagrange's equations become

$$\begin{aligned}
&(a^2-1)\dot{x}^2+a^2\dot{y}^2+a^2\dot{z}^2=0,\\
&b^2\dot{x}^2+(b^2-1)\dot{y}^2+b^2\dot{z}^2=0, \qquad \ldots (5)\\
&c^2\dot{x}^2+c^2\dot{y}^2+(c^2-1)\dot{z}^2=0.
\end{aligned}$$

These equations are consistent provided

$$\begin{vmatrix} a^2-1 & a^2 & a^2 \\ b^2 & b^2-1 & b^2 \\ c^2 & c^2 & c^2-1 \end{vmatrix}=0$$

$$\Rightarrow \qquad a^2+b^2+c^2=1.$$

Solving equations (5) we obtain

$$\dot{x}=\frac{a}{\sqrt{1-a^2-b^2}}\dot{z}, \qquad \ldots (6)$$

$$\dot{y}=\frac{b}{\sqrt{1-a^2-b^2}}\dot{z}, \quad \dot{z}\neq 0. \qquad \ldots (7)$$

Integrating equations (6) and (7) we obtain

$$x=c_1 z+\phi(y), \quad c_1=\frac{a}{\sqrt{1-a^2-b^2}}, \qquad \ldots (8)$$

$$y=c_2 z+\psi(x), \quad c_2=\frac{b}{\sqrt{1-a^2-b^2}}. \qquad \ldots (9)$$

where $\psi(x)$ and $\phi(y)$ are constants of integration and may be functions of x and y respectively. Equations (8) and (9) represent planes. The locus of the common points of these planes is the straight line. Hence the geodesic in 3-dimentional Euclidean space is the straight line.

**Unit 2:** ***Isoperimetric Problems:***

The problems in which the function which is eligible for the extremization of a given definite integral is required to confirm with certain restrictions that are given as the boundary conditions. Such problems are called isoperimetric problems. The method is exactly analogous to the method of finding stationary value of a function under certain conditions by Lagrange's multipliers method.

**Theorem 4** : Obtain the differential equation, which is satisfied by the functional $f(x, y, y')$ which extremizes the integral

$$I(y(x)) = \int_{x_1}^{x_2} f(x, y, y')dx$$

subject to the conditions $y(x_1) = y_1, \quad y(x_2) = y_2$, and the integral

$$J = \int_{x_1}^{x_2} g(x, y, y')dx = \text{constant.}$$

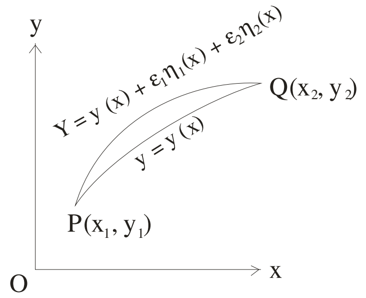
**Proof :** Consider the functional between two points $P(x_1, y_1)$ and $Q(x_2, y_2)$ given by

$$I(y(x)) = \int_{x_1}^{x_2} f(x, y, y')dx \qquad \ldots (1)$$

subject to the conditions

$$y(x_1) = y_1, \quad y(x_2) = y_2, \qquad \ldots (2)$$

and $$J = \int_{x_1}^{x_2} g(x, y, y')dx = \text{constant.} \qquad \ldots (3)$$




The points P and Q can be joined by infinitely many curves.

Accordingly the value of the integral $I$ will be different for different paths. Let all possible paths starting from P and ending at Q be given by two parameters family of curves

$$Y(x) = y(x) + \varepsilon_1\eta_1(x) + \varepsilon_2\eta_2(x) \qquad \ldots (4)$$

where $\varepsilon_1, \varepsilon_2$ are parameters and $\eta_1(x), \eta_2(x)$ are arbitrary differentiable functions of x such that

$$\begin{aligned} \eta_1(x_1) &= 0 = \eta_1(x_2), \\ \eta_2(x_1) &= 0 = \eta_2(x_2) \end{aligned} \qquad \ldots (5)$$

These conditions ensure us that the curves of the family that all pass through the points P and Q.

Note that, we can not however, express Y(x) as merely a one parameter family of curves, because any change in the value of the single parameter would in general alter the value of J, whose constancy must be maintained as prescribed. For this reason we introduce two parameter families of curves. We shall look for a curve along which the functional $I$ has an extremum value under the condition (3). Let c be such a curve between P and Q whose equation is given by y = y (x) such that the functional (1) along the curve c has extremum value. The values of the integrals (1) and (3) along the neighboring curve (4) are obtained by replacing y by Y in both the equations (1) and (3). Thus we have

$$I(\varepsilon_1, \varepsilon_2) = \int_{x_1}^{x_2} f(x, Y, Y')dx \qquad \ldots (6)$$

and $$J\left(\varepsilon_1,\varepsilon_2\right)=\int_{x_1}^{x_2} g\left(x,Y,Y'\right)dx=const. \qquad \ldots (7)$$

Equation (7) shows that $\varepsilon_1$ and $\varepsilon_2$ is not independent, but they are related by

$$J\left(\varepsilon_1,\varepsilon_2\right)=const. \qquad \ldots (8)$$

Thus the changes in the value of the parameters are such that the constancy of (7) is maintained. Thus our new problem is to extremizes (6) under the restriction (7). To solve the problem we use the method of Lagrange's multipliers.

Multiply equation (7) by $\lambda$ and adding it to equation (6) we get

$$I^*\left(\varepsilon_1,\varepsilon_2\right)=I+\lambda J=\int_{x_1}^{x_2} f^*\left(x,Y,Y'\right)dx, \qquad \ldots (9)$$

where $\lambda$ is Lagrange's undetermined multiplier and

$$f^*=f+\lambda g \qquad \ldots (10)$$

Thus extremization of (1) subject to the condition (3) is equivalent to the extremization of (9). Thus from differential calculus, the integral $I^*$ is extremum if

$$\left(\frac{\partial I^*}{\partial\varepsilon_j}\right)_{\varepsilon_1=0,\varepsilon_2=0}=0.$$

Thus from equation (9) we have

$$\left(\frac{\partial I^*}{\partial\varepsilon_j}\right)_{\varepsilon_1=0,\varepsilon_2=0}=0 \;\Rightarrow\; \int_{x_1}^{x_2}\left(\frac{\partial f^*}{\partial y}\eta_j\left(x\right)+\frac{\partial f^*}{\partial y'}\eta_j'\left(x\right)\right)dx=0.\ \text{J=1, 2} \qquad \ldots (11)$$

Note here that by setting $\varepsilon_1=\varepsilon_2=0$, we replace $Y$ and $Y'$ to $y, y'$.

Integrating the second integration by parts, we get

$$\int_{x_1}^{x_2}\frac{\partial f^*}{\partial y}\eta_j\left(x\right)dx+\left(\frac{\partial f^*}{\partial y'}\eta_j\left(x\right)\right)_{x_1}^{x_2}-\int_{x_1}^{x_2}\frac{d}{dx}\left(\frac{\partial f^*}{\partial y'}\right)\eta_j\left(x\right)dx=0. \qquad \ldots (12)$$

As any curve is prescribed at the end points, hence on using conditions (5) we obtain

$$\int_{x_1}^{x_2}\left(\frac{\partial f^*}{\partial y}-\frac{d}{dx}\left(\frac{\partial f^*}{\partial y'}\right)\right)\eta_j(x)\,dx=0,\ j=1,2\,.$$

By using the basic lemma of calculus of variation we get

$$\frac{\partial f^*}{\partial y}-\frac{d}{dx}\left(\frac{\partial f^*}{\partial y'}\right)=0\,. \qquad \ldots (13)$$

This is required Euler- Lagrange's differential equation to be satisfied by y (x) for which the functional $I$ has extremum value under the condition (3).

**Remark :** If y is not prescribed at either end point then from equation (12) we have

$$\frac{\partial f^*}{\partial y'}=0 \text{ at that end point.}$$

- ***Generalization of Theorem 4 : Euler-Lagrange's equations for several dependent variables :***

**Theorem 4a :** Obtain the differential equations which must be satisfied by the function which extremize the integral

$$I=\int_{x_1}^{x_2} f\left(x, y_1, y_2,\ldots, y_n, y_1', y_2',\ldots, y_n'\right)dx$$

with respect to the twice differentiable functions $y_1, y_2,\ldots, y_n$ for which

$$J=\int_{x_1}^{x_2} g\left(x, y_1, y_2,\ldots, y_n, y_1', y_2',\ldots, y_n'\right)dx=const.$$

and with $y, y_i'$ prescribed at points $x_1, x_2$.

**Proof :** The functional which is to be extremized can be written as

$$I=\int_{x_1}^{x_2} f\left(x, y_i, y_i'\right)dx, \quad i=1,2,\ldots,n$$

together with the conditions

$$J = \int_{x_1}^{x_2} g\left(x, y_i, y_i'\right) dx = const. \quad i = 1, 2, \ldots, n$$

and $y, y_i'$ prescribed at points $x_1, x_2$.

Repeating the procedure described in the Theorem (4) we arrive the following set of Euler-Lagrange's equations

$$\frac{\partial f^*}{\partial y_i} - \frac{d}{dx}\left(\frac{\partial f^*}{\partial y_i'}\right) = 0, \quad i = 1, 2, \ldots, n$$

where $f^* = f + \lambda g$.

**Theorem 5 :** Obtain the differential equation, which is satisfied by four times differential function y (x) which extremizes the functional

$$I\left(y(x)\right) = \int_{x_1}^{x_2} f\left(x, y, y', y''\right) dx$$

subject to the conditions that the integral

$$J = \int_{x_1}^{x_2} g\left(x, y, y', y''\right) dx = \text{constant.}$$

and both $y$ and $y'$ are prescribed at the end points.

**Proof:** Proof of the Theorem 5 runs exactly in the same manner as that of the proof of Theorem 3 and Theorem 4. The required Euler-Lagrange's differential equation in this case is given by

$$\frac{\partial f^*}{\partial y} - \frac{d}{dx}\left(\frac{\partial f^*}{\partial y'}\right) + \frac{d^2}{dx^2}\left(\frac{\partial f^*}{\partial y''}\right) = 0,$$

where $\qquad f^* = f + \lambda g$.

**Remarks :**

1. If y is not prescribed at either end point, then we have the condition $\frac{\partial f^*}{\partial y'} - \frac{d}{dx}\left(\frac{\partial f^*}{\partial y''}\right) = 0$ at that end point.

2. If $y'$ is not prescribed at either end point then we have $\frac{\partial f^*}{\partial y''} = 0$ at that point.

3. In general if

$$f = f\left(x, y, y', y'', ..., y^n\right)$$
$$g = g\left(x, y, y', y'', ..., y^n\right)$$

with the boundary conditions that $y, y', y'', ..., y^{n-1}$ are prescribed at both the ends, then in this case the Euler-Lagrange's equation is

$$\frac{\partial f^*}{\partial y} - \frac{d}{dx}\left(\frac{\partial f^*}{\partial y'}\right) + \frac{d^2}{dx^2}\left(\frac{\partial f^*}{\partial y''}\right) + ... + (-1)^n \frac{d^n}{dx^n}\left(\frac{\partial f^*}{\partial y^n}\right) = 0 .$$

## Worked Examples

**Example 20 :** Find the plane curve of fixed perimeter that encloses maximum area.

(The problem is supposed to have arisen from the gift of a king who was happy with a person and promised to give him all the land that he could enclose by running round in a day. The perimeter of his path was fixed.)

**Solution**: Let c : y = y (x) be a plane curve of fixed perimeter $l$.

$$l = \int_{x_1}^{x_2} ds , \qquad \text{. . . (1)}$$

where the infinitesimal distance between two points on the curve is given by

$$ds = \sqrt{1 + y'^2}\, dx, \quad y' = \frac{dy}{dx} .$$

Hence the total length of the curve between two points P and Q becomes

$$\int_{x_1}^{x_2} \sqrt{1+y'^2}\,dx = l . \qquad \ldots (2)$$

The area bounded by the curve c and the x-axis is given by

$$A\left(y(x)\right) = \int_{x_1}^{x_2} y\,dx . \qquad \ldots (3)$$

Thus we maximize (3) subject to the condition (2). Hence the required Euler-Lagrange's equation to be satisfied is

$$\frac{\partial f^*}{\partial y} - \frac{d}{dx}\left(\frac{\partial f^*}{\partial y'}\right) = 0 , \qquad \ldots (4)$$

where

$$\begin{aligned} f^* &= f + \lambda g, \\ f^* &= y + \lambda\sqrt{1+y'^2}. \end{aligned} \qquad \ldots (5)$$

Solving equation (4) we get

$$1 - \frac{d}{dx}\left(\frac{\lambda y'}{\sqrt{1+y'^2}}\right) = 0,$$

$$\Rightarrow \qquad x - \frac{\lambda y'}{\sqrt{1+y'^2}} = a .$$

Solving for $y'$ we get

$$(x-a)^2 = \frac{\lambda^2 y'^2}{\left(1+y'^2\right)},$$

or $$y' = \frac{x-a}{\sqrt{\lambda^2 - (x-a)^2}} . \qquad \ldots (6)$$

Integrating we get

$$y = \int \frac{(x-a)}{\left[\lambda^2 - (x-a)^2\right]^{\frac{1}{2}}} dx + b \qquad \text{. . . (7)}$$

Put

$$x - a = \lambda \sin t,$$
$$\Rightarrow \quad dx = \lambda \cos t dt \qquad \text{. . . (8)}$$

$$y = \int \lambda \sin t dt + b,$$

$$y = -\lambda \cos t + b \qquad \text{. . . (9)}$$

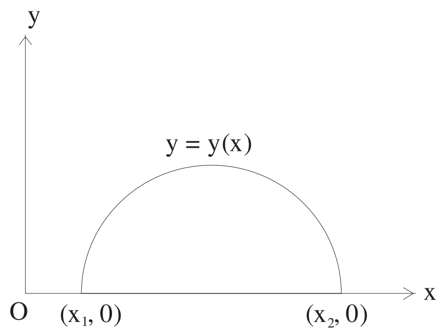
Squaring and adding equations (8) and (9) we get

$$\Rightarrow \quad (x-a)^2 + (y-b)^2 = \lambda^2. \qquad \text{. . . (10)}$$

This is a circle centered at (a, b) and of radius $\lambda$ and is to be determined. To determine $\lambda$, we know that the circumference of the circle is $2\pi\lambda = l \Rightarrow \lambda = \frac{l}{2\pi}$.

**Example 21 :** Find the shape of the plane curve of fixed length $l$ whose end points lie on the x-axis and area enclosed by it and the x-axis is maximum.

**Solution**: Let c: y = y(x) be a plane curve of fixed length $l$ whose end points lie on the x-axis and the curve lies in the upper half plane. The area bounded by the curve c and the x-axis is given by




$$A(y(x)) = \int_{x_1}^{x_2} y dx \qquad \text{. . . (1)}$$

such that the length of the curve is fixed and is given by

$$J = \int_{x_1}^{x_2} ds = \int_{x_1}^{x_2} \sqrt{1 + y'^2} dx = l \qquad \text{. . . (2)}$$

The area given in equation (1) is maximum under the condition (2), if

$$\frac{\partial f^*}{\partial y}-\frac{d}{dx}\left(\frac{\partial f^*}{\partial y'}\right)=0, \qquad \ldots (3)$$

where

$$\begin{aligned} f^* &= f+\lambda g, \\ f^* &= y+\lambda\sqrt{1+y'^2} \end{aligned} \qquad \ldots (4)$$

where $\lambda$ is Lagrange's multiplier. Solving equation (3) we get

$$1-\frac{d}{dx}\left(\frac{\lambda y'}{\sqrt{1+y'^2}}\right)=0,$$

$$\Rightarrow x-\frac{\lambda y'}{\sqrt{1+y'^2}}=a$$

Solving for $y'$ we get

$$(x-a)^2=\frac{\lambda^2 y'^2}{\left(1+y'^2\right)},$$

or

$$y'=\frac{x-a}{\sqrt{\lambda^2-(x-a)^2}}. \qquad \ldots (5)$$

Integrating we get

$$y=\int\frac{(x-a)}{\left[\lambda^2-(x-a)^2\right]^{\frac{1}{2}}}dx. \qquad \ldots (6)$$

Put

$$\begin{aligned} & x-a=\lambda\sin t, \\ \Rightarrow\quad & dx=\lambda\cos t\,dt \end{aligned} \qquad \ldots (7)$$

$$y=\int\lambda\sin t\,dt+b,$$

$$y=-\lambda\cos t+b. \qquad \ldots (8)$$

Squaring and adding equations (7) and (8) we get

$$\Rightarrow\quad (x-a)^2+(y-b)^2=\lambda^2. \qquad \ldots (9)$$

Thus the curve is a semi circle centered at (a, b) and of radius $\lambda$, and is to be determined.

Since from the condition (2) the perimeter of the semi-circle is $\pi\lambda = l \Rightarrow \quad \lambda = \dfrac{l}{\pi}$.

This is the radius of the curve of fixed length $l$ which encloses maximum area.

**Example 22** : Find the extremals for an isoperimetric problem

$$I\left(y(x)\right) = \int_0^1 \left(y'^2 + x^2\right) dx$$

subject to the conditions that

$$\int_0^1 y^2 dx = 2, \quad y(0) = 0, \quad y(1) = 0.$$

**Solution :** The functional, which is to be extremized, is given by

$$I\left(y(x)\right) = \int_0^1 \left(y'^2 + x^2\right) dx \qquad \text{. . . (1)}$$

such that

$$\int_0^1 y^2 dx = 2 \qquad \text{. . . (2)}$$

and

$$y(0) = 0, \quad y(1) = 0 \qquad \text{. . . (3)}$$

Thus for the extremizes of (1) under (2), we know the condition to be satisfied is

$$\frac{\partial f^*}{\partial y} - \frac{d}{dx}\left(\frac{\partial f^*}{\partial y'}\right) = 0, \qquad \text{. . . (4)}$$

where

$$\begin{aligned} f^* &= f + \lambda g, \\ f^* &= y'^2 + x^2 + \lambda y^2, \end{aligned} \qquad \text{. . . (5)}$$

where $\lambda$ is Lagrange's multiplier.

Solving equation (4) we get

$$y'' - \lambda y = 0. \qquad \text{. . . (6)}$$

This equation has roots $\pm\sqrt{\lambda}$ for $\lambda > 0$ or $\pm i\sqrt{\lambda}$ for $\lambda < 0$.

**<u>Case 1 :</u>** Let $\lambda > 0$.

The solution of equation (6) is

$$y = ae^{\sqrt{\lambda}x} + be^{-\sqrt{\lambda}x}, \qquad \text{. . . (7)}$$

Conditions (3) give

$$y(0) = 0 \Rightarrow \quad a + b = 0,$$
$$y(1) = 0 \Rightarrow \quad ae^{\sqrt{\lambda}} + be^{-\sqrt{\lambda}} = 0.$$

Solving for a and b we have for

$$b \neq 0, \quad e^{2\sqrt{\lambda}} = 1$$
$$\Rightarrow e^{2\sqrt{\lambda}+2in\pi} = 1$$
$$\Rightarrow 2\sqrt{\lambda} + 2in\pi = 0,$$
$$\Rightarrow \sqrt{\lambda} = -in\pi.$$

This is contradictory to $\lambda$ is positive, hence $b = 0$ and consequently $a = 0$ proving that the equation has only trivial solution.

**<u>Case 2 :</u>** Let $\lambda < 0$.

The solution of equation (6) is

$$y = a\cos\sqrt{\lambda}x + b\sin\sqrt{\lambda}x. \qquad \text{. . . (8)}$$

Boundary conditions (3) give

$$y(0) = 0 \Rightarrow \quad a = 0$$
$$y(1) = 0 \Rightarrow \quad b\sin\sqrt{\lambda}x = 0,$$
$$\Rightarrow \quad \sqrt{\lambda} = n\pi, \quad for \quad \lambda \neq 0.$$

Hence the required solution becomes we get

$$y = b\sin n\pi x. \qquad \text{. . . (9)}$$

Condition (2) gives

$$b = \pm 2 .$$

Therefore the required solution is $y = \pm 2 \sin n\pi x$ .

**Example 23 :** Find the extremals for an isoperimetric problem

$$I\left(y(x)\right) = \int_0^{\pi} \left(y'^2 - y^2\right) dx$$

subject to the conditions that

$$\int_0^{\pi} y dx = 1, \quad y(0) = 0, \quad y(\pi) = 1 .$$

**Solution :** It is given that

$$I\left(y(x)\right) = \int_0^{\pi} \left(y'^2 - y^2\right) dx \qquad \text{. . . (1)}$$

where the functional $I$ is to be extremized under the conditions

$$\int_0^{\pi} y dx = 1, \qquad \text{. . . (2)}$$

and $$y(0) = 0, \quad y(\pi) = 1 \qquad \text{. . . (3)}$$

The corresponding Euler-Lagrange's equation is

$$\frac{\partial f^*}{\partial y} - \frac{d}{dx}\left(\frac{\partial f^*}{\partial y'}\right) = 0 , \qquad \text{. . . (4)}$$

where

$$\begin{aligned} f^* &= f + \lambda g, \\ f^* &= y'^2 - y^2 + \lambda y \end{aligned} \qquad \text{. . . (5)}$$

Hence the equation (4) becomes

$$y'' + y = \frac{\lambda}{2} . \qquad \text{. . . (6)}$$

The C.F. of equation (6) is given by

$$y = a \cos x + b \sin x ,$$

where as the P.I. is given by

$$y = \frac{\lambda}{2}.$$

Hence the general solution becomes

$$y = a\cos x + b\sin x + \frac{\lambda}{2} \qquad \text{. . . (7)}$$

To determine the arbitrary constants of integration we use the boundary conditions (3)

$$y(0) = 0 \Rightarrow \quad a = -\frac{\lambda}{2},$$

$$y(\pi) = 1 \Rightarrow \quad \lambda = 1, a = -\frac{1}{2}$$

To determine other constant of integration we use

$$\int_0^\pi y dx = 1,$$

$$\Rightarrow \quad -\frac{1}{2}\int_0^\pi \cos x dx + b\int_0^\pi \sin x dx + \frac{1}{2}\int_0^\pi dx = 1.$$

This gives the value of b as

$$b = \left(\frac{2-\pi}{4}\right).$$

Hence the required curve is

$$y = \frac{1}{2}(1-\cos x) + \frac{1}{4}(2-\pi)\sin x. \qquad \text{. . . (8)}$$

**Example 24 :** Prove that the extremal of the isoperimetric problem

$$I = \int_1^4 y'^2 dx, \quad y(1) = 3, \quad y(4) = 24,$$

subject to the condition

$$\int_1^4 y\,dx = 36$$

is a parabola.

**Solution**: Here $f^* = y'^2 + \lambda y$.

The corresponding Euler-Lagrange's differential equation is

$$2y'' - \lambda = 0. \qquad \text{. . . (1)}$$

Integrating two times we get

$$y = \frac{\lambda}{4}x^2 + ax + b, \qquad \text{. . . (2)}$$

where the constants of integration are to be determined. Now the boundary conditions

$$\begin{aligned} y(1) = 3 &\Rightarrow \lambda + 4a + 4b = 12, \\ y(4) = 24 &\Rightarrow 4\lambda + 4a + b = 24. \end{aligned} \qquad \text{. . . (3)}$$

Also the condition

$$\int_1^4 y\,dx = 36,$$

gives

$$\int_1^4 \left(\frac{\lambda}{4}x^2 + ax + b\right)dx = 36,$$

$$\Rightarrow \quad 21\lambda + 30a + 12b = 144. \qquad \text{. . . (4)}$$

Solving equations (3) and (4) we obtain

$$a = 2, b = 0, \lambda = 4.$$

Thus the required curve is obtain by putting these values in equation (2) and is

$$y = x^2 + 2x.$$

We write this as $\qquad (x+1)^2 = y + 1$

Or equivalently for X=(x+1), Y=(y+1), we have

$$X^2 = Y .$$

Hence the curve is a parabola.

**Example 25 :** Find the extremals for the isoperimetric problem

$$I = \frac{1}{2}\int_{t_1}^{t_2} (x\dot{y} - y\dot{x})dt \qquad ...(1)$$

with the conditions that

$$J = \int_{t_1}^{t_2} \sqrt{\dot{x}^2 + \dot{y}^2}\, dt = l , \qquad ...(2)$$

$$\left.\begin{aligned} x(t_1) &= x(t_2) = x_0, \\ y(t_1) &= y(t_2) = y_0. \end{aligned}\right\} \qquad ...(3)$$

**Solution :** We wand to find the function for which equation (1) is extremum w. r. t. the functions x(t), y(t) satisfying the conditions (2) and (3). We know the conditions that the integral (1) is extremum under (2) if the following Euler –Lagrange's equations are satisfied.

$$\frac{\partial f^*}{\partial x} - \frac{d}{dt}\left(\frac{\partial f^*}{\partial \dot{x}}\right) = 0 , \qquad ...(4)$$

$$\frac{\partial f^*}{\partial y} - \frac{d}{dt}\left(\frac{\partial f^*}{\partial \dot{y}}\right) = 0 , \qquad ...(5)$$

where

$$f^* = f + \lambda g ,$$

$$f^* = \frac{1}{2}(x\dot{y} - y\dot{x}) + \lambda\sqrt{\dot{x}^2 + \dot{y}^2} . \qquad ...(6)$$

Hence the equations (4) and (5) reduce to

$$\frac{\dot{y}}{2} - \frac{d}{dt}\left(-\frac{y}{2} + \frac{\lambda \dot{x}}{\sqrt{\dot{x}^2 + \dot{y}^2}}\right) = 0$$

$$-\frac{\dot{x}}{2}-\frac{d}{dt}\left(\frac{x}{2}+\frac{\lambda\dot{y}}{\sqrt{\dot{x}^2+\dot{y}^2}}\right)=0.$$

Integrating these equations w. r. t. t we get

$$\frac{y}{2}-\left(-\frac{y}{2}+\frac{\lambda\dot{x}}{\sqrt{\dot{x}^2+\dot{y}^2}}\right)=a$$

$$\Rightarrow \quad y-a=\frac{\lambda\dot{x}}{\sqrt{\dot{x}^2+\dot{y}^2}}$$

and

$$\frac{x}{2}+\left(\frac{x}{2}+\frac{\lambda\dot{y}}{\sqrt{\dot{x}^2+\dot{y}^2}}\right)=b,$$

$$\Rightarrow \quad x-b=\frac{\lambda\dot{y}}{\sqrt{\dot{x}^2+\dot{y}^2}}.$$

Squaring and adding above equations we get

$$(x-b)^2+(y-a)^2=\lambda^2. \qquad \ldots (7)$$

This is a circle of radius $\lambda$ and centered at (b, a). Thus the closed curve for which the enclosed area is maximum is a circle. The length of the circle is $2\pi\lambda=l \Rightarrow \quad \lambda=\frac{l}{2\pi}$. This gives the radius of the curve.

**Exercise:**

1. Show that the shortest distance between two points along the curve $x=x(t), \quad y=y(t)$ in a Euclidean plane is a straight line.
2. Show that the geodesic defined by $r=r(t), \theta=\theta(t)$ in a plane is a straight line.
3. Show that the stationary (extremum) distance between two points along the curve $\theta=\theta(t), \quad \phi=\phi(t)$ on the sphere $x=r\sin\theta\cos\phi, \quad y=r\sin\theta\sin\phi, \quad z=r\cos\theta$ is an arc of the great circle.

4. Find the geodesic on the surface obtained by generating the parabola $y^2 = 4ax$ about x-axis.

**Ans.:** The surface of revolution is $y^2 + z^2 = 4ax$, whose parametric representation is

$$x = au^2, \quad y = 2au\sin v, \quad z = 2au\cos v$$

The geodesic on this surface is obtain by solving the integral

$$v = c_1 \int \frac{\sqrt{1+u^2}}{u\sqrt{u^2 - c_1^2}} du + c_2 .$$

5. Derive the Euler-Lagrange's equations that are to be satisfied by twice differential functions $x(t),\ y(t), ..., z(t)$, that extremize the integral

$$I = \int_{t_1}^{t_2} f\left(x, y, ..., z, \dot{x}, \dot{y}, ..., \dot{z}, t\right) dt, \quad \dot{x} = \frac{dx}{dt}$$

which achieve prescribed values at the fixed points $t_1, t_2$.

**Ans :** $\frac{\partial f}{\partial x} - \frac{d}{dt}\left(\frac{\partial f}{\partial \dot{x}}\right) = 0, \quad \frac{\partial f}{\partial y} - \frac{d}{dt}\left(\frac{\partial f}{\partial \dot{y}}\right) = 0, ..., \quad \frac{\partial f}{\partial z} - \frac{d}{dt}\left(\frac{\partial f}{\partial \dot{z}}\right) = 0,$

6. Find the curve which generates a surface of revolution of minimum area when it is revolved about x –axis.

**Ans :** Area of revolution of a curve about x-axis is

$$I = \int_{x_0}^{x_1} y\sqrt{1 + y'^2}\, dx .$$

The curve is a catenary given by $y = c\sec\psi,\ or\ y = a\cosh\left(\frac{x-b}{a}\right)$.

7. Find the function on which the functional can be extremized

$$I\left[y(x)\right] = \int_0^1 \left(y''^2 - 2xy\right) dx,\ y(0) = 0,\ y(1) = \frac{1}{120},$$

and $y'$ is not prescribed at both the ends.

**Ans:** $y=\dfrac{x^5}{120}-\dfrac{x^3}{36}+\dfrac{x}{36}$.

8. Find the stationary function of $\int_0^4 \left(xy'-y'^2\right)dx$, which is determined by the boundary conditions $y(0)=0, y(4)=3.$

**Ans :** $y=\dfrac{x^2}{4}-\dfrac{x}{4}$.

9. Find the extremum of $I\left[y(x)\right]=\int_1^2 \dfrac{x^3}{y'^2}dx, y(1)=1, y(2)=4.$

**Ans:** $y=x^2$.

10. Find the extremal of $I\left[y(x)\right]=\int_{x_0}^{x_1} \dfrac{y'^2}{x^3}dx.$

**Ans:** $y=c_1\dfrac{x^4}{4}+c_2$.

11. Find the extremal of $\int_0^{\pi}\left(y'^2-y^2+4y\cos x\right)dx, \quad y(0)=0, y(\pi)=0.$

**Ans:** $y=\left(c_2+x\sin x\right)$.

12. Find the extremal of the functional

$$\int_0^1\left(y'^2-12xy\right)dx, \quad y(0)=1, y(1)=2.$$

**Ans:** $y=-x^3+2x+1$.

13. Determine the curve $z=z(x)$ for which the functional

$$I=\int_1^2\left(z'^2-2xz\right)dx, z(1)=0, z(2)=-1 \text{ is extremal.}$$

**Ans:** $z=-\dfrac{x^3}{3}+\dfrac{x}{6}$.

14. Determine the curve $z = z(x)$ for which the functional

$$I = \int_0^{\frac{\pi}{2}} \left(z'^2 - z^2\right) dx, \quad z(0) = 0, z\left(\frac{\pi}{2}\right) = 1$$

is extremal.

**Ans:** $z = \sin x$.

15. Find the extremal of the functional

$$I\left[y(x)\right] = \int_0^{\frac{\pi}{2}} \left(y''^2 - y^2 + x^2\right) dx, \quad y(0) = 1, y\left(\frac{\pi}{2}\right) = 0, y'(0) = 0, y'\left(\frac{\pi}{2}\right) = -1.$$

**Ans :** $y = \cos x.$.

16. Obtain the differential equation in which the extremizing function makes the integral

$$I\left(y(x)\right) = \int_{x_1}^{x_2} f\left(x, y, y'\right) dx$$

extremum subject to the conditions $y(x_1) = y_1, \quad y(x_2) = y_2$, and

$$J_k = \int_{x_1}^{x_2} g_k\left(x, y, y'\right) dx = const. \quad k = 1, 2, \ldots, n$$

**Ans :** The problem would be extrmization of

$$I^* = \int_{x_1}^{x_2} f^*\left(x, y, y'\right) dx,$$

where $I^* = I + \sum_{k=1}^{n} \lambda_k J_k$ and $f^* = f + \sum_{k=1}^{n} \lambda_k g_k$.

The required differential equation is $\frac{\partial f^*}{\partial y} - \frac{d}{dx}\left(\frac{\partial f^*}{\partial y'}\right) = 0$.

17. Find the extremals of the isoperimetric problem

$$I = \int_{x_0}^{x_1} y'^2 dx, \; s.t. \quad \int_{x_0}^{x_1} y dx = c.$$

**Ans:** $y = \lambda \frac{x^2}{4} + ax + b$.

18. Find the extremal of the functional

$$I[y(x)] = \int_0^1 (1 + y''^2) dx, \quad y(0) = 0, y(1) = 1, y'(0) = 1, y'(1).$$

**Ans:** $y = x$.

19. Find the extremal of the functional

$$I[y(x), z(x)] = \int_{x_0}^{x_1} (2yz - 2y^2 + y'^2 - z'^2) dx.$$

**Ans:** $y = (c_1 x + c_2)\cos x + (c_3 x + c_4)\sin x, \; z = 2y + y''$.

20. Find the extremal of the functional

$$I[y(x)] = \int_{x_0}^{x_1} (y^2 + y'^2 + 2ye^x) dx.$$

**Ans:** $y = \frac{xe^x}{2} + c_1 e^x + c_2 e^{-x}$.